Uniqueness for a wave propagation inverse problem in a half space

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Abstract

This paper considers an inverse problem for wave propagation in a perturbed, dissipative half-space. The perturbation is assumed to be compactly supported. This paper shows that in dimension three, the perturbation is uniquely determined by knowledge of the Dirichlet-to-Neumann map on an open subset of the boundary.

1 Introduction

In this paper, we consider an inverse problem for wave propagation modeled by the equation

$$(\nabla^2 + q(x))u(x) = 0, \tag{1}$$

where x is in $\mathbf{R}^3$ and q can be complex-valued. This equation arises in the propagation of electromagnetic [1], [3] and acoustic [5], [8] waves.

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We consider the problem in the half-space $x_3 < 0$. In the lower half-space, $q(x)$ differs from the constant q_- only in a region of compact support. Here the imaginary part of $q(x)$ and q_- are assumed to be non-negative.

For this wave propagation problem we consider an inverse boundary value problem for the lower half-space. In particular, we specify Dirichlet data on the top surface:

$$u|_{x_3=0} = f, \quad (2)$$

together with an outgoing radiation condition in the lower half-space. We assume that the boundary data f is in the Sobolev space $H^{1/2}$ defined by

$$H^s = \{f : \int |\hat{f}(\xi)|^2 (1 + |\xi|^2)^{s/2} d\xi < \infty\}.$$

For boundary data in this space, it is known that the boundary value problem (1), (2), together with radiation conditions, has a unique solution [1].

Thus the normal derivative $\partial u / \partial x_3$ on the surface $x_3 = 0$ is uniquely determined. The mapping from $H^{1/2}$ to $H^{-1/2}$

$$\Lambda : u|_{x_3=0} \mapsto \frac{\partial u}{\partial x_3} \Big|_{x_3=0} \quad (3)$$

is called the Dirichlet-to-Neumann map. Such maps have been used a great deal recently in the study of inverse problems [13], [7], [10], [9], [11]. Knowledge of the Dirichlet-to-Neumann map is equivalent, in a certain sense, to scattering data [1].

The inverse boundary value problem is to determine $q(x)$ in the lower half-space from knowledge of Λ . In the case in which the Dirichlet-to-Neumann map is defined on the boundary of a compact region, it is known [13] that knowledge of Λ uniquely determines $q(x)$. The purpose of this paper is to extend this result to the half-space geometry. For the case in which q is purely real, the half-space uniqueness question was studied in the unpublished manuscripts [14] and [4]. Here we consider the complex case.

2 The theorem and proof

We consider two equations of the form (1). We denote the two q s by q_1 and q_2 , and we will use subscripts to denote the corresponding associated solutions, Green's functions, Dirichlet-to-Neumann maps, etc..

We assume that the general region of space containing the possible perturbations q_1 and q_2 is known. We denote by B a large open set containing the supports of q_1 and q_2 . We also assume that the perturbation is strictly contained in the lower half-space, so that the closure of B does not intersect the boundary $x_3 = 0$.

The proof of the theorem will make use of the Dirichlet Green's function, which can be defined as follows by the method of images. We use a tilde to denote the image point, so that $\tilde{y}$ is the point obtained by reflecting y across the $y_3 = 0$ plane. First we recall that the free-space outgoing Green's function corresponding to the medium parameters in the lower half-space is

$$G^0(x, y) = \frac{e^{i\sqrt{q_-}|x-y|}}{4\pi|x-y|}. \quad (4)$$

The full-space outgoing Green's function in the presence of a perturbation in the medium is then the unique solution [1] of the Lippmann-Schwinger equation $G = G^0 + \int G^0(q - q_-)G$. For a source located at the image point $\tilde{y}$, the corresponding quantities are $\tilde{G}^0(x, y) = G^0(x, \tilde{y})$ and $\tilde{G}$, which satisfies $\tilde{G} = \tilde{G}^0 + \int G^0(q - q_-)\tilde{G}$. Finally we define the Dirichlet Green's function G^D to be the difference of the perturbed free-space Green's functions: $G^D = G - \tilde{G}$. We note that $\tilde{G}^D$ satisfies the integral equation

$$G^D(x, y) = G^0(x, y) - \tilde{G}^0(x, y) + \int G^0(x, z)(q(z) - q_-)G^D(z, y) dz \quad (5)$$

This Dirichlet Green's function automatically satisfies the condition

$$G^D(x, y)|_{x_3=0} = 0. \quad (6)$$

Moreover, from equation (5) it follows that Green's function $G^D(\cdot, y)$ satisfies the radiation condition

$$\lim_{|x| \rightarrow \infty} |x| \left(\frac{\partial}{\partial |x|} - ik \right) G^D(x, y) = 0. \quad (7)$$

Proposition 2.1 *The kernel of the Dirichlet-to-Neumann map Λ is*

$$\frac{\partial}{\partial x_3} \frac{\partial}{\partial y_3} G^D(x, y)|_{x_3=0, y_3=0}. \quad (8)$$

Proof. We apply the divergence theorem and radiation condition (7) to $\int(u\nabla^2 G^D - G^D \nabla^2 u) = u$, where u denotes a solution of (1), (2), with radiation conditions in the lower half-space. $\square$

We denote by R_-^3 the lower half-space $\{x : x_3 < 0\}$, and by A the set $R_-^3 \setminus \overline{B}$. In the following we say that $\Lambda_1 = \Lambda_2$ on some open subset Γ of the boundary $x_3 = 0$ if the kernels of the operators coincide on $\Gamma \times \Gamma$, i.e.,

$$\frac{\partial}{\partial x_3} \frac{\partial}{\partial y_3} G_1^D(x, y) \Big|_{x_3=0, y_3=0} = \frac{\partial}{\partial x_3} \frac{\partial}{\partial y_3} G_2^D(x, y) \Big|_{x_3=0, y_3=0} \quad (9)$$

for x and y in Γ .

Theorem 2.1 *Suppose the set B containing the supports of $q_1 - q_-$ and $q_2 - q_-$ is strictly contained in the lower half-space. If $\Lambda_1 = \Lambda_2$ on some open subset Γ of the boundary $x_3 = 0$, then $q_1 = q_2$.*

Proof. The proof involves a series of five lemmas. The first two show that knowledge of the Dirichlet-to-Neumann map suffices to determine the Dirichlet Green's function outside the perturbation in q . The third and fourth lemmas establish a version of the Green's theorem identity that is often used for uniqueness arguments. The last lemma, which is roughly based on the uniqueness proof in [6], shows that linear combinations of the Dirichlet Green's functions can be used to approximate the Sylvester-Uhlmann solutions.

Lemma 2.2 *Suppose the hypotheses of the theorem are satisfied. Then $\partial_{y_3} G_1^D(x, y) = \partial_{y_3} G_2^D(x, y)$ for y in Γ and x in A .*

Proof. For a fixed y in Γ , we denote by $F_i(x)$ the function $\partial_{y_3} G_i^D(x, y)$ for $i = 1, 2$. Each F_i satisfies the Helmholtz equation $(\nabla^2 + q_-)F_i(x) = 0$ in the set A away from the perturbation. Because $\Lambda_1 = \Lambda_2$ on Γ , from (9) we have that the normal derivatives of F_i coincide there: $\partial_{x_3} F_1(x) = \partial_{x_3} F_2(x)$ for x on Γ . Moreover, by taking y_3 derivatives of both sides of the boundary condition (6) we see that for x on Γ , $F_1(x) = F_2(x) = 0$. Thus F_1 and F_2 satisfy the same unperturbed Helmholtz equation outside $\overline{B}$, and have identical Cauchy data on Γ . The Holmgren uniqueness theorem can be used to show that F_1 and F_2 are identical in a neighborhood of Γ . From this the claim follows from the unique continuation principle [2]. $\square$

Lemma 2.3 *Suppose the hypotheses of the theorem are satisfied. Then $G_1^D(x, y) = G_2^D(x, y)$ for x in A and y in $A \setminus \{x\}$.*

Proof. Let x in A be fixed. Then $G_i^D(x, y)$ satisfies the Helmholtz equation $(\nabla^2 + q_-)G_i^D(x, \cdot) = 0$ in the set $A \setminus \{x\}$. By the boundary condition (6) and reciprocity, $G_i^D(x, y) = 0$ for y in Γ . By Lemma 2.2, $\partial_{y_3} G_1^D(x, y) = \partial_{y_3} G_2^D(x, y)$ for y in Γ . Thus G_1^D and G_2^D satisfy the homogeneous Helmholtz equation in $A \setminus \{x\}$ and have the same Cauchy data on Γ . Again, by the Holmgren uniqueness theorem $G_1^D(x, \cdot)$ and $G_2^D(x, \cdot)$ are identical in a neighborhood of Γ . Thus by the unique continuation principle they coincide everywhere outside $\overline{B} \cup \{x\}$. $\square$

Lemma 2.4 *For x in B , and y and z in A ,*

$$\begin{aligned} & \int_B (q_i - q_j)(x) G_i^D(x, y) G_j^D(x, z) dx \\ &= \int_{\partial B} (G_i^D(x, y) \partial_{\nu_x} G_j^D(x, z) - G_j^D(x, z) \partial_{\nu_x} G_i^D(x, y)) dS_x, \end{aligned}$$

where ∂_{ν_x} denotes differentiation with respect to the outward unit normal to B .

Proof. This follows from the divergence theorem. $\square$

Lemma 2.5 *Suppose the hypotheses of the theorem are satisfied. Then for x in B , and y and z in A ,*

$$\int_B (q_1 - q_2)(x) G_1^D(x, y) G_2^D(x, z) dx = 0. \quad (10)$$

Proof. From Lemma 2.4 with $i = 1, j = 2$ we have

$$\begin{aligned} & \int_B (q_1 - q_2)(x) G_1^D(x, y) G_2^D(x, z) dx \\ &= \int_{\partial B} (G_1^D(x, y) \partial_{\nu_x} G_2^D(x, z) - G_2^D(x, z) \partial_{\nu_x} G_1^D(x, y)) dS_x. \end{aligned} \quad (11)$$

However, by Lemma 2.3, $G_1^D = G_2^D$. By applying Lemma 2.4 in the case $i = j$ see that the right hand side of (11) vanishes and thus we obtain (10). $\square$

Lemma 2.6 *The span of the set $U = \{G^D(\cdot, y)|_B : y \text{ in an open subset of } A\}$ is dense in the set $V = \{u|_B : (\nabla^2 + q)u = 0\}$.*

Proof. Suppose there exists an f , supported in B , that is orthogonal to all G^D in U , i.e., $\int_B G^D(x, y)f(x)dx = 0$ for all y in an open subset of A . Let $w(y) = \int_B G^D(x, y)f(x)dx$. Then $(\nabla^2 + q)w = f$. By assumption, w is identically zero in an open subset of A , and by unique continuation [2], must be zero everywhere in A .

This enables us to show as follows that f must also be orthogonal to every solution u in V :

$$\int_B uf = \int_B u(\nabla^2 + q)w = \int_B w(\nabla^2 + q)u = 0.$$

Here the second equality follows because w vanishes outside B , which implies the vanishing of the boundary terms arising from integrating by parts. $\square$

The proof of the theorem concludes with the observation that Lemma 2.6 guarantees that we can use linear combinations of the functions G^D to approximate the Sylvester-Uhlmann solutions [13], which are of the form

$$\psi(x, \zeta) = e^{i\zeta \cdot x} + O(1/|\zeta|)$$

for complex vectors ζ such that $\zeta \cdot \zeta = q_-$. Then for any real ξ , we use G_j^D s to approximate $\psi_j = \psi(x, \zeta_j)$ with $\zeta_1 + \zeta_2 = \xi$, and thus on the left side of (10) obtain the Fourier transform of $q_1 - q_2$. Since this Fourier transform can be made arbitrarily close to zero, $q_1 = q_2$. $\square$

References

- [1] M. Cheney and D. Isaacson, “Inverse Problems for a Perturbed Dissipative Half-Space”, *Inverse Problems* 11, 856–888 (1995).
- [2] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer, New York, 1992.
- [3] J.D. Jackson, *Classical Electrodynamics*, 2nd Ed., Wiley, New York, 1975.

- [4] G. Eskin and J. Ralston, “Inverse coefficient problems in perturbed halfspaces”, preprint, 1997.
- [5] A. Kirsch, “An Introduction To The Mathematical Theory Of Inverse Problems”, Springer, New York, 1996.
- [6] R. Melrose, “Geometric Scattering Theory”, Cambridge University Press 1995.
- [7] A. Nachman, “Reconstructions from boundary measurements,” *Annals of Math.* **128** (1988), pp. 531-556.
- [8] F. Natterer and F. Wübbeling, “A propagation-backpropagation method for ultrasound tomography”, *Inverse Problems* 11, 1225–1232 (1995).
- [9] P. Ola, L. Päiväranta, and E. Somersalo, “An inverse boundary value problem in electrodynamics”, *Duke Math. J.* 70 (1993) 617–653.
- [10] E. Somersalo, M. Cheney, D. Isaacson, and E. Isaacson, “Layer Stripping: A Direct Numerical Method for Impedance Imaging,” *Inverse Problems*, 7 (1991) 899–926 ; 8 (1992), 493.
- [11] E. Somersalo, “Layer stripping for time-harmonic Maxwell’s equations with fixed frequency”, *Inverse Problems* 10 (1994) 449–466.
- [12] J. Sylvester, “A convergent layer stripping algorithm for the radially symmetric impedance tomography problem”, to appear, *Comm. Partial Diff. Eqs.*
- [13] J. Sylvester and G. Uhlmann, “A uniqueness theorem for an inverse boundary value problem in electrical prospection”, *Comm. Pure Appl. Math.* 39 (1986) 91-112; “A global uniqueness theorem for an inverse boundary value problem in electrical prospection”, *Ann. of Math.*, 125 (1987) 153-169; “Inverse boundary value problems at the boundary—continuous dependence”, *Comm. Pure Appl. Math.* 41 (1988) 197–221.
- [14] R. Weder, “Uniqueness theorem for an inverse problem in geophysics”, preprint, Universidad Nacional Autónoma de México.