ASYMPTOTIC PROBLEMS IN DISTRIBUTED SYSTEMS

BY

J.L. LIONS

IMA Preprint Series # 147

May 1985

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455

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#	Author(s) Title
1	**Workshop** Summaries from the September 1982 Workshop on Statistical Mechanics, Dynamical Systems and Turbulence
2	**Raphael De laLlave**, A Simple Proof of C. Siegel's Center Theorem
3	**H. Simpson, S. Spector**, On Copositive Matrices and Strong Ellipticity for Isotropic Elastic Materials
4	**George R. Sell**, Vector Fields in the Vicinity of a Compact Invariant Manifold
5	**Milan Miklavcic**, Non-linear Stability of Asymptotic Suction
6	**Hans Weinberger**, A Simple System with a Continuum of Stable Inhomogeneous Steady States
7	**Bau-Sen Du**, Period 3 Bifurcation for the Logistic Mapping
8	**Hans Weinberger**, Optimal Numerical Approximation of a Linear Operator
9	**L.R. Angel, D.F. Evans, B. Ninham**, Three Component Ionic Microemulsions
10	**D.F. Evans, D. Mitchell, S. Mukherjee, B. Ninham**, Surfactant Diffusion; New Results and Interpretations
11	**Leif Arkeryd**, A Remark about the Final Aperiodic Regime for Maps on the Interval
12	**Luis Magalhaes**, Manifolds of Global Solutions of Functional Differential Equations
13	**Kenneth Meyer**, Tori in Resonance
14	**C. Eugene Wayne**, Surface Models with Nonlocal Potentials: Upper Bounds
15	**K.A. Pericak-Spector**, On Stability and Uniqueness of Fluid Flow Through a Rigid Porous Medium
16	**George R. Sell**, Smooth Linearization Near a Fixed Point
17	**David Wollkind**, A Nonlinear Stability Analysis of a Model Equation for Alloy Solidification
18	**Pierre Collet**, Local C^∞ Conjugacy on the Julia Set for some Holomorphic Perturbations of $z \to z^2$
19	**Henry C. Simpson, Scott J. Spector**, On the Modified Bessel Functions of the First Kind / On Barrelling for a Material in Finite Elasticity
20	**George R. Sell**, Linearization and Global Dynamics
21	**P. Constantin, C. Foias**, Global Lyapunov Exponents, Kaplan-Yorke Formulas and the Dimension of the Attractors for 2D Navier-Stokes Equations
22	**Milan Miklavcic**, Stability for Semilinear Parabolic Equations with Noninvertible Linear Operator
23	**P. Collet, H. Epstein, G. Gallavotti**, Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and their Mixing Properties
24	**J.E. Dunn, J. Serrin**, On the Thermodynamics of Interstitial Working
25	**Scott J. Spector**, On the Absence of Bifurcation for Elastic Bars in Uniaxial Tension
26	**W.A. Coppel**, Maps on an Interval
27	**James Kirkwood**, Phase Transitions in the Ising Model with Traverse Field
28	**Luis Magalhaes**, The Asymptotics of Solutions of Singularly Perturbed Functional Differential Equations: and Concentrated Delays are Different
29	**Charles Tresser**, Homoclinic Orbits for Flow in R^3
30	**Charles Tresser**, About Some Theorems by L.P. Sil'nikov
31	**Michael Aizenmann**, On the Renormalized Coupling Constant and the Susceptibility in ϕ_4^4 Field Theory and the Ising Model in Four Dimensions
32	**C. Eugene Wayne**, The KAM Theory of Systems with Short Range Interactions I
33	**M. Slemrod, J. E. Marsden**, Spatial Chaos in a Van der Waals Fluid Due to Periodic Thermal Fluctuations
34	**J. Kirkwood, C.E. Wayne**, Percolation in Continuous Systems
35	**Luis Magalhaes**, Invariant Manifolds for Functional Differential Equations Close to Ordinary Differential Equations
36	**C. Eugene Wayne**, The KAM Theory of Systems with Short Range Interactions II
37	**Jean De Canniere**, Passive Quasi-Free States of the Noninteracting Fermi Gas
38	**Elias C. Aifantis**, Maxwell and van der Waals Revisited
39	**Elias C. Aifantis**, On the Mechanics of Modulated Structures
40	**William Ruckle**, The Strong ϕ Topology on Symmetric Sequence Spaces
41	**Charles R. Johnson**, A Characterization of Borda's Rule Via Optimization
42	**Hans Weinberger, Kazuo Kishimoto**, The Spatial Homogeneity of Stable Equilibria of Some Reaction-Diffusion Systems on Convex Domains
43	**K.A. Pericak-Spector, W.O. Williams**, On Work and Constraints in Mixtures
44	**H. Rosenberg, E. Toubiania**, Some Remarks on Deformations of Minimal Sufaces
45	**Stephan Pelikan**, The Duration of Transients
46	**V. Capasso, K.L. Cooke, M. Witten**, Random Fluctuations of the Duration of Harvest
47	**E. Fabes, D. Stoock**, The L^P-Intergrability of Green's Functions and Fundamental Solutions for Elliptic and Parabolic Equations
48	**H. Brezis**, Semilinear Equations in R^N without Conditions at Infinity
49	**M. Slemrod**, Lax-Friedrichs and the Viscosity-Capillarity Criterion
50	**C. Johnson, W. Barrett**, Spanning Tree Extensions of the Hadamard-Fischer Inequalities
51	**Andrew Postlewaite, David Schmeidler**, Revelation and Implementation under Differential Information
52	**Paul Blanchard**, Complex Analytic Dynamics on the Riemann Sphere
53	**G. Levitt, H. Rosenberg**, Topology and Differentiability of Labyrinths in the Disc and Annulus
54	**G. Levitt, H. Rosenberg**, Symmetry of Constant Mean Curvature Hyper-surfaces in Hyperbolic Space
55	**Ennio Stacchetti**, Analysis of a Dynamic, Decentralized Exchange Economy
56	**Henry Simpson, Scott Spector**, On Failure of the Complementing Condition and Nonuniqueness in Linear Elastostatics
57	**Craig Tracy**, Complete Integrability in Statistical Mechanics and the Yang-Baxter Equations
58	**Tongren Ding**, Boundedness of Solutions of Duffing's Equation
59	**Abstracts for the Workshop** on Price Adjustment, Quantity Adjustment, and Business Cycles
60	**Rafael Rob**, The Coase Theorem an Informational Perspective
61	**Joseph Jerome**, Approximate Newton Methods and Homotopy for Stationary Operator Equations
62	**Rafael Rob**, A Note on Competitive Bidding with Asymmetric Information
63	**Rafael Rob**, Equilibrium Price Distributions
64	**William Ruckle**, The Linearization Projection, Global Theories
65	**Russell Johnson, Kenneth Palmer, George R. Sell**, Ergodic Properties of Linear Dynamical Systems
66	**Stanley Reiter**, How a Network of Processors can Schedule Its Work
67	**R.N. Goldman, D.C. Heath**, Linear Subdivision is Strictly a Polynomial Phenomenon
68	**R. Glachett, R. Johnson**, The Floquet Exponent for Two-dimensional Linear Systems with Bounded Coefficients
69	**Steve Williams**, Realization and Nash Implementation: Two Aspects of Mechanism Design
70	**Steve Williams**, Sufficient Conditions for Nash Implementation
71	**Nicholas Yannelis, William R. Zame**, Equilibria in Banach Lattices Without Ordered Preferences
72	**W. Harris, Y. Sibuya**, The Reciprocals of Solutions of Linear Ordinary Differential Equations
73	**Steve Pelikan**, A Dynamical Meaning of Fractal Dimension
74	**D. Heath, W. Sudderth**, Continuous-Time Portfolio Management: Minimizing the Expected Time to Reach a Goal
75	**J.S. Jordan**, Information Flows Intrinsic to the Stability Economic Equilibrium
76	**J. Jerome**, An Adaptive Newton Algorithm Based on Numerical Inversion: Regularization Post Condition
77	**David Schmeidler**, Integral Representation Without Additivity

ASYMPTOTIC PROBLEMS IN DISTRIBUTED SYSTEMS

Riviere Memorial Lecture 1985

J.L. Lions

Introduction

Distributed systems are systems governed by Partial Differential Equations; this terminology is classical in the framework of control theory; we use this terminology in order to emphasize the following: we are concerned, in this paper, with some asymptotic questions which arise in connection with the optimal control of distributed systems.

The main difficulty in dealing with problems of optimal control of distributed systems is the complexity; this complexity may be due to the complexity of the materials which constitute the system, or to the complexity of the model, or to the complexity of the geometry etc.

In general these questions are the same for the analysis of problems without control as for the control of the systems. We want here to give some examples where the "control aspect" leads to some slightly unusual questions.

The examples are chosen among those leading to open questions.

The plan is as follows:

1. Composite materials and boundary control.

1.1 Statement of the problem.

Let $\Omega \in \mathbb{R}^3$ (dimension is taken equal 3 to fix ideas) a domain which consists of a composite material which can be modelled as follows.

Let $a_{ij}(\eta)$, i,j = 1,2,3, be a family of (smooth) functions which are y-periodic ($y =]0,1[^3$; a_{ij} admits period 1 in all variables) and which are such that

(1.1)
$$\left| \begin{array}{ll} a_{ij} = a_{ji} & \forall i,j \\ a_{ij}\zeta_i\zeta_j \geq \alpha\zeta_i\zeta_i & \forall \zeta_i \in R, \ \alpha>0 \end{array} \right. \quad (^1) .$$

We consider the system with the state equation given by

(1.2)
$$\frac{\partial^2 y_\varepsilon}{\partial t^2} + A_\varepsilon y_\varepsilon = 0 \qquad \text{in } \Omega \times]0,\tau[\ ,$$

where

(1.3)
$$A_\varepsilon = - \frac{\partial}{\partial x_i}(a_{ij} \ (x/\varepsilon) \ \frac{\partial}{\partial x_j}) \ .$$

The initial conditions are

(1.4)
$$y_\varepsilon(x,0) = \frac{\partial y_\varepsilon}{\partial t} (x,0) = 0 \qquad \text{in } \Omega$$

and one wants to control the system by a boundary control i.e.

(1.5)
$$y_\varepsilon = v \quad \text{on } \Sigma = \Gamma \times]0,T[\ , \ \Gamma = \partial\Omega \ .$$

Let us assume that the cost function is given by

(1.6)
$$J_\varepsilon(v) = \int_\Omega [y_\varepsilon(x,T;v) - z_d(x)]^2 \, dx + N\int_\Sigma v^2 \, d\Sigma$$

where:

z_d is given in $L^2(\Omega)$,

N is given > 0 .

---

(1) We use the summation convention of repeated indices

The structure of (1.6) shows that we have to take

(1.7) $$v \in L^2(\Sigma) .$$

Given v in $L^2(\Sigma)$, (1.2) (1.6) (1.5) admits a unique solution, which is denoted by $y_\varepsilon(x,t;v) = y_\varepsilon(v)$, and it is this function which is used in the first integral in (1.6). But one has to make this precise, cf. section 1.2 below. Assuming for the time being that the formulation (1.6) makes sense, the problem of optimal control is to find

(1.8) $$\inf_{v \in U_{ad}} J_\varepsilon(v) ,$$

where

(1.9) $$U_{ad} = \text{closed convex subset of } L^2(\Sigma) .$$

A few remarks are in order:

Remark 1.1

Given v the computation of $y_\varepsilon(v)$ is - if we do not use asymptotic methods - very complicated due to the rapid oscillations of $a_{ij}(x/\varepsilon)$. □

Remark 1.2

The goal of this section 1 is to seek an asymptotic expansion for

(1.10) $$\mathcal{Y}_\varepsilon = \inf_{v \in U_{ad}} J_\varepsilon(v)$$

as $\varepsilon \to 0$ and the solution $v = u_\varepsilon$ of (1.8). As we will see this question is essentially open! □

1.2 Solution of the problem for ε fixed.

One can show that, given $v \in L^2(\Sigma)$, there exists a unique function $y_\varepsilon(v)$, soution of (1.2) (1.6) (1.5), and which satisfies

(1.11) $$y_\varepsilon(v) \text{ is continuous from } [0,T] \to L^2(\Omega) . \qquad \square$$

The proof (cf. J.L. Lions [1]) is obtained by transposition of a result of regularity .

Let us consider ϕ_ε, the solution of

(1.12) $$\left| \begin{array}{l} \dfrac{\partial^2 \phi_\varepsilon}{\partial t^2} + A_\varepsilon \phi_\varepsilon = f \quad \text{in} \quad \Omega\times]0,T[\, , \\ \phi_\varepsilon(x,0) = \dfrac{\partial \phi_\varepsilon}{\partial t}(x,0) = 0 \quad \text{in} \quad \Omega \, , \\ \phi_\varepsilon = 0 \quad \text{on} \quad \Sigma \, ; \end{array} \right.$$

let us assume that

(4.13) $$f \in L^1(0,T; L^2(\Omega)) .$$

Then there exists a unique solution which satisfies

(4.14) $$\phi_\varepsilon \in C([0,T]; H_0^1(\Omega)) , \ \frac{\partial \phi_\varepsilon}{\partial t} \in C([0,T]; L^2(\Omega)) \qquad (^1) ;$$

---

(1) $C([0,T]; X)$ = continuous functions from $[0,T] \to X$.
$H_0^1(\Omega) = \{\phi\} \ \phi, \frac{\partial \phi}{\partial x_i} \in L^2(\Omega), \ \phi=0 \ \text{ on } \ \Gamma \}$.

this is classical; but an interesting regularity result is that [1]

(1.15) $$\frac{\partial \phi_\varepsilon}{\partial \nu_{A_\varepsilon}} \in L^2(\Sigma) ,$$

where $\frac{\partial}{\partial \nu_{A_\varepsilon}}$ stands for the normal derivative associated to A_ε .

Remark 1.3

The main difficulty for what follows is that the estimate

(1.16) $$\| \frac{\partial \phi_\varepsilon}{\partial \nu_{A_\varepsilon}} \|_{L^2(\Sigma)} < C(\varepsilon) \|f\|_{L^1(0,T;L^2(\Omega))}$$

contains a constant $C(\varepsilon)$ which increases as $1/\varepsilon$ as $\varepsilon \to 0$

($C(\varepsilon)$ depends on $\frac{\partial}{\partial x_k}(a_{ij}(x/\varepsilon))$) □

If we return to problem (1.8) one sees that, by virtue of (1.11), $v \to J_\varepsilon(v)$ is continuous from $L^2(\Sigma) \to \mathbb{R}$, so that there exists a unique element $u_\varepsilon \in U_{ad}$ such that

(1.17) $$J_\varepsilon(u_\varepsilon) = \inf \cdot J_\varepsilon(v) , v \in U_{ad} .$$

This is the optimal control which is characterized by the optimality system- which can be written as follows:

---

(1) cf. I. Lasiecka, J.L. Lions, R. Triggiani [1] for other results along these lines.

$$(1.18)\quad \begin{cases} \dfrac{\partial^2 y_\varepsilon}{\partial t^2} + A_\varepsilon y_\varepsilon = 0 \\ \dfrac{\partial^2 p_\varepsilon}{\partial t^2} + A_\varepsilon p_\varepsilon = 0 \quad \text{in} \quad \Omega\times]0,T[\ , \\ y_\varepsilon(x,0) = \dfrac{\partial y_\varepsilon}{\partial t}(x,0) = 0 \ , \\ p_\varepsilon(x,T) = 0 \ , \ \dfrac{\partial p_\varepsilon}{\partial t}(x,T) = y_\varepsilon(x,T) - z_d(x) \quad \text{in} \quad \Omega \ , \\ y_\varepsilon = u_\varepsilon \ \text{on} \ \Sigma \ , \ p_\varepsilon = 0 \ \text{on} \ \Sigma \end{cases}$$

and

$$(1.19)\qquad \int_\Sigma \left(\frac{\partial p_\varepsilon}{\partial \nu_{A_\varepsilon}} + Nu\right)(v-u)\,d\Sigma \geq 0 \quad \forall\, v \in U_{ad} \ .$$

<u>Remark</u> 1.4

The "adjoint state" p_ε is given by the solution of the backward wave equation, with $\frac{\partial p_\varepsilon}{\partial t}(x,T) = y_\varepsilon(x,T) - z_d(x) \in L^2(\Omega)$; then $\frac{\partial p_\varepsilon}{\partial \nu_{A_\varepsilon}} \in L^2(\Sigma)$ so that the integrals in (1.19) make sense. □

The main question is now: <u>is it possible to simplify the problem by using asymptotic expansions</u>?

<u>Remark 1.5</u>

There are many situations where this is indeed possible, as shown in J.L. Lions [2] [3]. We have chosen here to present an (apparently) tricky situation.

□

<u>1.3 Homogenization theory</u>.

Let us return to problem (1.12), where f is given <u>fixed</u>. Then, <u>as</u> $\varepsilon \to 0$, one has:

$$(1.20)\qquad \left|\begin{array}{l} \phi_\varepsilon \to \phi \quad \text{in} \quad L^\infty(0,T;H_0^1(\Omega)) \text{ weak star } \quad (^1) \\ \\ \dfrac{\partial \phi_\varepsilon}{\partial t} \to \dfrac{\partial \phi}{\partial t} \quad \text{in} \quad L^\infty(0,T\ ;\ L^2(\Omega)) \text{ weak star } \quad , \end{array}\right.$$

where ϕ is the solution of

$$(1.21)\qquad \left|\begin{array}{l} \dfrac{\partial^2 \phi}{\partial t^2} + \mathcal{A}\phi = f \quad \text{in } \Omega\times]0,T[\ , \\ \\ \phi(x,0) = \dfrac{\partial \phi}{\partial t}(x,0) = 0 \quad \text{in } \Omega\ , \\ \\ \phi = 0 \quad \text{on} \quad \Sigma\ ; \end{array}\right.$$

in (1.21) $\mathcal{A}$ is the homogenized operator, which is given by

$$(1.22)\qquad \mathcal{A} = -\, q_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\ ;$$

the q_{ij}' s are constants - the effective coefficients - which are given by explicit (constructive) formulas. We refer to E. Sanchez-Palencia [1], A. Bensoussan, J.L. Lions and G. Papanicolaou [1] for the formulas.

Remark 1.6

The homogenized operator $\mathcal{A}$ is elliptic . □

Remark 1.7

The coefficients q_{ij} do not depend on Ω but only on the material. There are codes to compute the q_{ij}'s . □

$(^1)$ I.e. $\int_0^T (\phi_\varepsilon,\psi)_{H_0^1} g(t)dt \to \int_0^T (\phi,\psi)_{H_0^1} g(t)\, dt \quad \forall\psi \in H_0^1(\Omega),\ \forall g \in L^1(0,T).$

1.4. A natural conjecture....

At this stage, it is very natural to introduce the "homogenized control problem", defined as follows. The state equation is given by

(1.23) $$\frac{\partial^2 y}{\partial t^2} + \mathcal{A} y = 0 \quad \text{in} \quad \Omega \times]0,T[\ ,$$

(1.24) $$y(x,0) = \frac{\partial y}{\partial t}(x,0) = 0 \quad \text{in} \quad \Omega \ ,$$

(1.25) $$y = v \quad \text{on} \quad \Sigma = \Gamma \times]0,T[\ .$$

This system admits a unique solution $y(v)$, which is continuous from $[0,T] \to L^2(\Omega)$; we consider the cost function

(1.26) $$\mathcal{J}(v) = \int_\Omega [y(x,T;v) - z_d(x)]^2 \, dx + N \int_\Sigma v^2 d\Sigma$$

and the problem

(1.27) $$\inf \mathcal{J}(v) \quad , \quad v \in U_{ad} \ .$$

It seems natural to conjecture that

(1.28) $$\inf_v J(v) \underset{\varepsilon \to 0}{\to} \inf \mathcal{J}(v) \ , \ v \in U_{ad}$$

and that, if u_ε denotes the unique solution of (1.27), then

(1.29) $$u_\varepsilon \to u \quad \text{in} \quad L^2(\Sigma) \quad \text{weakly} \ . \qquad \square$$

Remark 1.8.

Of course problem (1.27) is significantly simpler than the initial problem, since $\mathcal{A}$ is much simpler than A_ε. This is the interest of homogenization theory! $\square$

Remark 1.9

Of course one can raise similar problems in more general situations where we have non periodical structures cf. "Abstracts of the Workshop on Homogenization", I.M.A., University of Minnesota, #115, November 1984. □

Remark 1.10

The difficulty arises from Remark 1.3 . It is not known, whether or not, for fixed v, $y_\varepsilon(v) \to y(v)$ (in, say, $L^\infty(0,T;\, L^2(\Omega))$ weakly star), where $y(v)$ is the solution of (1.23) (1.24)(1.25). □

Remark 1.11

In this direction we also mention the following open question: let ϕ_ε be the solution of the stationary problem

$$(1.30) \qquad \left| \begin{array}{l} A_\varepsilon \phi_\varepsilon = 0 \quad \text{in } \Omega , \\ \phi_\varepsilon = g \quad \text{on } \Gamma \end{array} \right.$$

where g is given in $L^2(\Gamma)$. Is it true that $\phi_\varepsilon \to \phi$ in $L^2(\Omega)$ weakly, where ϕ is the solution of

$$(1.31) \qquad \left| \begin{array}{l} \mathcal{A} \phi = 0 \quad \text{in } \Omega , \\ \phi = g \quad \text{on } \Gamma . \end{array} \right.$$

□

Remark 1.12

Similar questions will arise for parabolic systems, with boundary control in $L^2(\Sigma)$. □

2. A thin domain

2.1. A preliminary problem.

Let Ω be an open set in $\mathbb{R}^n$ with boundary

$$\partial\Omega = \Gamma_0 \cup \Gamma_1$$

as represented on Fig. 1.

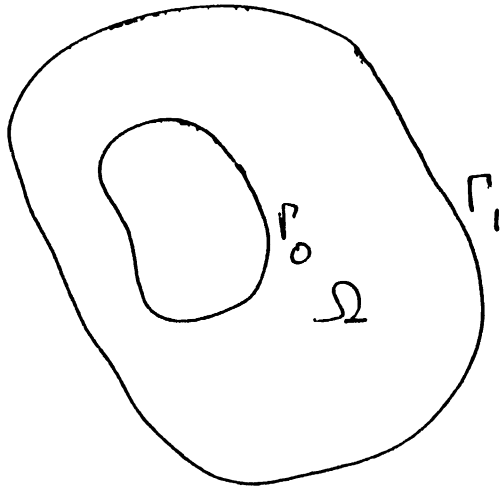

Fig. 1

We are interested in the problem

(2.1) $\Delta^2 u = f$ in Ω ,

(2.2)
$$\left| \begin{array}{l} u = \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0 , \\ \Delta u = \dfrac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma_1 . \end{array} \right.$$

A natural approach is to introduce the space

(2.3) $$E_0 = \{\phi | \ \phi \in C^2(\bar{\Omega}), \ \phi = \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Gamma_0\} ;$$

we provide E_0 with

(2.4) $$\|\phi\|_E = \|\Delta\phi\|_{L^2(\Omega} ;$$

this defines a norm on E_0 , since if $\Delta\phi = 0$ in Ω and $\phi = \frac{\partial\phi}{\partial\nu} = 0$ on Γ_0 then $\phi = 0$ in Ω . We then introduce

(2.5) E = completion of E_0 for $\|\phi\|_E$.

Then if $v \to (f,v)$ defines a continuous linear form on E , problem (2.1)(2.2) admits a unique solution in E , defined by

(2.6) $$(\Delta u, \Delta v) = (f,v) \ \ \forall \, v \in E .$$ □

A question which does not seem to be settled is the following: what are the properties of $u \in E$ near Γ_1 ? Another form of the same question is: what are the properties needed on f near Γ_1 in order for $v \to (f,v)$ to define a continuous linear form on E ? □

We are now going to consider a problem of this type in a thin structure.

2.2 An asymptotic problem in a thin domain

Let $\mathcal{O} \subset \mathbb{R}^2$, $\partial\mathcal{O} = S$,

$\Omega_\varepsilon = \mathcal{O} \times]0,\varepsilon[\in \mathcal{O}$

$x' = \{x_1, x_2\} \in \mathcal{O}$,

$x_3 = \ ]0,\varepsilon[$,

$S_\varepsilon = S \times]0,\varepsilon[$.

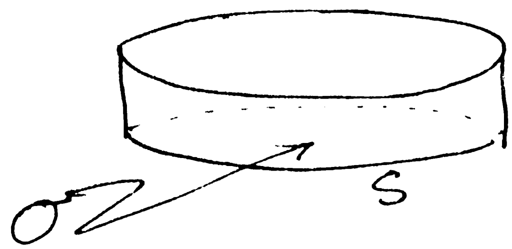

Fig. 2

In Ω_ε we consider the problem

$$\Delta^2 u_\varepsilon = f \quad \text{in} \quad \Omega_\varepsilon \tag{2.7}$$

$$u_\varepsilon = \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on} \quad S_\varepsilon \tag{2.8}$$

$$\left| \begin{array}{l} \Delta u_\varepsilon = \dfrac{\partial}{\partial \nu} \Delta u_\varepsilon = 0 \quad \text{on} \quad \partial_\pm \Omega_\varepsilon \quad , \text{ where} \\ \partial_+ \Omega_\varepsilon = \mathcal{O} \times \{\varepsilon\}, \quad \partial_- \Omega_\varepsilon = \mathcal{O} \times \{0\} \ . \end{array} \right. \tag{2.9}$$

Remark 2.1

We are going to work in a formal fashion; problem (2.7)(2.8)(2.9) is a variant of the problem considered in Section 2.1 and in order this problem to make sense it seems that f should satisfy "some conditions" near $x_3 = 0$ and near $x_3 = \varepsilon$.

Our goal is to have some kind of indication on the conditions by a (formal) asymptotic expansion.

2.3 An ansatz.

We introduce

(2.10) $$y = x_3/\varepsilon \ ;$$

in the new variables x', y, Ω_ε is replaced by

$$\Omega_1 = 0 \times \,]0,1[\ ;$$

we look for u_ε in the form

(2.11) $$u_\varepsilon = u_0 + \varepsilon^2 u_1 + \varepsilon^4 u_2 + \dots ,$$

where

$$u_j = u_j(x', y) \text{ is defined in } \Omega_1$$

and where at the end of the computation we replace y by x_3/ε, and with $u_j = \frac{\partial u_j}{\partial \nu} = 0$ if $x' \in S = \partial 0$.

We set

$$\Delta' = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} , \ D = \partial/\partial y \ .$$

Then (2.7) becomes

(2.12) $$(\varepsilon^{-2}D^2 + \Delta')^2 (u_0 + \varepsilon^2 u_1 + \dots) = f_\varepsilon = f(x', \varepsilon y)$$

and the boundary conditions (2.9) become

(2.13) $$\begin{aligned} &(\varepsilon^{-2}D^2 + \Delta')(u_0 + \varepsilon^2 u_1 + \dots) = 0 \quad \text{for} \quad y = 0 , 1 \\ &(\varepsilon^{-2}D^3 + D\Delta')(u_0 + \varepsilon^2 u_1 + \dots) = 0 \quad \text{for} \quad y = 0, 1. \end{aligned}$$

Identifying in (2.12) the powers of ε gives

$$(2.14)\quad \left| \begin{array}{l} D^4 u_0 = 0 , \\ D^4 u_1 + 2D^2 \Delta' u_0 = 0 , \\ D^4 u_2 + 2D^2 \Delta' u_1 + \Delta'^2 u_0 = f_\varepsilon \end{array} \right.$$

and the boundary conditions are

$$(2.15)\quad \left| \begin{array}{l} D^2 u_0 = 0 , \ D^3 u_0 = 0 \quad \text{for } y=0 , 1 \\ D^2 u_1 + \Delta' u_0 = 0 , \ D^3 u_1 + D\Delta' u_0 = 0 \ \text{ for } y = 0 , 1 \\ D^2 u_2 + \Delta' u_1 = 0 , \ D^3 u_2 + D\Delta' u_1 = 0 \ \text{ for } y = 0, 1, \end{array} \right.$$

It follows from $(2.16)_1$ $(2.15)_1$ that

$$(2.16)\quad D^2 u_0 = 0 \quad \text{in } \mathcal{O}\times]0,1[,$$

i.e.,

$$(2.17)\quad u_0 = u_0(x') + yw_0(x').$$

Then $(2.16)_2$ $(2.15)_2$ reduces to

$$D^4 u_1 = 0 , \quad D^2 u_1 + \Delta' u_0 = 0 \ \text{ for } y = 0,1,$$

$$D^3 u_1 + D\Delta' u_0 = 0 \ \text{ for } y = 0,1$$

i.e.

$$(2.18)\quad D^2 u_1 + \Delta' u_0 = 0 \ \text{ in } \mathcal{O} \times]0,1[$$

i.e.

$$(2.19)\quad u_1 + \Delta'(\frac{y^2}{2} v_0 + \frac{y^3}{6} w_0) = v_1(x') + yw_1(x').$$

Equations $(2.16)_3$ $(2.15)_3$ reduce to

$$(2.20)\quad \left| \begin{array}{l} D^4 u_2 - \Delta'^2 u_0 = f_\varepsilon(x',y) \\ D^2 u_2 + \Delta' u_1 = 0 , \ D^3 u_2 + D\Delta' u_1 = 0 \ \text{ for } y=0, 1. \end{array} \right.$$

In order (2.20) to admit a solution u_2 one has to have compatibility conditions which are obtained by writing that

$$\int_0^1 (D^4 u_2 - \Delta'^2 u_0) dy = \int_0^1 f_\varepsilon(x',y)\, dy$$

and

$$\int_0^1 y (D^4 u_2 - \Delta'^2 u_0) dy = \int_0^1 y\, f_\varepsilon(x',y)\, dy\ .$$

Using the boundary conditions, one verifies that these conditions reduce to

$$\int_0^1 f_\varepsilon(x',y) dy = 0\ , \quad \int_0^1 y f_\varepsilon(x',y) dy = 0$$

i.e.

$$(2.21) \qquad \int_0^\varepsilon f(x',x_3) dx_3 = 0\ , \quad \int_0^\varepsilon x_3 f(x',x_3) dx_3 = 0\ .$$

This leads to the following question: are they any connections between (2.21) and the condition that f belongs to the dual of the analog of the space E introduced in Section 2.1?

Remark 2.2

The control of thin structures, or of structures which contain some parts which are thin, is quite an important problem in the applications. □

3. Singular perturbations.

3.1. Optimal control and regular approximation.

Let us consider the system whose state is given by

$$(3.1) \qquad \frac{\partial^2 y}{\partial t^2} - \Delta y = v \quad \text{in} \quad Q = \Omega \times]0,T[,$$

subject to

(3.2) $$y(x,0) = \frac{\partial y}{\partial t}(x,0) = 0 \quad \text{in} \quad \Omega ,$$

(3.3) $$y = 0 \quad \text{on} \quad \Sigma = \Gamma \times]0,T[, \qquad \Gamma = \partial\Omega \quad .$$

The cost function is given by

(3.4) $$J(v) = \int_\Sigma \left| \frac{\partial y}{\partial \nu}(v) - z_d \right|^2 d\Sigma + N \int_Q v^2 dx dt \; .$$

We remark that given v in $L^2(Q)$, the unique solution $y(v)$ of (3.1) (3.2)(3.3) satisfies $\frac{\partial y}{\partial \nu}(v) \in L^2(\Sigma)$ (1) so that (3.4) defines a continuous function on $L^2(Q)$.

The problem of optimal control

(3.5) $$\inf \; J(v), \; v \in U_{ad} = \text{closed convex subset of } L^2(Q)$$

admits a unique solution, denoted by u.

We do not write here the optimality system which characterizes u. The question we want to raise is the following: in looking for numerical approximations schemes, we shall need approximations y_h of y which give "good" approximations of $\frac{\partial y_h}{\partial \nu}$ under the hypothesis $v \in L^2(Q)$.

This type of question - which does not seem to have been considered in the literature - leads to the problem considered in the following section.

3.2. A singular perturbation problem.

Let us consider the equation

(3.6) $$\frac{\partial^2 u_\varepsilon}{\partial t^2} + \varepsilon \Delta^2 u_\varepsilon - \Delta u_\varepsilon = f \quad \text{in} \quad Q = \Omega \times]0,T[, \; \varepsilon > 0$$

---

(1) Cf. (1.15). It would be sufficient to have $v \in L^1(0,T;L^2(\Omega))$ to obtain the same conclusion.

where u_ε is subject to

(3.7) $$u_\varepsilon(x,0) = \frac{\partial u_\varepsilon}{\partial t}(x,0) = 0 \text{ in } \Omega ,$$

(3.8) $$\left| \begin{array}{l} u_\varepsilon = 0 \text{ on } \Sigma , \\ \Delta u_\varepsilon = 0 \text{ on } \Sigma . \end{array} \right.$$

We assume that

(3.9) $$f \in L^1(0,T;L^2(\Omega)).$$

This problem admits a unique solution for every $\varepsilon>0$, which satisfies

(3.10) $$\left| \begin{array}{l} u_\varepsilon \in C([0,T];\ H^2(\Omega) \cap H_0^1(\Omega)) , \\ \frac{\partial u_\varepsilon}{\partial t} \in C([0,T];\ L^2(\Omega)). \end{array} \right.$$

Moreover:

(3.11) $$\left| \begin{array}{l} \|u_\varepsilon\|_{C([0,T];\ H_0^1(\Omega))} \leq C \text{ (independent of } \varepsilon) \\ \|\frac{\partial u_\varepsilon}{\partial t}\|_{C([0,T];\ L^2(\Omega))} \leq C , \end{array} \right.$$

and

(3.12) $$\sqrt{\varepsilon}\ \|u_\varepsilon\|_{C([0,T];H^2(\Omega))} \leq C .$$

One can show easily that, as $\varepsilon \to 0$,

$$(3.13)\quad \begin{cases} u_\varepsilon \to u \quad \text{in } L^\infty(0,T;\, H_0^1(\Omega)) \quad \text{weak star} \\ \dfrac{\partial u_\varepsilon}{\partial t} \to \dfrac{\partial u}{\partial t} \quad \text{in } L^\infty(0,T;\, L^2(\Omega)) \quad \text{weak star} \end{cases}$$

where u is the solution of

$$(3.14)\quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f\,, \\ u(x,0) = \dfrac{\partial u}{\partial t}(x,0) = 0 \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \Sigma\,. \end{cases}$$

□

A (seemingly) more difficult question is the following. It follows from $(3.10)_1$ that

$$(3.15)\quad \frac{\partial u_\varepsilon}{\partial \nu} \in C([0,T];\, H^{1/2}(\Gamma))$$

so that <u>in particular</u>

$$(3.16)\quad \frac{\partial u_\varepsilon}{\partial \nu} \in L^2(\Sigma)\,.$$

On the other hand we know that under the assumptions (3.19) then

$$(3.17)\quad \frac{\partial u}{\partial \nu} \in L^2(\Sigma)\,,$$

hence the natural question: do we have

$$(3.18)\quad \frac{\partial u_\varepsilon}{\partial \nu} \to \frac{\partial u}{\partial \nu} \quad \text{in } L^2(\Sigma)\ ?$$

Remark 3.1

If (3.18) was true, then we could use standard (finite elements) approximations for $\frac{\partial^2 y}{\partial t^2} + \varepsilon \Delta^2 y - \Delta y = v$, $y = 0$ $\Delta y = 0$ on Σ , in numerical algorithms for computing inf. J(v) as given by (3.4). □

Remark 3.2.

Of course the question (3.18) amounts to find an a priori estimate of the type

$$\left\| \frac{\partial u_\varepsilon}{\partial \nu} \right\|_{L^2(\Sigma)} < C ,$$

but this is not known.

A priori estimates of this sort - but for different boundary conditions - have been given in J.L. Lions [4]. □

4. A fourth problem.

We end up by mentioning a problem which we have alredy raised since several years and which may be of some relevance in vibrations problems.

We consider a perforated domain Ω_ε which consists of a domain Ω where we take out "holes" of size ε in a periodic manner, with period ε .

The boundary of Ω_ε consists in two parts:

$$\partial\Omega_\varepsilon = \Gamma_\varepsilon \cup S_\varepsilon$$

where

Γ_ε = what remains of $\Gamma = \partial\Omega$ after taking out the holes,

S_ε = union of the boundaries of the holes which intersect Ω .

Let us consider now the spectral problem

(4.1) $$\Delta^2 u_\varepsilon = \lambda(\varepsilon) u_\varepsilon \quad \text{in} \quad \Omega_\varepsilon ,$$

(4.2) $$u_\varepsilon = \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_\varepsilon \cup S_\varepsilon .$$

What is the behaviour of - in particular - the discrete spectrum

(4.3) $$0 < \lambda_1(\varepsilon) \leq \lambda_2(\varepsilon) \leq \dots \leq \lambda_m(\varepsilon) \leq \dots$$

of (4.1) (4.2) as $\varepsilon \to 0$?

Remark 4.1

The analogous problem for $- \Delta$:

(4.4) $$- \Delta u_\varepsilon = \lambda(\varepsilon) u_\varepsilon \quad \text{in} \quad \Omega_\varepsilon ,$$

(4.5) $$u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\varepsilon$$

has been soled by S. Vanninathan [1] and L. Tartar [1] (cf. a presentation of the results in J.L. Lions [5]). The solution uses in a seemingly essential manner the fact that the first eigenfunction of $- \Delta$ is a positive function and does not seem to extend to problem (4.1)(4.2). □

Bibliography

A. Bensoussan, J.L. Lions and G. Papanicolaou [1] Asymptotic Analysis for periodic structures - Studies in Math. and its Applications - 5 - (1978) - North Holland.

I. Lasiecka, J.L. Lions, R. Triggiani [1] Boundary control of hyperbolic systems - J.M.P.A. 1985.

J.L. Lions [1] Controle des systemes distribués Singuliers. Paris, Gauthier Villars, 1983.

[2] Some methods in the Analysis of systems and their control. Science Press, Beijing (1981). Gordon and Breach.

[3] Some asymptotic problems in the optimal control of distributed systems. NASA-IEEE conference in San Diego, 1984.

[4] Un resultat de régularité pour l'operateur $\frac{\partial^2}{\partial t^2} + \Delta^2$. Dedicated to S. Mizohata for his 60th birthday. To appear - Kyoto Univ.

[5] Lectures at the course CEA-EDF-INRIA, Breau Sans Nappes, Summer 1983. CEA Publication, 1985.

E. Sanchez-Palencia [1] Non homogeneous media and vibration theory. Lecture Notes in Physics 127 (1980), Springer Verlag.

L. Tartar [1] Unpublished.

S. Vanninathan [1] C.R. Acad. Sc. 1979. 1980.

J.L. Lions
College de France
and CNES

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