**QUANTITATIVE IMAGING OF CORROSION IN PLATES
BY EDDY CURRENT METHODS**

By

Bruno Luong

and

Fadil Santosa

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**INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA**

**514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455**

Quantitative Imaging of Corrosion in Plates by Eddy Current Methods

Bruno Luong ^{*} and Fadil Santosa [†]

School of Mathematics
University of Minnesota
Minneapolis, MN 55455

Abstract

We consider the problem of determining material loss due to corrosion in a plate. Eddy current data are collected on one side of the plate with corrosion damage occurring on the opposite side. Our work is based on linearizing the relationship between eddy current loop impedance change with the loss profile. In this context, we show that eddy current data are sufficient to uniquely determine the material loss profile. Because of the illposedness of the reconstruction problem, we propose 3 regularization strategies in devising computational methods. Numerical results using consistent data, and with added noise, indicate that the computational methods are quite capable of obtaining accurate quantitative estimates of the loss profile.

1 Introduction

Eddy current method for nondestructive evaluation is a technique to detect flaws in conducting medium such as metals. It has found prevalent use in the aircraft industry, and in the vessels and piping industries. The method uses a probe to generate an electrical field. The field penetrates the specimen that is being inspected, and when it encounters an anomaly, it is altered by the anomaly. The change in the field is monitored as the probe scans over

^{*}Department of Mathematical Science, University of Delaware, Newark, DE 19716

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the surface of the specimen. The problem we analyze in this work is the determination of a flaw from the measured field.

While this method of nondestructive evaluation has been shown to be quite effective in detecting flaws, it has not been successfully used to determine quantitative information about the flaws. There has been some ad hoc methods which are simply based on empirical relation between the amplitude of the changes in the field and the and some quantities characterizing the properties of the flaw [4].

Our work attempts to understand the relationship between parameters describing the flaw and the measured data. Some work in this direction can be found in the literature (see [2], [3], and [14]). By focusing our attention to a specific problem geometry, we are able to obtain analytical results that can be used in practical inversion procedure.

The specific problem we consider is the determination of a material loss profile in a plate geometry. It is assumed that we have access to the undamaged side of the plate. This model for corrosion damage is appropriate for inspection of aircraft skins [16].

The main results in the paper concerns the case when material loss is small in comparison to the plate thickness. This allows for linearization of the relationship between the material loss profile and the measured data. The relationship is derived formally, and is shown to be a simple convolution. Finally, we prove that the linearized problem has a unique solution. Computational aspects are also addressed. In particular, we show that by adding appropriate regularization, we can construct methods that are quite effective at giving quantitative information about corrosion.

The organization of the paper is as follows. In Section 2, we describe a model for the problem. We derive an integral identity that relates the data to the desired unknown in Section 3. The identity will play a central role in understanding the inverse problem. Section 4 contains a derivation that leads to explicit formulas for the electromagnetic fields in the case of an undamaged plate. A formal linearization of the relationship between impedance change and material loss is described in Section 5. The results will be used in Section 6 to show a uniqueness result for the linearized inverse problem. In Section 7 we develop computational strategies for solving the inverse problems, paying close attention to the illposedness. The main ideas of the strategies are illustrated in numerical examples. A final section gives some closing remarks.

Figure 1: A sketch of the experimental setup. A plate of thickness h has corrosion damage which has caused material loss, given by the profile $z = \varepsilon\theta(x, y)$. The probe is a circular loop of radius r_0 with the plane of the circle parallel to, and hovering at a distance t above, the plate surface. The loop causes electromagnetic field to develop in the plate and surrounding air. The intensity of the field is measured in the loop as voltage potential. We record the impedance, $Z(x_0, y_0)$ which is the ratio of the voltage and the input current. The problem is to find the loss profile from the measured impedance.

2 Model of problem

Corrosion is a complicated phenomena that involves material and geometry changes. What is of interest to the practicing engineer is the damage caused by corrosion, and its effect on structural integrity. We introduce a simple corrosion damage model which is characterized by material that is lost to corrosive processes. The problem we wish to solve is to determine material loss from eddy current data.

For simplicity, we will consider a plate geometry for the specimen. Let $\mathbf{x} = (x, y, z)$ denote the coordinates of a point in $\mathbb{R}^3$. Then an undamaged plate is identified by the set

$$\Omega^0 = \{\mathbf{x} : -\infty < x, y < \infty, 0 < z < h\}. \quad (1a)$$

We assume we have access to the plane $z = h$ and corrosion has occurred on the side closest to the plane $z = 0$. To characterize the damage, which

is material loss, we introduce a real $\mathcal{C}^2$ function $\theta(x, y)$. We let $\epsilon > 0$ with $\epsilon\theta < h$. The damaged plate is

$$\Omega^\epsilon = \{\mathbf{x} : -\infty < x, y < \infty, \epsilon\theta(x, y) < z < h\}. \quad (1b)$$

The quantity $\epsilon\theta(x, y)$ will be referred to as the *loss profile*. For notational convenience, we denote by $\Omega_+^\epsilon = \mathbb{R}^3 \setminus \overline{\Omega^\epsilon}$.

The eddy current probe generates an electromagnetic field in the plate as well as in the surrounding air. The plate is assumed to be conducting with conductivity σ_0 , whereas the air has zero conductivity. It is convenient to denote the conductivity in all of $\mathbb{R}^3$ with the indicator function

$$\sigma \stackrel{def}{=} \sigma_0 \mathbb{I}_{\Omega^\epsilon}.$$

For simplicity, we shall assume that the electric permittivity and magnetic permeability are identical in the plate and the air, and denote them by ϵ and μ . We will use Maxwell's equation for time-harmonic fields to model the interaction of the probe with its surrounding.

Let (E^ϵ, H^ϵ) represent the electric and magnetic fields generated by the probe when the plate is damaged¹. The fields (E^ϵ, H^ϵ) satisfy Maxwell's equations for time harmonic electromagnetic fields

$$\nabla \times E^\epsilon = i\omega\mu H^\epsilon, \quad (2a)$$

$$\nabla \times H^\epsilon = (-i\omega\epsilon + \sigma)E^\epsilon + J, \quad (2b)$$

where ω is the frequency of the fields. The current source J is caused by the probe. The probe is modeled by a single loop parallel to the top surface of the plate, and hovering at a fixed distance t above it. The loop radius is r_0 , and the amplitude of current passing through the circuit is I (alternating current at frequency ω). To model this probe, we set

$$J(x, y, z; x_0, y_0) = \frac{I}{r_0} \delta(z - (h + t)) \delta\left(\left((x - x_0)^2 + (y - y_0)^2\right)^{\frac{1}{2}} - r_0\right) \begin{pmatrix} -(y - y_0) \\ (x - x_0) \\ 0 \end{pmatrix}. \quad (3)$$

On the boundary of the plate, we have continuity of the tangential components of (E^ϵ, H^ϵ)

$$[n \times E^\epsilon] = [n \times H^\epsilon] = 0 \quad \text{on } \partial\Omega^\epsilon, \quad (4)$$

¹Note that later, we will also consider the fields in the undamaged plate and they will be denoted by (E^0, H^0) .

where the symbol $[\cdot]$ means the jump in value across the boundary, and n designates the outward normal to Ω^ε . In the farfield, we have the Silver-Müller radiation boundary conditions (not written here).

Remark 2.1 We do not assert that the Maxwell's equation (2) with the source in (3), with the continuity condition (4) and the Silver-Müller radiation conditions, lead to a wellposed problem for the determination of E^ε and H^ε . We expect this to be the case. For a finite body, the work of Abboud and Nedelec [1] has addressed this issue. Unfortunately, we cannot apply their result to the present case. ■

The situation is depicted in Figure 1. The electromagnetic field generated by the loop probe is also measured at the probe as electrical potential. The electrical potential can be calculated using Faraday's law [11]. What is in fact recorded is the circuit impedance, which is the ratio of the voltage and the input current. Therefore, from Faraday's law, we obtain a formula for the measured impedance when the probe is at (x_0, y_0)

$$Z^\varepsilon(x_0, y_0) = \frac{i\omega\mu}{I} \int_{(x-x_0)^2+(y-y_0)^2 \leq r_0^2} H_z^\varepsilon(x, y, h+t) \, dx dy. \quad (5)$$

H_z is the z -component of the magnetic field.

The inverse problem can be stated as

Find the loss profile $\varepsilon\theta(x, y)$ from the impedance measurement $Z^\varepsilon(x_0, y_0)$ for $-\infty < x_0, y_0 < \infty$.

Remark 2.2 It is possible to write the loop impedance as a function of the electric field. The integrand in (5) is $H \cdot \hat{z}$, where $\hat{z}$ is the unit vector in the z -direction. Using Maxwell's equation (2a), we rewrite

$$\begin{aligned} Z^\varepsilon(x_0, y_0) &= \frac{1}{I} \int_{(x-x_0)^2+(y-y_0)^2 \leq r_0^2} \nabla \times E^\varepsilon \cdot \hat{z} \, dx dy \\ &= \frac{1}{I} \int_{(x-x_0)^2+(y-y_0)^2 = r_0^2} E^\varepsilon(x, y) \cdot \tau \, dl, \end{aligned}$$

after using Stoke's theorem. The line integration is now over the the circular loop, with τ being the unit tangent along the circle oriented so that the interior of the circle is on the left in the xy -plane. ■

The relationship between the eddy current data $Z^\varepsilon(x_0, y_0)$ and the unknown material loss profile $\varepsilon\theta(x, y)$ will be explored in detail in the subsequent sections.

Figure 2: The four subdomains associated with the damaged specimen Ω^ϵ and the undamaged plate Ω^0 .

3 Flaw and field interaction

In this section, we concern ourselves with the interaction between a flaw caused by corrosion and the electromagnetic field generated by the probe. A way to describe the effect of the flaw is to compare the difference between the field when the flaw is present with the one in the undamaged plate. That is, we consider

$$(E^p, H^p) \stackrel{def}{=} (E^\epsilon, H^\epsilon) - (E^0, H^0).$$

The fields (E^p, H^p) and (E^0, H^0) are referred to as perturbational and background fields, respectively.

Recall the difference in the two fields is due to the difference in the domains of the plates Ω^ϵ and Ω^0 . To calculate the difference, we write

$$\begin{cases} D_1^\epsilon &= \Omega^0 \cap \Omega^\epsilon, \\ D_2^\epsilon &= \Omega^\epsilon \setminus \overline{\Omega^0}, \\ D_3^\epsilon &= \Omega^0 \setminus \overline{\Omega^\epsilon}, \\ D_4^\epsilon &= \Omega_+^0 \cap \Omega_+^\epsilon. \end{cases}$$

Figure 2 illustrates the four subdomains. Note that we have allowed $\theta(x, y)$ to be positive and negative even though material loss literally should mean $\theta \geq 0$.

For $i = 1, \dots, 4$, we designate by E_i^p the restriction of E^p on the subdomain D_i^ϵ . We apply similar designation for H^p . Note that both (E^ϵ, H^ϵ) and (E^0, H^0) satisfy Maxwell's equation (2) with the same current distribution.

Therefore, we can verify that (E^p, H^p) satisfy

$$\left\{ \begin{array}{ll} \nabla \times E_i^p &= i\omega\mu H_i^p \quad \text{in } D_i^\varepsilon \quad \forall i = 1, \dots, 4, \\ \nabla \times H_i^p &= (-i\omega\epsilon + \sigma_i)E_i^p + K_i \quad \text{in } D_i^\varepsilon \quad \forall i = 1, \dots, 4, \\ [n \times E^p] &= 0 \quad \text{on } \partial\Omega^0 \cup \partial\Omega^\varepsilon, \\ [n \times H^p] &= 0 \quad \text{on } \partial\Omega^0 \cup \partial\Omega^\varepsilon, \\ &\text{the Silver-Müller radiation conditions;} \end{array} \right. \quad (6)$$

with

$$\sigma_i = \begin{cases} \sigma_0 & \text{in } D_1^\varepsilon, \\ 0 & \text{in } D_2^\varepsilon, \\ \sigma_0 & \text{in } D_3^\varepsilon, \\ 0 & \text{in } D_4^\varepsilon, \end{cases} = \begin{cases} \sigma_0 & \text{in } \Omega^0, \\ 0 & \text{in } \Omega_+^0, \end{cases} \quad (7)$$

and “pseudo sources”

$$K_i = \begin{cases} 0 & \text{in } D_1^\varepsilon, \\ \sigma_0 E^\varepsilon & \text{in } D_2^\varepsilon, \\ -\sigma_0 E^\varepsilon & \text{in } D_3^\varepsilon, \\ 0 & \text{in } D_4^\varepsilon. \end{cases} \quad (8)$$

Equations (6), (7), and (8) serve no other purpose than to provide the partial differential equations satisfied by the perturbational field. Observe that the relation between the damage, characterized by $\varepsilon\theta(x, y)$, and the perturbational field is *nonlinear*.

We now derive an expression that relates the damage to the change in the measured impedance. The derivation is based on considering a test electromagnetic field and performing integrations-by-parts. Let (F, G) be complex vector functions such that both $F|_{D_i^\varepsilon}$ and $G|_{D_i^\varepsilon} \in \mathcal{C}^1(D_i^\varepsilon)$ for $i = 1, \dots, 4$. Moreover, we assume that (F, G) have continuous tangential components across ∂D_i^ε . Let R be a real number representing the radius of an open ball $B(0, R)$ with its center at the origin. We assume (for simplicity only) that D_i^ε for $i = 1, \dots, 4$ are contained in $B(0, R)$.

We take the scalar product of the first and second equations in (6) with G and F , and integrate the expressions over the subdomains $\mathcal{O}_i = D_i^\varepsilon \cap B(0, R)$. Adding the resulting expressions and rearranging, we arrive at

$$\begin{aligned} & \int_{\mathcal{O}_i} G \cdot (\nabla \times E_i^p - i\omega\mu H_i^p) \, d\mathbf{x} + \\ & \int_{\mathcal{O}_i} F \cdot \{\nabla \times H_i^p - (-i\omega\epsilon + \sigma_i)E_i^p\} \, d\mathbf{x} - \int_{\mathcal{O}_i} K_i \cdot F \, d\mathbf{x} = 0. \end{aligned} \quad (9)$$

Applying the vector identity

$$\nabla \cdot (a \times b) = b \cdot \nabla \times a - a \cdot \nabla \times b;$$

and Gauss's formula

$$\int_{\mathcal{O}_i} \nabla \cdot c \, d\mathbf{x} = \int_{\partial \mathcal{O}_i} c \cdot n \, ds;$$

with $a = E^p$, $b = G$, and $c = a \times b$, it is easy to see that

$$\int_{\mathcal{O}_i} G \cdot (\nabla \times E^p) \, d\mathbf{x} = \int_{\mathcal{O}_i} E^p \cdot (\nabla \times G) \, d\mathbf{x} + \int_{\partial \mathcal{O}_i} (E^p \times G) \cdot n \, ds.$$

Next, we sum the identities above over the index i and note that by continuity of the tangential field for E^p across the boundaries ∂D_i^ε , we are left with the surface integral over $\partial B(0, R)$ only. That is, we obtain

$$\int_{B(0, R)} G \cdot (\nabla \times E^p) \, d\mathbf{x} = \int_{B(0, R)} E^p \cdot (\nabla \times G) \, d\mathbf{x} + \int_{\partial B(0, R)} (E^p \times G) \cdot n \, ds,$$

where it is understood that $E^p = E_i^p$ and $H^p = H_i^p$ in the appropriate subdomains $\mathcal{O}_i$.

Similarly, we obtain

$$\int_{B(0, R)} F \cdot (\nabla \times H^p) \, d\mathbf{x} = \int_{B(0, R)} H^p \cdot (\nabla \times F) \, d\mathbf{x} + \int_{\partial B(0, R)} (H^p \times F) \cdot n \, ds.$$

The relation (9) is now summed over i and using the above identities, we arrive at

$$\begin{aligned} & \int_{B(0, R)} E^p \cdot \{ \nabla \times G - (-i\omega\epsilon + \sigma_i)F \} \, d\mathbf{x} + \\ & \int_{B(0, R)} H_i^p \cdot \{ \nabla \times F - i\omega\mu G \} \, d\mathbf{x} + \\ & \int_{\partial B(0, R)} \{ (H^p \times F) \cdot n + (E^p \times G) \cdot n \} \, ds - \\ & \int_{D_2^\varepsilon} \sigma_0 E^\varepsilon \cdot F \, d\mathbf{x} + \int_{D_3^\varepsilon} \sigma_0 E^\varepsilon \cdot F \, d\mathbf{x} = 0. \end{aligned} \tag{10}$$

We will now need to identify equations satisfied by (F, G) in order to obtain an expression for the change in the impedance. Recall that because we have defined $H^p = H^\varepsilon - H^0$, it follows that the corresponding difference in the measured impedance at (x_0, y_0) is $\delta Z^\varepsilon(x_0, y_0) = Z^\varepsilon(x_0, y_0) - Z^0(x_0, y_0)$ (see (5)). By choosing the test field (F, G) to satisfy Maxwell's equation with

specific forcing terms, we can force the second integral in (10) to become the impedance difference. Specifically, let (F, G) satisfy

$$\left\{ \begin{array}{ll} \nabla \times F &= i\omega\mu G + L \quad \text{in } \mathbb{R}^3, \\ \nabla \times G &= (-i\omega\epsilon + \sigma)F \quad \text{in } \mathbb{R}^3, \\ [n \times F] &= 0 \quad \text{on } \partial\Omega^0, \\ [n \times G] &= 0 \quad \text{on } \partial\Omega^0, \\ &\text{the Silver-Müller radiation conditions;} \end{array} \right. \quad (11)$$

with

$$L(x, y, z; x_0, y_0) = \delta(z - (h + t)) \mathbb{I}_{(x-x_0)^2 + (y-y_0)^2 < r_0^2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using (5) and (11), the identity (10) becomes²

$$\begin{aligned} \frac{I}{i\omega\mu} \delta Z^\epsilon(x_0, y_0) &= \int_{D_2^\epsilon} \sigma_0 E^\epsilon \cdot F \, d\mathbf{x} - \int_{D_3^\epsilon} \sigma_0 E^\epsilon \cdot F \, d\mathbf{x} \\ &\quad - \int_{\partial B(0, R)} \{ (H^p \times F) \cdot n + (E^p \times G) \cdot n \} \, ds. \end{aligned} \quad (12)$$

Now we let the radius of the ball R approach ∞ . The Silver-Müller radiation conditions (S.M.) are used to show that

$$\begin{aligned} &\int_{\partial B(0, R)} \{ (H^p \times F) \cdot n + (E^p \times G) \cdot n \} \, ds \\ &= \int_{\partial B(0, R)} \{ (n \times H^p) \times (n \times F) \cdot n + (E^p \times G) \cdot n \} \, ds \\ &\stackrel{\text{S.M.}}{=} \int_{\partial B(0, R)} \left\{ \left(-E^p + o\left(\frac{1}{R}\right) \right) \times \left(G + o\left(\frac{1}{R}\right) \right) \cdot n + (E^p \times G) \cdot n \right\} \, ds \\ &= \int_{\partial B(0, R)} \left\{ \left(-E^p \times o\left(\frac{1}{R}\right) \right) \cdot n + \left(o\left(\frac{1}{R}\right) \times G \right) \cdot n + o\left(\frac{1}{R^2}\right) \right\} \, ds. \end{aligned}$$

Moreover, each component of E^p and G satisfy the usual Sommerfeld radiation condition which imply that they are $O\left(\frac{1}{R}\right)$ for $R \rightarrow \infty$ uniformly in all directions. Consequently, one has

$$\int_{\partial B(0, R)} \{ (H^p \times F) \cdot n + (E^p \times G) \cdot n \} \, ds = \int_{\partial B(0, R)} o\left(\frac{1}{R^2}\right) \, ds \longrightarrow 0.$$

²The fields (F, G) have continuous tangential components across ∂D_i^ϵ .

Using this in (12), we arrive at an identity relating impedance difference

$$\delta Z^\varepsilon(x_0, y_0) = \frac{i\omega\mu\sigma_0}{I} \left(\int_{D_2^\varepsilon} E^\varepsilon \cdot F \, d\mathbf{x} - \int_{D_3^\varepsilon} E^\varepsilon \cdot F \, d\mathbf{x} \right). \quad (13)$$

A way to interpret this identity is that given a material loss profile $\varepsilon\theta(x, y)$, we can (in principle) calculate the fields $(E^\varepsilon, H^\varepsilon)$ using (2) and (F, G) using (11), which when inserted in (13) produces the impedance change at the measurement point (x_0, y_0) . Therefore, the formula (13) can be viewed as another characterization of the forward problem.

The test field F is in fact proportional to the background electric field E^0 . To see this, we take the curl of the equations satisfied by both fields and eliminate, respectively, G and H^0 . When this is done, one can make the identification that

$$E^0 = i\omega\mu IF.$$

Using this observation in (13), we arrive at the formula

$$\delta Z^\varepsilon(x_0, y_0) = \frac{\sigma_0}{I_0^2} \left(\int_{D_2^\varepsilon} E^\varepsilon \cdot E^0 \, d\mathbf{x} - \int_{D_3^\varepsilon} E^\varepsilon \cdot E^0 \, d\mathbf{x} \right). \quad (14)$$

The dependence on (x_0, y_0) of the change in the impedance is through the fields E^ε and E^0 , which are dependent on the probe position (x_0, y_0) as indicated by the source $J(x, y, z)$ given in (3).

Finally, recall that using the definition of the plate domains in (1), the subdomains are easily found to be

$$\begin{aligned} D_2^\varepsilon &= \{(x, y, z) : \varepsilon\theta(x, y) < z < 0\}, \\ D_3^\varepsilon &= \{(x, y, z) : \varepsilon\theta(x, y) > z > 0\}. \end{aligned}$$

Therefore, (14) can be simplified to

$$\delta Z^\varepsilon(x_0, y_0) = -\frac{\sigma_0}{I^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\varepsilon\theta(x, y)} E^\varepsilon \cdot E^0 \, dz dx dy. \quad (15)$$

This formula, while the derivation is formal, is exact. In order to justify its derivation, we must establish that the fields involved are sufficiently regular as to permit integration by parts. In this work, we simply assume that the fields are indeed regular enough to justify all the steps leading to (14). In Section 5, we will further simplify the relationship by linearizing it.

4 The field for an undamaged plate

Due to the central importance the field for the undamaged plate plays in the development leading to the results of the present work, we will now derive an explicit formula for the field. In fact, what we intend to do is to solve (2), (3), and (4) for the case of $\varepsilon = 0$.

It can be shown that the electric field $E^0(x, y, z)$ satisfies a vector Helmholtz equation

$$\begin{cases} \Delta E^0 + k^2 E^0 = -i\omega\mu J \\ \text{Sommerfeld radiation conditions,} \end{cases} \quad (16a)$$

where

$$k^2 = \begin{cases} \omega^2\mu\epsilon & \text{if } z > h \text{ or } z < 0, \\ \omega^2\mu(\epsilon + i(\sigma_0/\omega)) & \text{if } 0 < z < h. \end{cases} \quad (16b)$$

The wave number k must be such that $\text{Im } k \geq 0$. The source $J(x, y, z)$ is as specified in (3) but observe that because of the geometry of the problem, the field will be translation invariant in the (x, y) plane. That is, the field with the loop at (x_0, y_0) is simply the translation of the field with the loop at $(0, 0)$. Therefore, we need only to concern ourselves with

$$J(x, y, z) = \frac{I}{r_0} \delta(z - (h + t)) \delta((x^2 + y^2)^{\frac{1}{2}} - r_0) \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad (17)$$

The tangential components of the electric and magnetic fields must be continuous across the planes $z = 0$ and $z = h$. These conditions imply that

$$\begin{cases} [E_x^0] = [E_y^0] = 0 \\ \left[\frac{\partial E_x^0}{\partial z} \right] = \left[\frac{\partial E_y^0}{\partial z} \right] = 0 \end{cases} \quad \text{on } \{z = 0 \text{ or } z = h\}. \quad (18)$$

In order to solve (16), (17) and (18), we use the Fourier transform in the directions x and y . The notation used for the transform and its inverse are

$$\widehat{f}(\xi, \eta, z) \stackrel{\text{def}}{=} \frac{1}{2\pi} \iint_{\mathbb{R}^2} f(x, y, z) e^{-i(x\xi + y\eta)} dx dy;$$

$$f(x, y, z) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \widehat{f}(\xi, \eta, z) e^{i(x\xi + y\eta)} d\xi d\eta.$$

From (16) the transformed electric field satisfies

$$\frac{\partial^2 \widehat{E}^0}{\partial z^2} + (k^2 - (\xi^2 + \eta^2)) \widehat{E}^0 = -i\omega\mu \widehat{J}$$

We proceed by calculating the Fourier transform of the source. Applying the transform to (17), and changing from rectangular coordinates (x, y) to polar coordinates (r, ϕ) ,

$$\begin{aligned}\hat{J}(\xi, \eta, z) &= \frac{I}{2\pi r_0} \delta(z - (h + t)) \\ &\quad \int_0^\infty \int_0^{2\pi} \delta(r - r_0) \begin{pmatrix} -r \sin \phi \\ r \cos \phi \\ 0 \end{pmatrix} e^{-ir(\xi \cos \phi + \eta \sin \phi)} r d\phi dr.\end{aligned}$$

Next, we use polar coordinates in the (ξ, η) plane as well with $\xi = \rho \cos \psi$ and $\eta = \rho \sin \psi$. The expression above becomes

$$\hat{J}(\xi, \eta, z) = \frac{r_0 I}{2\pi} \delta(z - (h + t)) \int_0^{2\pi} \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} e^{-ir_0 \rho \cos(\phi - \psi)} d\phi. \quad (19)$$

Let us examine the first component on the righthand side of (19). By a change of coordinates $\phi' = \phi - \psi$, we see that

$$\begin{aligned}& \frac{1}{2\pi} \int_0^{2\pi} -\sin \phi e^{-ir_0 \rho \cos(\phi - \psi)} d\phi; \\ &= \frac{1}{2\pi} \int_0^{2\pi} -\sin(\phi' + \psi) e^{-ir_0 \rho \cos \phi'} d\phi'; \\ &= -\frac{\cos \psi}{2\pi} \int_0^{2\pi} \sin \phi' e^{-ir_0 \rho \cos \phi'} d\phi' - \frac{\sin \psi}{2\pi} \int_0^{2\pi} \cos \phi' e^{-ir_0 \rho \cos \phi'} d\phi'.\end{aligned}$$

Next, split the integration interval to $[0, \pi]$ and $[\pi, 2\pi]$. The following identities can be verified by a change of variable $\phi' \rightarrow 2\pi - \phi'$,

$$\begin{aligned}\int_\pi^{2\pi} \sin \phi' e^{-ir_0 \rho \cos \phi'} d\phi' &= -\int_0^\pi \sin \phi' e^{-ir_0 \rho \cos \phi'} d\phi'; \\ \int_\pi^{2\pi} \cos \phi' e^{-ir_0 \rho \cos \phi'} d\phi' &= \int_0^\pi \cos \phi' e^{-ir_0 \rho \cos \phi'} d\phi'.\end{aligned}$$

Summarizing, we have

$$\frac{1}{2\pi} \int_0^{2\pi} -\sin \phi e^{-ir_0 \rho \cos(\phi - \psi)} d\phi = -\frac{\sin \psi}{\pi} \int_0^\pi \cos \phi' e^{-ir_0 \rho \cos \phi'} d\phi'. \quad (20)$$

The integral on the righthand side of (20) is recognized to be a constant multiple of the integral representation for Bessel function of the first kind of order 1, J_1 . Hence we can write

$$\frac{1}{2\pi} \int_0^{2\pi} -\sin \phi e^{-ir_0 \rho \cos(\phi - \psi)} d\phi = i \sin \psi J_1(r_0 \rho).$$

Similarly, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \cos \psi e^{-ir_0 \rho \cos(\phi-\psi)} d\phi = -i \cos \psi J_1(r_0 \rho).$$

We conclude that the Fourier transform of the source $J(x, y, z)$ is given by

$$\hat{J}(\xi, \eta, z) = \delta(z - (h + t)) i r_0 I J_1(r_0 \rho) \begin{pmatrix} \sin \psi \\ -\cos \psi \\ 0 \end{pmatrix}, \quad (21)$$

where $\rho^2 = \xi^2 + \eta^2$ and $\tan \psi = \eta/\xi$.

We continue by seeking an explicit expression for $\widehat{E}^0(\xi, \eta, z)$. Let us define

$$\kappa^2 = \begin{cases} \kappa_+^2 \stackrel{\text{def}}{=} \omega^2 \mu \epsilon - \rho^2 & \text{if } z < 0 \text{ if } z > h, \\ \kappa_-^2 \stackrel{\text{def}}{=} \omega^2 \mu (\epsilon + i(\sigma_0/\omega)) - \rho^2 & \text{if } 0 < z < h, \end{cases} \quad (22a)$$

with κ_+ et κ_- having nonnegative imaginary parts. We seek a solution $\widehat{E}$ satisfying

$$\frac{\partial^2 \widehat{E}^0}{\partial z^2} + \kappa^2 \widehat{E}^0 = -i\omega \mu \widehat{J}, \quad (22b)$$

with the source $\widehat{J}(\xi, \eta, z)$ given by (21), continuity conditions given by the Fourier transform of (18), and radiation conditions

$$\frac{\partial \widehat{E}^0}{\partial z} \mp i \kappa_{\pm} \widehat{E}^0 = o\left(\frac{1}{|z|}\right) \text{ when } z \rightarrow \pm\infty.$$

This can be done by observing that what we are seeking is in fact a multiple of the Green's function of a second order ordinary differential equation:

$$\begin{aligned} g'' + \kappa^2 g &= \delta(z - z_0); \\ g(z) \text{ and } g'(z) &\text{ continuous across } z = 0, h; \\ g' \mp i \kappa_{\pm} g &= o\left(\frac{1}{|z|}\right) \text{ } z \rightarrow \pm\infty. \end{aligned}$$

It can be verified that $g(z, z_0)$, for $z_0 \geq h$, is given by the following formula:

$$g(z, z_0) = \begin{cases} 2\kappa_- \gamma e^{-i\kappa_+ z} & \text{if } z \leq 0 \\ \gamma \left((\kappa_- - \kappa_+) e^{i\kappa_- z} + (\kappa_- + \kappa_+) e^{-i\kappa_- z} \right) & \text{if } z \in]0, h[\\ \frac{i(\kappa_-^2 - \kappa_+^2) \gamma \sin(\kappa_- h)}{\kappa_+} e^{i\kappa_+(z-h)} - \frac{i}{2\kappa_+} e^{i\kappa_+|z-z_0|} & \text{if } z > h; \end{cases} \quad (23a)$$

with

$$\gamma \stackrel{\text{def}}{=} \frac{i e^{i\kappa_+(z_0-h)}}{(\kappa_- - \kappa_+)^2 e^{i\kappa_-h} - (\kappa_- + \kappa_+)^2 e^{-i\kappa_-h}}. \quad (23b)$$

We can conclude from (22b), (23) and (21) that the Fourier transform of the electric field is given by

$$\widehat{E}^0(\xi, \eta, z) = \omega \mu r_0 I J_1(r_0 \rho) f(\rho, z) \begin{pmatrix} \sin \psi \\ -\cos \psi \\ 0 \end{pmatrix}; \quad (24a)$$

where f is given by

$$f(\rho, z) = \begin{cases} 2\kappa_- \gamma e^{-i\kappa_+z} & \text{if } z \leq 0 \\ \gamma \left((\kappa_- - \kappa_+) e^{i\kappa_-z} + (\kappa_- + \kappa_+) e^{-i\kappa_-z} \right) & \text{if } z \in]0, h] \\ \frac{-\sigma_0 \mu \omega \gamma \sin(\kappa_- h)}{\kappa_+} e^{i\kappa_+(z-h)} - \frac{i}{2\kappa_+} e^{i\kappa_+|z-(h+t)|} & \text{if } z > h; \end{cases} \quad (24b)$$

and γ is given by

$$\gamma = \frac{i e^{i\kappa_+t}}{(\kappa_- - \kappa_+)^2 e^{i\kappa_-h} - (\kappa_- + \kappa_+)^2 e^{-i\kappa_-h}}. \quad (24c)$$

The explicit form of $E^0(x, y, z)$ as given through (24) will play a central role in the remaining sections.

5 Formal linearization

Returning to the relation (15), we now formally derive a linearized relation between the observed impedance change and the loss profile. Let us denote the electric field generated by a probe at (x_0, y_0) with the damaged plate by $E_{(x_0, y_0)}^\varepsilon$. We begin by assuming that the field can be expanded in a convergent perturbation series

$$E_{(x_0, y_0)}^\varepsilon = E_{(x_0, y_0)}^0 + \varepsilon E_{(x_0, y_0)}^1 + \varepsilon^2 E_{(x_0, y_0)}^2 + \dots$$

Here $E_{(x_0, y_0)}^0$ is the field for the undamaged plate with the probe at the same location (x_0, y_0) . Inserting the expression in (15) and keeping only terms of higher order, we arrive at

$$\delta Z^\varepsilon(x_0, y_0) = -\frac{\sigma_0}{I^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\varepsilon \theta(x, y)} E_{(x_0, y_0)}^0 \cdot E_{(x_0, y_0)}^0 \, dz dx dy + O(\varepsilon^2). \quad (25)$$

The remainder is $O(\varepsilon^2)$ because the range of integration in the z variable is linear in ε .

Next, we know that the field $E_{(x_0, y_0)}^0(x, y, z)$ is sufficiently regular in the neighborhood of $z = 0$ from the explicit formula in (24). In particular, a Taylor series expansion of $E_{(x_0, y_0)}^0$ around $z = 0$ is justified. Therefore, we can write

$$E_{(x_0, y_0)}^0(x, y, z) = E_{(x_0, y_0)}^0(x, y, 0) + z \frac{\partial E_{(x_0, y_0)}^0}{\partial z}(x, y, 0) + \dots$$

Substituting this in (25) and integrating, we get

$$\delta Z^\varepsilon(x_0, y_0) = -\varepsilon \frac{\sigma_0}{I^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_{(x_0, y_0)}^0 \cdot E_{(x_0, y_0)}^0 \Big|_{z=0} \theta(x, y) dx dy + O(\varepsilon^2). \quad (26)$$

In (24), we solved for the Fourier transform of the electric field when the loop is centered at the origin, $\widehat{E}_{(0,0)}^0$ (see (17)). Because of the plate is infinite in the xy -plane, the field when the loop is at (x_0, y_0) is simply the field obtained translated by (x_0, y_0) . Therefore, if we designate $E^0(x, y, z)$ to be the field when the loop is at the origin, the relationship in (26) after dropping the ε^2 term is in fact

$$\delta Z^\varepsilon(x_0, y_0) = -\varepsilon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x - x_0, y - y_0) \theta(x, y) dx dy, \quad (27a)$$

where the kernel is given by

$$K(x, y) = \frac{\sigma_0}{I^2} E^0(x, y, 0) \cdot E^0(x, y, 0).$$

Next, we observe that the kernel K is even, which means that we can rewrite (27a) as a true convolution

$$\delta Z^\varepsilon(x_0, y_0) = -\varepsilon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x_0 - x, y_0 - y) \theta(x, y) dx dy. \quad (27b)$$

The linearized inverse problem then amounts to solving a convolutional problem in two dimensions. A similar convolutional problem arises in a linearized problem of thermal imaging [5].

Given the convolutional nature of the problem, we can pose it in the Fourier transform domain as

$$\widehat{\delta Z^\varepsilon}(\xi, \eta) = -\varepsilon \widehat{K}(\xi, \eta) \widehat{\theta}(\xi, \eta),$$

where $\widehat{\theta}$ is the Fourier transform, in x and y , of the loss profile $\theta(x, y)$.

Remark 5.1 In order to give the relation (26) rigorous justification, we must establish an estimate of the type

$$\|E^\varepsilon - E^0\| \leq C\varepsilon^\alpha,$$

for some positive $\alpha \geq 1/2$. Such a result would imply that (26) can be interpreted as a Frechét derivative. ■

6 Uniqueness of the linearized problem

Let us examine the situation where impedance data δZ^ε are given for all (x_0, y_0) . We wish to establish that such data are sufficient to determine the loss profile $\theta(x, y)$ uniquely. The question amounts to whether the linearized relation (26) can be inverted. For computational purposes, we will not have impedance reading over the entire plate's upper surface. Rather, we would have readings of δZ^ε over a set of points. We will address this practical issue in the next section.

The procedure for proving uniqueness goes as follows. We start by studying the Fourier transform of the kernel $K(x, y)$ in (27b). We will establish that it is an analytic function of the Fourier transform variables. By the analyticity property, the Fourier transform of $K(x, y)$ cannot vanish over a set of positive measure in the Fourier domain. Therefore the convolution equation (27b) must be invertible.

In Section 3, we obtained an explicit formula (24) for the Fourier transform of E^0 . We will be paying close attention to its value at $z = 0$, given by

$$\hat{E}^0(\xi, \eta, 0) = 2\omega\mu r_0 I J_1(r_0\rho) \kappa_- \gamma \begin{pmatrix} \sin \psi \\ -\cos \psi \\ 0 \end{pmatrix}, \quad (28)$$

where

$$\gamma = \frac{i e^{i\kappa_+ t}}{(\kappa_- - \kappa_+)^2 e^{i\kappa_- h} - (\kappa_- + \kappa_+)^2 e^{-i\kappa_- h}}.$$

Analyticity of $\hat{E}^0$ is verified by a classical result which states that the ratio of two analytic functions with a nonvanishing denominator is analytic. A statement of such a result can be found in Strichartz [15]. Note that our function $\hat{E}^0$ is a complex vector function of real variables ξ and η . However, the results stated can be applied to assert analyticity of the real and imaginary components.

Observe that $\hat{E}^0$ is already a ratio of two analytic functions of (ξ, η) . We need only to show that the denominator of γ is nonvanishing.

To this end, we need to establish certain properties about complex variables.

Lemma 1 *Let $a, b \in \mathbb{C}$ such that $a \neq 0$ and $\operatorname{Re}\left(\frac{b}{a}\right) \geq 0$ or $b \neq 0$ and $\operatorname{Re}\left(\frac{a}{b}\right) \geq 0$. Then $|a + b| \geq |a - b|$.*

Proof:

The roles of a and b are symmetric. We will the result only for the case where $a \neq 0$. Define

$$\beta_r \stackrel{\text{def}}{=} a \operatorname{Re} \left(\frac{b}{a} \right) \quad \text{and} \quad \beta_i \stackrel{\text{def}}{=} ia \operatorname{Im} \left(\frac{b}{a} \right).$$

It is easily shown that

$$a \bar{\beta}_i \text{ and } \beta_r \bar{\beta}_i \text{ are imaginary,} \quad (29a)$$

and

$$a \bar{\beta}_r \text{ is real with } a \bar{\beta}_r \geq 0. \quad (29b)$$

Computing directly, we have

$$\begin{aligned} |a + b|^2 &= |a + \beta_r + \beta_i|^2 \\ &= |a + \beta_r|^2 + |\beta_i|^2 + 2\operatorname{Re}(a \bar{\beta}_i) + 2\operatorname{Re}(\beta_r \bar{\beta}_i) \\ &= |a + \beta_r|^2 + |\beta_i|^2, \end{aligned}$$

by (29a). Similarly, we have

$$|a - b|^2 = |a - \beta_r|^2 + |\beta_i|^2.$$

Therefore

$$\begin{aligned} |a + b|^2 - |a - b|^2 &= |a + \beta_r|^2 - |a - \beta_r|^2 \\ &= 2\operatorname{Re}(a \bar{\beta}_r) \geq 0. \end{aligned}$$

The last inequality has been shown in (29b). ■

The next result is true because for $z \in \mathbb{C}$ and $z \neq 0$, both $\operatorname{Re}(z)$ and $\operatorname{Re}\left(\frac{1}{z}\right)$ have the same sign.

Lemma 2 *If $z \in \mathbb{C} \setminus \{0\}$. Then $\operatorname{Re}(z) \geq 0 \iff \operatorname{Re}\left(z + \frac{1}{z}\right) \geq 0$.*

Lemma 3 *Let $k \in \mathbb{C}$ such that $\operatorname{Im}(k^2) \geq 0$. Let $\rho \in \mathbb{R}$ and $\kappa \in \mathbb{C}$ such that $\kappa^2 = k^2 - \rho^2$. Then $\operatorname{Im}(\kappa) \geq 0 \Rightarrow \operatorname{Re}(\kappa) \geq 0$. The result holds for strict inequalities as well.*

Proof:

From $\kappa^2 = k^2 - \rho^2$ with $k \in \mathbb{C}$ and $\rho \in \mathbb{R}$, we know that $\operatorname{Im}(\kappa^2) = \operatorname{Im}(k^2) \geq 0$ by hypothesis. Therefore, the argument of κ^2 must take value in $[0, \pi]$.

If we choose κ such that $\operatorname{Im}(\kappa) \geq 0$, then it is clear that the argument of κ must take value in $[0, \pi/2]$, and so, $\operatorname{Re}(\kappa) \geq 0$. ■

Lemma 4 *Let κ_+ et κ_- be defined as in (22a). Then $\operatorname{Re}\left(\frac{\kappa_+}{\kappa_-}\right) \geq 0$.*

Proof:

We write $\kappa_+ = a + ib$ and $\kappa_- = c + id$ with a, b, c and $d \in \mathbb{R}$. Next, we check that $\operatorname{Im}(\omega^2 \mu \epsilon) = 0$, and $\operatorname{Im}(\omega^2 \mu (\epsilon + i(\sigma_0/\omega))) = \omega \mu \sigma \geq 0$. Because both κ_+ and κ_- both have positive imaginary parts, we must have b et $d \geq 0$. Now, apply Lemma 3 and we can conclude that $\operatorname{Re}(\kappa_+) \geq 0$ and $\operatorname{Re}(\kappa_-) \geq 0$. Hence,

$$\operatorname{Re}\left(\frac{\kappa_+}{\kappa_-}\right) = \frac{ac + bd}{c^2 + d^2} \geq 0. \quad \blacksquare$$

Lemma 5 *Let κ_+ et κ_- be defined as in (22a). Then*

$$\left|(\kappa_- + \kappa_+)^2\right| \geq \left|(\kappa_- - \kappa_+)^2\right|.$$

Proof:

It is clear that the inequality holds for κ_- or $\kappa_+ = 0$. Suppose that neither is zero. We have

$$\begin{aligned} (\kappa_- + \kappa_+)^2 &= \kappa_-^2 + \kappa_+^2 + 2\kappa_- \kappa_+ \\ (\kappa_- - \kappa_+)^2 &= \kappa_-^2 + \kappa_+^2 - 2\kappa_- \kappa_+. \end{aligned}$$

If we write $a = \kappa_-^2 + \kappa_+^2$ and $b = 2\kappa_- \kappa_+$, then

$$\frac{a}{b} = \frac{1}{2} \left(\frac{\kappa_+}{\kappa_-} + \frac{\kappa_-}{\kappa_+} \right).$$

From Lemma 4, we know that $\operatorname{Re}(\kappa_+/\kappa_-) \geq 0$. Using this fact in Lemma 2, we can establish that

$$\operatorname{Re}\left(\frac{\kappa_+}{\kappa_-} + \frac{\kappa_-}{\kappa_+}\right) \geq 0,$$

hence, $\operatorname{Re}(a/b) \geq 0$. Finally, using this fact in Lemma 1, we can conclude that $|a + b| \geq |a - b|$, which imply the inequality of the lemma. $\blacksquare$

Proposition 1 *The denominator of γ as defined in (28) [also (24)]*

$$(\kappa_- - \kappa_+)^2 e^{i\kappa_- h} - (\kappa_- + \kappa_+)^2 e^{-i\kappa_- h}$$

is never zero for any $\rho > 0$.

Proof:

The proof hinges on the fact that because κ_-^2 has a positive imaginary part, $\text{Im}(\kappa_-) > 0$ for any ρ . This implies that

$$|e^{-i\kappa_-h}| > 1 \text{ and } |e^{i\kappa_-h}| < 1.$$

Using the result of Lemma 5, we conclude that

$$|(\kappa_- + \kappa_+)^2 e^{-i\kappa_-h}| > |(\kappa_- - \kappa_+)^2 e^{i\kappa_-h}|.$$

Consequently,

$$(\kappa_- - \kappa_+)^2 e^{i\kappa_-h} - (\kappa_- + \kappa_+)^2 e^{-i\kappa_-h} \neq 0. \quad \blacksquare$$

We have now establish that $\gamma(\rho)$ where $\rho^2 = \xi^2 + \eta^2$ has nonvanishing denominator. That it is analytic can be seen by the fact that it is a combination, by addition and multiplication, of analytic functions. Returning to the expression for $\hat{E}^0(\xi, \eta, 0)$ in (28), we can now conclude that the real and imaginary components of $\hat{E}_x^0(\xi, \eta, 0)$ and $\hat{E}_y^0(\xi, \eta, 0)$ are analytic functions of (ξ, η) .

Next, consider the kernel in the convolutional equation (27b). The kernel is given by $K(x, y) = \frac{\sigma_0}{I^2} \|E(x, y, 0)\|^2$, therefore, its Fourier transform must be

$$\widehat{K} = \frac{\sigma_0}{I^2} \left(\hat{E}_x^0(\cdot, \cdot, 0) * \hat{E}_x^0(\cdot, \cdot, 0) + \hat{E}_y^0(\cdot, \cdot, 0) * \hat{E}_y^0(\cdot, \cdot, 0) \right);$$

where $*$ designates the convolution operator in $2D$. To show that $\widehat{K}$ is analytic, we need to show that the convolution of an analytic function with itself is analytic.

In this context, let us recall the definition of a real, scalar analytic function of several variables.

Definition 1 *A function f is said to be analytic over an open connected set D in $\mathbb{R}^n$ if for all $x_0 \in D$, there exists a neighborhood U_{x_0} and 2 positive constants β_{f,x_0} and M_{f,x_0} such that the following inequality holds:*

$$\|D^\alpha f\|_{L^\infty(U_{x_0})} \leq \beta_{f,x_0} M_{f,x_0}^{|\alpha|} (|\alpha|!) \quad \forall \alpha \in \mathbb{N}^n. \quad (30)$$

The bound on the derivatives allows one to assert convergence of the Taylor series of f

$$f(x_0 + h) = \sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha|=k}} \frac{1}{k!} D^\alpha f(x_0) h^\alpha \quad \forall x_0 + h \in U_{x_0} \cap B(x_0, M_{f,x_0}^{-1}).$$

In the context of this work, the functions in question depend on 2 variables ξ and η . However, they are complex functions, and therefore, we treat their real and imaginary parts separately.

Let us consider the convolution of $\widehat{E}_x(.,.,0)$ with itself. From the property of convolutions

$$D^\alpha(\widehat{E}_x * \widehat{E}_x) = D^\alpha(\widehat{E}_x) * \widehat{E}_x,$$

and therefore

$$\|D^\alpha(\widehat{E}_x * \widehat{E}_x)\|_{L^\infty} \leq \|D^\alpha(\widehat{E}_x)\|_{L^\infty} \|\widehat{E}_x\|_{L^1}.$$

Because $\widehat{E}_x(.,.,0)$ is analytic, by (30) its derivatives are bounded and must satisfy, at each point (ξ_0, η_0) ,

$$\|D^\alpha \widehat{E}_x\|_{L^\infty(U_{(\xi_0, \eta_0)})} \leq \beta_{(\xi_0, \eta_0)} M_{(\xi_0, \eta_0)}^{|\alpha|} (|\alpha|!) \quad \forall \alpha \in \mathbb{N}^2, \quad (31)$$

where $\beta_{(\xi_0, \eta_0)}$ and $M_{(\xi_0, \eta_0)}$ are constants, and $U_{(\xi_0, \eta_0)}$ is an open neighborhood of (ξ_0, η_0) .

Next, we show that $\widehat{E}_x(.,.,0) \in L^1(\mathbb{R}^2)$. To this end, recall that if we write $\kappa_+ = c + id$ with c and $d \in \mathbb{R}$, then when $\rho \rightarrow \infty$, we have $d \sim \rho$. Also, note that for large ρ , $|\kappa_-| \sim \rho$, $\gamma \sim i/4\rho^{-2}e^{-\rho h}$, and $J_1(r_0\rho) \leq C\rho^{-\frac{1}{2}}$. Therefore, one can conclude that for R large,

$$\begin{aligned} \int_{B(0,R) \setminus B(0,1)} |\widehat{E}_x(\xi, \eta, 0)| d\xi d\eta &\leq C \int_1^R \rho^{-\frac{1}{2}} \frac{\rho e^{-\rho t}}{4\rho^2 e^{\rho h}} \rho d\rho \\ &\leq C \int_1^R \rho^{-\frac{1}{2}} e^{-(h+t)\rho} d\rho. \end{aligned}$$

The last integral converges when $R \rightarrow \infty$, and moreover, $\widehat{E}_x(\xi, \eta, 0)$ is bounded for ρ small. Therefore, we have established that

$$\int |\widehat{E}_x(\xi, \eta, 0)| d\xi d\eta < \infty.$$

This fact together with (31) imply that

$$\|D^\alpha(\widehat{E}_x * \widehat{E}_x)\|_{L^\infty(U_{(\xi_0, \eta_0)})} \leq \tilde{\beta}_{(\xi_0, \eta_0)} M_{(\xi_0, \eta_0)}^{|\alpha|} (|\alpha|!) \quad \forall \alpha \in \mathbb{N}^2,$$

where $\tilde{\beta} = \beta \|\widehat{E}_x\|_{L^1(U_{(\xi_0, \eta_0)})}$. Therefore, $\widehat{E}_x * \widehat{E}_x|_{z=0}$ is analytic.

Similarly, one can show that $\widehat{E}_y * \widehat{E}_y|_{z=0}$ is analytic. Therefore, we have established that

Proposition 2 *The Fourier transform of the convolution kernel in the linearized relation (27b), $\widehat{K}(\xi, \eta)$, is analytic in $\mathbb{R}^2$.*

With the last result, we obtain the main result of this section

Theorem 1 (Uniqueness of the linearized problem) *Given impedance data $\delta Z^\varepsilon(x_0, y_0)$ for all (x_0, y_0) , the linearized inverse problem (27b) admits a unique loss profile $\varepsilon\theta(x, y)$ in $L^2(\{z = 0\}) \cap L^\infty(\{z = 0\})$.*

Proof:

We only need to show that if $\delta Z^\varepsilon(x_0, y_0) = 0$, then the only solution to (27b) is $\theta(x, y) = 0$. In the Fourier transform domain, what we must show is that if $\widehat{K}(\xi, \eta)\widehat{\theta}(\xi, \eta) = 0$, then $\widehat{\theta}(\xi, \eta) = 0$.

Suppose that $\theta(x, y) \neq 0$, and $\widehat{K}(\xi, \eta)\widehat{\theta}(\xi, \eta) = 0$. Then it means that $\widehat{K}(\xi, \eta)$ must vanish for some sets of nonzero measure in the (ξ, η) -plane. But this is a contradiction with the analyticity property about $\widehat{K}(\xi, \eta)$ we have just proven in Proposition 2: if $\widehat{K}(\xi, \eta) = 0$ in some set of positive measure, then it must vanish everywhere. Therefore, we must conclude that the only way for $\widehat{K}(\xi, \eta)\widehat{\theta}(\xi, \eta) = 0$ for all (ξ, η) is for $\theta \equiv 0$. ■

7 Numerical calculations

From the representation of the convolution kernel $\widehat{K}(\xi, \eta)$, we expect the linearized inverse problem in (27b) to be illposed. Note that $\widehat{K}$ is the sum of the convolutions of $\widehat{E}_x^0$ and $\widehat{E}_y^0$ with themselves. In the analysis leading to Proposition 2, we showed that both $\widehat{E}_x^0$ and $\widehat{E}_y^0$ decay like $\rho^{-\frac{1}{2}}e^{-(h+t)\rho}$. This means that $K(x, y)$ is a very smooth kernel. We expect small high wave number error in the data δZ^ε would lead to large errors in $\theta(x, y)$ when inverting the relation (27b). For this reason, it will be important to carefully examine the shape of $K(x, y)$ and also to incorporate regularization strategy in solving the inverse problem.

Before we proceed, we introduce scalings that would make the problem easier to describe and understand. For an operating frequency of ω , skin depth is defined as

$$d_s = \sqrt{\frac{2}{\omega\mu\sigma_0}}.$$

This is a measure of how far an electromagnetic field can penetrate a conductive material. We will normalize distances by $1/\sqrt{\omega\mu\sigma_0}$. Hence,

$$\begin{aligned} (x, y, z) &\rightarrow \frac{(x, y, z)}{\sqrt{\omega\mu\sigma_0}} \\ \kappa_\pm &\rightarrow \sqrt{\omega\mu\sigma_0} \kappa_\pm \\ (\xi, \eta) &\rightarrow \sqrt{\omega\mu\sigma_0} (\xi, \eta) \end{aligned}$$

Figure 3: The real and imaginary parts of the convolution kernel. Due to radial symmetry, the graphs are shown in polar coordinates.

The field $\hat{E}^0$ in (28) can now be redefined as

$$\hat{E}^0(\xi, \eta, 0) = 2r_0\kappa_- \gamma J_1(r_0\rho) \begin{pmatrix} \sin \psi \\ -\cos \psi \\ 0 \end{pmatrix},$$

where

$$\gamma = \frac{i e^{i\kappa_+ t}}{(\kappa_- - \kappa_+)^2 e^{i\kappa_- h} - (\kappa_- + \kappa_+)^2 e^{-i\kappa_- h}},$$

and

$$\kappa_+^2 = \lambda - \rho^2 \quad \kappa_-^2 = \lambda + i - \rho^2.$$

Another physical parameter

$$\lambda = \frac{\omega\epsilon}{\sigma_0},$$

is needed to specify a problem. Finally, the kernel $K(x, y)$ has been normalized to

$$K(x, y) = -\frac{1}{\sigma_0} E^0(x, y, 0) \cdot E^0(x, y, 0).$$

Thus, from now on, distances are measured roughly in units of skin depth.

To get a sense of the illconditioning of the convolution equation (27b) we must take a look at the kernel. In Figure 3, we display the real and imaginary parts of the convolution kernel $K(x, y)$ (without the factor of $1/\sigma_0$) for $\lambda = 0.001$. The normalized radius of the loop is $r_0 = 0.35$, and the air gap is $t = 0.2$. The plate has normalized thickness $h = 1$. The plots are shown in polar coordinates since $K(x, y)$ is radially symmetric.

Figure 4: The setup for our numerical experiments. The scan area is indicated where impedance values are sampled regularly. The loss profile is in the form of the letter ‘M’, and is assumed to be inside the square of size 8.

As can be seen in the figures, the kernel is very smooth. We therefore expect inversion of the convolution equation (27b) to be illconditioned. To ameliorate the illposedness, we will consider 3 inversion strategies:

1. Least squares inversion,
2. Least squares with positivity constraints,
3. Least squares with minimum total variation penalty,

and compare the results. We generate data that are consistent with the linearization, however, we will consider the effect of data noise. The *only* reason why we have not used nonlinear data is that we do not have access to computer programs for solving the full three-dimensional Maxwell equations. We hope to try out methods on nonlinear data as well as on real data in the future. For now, we will only simulate real, nonlinear data by adding noise to numerically calculated, linearized data.

The problem we will attempt to solve involves determination of a loss profile in the form of the letter ‘M’ as shown in Figure 4. The height of the loss profile is $0.03125 (= 1/32)$. We will absorb the factor of ε in $\theta(x, y)$ in the convolution equation (27b). The geometry of the plate and the probe are: $h = 1$, $t = 0.2$ and $r_0 = 0.35$. We again assume $\lambda = 0.001$. Data are collected over a finite region of size 6×6 at a sampling interval of 0.25 for a total of 625 complex numbers. The unknown loss profile is assumed to occur inside a square of 8×8 , and has been discretized as piecewise constant with pixel size of $0.0625 (= 1/16)$ for a total of 16384 unknowns.

Figure 5: Data for the numerical experiments. The figures in the left column represent the real part of the impedance change, while those on the right represent the imaginary part. The unknown profile is in the shape of the letter ‘M’, and is outlined in the greylevel plots for reference. Observe that while the greylevel images of the real and imaginary parts of impedance indicate the presence of material loss, the exact shape and size of the loss profile is not discernible from the raw data.

The integral in the convolution with the kernel $K(x, y)$ will be evaluated by using piecewise constant approximations as well. This approximation is justified by the smoothness of the kernel.

The data for the problem are shown in Figure 5. Displayed are the real and imaginary parts of the impedance and their surface plots. Overlaid in the greylevel plots are the true locations of the loss profile. Note that the data do indicate the presence of the material loss, but the shape and the size of the loss profile cannot be easily determined from the data.

We provide brief descriptions of the methods used in inversion. Because of the large number of unknowns in the loss profile, we chose methods that are iterative, and thus requiring minimal storage.

1. Least squares inversion In this approach, we simple attempt to minimize the misfit

$$J(\theta) := \sum_j \left| \int \int K(x_{0j} - x, y_{0j} - y) \theta(x, y) dx dy - g_j \right|^2.$$

Here g_j represent the (complex) impedance reading at location (x_{0j}, y_{0j}) . The computational approach we use is conjugate gradients method [9]. Because we have more unknowns than equations, iterations are stopped after a fixed number.

2. Least squares with positivity constraints We further assume that the material loss is the only possible type of unknown, thus, the loss profile satisfy $\theta(x, y) \geq 0$. The use of this constraint has the effect of stabilizing the inversion. We use a conjugate projected gradient method proposed by Dembo and Tulowitzki [6].

3. Least squares with minimum total variation penalty Successful use of total variation in image restoration and inverse problems have been documented in various publications [12], [13], [10], [17], [7], [18]. Its theoretical properties have been analyzed in [8]. For our problem, we simply add a penalty term

$$R(\theta) := \alpha \int |\nabla \theta(x, y)| dx dy.$$

We use the method proposed by Vogel [18] because of its iterative nature. In this method, the total variation is further approximated with a smooth function (see reference for details). We did not implement a multigrid approach as suggested in the cited work.

The method requires the user to choose two parameters: the penalty parameter α and the smoothing parameter in the approximation of the total variation. Our experience with the method indicates that the choice of α is crucial for the success of the method. As there is no automatic way of

finding the optimal value, we experimented with several values until we were able to obtain satisfactory results. Of the three methods, this is the most computationally intensive.

Reconstructions of the loss profile are shown in Figure 6. Displayed starting on the top left is the true loss profile, on the top right is the profile obtained by leastsquares inversion, on the bottom left is the profile with the added positivity constraints, and on the bottom right is the profile obtained with the added total variation penalty. The number of iterations for the first two is 300, whereas for the last method, it is 20 outer iterations with 100 inner iterations, amounting to an equivalent of 2000 conjugate gradient iterations. The penalty parameter $\alpha = 10^{-16}$ appears to work well for this case. Note that in all cases, we are able to obtain a clear picture of the shape of the material loss, and that the height of the material loss, is well estimated: the maximum heights in the reconstructions are 0.0385 for Method 1, 0.0388 for Method 2, and 0.0316 for Method 3.

Next, we wish to assess the behavior of our reconstruction algorithms when the data are noisy. To this end, we added 5% RMS pseudo-random noise to the data. The results of our reconstructions are shown in Figure 7. On the top left is the true loss profile, on the top right, the profile obtained by the least squares method. The conjugate gradient iterations were halted after 50 iterations. On the bottom left, we show the reconstruction with positivity constraints after 100 iterations, and on the bottom right, the reconstruction with total variation penalty ($\alpha = 2 \times 10^{-15}$). The estimate for the depth of the loss profile was not very accurate in Methods 1 and 2, however, it is still quite good for Method 3.

It appears that total variation is the best of the 3 methods we considered so far. However, it is also the most intensive computationally. Perhaps other ideas such as the multigrid [18] should be implemented in order to speed up computations.

8 Conclusions

We analyzed the problem of determining material loss in a plate using eddy current methods. By linearizing the relation between impedance change and the material loss profile, we arrive at a convolutional model of the problem. We show that impedance data over the entire plate uniquely determine the loss profile in this model. Nevertheless, the linearized inverse problem is illposed, and regularization strategies must be employed in order to be able to robustly invert data. We implemented 3 methods and did extensive numerical

Figure 6: Numerical reconstructions from synthetic data. On the top left is the true loss profile, on the top right is the profile obtained using least squares (Method 1), and on the bottom row, the effects of adding positivity (Method 2), and total variation penalty (Method 3). In all the cases, good estimate of material loss depth are obtained.

Figure 7: Numerical reconstructions from noisy synthetic data. On the top left is the true loss profile, on the top right is the profile obtained using least squares (Method 1), and on the bottom row, the effects of adding positivity (Method 2), and total variation penalty (Method 3).

tests with linearized data, both noisy and noiseless. The tests show that a regularized inversion can indeed produce accurate estimates of the shape and size of material loss from eddy current data. It would be interesting to attempt to invert real data using some of the computational approaches presented in this work.

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