THE POROUS MEDIUM EQUATION AS A FINITE-SPEED APPROXIMATION TO
A HAMILTON-JACOBI EQUATION

BY
D.G. ARONSON
AND
J.L. VAZQUEZ

IMA Preprint Series # 143

May 1985

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455

#	Author(s)	Title	#	Author(s)	Title
1	Workshop	Summaries from the September 1982 Workshop on Statistical Mechanics, Dynamical Systems and Turbulence	40	William Ruckle,	The Strong ϕ Topology on Symmetric Sequence Spaces
2	Raphael De laLave,	A Simple Proof of C. Siegel's Center Theorem	41	Charles R. Johnson,	A Characterization of Borda's Rule Via Optimization
3	H. Simpson, S. Spector,	On Copositive Matrices and Strong Ellipticity for Isotropic Elastic Materials	42	Hans Weinberger, Kazuo Kishimoto,	The Spatial Homogeneity of Stable Equilibria of Some Reaction-Diffusion Systems on Convex Domains
4	George R. Sell,	Vector Fields in the Vicinity of a Compact Invariant Manifold	43	K.A. Perlick-Spector, W.O. Williams,	On Work and Constraints in Mixtures
5	Milan Miklavic,	Non-linear Stability of Asymptotic Suction	44	H. Rosenberg, E. Toubiana,	Some Remarks on Deformations of Minimal Surfaces
6	Hans Weinberger,	A Simple System with a Continuum of Stable Inhomogeneous Steady States	45	Stephan Pelikan,	The Duration of Transients
7	Bau-Son Du,	Period 3 Bifurcation for the Logistic Mapping	46	V. Capasso, K.L. Cooke, M. Witten,	Random Fluctuations of the Duration of Harvest
8	Hans Weinberger,	Optimal Numerical Approximation of a Linear Operator	47	E. Fabes, D. Stroock,	The L^p -Intergrability of Green's Functions and Fundamental Solutions for Elliptic and Parabolic Equations
9	L.R. Angel, D.F. Evans, B. Ninham,	Three Component Ionic Microemulsions	48	H. Brezis,	Semilinear Equations in R^N without Conditions at Infinity
10	D.F. Evans, D. Mitchell, S. Mukherjee,	B. Ninham, Surfactant Diffusion; New Results and Interpretations	49	M. Stenrod,	Lax-Friedrichs and the Viscosity-Capillarity Criterion
11	Leif Arkerlyd,	A Remark about the Final Aperiodic Regime for Maps on the Interval	50	C. Johnson, W. Barrett,	Spanning Tree Extensions of the Hadamard-Fischer Inequalities
12	Luis Magalhães,	Manifolds of Global Solutions of Functional Differential Equations	51	Andrew Postlewaite, David Schmüdler,	Revelation and Implementation under Differential Information
13	Kenneth Meyer,	Tori in Resonance	52	Paul Blanchard,	Complex Analytic Dynamics on the Riemann Sphere
14	C. Eugene Wayne,	Surface Models with Nonlocal Potentials: Upper Bounds	53	G. Levitt, H. Rosenberg,	Topology and Differentiability of Labyrinths in the Disc and Annulus
15	K.A. Perlick-Spector,	On Stability and Uniqueness of Fluid Flow Through a Rigid Porous Medium	54	G. Levitt, H. Rosenberg,	Symmetry of Constant Mean Curvature Hyper-surfaces in Hyperbolic Space
16	George R. Sell,	Smooth Linearization Near a Fixed Point	55	Ennio Stecchetti,	Analysis of a Dynamic, Decentralized Exchange Economy
17	David Wolkind,	A Nonlinear Stability Analysis of a Model Equation for Alloy Solidification	56	Henry Simpson, Scott Spector,	On Failure of the Complementing Condition and Nonuniqueness in Linear Elastostatics
18	Pierre Collet,	Local C^∞ Conjugacy on the 2-torus Set for some Holomorphic Perturbations of $z \mapsto z^2$	57	Craig Tracy,	Complete Integrability in Statistical Mechanics and the Yang-Baxter Equations
19	Henry C. Simpson, Scott J. Spector,	On the Modified Bessel Functions of the First Kind / On Barrelling for a Material in Finite Elasticity	58	Tongren Ding,	Boundedness of Solutions of Duffing's Equation
20	George R. Sell,	Linearization and Global Dynamics	59	Abstracts for the Workshop on Price Adjustment, Quantity Adjustment, and Business Cycles	
21	P. Constantin, C. Fofas,	Global Lyapunov Exponents, Kaplan-Yorke Formulas and the Dimension of the Attractors for 2D Navier-Stokes Equations	60	Rafael Rob,	The Coase Theorem an Informational Perspective
22	Milan Miklavic,	Stability for Semilinear Parabolic Equations with Noninvertible Linear Operator	61	Joseph Jerome,	Approximate Newton Methods and Homotopy for Stationary Operator Equations
23	P. Collet, H. Epstein, G. Gallavotti,	Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and their Mixing Properties	62	Rafael Rob,	A Note on Competitive Bidding with Asymmetric Information
24	J.E. Dunn, J. Serrin,	On the Thermodynamics of Interstitial Working	63	Rafael Rob,	Equilibrium Price Distributions
25	Scott J. Spector,	On the Absence of Bifurcation for Elastic Bars in Uniaxial Tension	64	William Ruckle,	The Linearization Projection, Global Theories
26	M.A. Coppel,	Maps on an interval	65	Russell Johnson, Kenneth Palmer, George R. Sell,	Ergodic Properties of Linear Dynamical Systems
27	James Kirkwood,	Phase Transitions in the Ising Model with Transverse Field	66	Stanley Reiter,	How a Network of Processors can Schedule its Work
28	Luis Magalhães,	The Asymptotics of Solutions of Singularly Perturbed Functional Differential Equations: and Concentrated Delays are Different	67	R.N. Goldman, D.C. Heath,	Linear Subdivision Is Strictly a Polynomial Phenomenon
29	Charles Tresser,	Homoclinic Orbits for Flow in R^3	68	R. Gluckert, R. Johnson,	The Floquet Exponent for Two-dimensional Linear Systems with Bounded Coefficients
30	Charles Tresser,	About Some Theorems by L.P. Sil'nikov	69	Steve Williams,	Realization and Nash Implementation: Two Aspects of Mechanism Design
31	Michael Alzenmann,	On the Renormalized Coupling Constant and the Susceptibility in ϕ_4 Field Theory and the Ising Model in Four Dimensions	70	Steve Williams,	Sufficient Conditions for Nash Implementation
32	C. Eugene Wayne,	The KAM Theory of Systems with Short Range Interactions I	71	Nicholas Yannellis, William R. Zame,	Equilibria in Banach Lattices without Ordered Preferences
33	M. Slemrod, J. E. Marsden,	Spatial Chaos in a Van der Waals Fluid Due to Periodic Thermal Fluctuations	72	W. Harris, Y. Sibuya,	The Reciprocals of Solutions of Linear Ordinary Differential Equations
34	J. Kirkwood, C.E. Wayne,	Percolation in Continuous Systems	73	Steve Pelikan,	A Dynamical Meaning of Fractal Dimension
35	Luis Magalhães,	Invariant Manifolds for Functional Differential Equations Close to Ordinary Differential Equations	74	D. Heath, W. Sudderth,	Continuous-Time Portfolio Management: Minimizing the Expected Time to Reach a Goal
36	C. Eugene Wayne,	The KAM Theory of Systems with Short Range Interactions II	75	J.S. Jordan,	Information Flows Intrinsic to the Stability Economic Equilibrium
37	Jean De Canniere,	Passive Quasi-Free States of the Noninteracting Fermi Gas	76	J. Jerome,	An Adaptive Newton Algorithm Based on Numerical Inversion: Regularization Post Condition
38	Elias C. Alifantis,	Maxwell and van der Waals Revisited	77	David Schmüdler,	Integral Representation Without Additivity
39	Elias C. Alifantis,	On the Mechanics of Modulated Structures			

THE POROUS MEDIUM EQUATION AS A FINITE-SPEED APPROXIMATION TO
A HAMILTON-JACOBI EQUATION

by

D.G. Aronson^{1,2}
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

and

J.L. Vazquez^{1,3}
Division de Matematicas
Universidad Autonoma de Madrid
28049 Madrid, SPAIN

¹Partially supported by USA-Spain Cooperation Agreement under Joint Research Grant CCB-8402023.

²Partially supported by National Science Foundation Grant DMS 83-01247.

³Long-term visitor at the Institute for Mathematics and its Applications, 1985.

Abstract

It is known that solutions of the porous medium equation $u_t = (u^m)_{xx}$ converge to solutions of the heat equation $u_t = u_{xx}$ as $m \rightarrow 1$ if the initial datum $u(x,0)$ is kept fixed. For porous medium flow u represents a suitably scaled density and $v = mu^{m-1}/(m-1)$ represents the pressure. We prove that v converges to a solution of the Hamilton-Jacobi equation $v_t = (v_x)^2$ as $m \rightarrow 1$ if $v(x,0)$ is fixed. Moreover, if $v(x,0)$ has compact support the interface for the porous medium equation tends to the interface for the latter equation. The limit $m \rightarrow 1$ is also discussed. In this fast-diffusion case no interfaces appear.

INTRODUCTION AND RESULTS

The density $u = u(x,t)$ of an ideal gas flowing isentropically through a one-dimensional ($x \in \mathbb{R}$) porous medium obeys the equation

$$u_t = (u^m)_{xx} \quad \text{in } Q = \mathbb{R} \times \mathbb{R}^+ \quad (0.1)$$

where $m > 1$ is a constant. It is known, [BC1], that the solutions u to (0.1) depend continuously in the $C(\mathbb{R}^+; L^1(\mathbb{R}))$ - norm on both the initial datum $u(\cdot, 0) = u_0 \in L^1(\mathbb{R})$ and on m . In particular if u_0 is kept fixed and $m \downarrow 1$, then $u = u(x, t; m)$ converges to a solution of the heat conduction equation

$$u_t = u_{xx} \quad (0.2)$$

with initial datum u_0 . Thus, for m near 1, the porous medium equation can be regarded as a perturbation of the heat equation.

Despite the convergence of solutions of the nonlinear equation (0.1) to solutions of the linear equation (0.2), there is a marked difference in the behaviour of solutions of these equations stemming from the fact that (0.1) is of degenerate parabolic type. Perhaps the most striking consequence of that degeneracy is the finite speed of propagation of disturbances from rest for the porous medium equation as opposed to the infinite speed associated with the heat equation. Specifically, suppose that u_0 is a continuous, nonnegative and bounded real function such that $u_0 = 0$ on $\mathbb{R}^+$ and $u_0 \not\equiv 0$. Then there exists a unique continuous, nonnegative and bounded function $u(x, t)$ in Q which solves (0.1) in a generalized sense and is such that $u(\cdot, 0) = u_0$, cf. [OKC], [AB]. Moreover the support of $u(\cdot, t)$ is bounded away from $x = \infty$ for every $t > 0$ and the finite function

$$\zeta(t) = \sup\{x \in \mathbb{R} : u(x, t) > 0\} \quad (0.3)$$

is continuous and nondecreasing in $\mathbb{R}^+$. The curve $x = \zeta(t)$ is called the (right-hand) interface of u . If the support of u_0 is not an interval of the form $(-\infty, a)$ other interfaces will exist, but their properties are essentially the

same as those of the right-hand interface. The interfaces have been the object of intense study in recent years and their behaviour is now reasonably well understood. Detailed results and references can be found in [V3]. In particular the interface exists and has the properties described above as long as the initial datum is a nonnegative function in $L^1_{loc}(\mathbb{R})$ which grows at most like $O(|x|^{2/(m-1)})$ as $|x| \rightarrow \infty$, cf. [AC], [BCP], [DK].

The finite speed of propagation associated with the porous medium equation is, of course, reminiscent of hyperbolic equations. Our purpose in this paper is to investigate the precise nature of the relationship between (0.1) and the Hamilton-Jacobi equation

$$v_t = (v_x)^2, \quad (0.4)$$

sometimes called the nonstationary eikonal equation. To obtain (0.4) as the limit of (0.1) we proceed as follows. In view of the application in mind it is natural to restrict attention to nonnegative solutions of (0.1). We can then replace the variable u by the corresponding scaled pressure

$$v = \frac{m}{m-1} u^{m-1}. \quad (0.5)$$

Formally v satisfies the equation

$$v_t = (m-1)vv_{xx} + (v_x)^2. \quad (0.6)$$

The local velocity of the flow at any point $(x,t) \in Q$ is given by the function $-v_x$ (Darcy's law) and the interface is characterized by the relationship

$$\zeta'(t) = -v_x(\zeta(t), t), \quad (0.7)$$

where $v_x(\zeta(t), t)$ means $\lim v_x(x, t)$ as $x \rightarrow \zeta(t)$ with $t > 0$, cf [A2], [Kn]⁽¹⁾.

Equations (0.4) and (0.6) share the property of finite propagation speed. Thus we can view (0.6) as a finite-speed viscous approximation to (0.4). In fact (0.4) and (0.6) formally agree on the interfaces. This agreement has been rigorously established in [CF] where it is shown that $v_t - (v_x)^2 \rightarrow 0$ as $(x, t) \rightarrow (\zeta(t_0), t_0)$ with $t_0 > 0$ and $v(x, t) > 0$ ⁽²⁾.

On the other hand, equations (0.4) and (0.6) formally agree everywhere when $m \rightarrow 1$ and it is this limit which is our main concern here. Set

$$\varepsilon = m-1.$$

For every $\varepsilon > 0$ let $v_{\varepsilon 0}$ be a continuous, nonnegative and bounded real function and consider the initial-value problem

$$v_t = \varepsilon v v_{xx} + (v_x)^2 \quad \text{in } Q, \quad (P_\varepsilon)$$

$$v(x, 0) = v_{\varepsilon 0}(x) \quad \text{in } \mathbb{R}.$$

For $\tau > 0$ let $Q_\tau = \{(x, t) \in Q : t > \tau\}$. The following is our main convergence result:

THEOREM 1. (A) Suppose that for every small $\varepsilon > 0$, $v_{0\varepsilon}$ is a continuous real function such that

$$0 < v_{\varepsilon 0} < N \quad (0.8)$$

⁽¹⁾ If the time t^* at which the interface starts to move (waiting time) is positive then ζ may not be differentiable. In that case formula (0.7) holds at $t = t^*$ with $\zeta'(t)$ replaced by the right-hand derivative $D^+\zeta(t^*)$, cf. [CF], [ACK], [ACV].

⁽²⁾ At $t_0 = t^*$ we also need $t > t_0$.

for some constant $N > 0$ and $\{v_{\epsilon 0}\}$ converges to a function v_0 uniformly on compact subsets of $\mathbb{R}$ as $\epsilon \rightarrow 0$. Let $v_{\epsilon} = v_{\epsilon}(x,t)$ denote the solution to (P_{ϵ}) . Then as $\epsilon \rightarrow 0$, the family $\{v_{\epsilon}\}$ converges uniformly on compact subsets of $\bar{Q}$ to a function $v \in C(\bar{Q})$ such that

- i) $v \in \text{Lip}(Q_{\tau})$ for every $\tau > 0$ and $v_t = (v_x)^2$ a.e. in Q ,
- ii) $v(x,0) = v_0(x)$ for all $x \in \mathbb{R}$,
- iii) $v_{xx} > -1/2t$ in $D'(Q)$.

Moreover $v_{\epsilon x} \rightarrow v_x$ in $L^p_{\text{loc}}(Q)$ for every $1 < p < \infty$.

(B) The limit function $v = v(x,t)$ is uniquely characterized as a solution in $C(\bar{Q})$ of the initial-value problem

$$\begin{cases} v_t = (v_x)^2 & \text{in } Q, \\ v(\cdot, 0) = v_0 & \text{in } \mathbb{R} \end{cases} \quad (P_0)$$

by the semiconvexity property (iii).

The proof of Theorem 1 is given in section 2 after the required estimates for v_{ϵ} and its derivatives are derived in Section 1. Here we shall discuss some relevant aspects of our result. First, note that the convergence of v_{ϵ} to v is compatible with the statement that u_{ϵ} converges to a solution of the heat equation. This is possible since (0.5) implies that $u_{\epsilon} \rightarrow 0$ uniformly in Q as $\epsilon \rightarrow 0$ if v_{ϵ} is bounded.

The semiconvexity property (iii), which selects the "correct" solution of problem (P_0) , is an immediate consequence of the estimate

$$v_{xx} > -\frac{1}{(\epsilon+2)t}, \quad (0.9)$$

valid for all nonnegative solutions of (0.1) ([AB]). A similar estimate, $\Delta u \geq -k/t$, also holds in several space dimensions, the equation being then $u_t = \Delta(u^m)$ in $Q = \mathbb{R}^d \times \mathbb{R}^+$ with $d > 1$, and has played a key role in the general theory of the porous medium equation, cf. [AC], [BCP]. Theorem 1(B) sheds a new light on its significance, but it would be interesting to better understand its physical meaning (in this connection see (0.13) below).

Crandall and P.L. Lions, [CL], (see also [CEL]) have recently introduced the notion of viscosity solution to characterize the "good" solutions of Hamilton-Jacobi equations. To be specific, if the equation is

$$u_t + H(Du) = 0 \quad \text{in } Q = \mathbb{R}^d \times (0, T) \quad (0.10)$$

with $H: \mathbb{R}^d \rightarrow \mathbb{R}$ continuous and $Du = (u_{x_1}, \dots, u_{x_d})$, then a function $u \in C(Q)$ is a viscosity solution of (0.10) if for every $\phi \in C^1(Q)$ we have

$$\phi_t + H(D\phi) \leq 0 \quad (0.11.a)$$

at all local maxima of $u - \phi$ and

$$\phi_t + H(D\phi) \geq 0 \quad (0.11.b)$$

at all local minima of $u - \phi$. Our limit function $v(x, t)$ of Theorem 1 is a viscosity solution of $v_t = (v_x)^2$ because v_x is locally bounded and v_{xx} is locally bounded below in Q . It then follows from Theorem 10.2 of [Li].

It should also be noted that since our estimates break down at $t=0$ the current uniqueness proofs for viscosity solutions do not apply. In fact our uniqueness result, Theorem 1(B), is an improvement of Lemma 2.1 of [B] to cover the case in which the bounds for the derivatives are not uniform in x, t . As explicit examples show, the estimates which we derive i.e. $v_x = O(t^{-1/2})$, $v_{xx} \geq O(1/t)$, are actually attained for general initial data.

The idea that solutions of $v_t = (v_x)^2$ must approximate to leading order solutions of $v_t = \epsilon v v_{xx} + (v_x)^2$ for very small $\epsilon > 0$ is used by Kath and Cohen in [KC]

where they study shock formation at the waiting-time on the interfaces of solutions of (0.6) for ε small using singular perturbation methods.

Since the equation $v_t = (v_x)^2$ is invariant under translations, in particular in v , the restriction to positive bounded solutions is equivalent to working with just bounded solutions. However, setting the lower bound at $u=v=0$ plays an important role in the convergence discussed above because of the degeneracy of the diffusion term εv_{xx} when v vanishes.

We also prove convergence of the interfaces which appear when $v_{\varepsilon 0}$ and v_0 vanish in some interval. To make things simple suppose that $v_{\varepsilon 0} \equiv v_0$ and v_0 vanishes in $\mathbb{R}^+$, but $v_0 \not\equiv 0$. Let $\zeta_\varepsilon(t)$ denote the (right-hand) interface for the solution v_ε of (P_ε) and let $\zeta(t)$ denote the interface for the solution v to (P_0) . Then $\zeta_\varepsilon(t)$ and $\zeta(t)$ are Lipschitz continuous, nondecreasing functions of t for $0 \leq t < \infty$ and we have the following convergence result.

THEOREM 2. As $\varepsilon \rightarrow 0$ we have $\zeta_\varepsilon(t) \rightarrow \zeta(t)$ uniformly on $[0, T]$ for every $T > 0$ and $\zeta'_\varepsilon(t) \rightarrow \zeta'(t)$ a.e and in $L^p_{loc}(\mathbb{R}^+)$ for every $p \in [1, \infty)$.

Further properties of the convergence of the interfaces, as well as proof of these results are given in Section 3.

The convergence results of Theorems 1 and 2 cannot be substantially improved because of the lack of regularity of the solution v to (0.4). In fact v_x is in general discontinuous and so is $\zeta'(t)$, cf. [D] or [La]. Stronger convergence results are obtained in Section 4 under the further assumption that the initial data $v_{\varepsilon 0}$ are concave on their support, cf. Theorem 3. The result depends on the fact that, for concave data, v is a C^1 function on the set where it does not vanish and $\zeta \in C^1[0, \infty)$. We also prove that if $v_{0\varepsilon} \rightarrow v_0$ as $\varepsilon \rightarrow 0$ then v_ε also converges monotonically to v (Theorem 4).

As we noted above the local velocity of propagation of solutions of (0.1) is given by $w = -v_x$. As a consequence of Theorem 1 it follows that the family

$w_\varepsilon = v_{\varepsilon x}$, $\varepsilon > 0$, converges in $L^p_{loc}(Q)$ to a solution w of the conservation law

$$w_t + (w^2)_x = 0 \quad \text{in } D'(Q) \quad (0.12)$$

and w satisfies the entropy condition

$$w_x \leq \frac{1}{2t} \quad (0.13)$$

If in addition v_{0x} exists in a suitable sense then w is the unique distributional solution of (0.12) satisfying (0.13) and taking the initial value $w(x,0) = -v'_0(x)$, cf. [U], [LP].

It is also of some interest to consider what happens if we take the limit $m \rightarrow 1$ for $m < 1$ in (0.6). For $m \in (0,1)$ the initial value problem for (0.1) has a unique solution provided that $u(x,0)$ is nonnegative and locally integrable ([AB], [HP]). Moreover $u \in C^\infty(Q)$ and is positive everywhere in Q so that, in particular, there are no interfaces. The analog of Theorem 1 with $m \rightarrow 1$ is proved in Section 5. Note that in this case v , which is still defined by (0.5), is negative.

Before we turn to the proofs of our results we pause to describe an important connection between equations (0.1) or (0.6) and (0.4). Let $m > 1$. Recall that if u is a nonnegative solution of (0.1) then v satisfies

$$v_t = (m-1)vv_{xx} + (v_x)^2$$

and

$$v_{xx} \geq -\frac{1}{(m+1)t} \cdot \quad (0.15)$$

It follows that

$$v_t + \frac{m-1}{(m+1)t} v \geq (v_x)^2 \quad (0.16)$$

Now define

$$V(x, \tau) = v(x, t) t^{\frac{m-1}{m+1}}, \quad (0.17.a)$$

where t and τ are related by

$$\tau = \frac{m+1}{2} t^{\frac{2}{m+1}}. \quad (0.17.b)$$

With this change of variables (0.16) and (0.15) become

$$V_\tau \geq (V_x)^2 \quad \text{and} \quad V_{xx} \geq -\frac{1}{2\tau}. \quad (0.18)$$

Thus if u is a solution of porous medium equation (0.1) then the change of variables (0.5), (0.17) transforms it into a viscosity supersolution of the equation (0.4). (1)

The equalities in (0.18) hold for the Barenblatt solutions

$$\bar{u}(x, t) = t^{-\frac{1}{m+1}} \left(C - \frac{m-1}{2m(m+1)} \frac{x^2}{t^{2/(m+1)}} \right)_+^{\frac{1}{m-1}}, \quad (0.19)$$

where $(\cdot)_+$ means $\max(\cdot, 0)$. For each $C > 0$ this function is a solution of (0.1) with initial data which is a multiple of the Dirac measure concentrated at $x=0$. If $\bar{v}$ is defined through (0.5) from $\bar{u}$ then

$$\bar{v}_{xx} = -\frac{1}{(m+1)t}$$

whenever $\bar{u} > 0$. The change of variables (0.17) gives

$$V(x, \tau) = \left(K - \frac{x^2}{4\tau} \right)_+ \quad \text{with} \quad K = \frac{Cm}{m-1}. \quad (0.20)$$

The functions (0.20) are the well-known bounded self-similar solutions to $V_\tau = (V_x)^2$.

Note that the initial value is

$$V(x, 0) = \begin{cases} 0 & \text{if } x \neq 0, \\ K & \text{if } x = 0. \end{cases} \quad (0.21)$$

(1) We say that v is a viscosity supersolution of (0.10) if the condition (0.11.b) is satisfied. For the proof that v satisfying (0.18) is a viscosity supersolution see [Li], Theorem 10.2.

This rather striking correspondence has some deep consequences. It is known that as $t \rightarrow \infty$ every solution of (0.1) with $u(x,0) \in L^1(\mathbb{R})$, $u_0 > 0$ and $u_0 \not\equiv 0$, converges with the appropriate scaling to the Barenblatt solution $\bar{u}$ with the same mass, $\int u(x,t) dx = \int \bar{u}(x,t) dx = M$. Specifically

$$t^{1/(m+1)} \|u(\cdot, t) - \bar{u}(\cdot, t)\|_{\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

(cf. [K] and also [V1], [V2] for further details). The self-similar solutions (0.20) play the same role for bounded solutions of (0.4). Thus we see that the asymptotic behaviour of these classes of solutions for equations (0.1) and (0.4) coincide under the transformation (0.5), (0.17). This asymptotic similarity was first observed in [V2] where, in particular, it was shown that if $m > 1$ then the velocity $w = -v_x$ associated with the solutions of (0.1) behaves as $t \rightarrow \infty$ like the finite N-waves that are the typical profiles of solutions of first-order conservation laws if $m > 1$. Corresponding results hold for $m \leq 1$ as we show in Section 5.

The convergence results Theorem 1 and its analog for $m \leq 1$, Theorem 6, also hold for $x \in \mathbb{R}^d$ for any $d > 1$. While our discussion of the case $m \leq 1$ in Section 5 is valid in several space dimensions, the case $m > 1$ requires new estimates and will be studied in [LSV].

1. ESTIMATES

In this section we collect various results for the solutions of problem (P_{ϵ}) that are needed in the sequel. In particular we obtain several estimates, none of which is entirely new, but we shall give new proofs in some cases in order to get the precise dependence on ϵ .

We consider a family of measurable, nonnegative and bounded initial functions $\{v_{0\epsilon} : \epsilon > 0\}$ defined on the real line. We further assume that they are bounded uniformly in ϵ , i.e., there exists a constant $N > 0$ such that

$$0 \leq v_{\varepsilon 0}(x) \leq N \quad (1.1)$$

for a.e. $x \in \mathbb{R}$ and every $\varepsilon > 0$. We define the corresponding initial densities by

$$u_{\varepsilon 0}(x) = \left(\frac{\varepsilon}{\varepsilon+1} v_{\varepsilon 0}(x) \right)^{1/\varepsilon}$$

For every $\varepsilon > 0$ there exists a unique bounded continuous function u_{ε} defined in Q which satisfies

$$\begin{aligned} u_t &= (u^{1+\varepsilon})_{xx} \quad \text{in } \mathcal{D}'(Q), \\ u_{\varepsilon}(\cdot, t) &\rightarrow u_{\varepsilon 0} \quad \text{in } L^1_{loc}(\mathbb{R}) \text{ as } t \downarrow 0. \end{aligned}$$

This follows from the known existence and uniqueness theory for the porous medium equation, cf. [OKC], [AC], [BCP], [DK]. If we recover the pressure v_{ε} by inverting the change of variables, i.e.

$$v_{\varepsilon}(x, t) = \frac{\varepsilon+1}{\varepsilon} u(x, t)^{\varepsilon},$$

then v_{ε} is a solution of (P_{ε}) :

$$v_t = \varepsilon v v_{xx} + (v_x)^2 \quad \text{in } Q, \quad (1.2.a)$$

$$v(x, 0) = v_{\varepsilon 0}(x) \quad \text{in } \mathbb{R}. \quad (1.2.b)$$

in the sense that v is continuous, nonnegative and bounded in Q , and satisfies

$$\iint_Q \{ \varepsilon v v_x \psi_x + (\varepsilon-1)(v_x)^2 \psi - v \psi_t \} = \int_{\mathbb{R}} v_{\varepsilon 0}(x) \psi(x, 0) dx$$

for every test function $\psi \in C^{\infty}_0(\overline{Q})$, cf. [A2]. Moreover u_{ε} and v_{ε} are C^{∞} functions on the open subset of Q where they are positive.

The solutions satisfy the maximum principle: if $v_{\varepsilon}^{(1)}$ and $v_{\varepsilon}^{(2)}$ are two solutions of (P_{ε}) with initial data $v_{\varepsilon 0}^{(1)}$ and $v_{\varepsilon 0}^{(2)}$ and $v_{\varepsilon 0}^{(1)} \geq v_{\varepsilon 0}^{(2)}$ a.e. then

$v_\varepsilon^{(1)} > v_\varepsilon^{(2)}$ everywhere in Q , cf. [OKC],[BCP],[DK]. From this we obtain our first estimate.

LEMMA 1.1. For every $(x,t) \in Q$ we have

$$0 < v_\varepsilon(x,t) < \|v_{\varepsilon 0}\|_{L^\infty(\mathbb{R})} < N. \quad (1.3)$$

The following remark also follows from the maximum principle. If $x_0 \in \mathbb{R}$ is such that $\text{ess lim inf } v_{\varepsilon 0}(x)$ is positive as $x \rightarrow x_0$ then for every $t > 0$ $v_\varepsilon(x_0, t) > 0$. (To prove this compare with a small Barenblatt solution centered at $x = x_0$). Using this remark we conclude that if $v_{\varepsilon 0}(x)$ is C^∞ and positive everywhere then u_ε is C^∞ in Q , cf [OKC]. Since the solutions u_ε depend continuously on the initial data $u_{\varepsilon 0}$, ([BC1], [BCP], [DK]), every solution can be suitably approximated by C^∞ solutions.

Our next estimates are taken from [AB]:

LEMMA 1.2. i) For every $t > 0$, $v_{\varepsilon xx}(\cdot, t)$ is a locally bounded measure on $\mathbb{R}$ and

$$v_{\varepsilon xx}(\cdot, t) > -\frac{1}{(\varepsilon+2)t} \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (1.4)$$

ii)

$$v_{\varepsilon t}(\cdot, t) > -\frac{\varepsilon v(\cdot, t)}{(\varepsilon+2)t} \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (1.5)$$

We remark that both estimates are sharp as can be seen by checking them on the Barenblatt solutions.

It was proved in [A1] that the velocity $v_{\varepsilon x}$ is bounded in $Q_\tau = \{(x,t) \in Q : t > \tau\}$ for every $\tau > 0$. We give a new and simple proof of this bound using Lemmas 1.1 and 1.2, which exhibits the explicit dependence of the bound on N, t and ε :

LEMMA 1.3. For every $t > 0$ the function $v_\varepsilon(\cdot, t)$ is Lipschitz continuous and satisfies

$$|v_{\varepsilon x}(x,t)|^2 \leq \frac{2}{(\varepsilon+2)t} \|v(\cdot, t)\|_{\infty} \leq \frac{2N}{(\varepsilon+2)t} \quad (1.6)$$

a.e. in Q .

REMARK. Again the bound is attained by the Barenblatt solution.

Proof. The argument proceeds at fixed time $t > 0$ by applying the estimates

$0 \leq v_{\varepsilon} \leq N$ and $v_{\varepsilon xx} \geq -((\varepsilon+2)t)^{-1}$ to the function $x \mapsto v(x,t)$.

For $t > 0$ and $y \in \mathbb{R}$ we define

$$\phi(x) = v(x+y,t) + \frac{x^2}{2(\varepsilon+2)t}.$$

Then ϕ is continuous, nonnegative and convex ($\phi'' \geq 0$ in $\mathcal{D}'(\mathbb{R})$). Therefore for every $h > 0$ we have

$$\phi(x+h) \geq \phi(x-h) - \phi(x) \geq \pm \phi'(x)h$$

Assume that $\phi'(x) \neq 0$. In the above inequality take the sign which gives

$|\phi'(x)|h$ in the right-hand member to get

$$|\phi'(x)| \leq \frac{1}{h} \|\phi\|_{L^{\infty}(x-h, x+h)}.$$

Letting $x = 0$ and taking into account the definition of ϕ this means that for every $y \in \mathbb{R}$

$$|v_{\varepsilon x}(y,t)| \leq \frac{1}{h} (\|v_{\varepsilon}(\cdot, t)\|_{L^{\infty}(\mathbb{R})} + \frac{1}{2(\varepsilon+2)t} h^2).$$

The result now follows by choosing h so as to minimize the right-hand member of the inequality, i.e.

$$h^2 = 2(\varepsilon+2)t \|v_{\varepsilon}(\cdot, t)\|_{L^{\infty}(\mathbb{R})}. \quad \#$$

In view of Lemma 1.3 the family $\{v_\epsilon\}$ is uniformly Lipschitz continuous with respect to x in Q_τ for $\tau > 0$ with Lipschitz constant $L_\tau = (2N/(\epsilon+2)\tau)^{1/2}$ independent of ϵ . Using the results of [G] the solutions are then Hölder-continuous in Q_τ with respect to the variable t with exponent $1/2$. We give a simple direct proof of this fact that shows the dependence on ϵ .

LEMMA 1.4. The family $\{v_\epsilon\}$ is uniformly Hölder-continuous with respect to t in Q_τ , $\tau > 0$, with exponent $1/2$. More precisely, for every $x \in \mathbb{R}$ and $t_1 > t_0 > \tau > 0$ we have

$$|v_\epsilon(x, t_1) - v_\epsilon(x, t_0)| \leq A\epsilon^{1/2} N^{1/2} L_\tau (t_1 - t_0)^{1/2} + B L_\tau^2 (t_1 - t_0) \quad (1.7)$$

for some constants $A, B > 0$ independent of ϵ, N, L_τ .

Proof. We may assume that $v_\epsilon \in C^\infty(Q)$. For convenience we temporarily drop the subscript ϵ . If x, t_1, t_0 and τ are as above, it follows from $|v_x| \leq L_\tau$ in Q_τ that for every $y \in \mathbb{R}$ such that $|y-x| \leq \lambda$ we have

$$|v(y, t_i) - v(x, t_i)| \leq L|y-x| \leq L\lambda, \quad (1.8)$$

where $L = L_\tau$. We want to estimate $h = v(x, t_1) - v(x, t_0)$ in terms of $\delta = t_1 - t_0$. Suppose, for example, that $h > 0$. Then take $\lambda < h/(2L)$ and set $I = [x-\lambda, x+\lambda]$ and $S = I \times [t_0, t_1]$. Integrating the equation $v_t = \epsilon(vv_x)_x + (1-\epsilon)v_x^2$ in S we obtain

$$\begin{aligned} \int_I (v(y, t_1) - v(y, t_0)) dy &= \int_I dy \int_{t_0}^{t_1} v_t(y, t) dt = \\ &= \epsilon \int_{t_0}^{t_1} \{(vv_x)(x+\lambda, t) - (vv_x)(x-\lambda, t)\} dt + (1-\epsilon) \iint_S (v_x)^2(y, t) dy dt. \end{aligned}$$

In view of (1.8) we have

$$v(y, t_1) - v(y, t_0) \geq v(x, t_1) - v(x, t_0) - 2L\lambda = h - 2\lambda L.$$

Therefore, using Lemmas 1 and 2, we find

$$2(h-2\lambda L)\lambda \leq 2\epsilon N\delta L + 2(1-\epsilon)L^2\lambda\delta$$

Now if we set $\lambda = \frac{h}{4L}$ we get

$$h^2 \leq 2\epsilon N L^2 \delta + 2(1-\epsilon)L^2 H \delta ,$$

from which it follows that

$$h \leq 2(1-\epsilon) L^2 \delta + (8\epsilon)^{1/2} N^{1/2} L \delta^{1/2} , \quad (1.9)$$

and the assertion is proved. #

In Lemma 1.3 we derived a bound for $v_{\epsilon x}$ which does not depend on smoothness of the initial data but which is only useful for t bounded away from zero. We shall also need a bound which is valid down to $t=0$. For this we employ the Bernstein method as in reference [A1].

LEMMA 1.5. Suppose that $v'_{\epsilon 0}$ is bounded in some interval $(a,b) \in \mathbb{R}$. For any $T \in \mathbb{R}^+$ and $\delta \in (0, \frac{b-a}{2})$ let $R = (a,b) \times (0,T]$ and $R^* = (a+\delta, b-\delta) \times (0,T]$. There exists a constant $C > 0$ independent of a,b,ϵ,δ,T and $v_{\epsilon 0}$ such that

$$|v_{\epsilon x}(x,t)| \leq 2\|v'_{\epsilon 0}\|_{L^\infty(a,b)} + C\delta^{-1} \|v_\epsilon\|_{L^\infty(R)} \quad (1.10)$$

for all $(x,t) \in \bar{R}^*$.

Proof. We may assume that v_0 is positive and smooth in R . Set

$$\phi(r) = Mr(4-r)/3$$

where $M = \|v_\epsilon\|_{L^\infty(R)}$ and define w by $v_\epsilon = \phi(w)$. Note that $w \in [0,1]$. Let $p = w_x$ and $z = \zeta^2 p^2$, where $\zeta = \zeta(x)$ is a $C^\infty([a,b])$ function with values in $[0,1]$ which vanishes in a neighborhood of $x = a$ and $x = b$. Then, as in [A1], at points of R where z has a relative maximum we have

$$\begin{aligned}
 - \{m\phi'' + (m-1)\phi(\phi''/\phi')'\} p^4 \zeta^2 &\leq - \{(m+1)\phi' + 2(m-1)\phi \frac{\phi''}{\phi'}\} p^3 \zeta \zeta' \quad (1.11) \\
 &+ \{2(m-1)\phi |\zeta'|^2 - (m-1)\phi \zeta \zeta''\} p^2.
 \end{aligned}$$

Now let $\bar{\zeta} = \bar{\zeta}(x)$ be a $C^\infty(\mathbb{R})$ function with $\bar{\zeta}(0) = 1$ and $\bar{\zeta} = 0$ for all $|x| > 1$ such that $\bar{\zeta} \in [0,1]$, $|\bar{\zeta}'| \leq 2$, and $|\bar{\zeta}''| \leq 4$ in $\mathbb{R}$. For an arbitrary fixed $\bar{x} \in [a+\delta, b-\delta]$ set $\zeta(x) = \bar{\zeta}(\frac{x-\bar{x}}{\delta})$. Then substituting in (1.11) and taking into account the estimates $\frac{2M}{3} \leq \phi' \leq \frac{4M}{3}$, $\phi'' = -\frac{2m}{3}$, $|\frac{\phi''}{\phi'}| \leq 1$ and $(\frac{\phi''}{\phi'})' \leq 0$ we obtain

$$\zeta^2 p^4 \leq C_1 \delta^{-2} p^2 + C_2 \epsilon^{-1} \zeta |p|^3,$$

where the constants C_1 and C_2 are independent of $a, b, \epsilon, \delta, M$ and T . Since $2C_2 \delta^{-1} \zeta p \leq \zeta^2 p^2 + C_2^2 \delta^{-2}$ we conclude that

$$\max z \leq (2C_1 + C_2^2) \delta^{-2} \equiv C_3^2 \delta^{-2}. \quad (1.12)$$

Suppose now that

$$\|v_{\epsilon X}\|_{L^\infty(\mathbb{R}^*)} > 2\|v'_{\epsilon 0}\|_{L^\infty(a,b)}.$$

Let $(\tilde{x}, \tilde{t}) \in \mathbb{R}^*$ be a point where the maximum value of $|v_{\epsilon X}|$ is achieved.

Clearly $\tilde{t} > 0$. Moreover, since $v_{\epsilon X} = \phi'(w)w_X$ we have

$$\|w_X\|_{L^\infty(\mathbb{R}^*)} > |w_X(\tilde{x}, \tilde{t})| > \frac{2}{|\phi'(w)|} \|v'_{\epsilon 0}\|_{L^\infty(a,b)} > \frac{3}{2M} \|v'_{\epsilon 0}\|_{L^\infty(a,b)} > \|w_X(\cdot, 0)\|_{L^\infty(a,b)}.$$

Therefore if $(\bar{x}, \bar{t}) \in \mathbb{R}^*$ is a point at which $|w_X|$ attains its maximum value then $\bar{t} > 0$. Set

$$z(x, t) = w_X^2(x, t) \bar{\zeta}\left(\frac{x-\bar{x}}{\delta}\right)$$

where $\bar{\zeta}$ is the function described in the last paragraph. Since

$$\|z\|_{L^\infty(R)} \geq z(\bar{x}, \bar{t}) = |w_x|^2(\bar{x}, \bar{t}) = \|w_x\|_{L^\infty(R^*)}^2 \geq \|w_x(\cdot, 0)\|_{L^\infty(a,b)}^2 \geq \|z(\cdot, 0)\|_{L^\infty(a,b)}$$

it follows that for any point $(x, t) \in R$ where z achieves its maximum we must have $t > 0$. Thus we can apply (1.12) to conclude that

$$w_x^2(\bar{x}, \bar{t}) = z(\bar{x}, \bar{t}) \leq \|z\|_{L^\infty(R)} \leq C_3^2 \delta^{-2}.$$

Therefore

$$|v_x(x, t)| = |\phi'(w)w_x| \leq \frac{4}{3} MC_3 \delta^{-1}$$

and (1.10) holds with $C = \frac{4}{3} C_3$.

REMARK. In case $(a, b) = \mathbb{R}$, the maximum principle holds for $v_{\epsilon x}$ and we have

$$|v_x(x, t)| \leq \|v'_{\epsilon u}\|_{L^\infty(\mathbb{R})}. \quad (1.13)$$

This can be proved by a slight alteration of the above argument. A proof for more general equations of the form $u_t = \phi(u)_{xx}$, with ϕ a continuous increasing real function, can be found in [V2] along with a discussion of the appropriate concept of velocity and its behaviour.

By essentially the same argument used to prove Lemma 1.4 we can derive the following consequences of Lemma 1.5.

COROLLARY 1.1. Under the hypothesis of Lemma 5, v_ϵ is Hölder continuous with respect to t in R^* with exponent $1/2$ and Hölder constant independent of ϵ .

2. PROOF OF THEOREM 1

2.1 Passage to the limit $\varepsilon \rightarrow 0$: In view of Lemma 1 the family $\{v_\varepsilon\}$ of solutions in Theorem 1 is uniformly bounded in Q . Moreover, by Lemmas 1.3 and 1.4, $\{v_\varepsilon\}$ is equicontinuous in Q_τ for any $\tau > 0$. Therefore there exists a sequence $\varepsilon_n \rightarrow 0$ such that $v_n \equiv v_{\varepsilon_n} \rightarrow v \in C(Q)$ uniformly on Q_τ for every $\tau > 0$. It is clear from Lemmas 1 and 2 that

$$0 < v < \|v_0\|_\infty \quad \text{in } Q, \quad (2.1)$$

$$v_{xx} > -\frac{1}{2\tau} \quad \text{in } D'(Q) \quad (2.2)$$

and

$$v_t > 0 \quad \text{in } D'(Q).$$

By Lemma 3, v is uniformly Lipschitz continuous with respect to x in Q_τ for any $\tau > 0$. Letting $\varepsilon \rightarrow 0$ in estimate (1.7) it follows that v is also uniformly Lipschitz continuous with respect to t in Q_τ for any $\tau > 0$. Moreover, the sequence $\{v_{nx}\}$ is relatively compact in $L^p_{loc}(Q)$. This is a consequence of the following compactness result, which is a variation of Lemma 10.1 of [Li]

LEMMA 2.1. Let $\{V_n\}_{n \geq 1}$ be a sequence in $C(Q)$ such that on every compact subset $Q' \subset\subset Q$ we have

- i) $\{V_n\}$ converges uniformly,
- ii) $\{V_{nx}\}$ is bounded in $L^\infty(Q')$,
- iii) There exists a constant $C = C(Q') > 0$ such that for every $n \geq 1$

$$V_{nxx} > -C(Q') \quad \text{in } D'(Q'). \quad (2.3)$$

Then $\{V_{nx}\}$ is relatively compact in $L^p_{loc}(Q)$ for every $p \in [1, \infty)$.

The main idea of the proof is that an estimate of the type (2.3) implies a bound for v_{nxx} in $\mathcal{M}_b(Q')$, the space of bounded measures on Q' , cf. also [Li], Lemma 3.1.

In view of the relative compactness of $\{v_{xx}\}$ we have, after passing to a subsequence if necessary,

$$v_{nx} \rightarrow v_x \text{ in } L^p_{loc}(Q) \text{ and a.e.} \quad (2.4)$$

Since the v_n satisfy

$$\int_Q (\epsilon_n v_n v_{nx} \psi_x + (\epsilon_n - 1)(v_{nx})^2 \psi - v_n \psi_t) = 0 \quad (2.5)$$

for all functions $\psi \in C_0^1(Q)$, letting $n \rightarrow \infty$ we obtain

$$\int (v_x)^2 \psi + v \psi_t = 0.$$

Thus v satisfies (0.4) $v_t = (v_x)^2$ in $\mathcal{D}'(Q)$ and almost everywhere. Since it also satisfies (2.2), it is a viscosity solution of (0.4), cf. [Li], Theorem 10.2.

We now consider the convergence at $t = 0$. Assume to begin with that $v_{0\epsilon} = v_0 \in C^1(\mathbb{R})$ and $\|v'\|_\infty = L$. By Lemmas 1.4 and 1.5 and the subsequent Remark, for any $\delta > 0$ we can find a $\tau = \tau(N, L)$ such that if $\epsilon \in (0, 1)$ we have

$$|v_\epsilon(x, t) - v_0(x)| < \delta/2 \quad (2.6)$$

for $0 < t < \tau$ and $x \in \mathbb{R}$. Letting $\epsilon \rightarrow 0$ we get

$$|v(x, t) - v_0(x)| < \delta/2, \quad (2.7)$$

hence $|v_\epsilon(x, t) - v(x, t)| < \delta$ and the convergence is uniform in x near $t=0$. Thus in this case $v \in W^{1, \infty}(Q)$.

For general $v_{0\epsilon}$ we use a barrier argument, taking advantage of the above result. Given $I = (a, b)$, take $\delta > 0$ and construct functions $\bar{v}_0, \underline{v}_0 \in C^1(\mathbb{R})$

with bounded derivative such that for a certain ϵ_0

$$0 < \underline{v}_0(x) \leq v_{\epsilon_0}(x) \leq \overline{v}_0(x), \quad (2.7)$$

$$\overline{v}_0(x) - \underline{v}_0(x) < \delta/2, \text{ on } I, \quad (2.8)$$

if $x \in I$ and $0 < \epsilon < \epsilon_0$. Denote by $\underline{v}_\epsilon(x,t)$, $\overline{v}_\epsilon(x,t)$ the solutions to (P_ϵ) with initial data $v_0, \underline{v}_0$ resp. and by $\overline{v}(x,t)$, $\underline{v}(x,t)$ their limits as $\epsilon \rightarrow 0$. Using (2.6), (2.7) on $\overline{v}_0$ and $\underline{v}_0$, there exists a time $\tau > 0$ such that if $x \in I$ and $0 < t < \tau$ then

$$v_\epsilon(x,t) \leq \overline{v}_\epsilon(x,t) \leq \overline{v}_0(x) + \delta/2 \leq v_0(x) + \delta$$

and

$$v(x,t) \geq \underline{v}(x,t) \geq \underline{v}_0(x) - \delta/2 \geq v_0(x) - \delta.$$

Therefore $v_\epsilon(x,t) - v(x,t) \leq 2\delta$. In a similar way we obtain

$v_\epsilon(x,t) - v(x,t) \geq -2\delta$. It follows that $v_\epsilon \rightarrow v$ uniformly on compact subsets of $\overline{Q}$. We conclude that $v \in C(\overline{Q})$ and $v(x,0) = v_0(x)$.

2.2 Uniqueness: We have just shown the existence of a sequence $\{v_{\epsilon_n}\}$ from the family $\{v_\epsilon\}$ which converges to a function $v \in C(Q)$ with the properties i), ii), iii) of Theorem 1(A). We shall now show that the whole family $\{v_\epsilon\}$ converges to v as $\epsilon \rightarrow 0$ by showing the uniqueness of the solution of problem (P_0) in that class of functions. In fact, part (B) of Theorem 1 follows from the following result which extends a result of Benton [B].

PROPOSITION 2.1. For $i = 1, 2$ let $v_i \in C(\overline{Q})$ be solutions of $v_t = (v_x)^2$ in $\Delta'(Q)$ which satisfy in the sense of distributions in Q

$$(i) \quad v_{ixx} \geq -A/t$$

and

$$(ii) \quad |v_{ix}| \leq B/t^{1/2}$$

for some constants $A, B > 0$. For arbitrary $a, b \in \mathbb{R}$ with $a < b$, let

$$D = \{(x, t) \in \mathbb{R}^2: a + 4Bt^{1/2} < x < b - 4Bt^{1/2}, 0 < t < T\} ,$$

where $T = \{(b-a)/8B\}^2$ and let $D_t = \{x \in \mathbb{R}: (x, t) \in D\}$. Then for every
 $\mu > 0$ the function

$$\Phi_\mu(t) = t^{-2A} \int_{D_t} \{(v_2 - v_1)^+\}^\mu dx \quad (2.8)$$

is nonincreasing on $[0, T]$. Moreover for every $t \in [0, T]$

$$\max_{x \in D_t} \{v_2(x, t) - v_1(x, t)\}^+ < \max_{a < x < b} \{v_2(x, 0) - v_1(x, 0)\}^+ . \quad (2.9)$$

REMARKS: (1) As we have shown in the proof of Lemma 3, condition (ii) is a consequence of (i) and bound for the v_i . Indeed if $0 < v_i < N$ then $B < \sqrt{2NA}$. Thus the essential assumption is the semiconvexity assumption (i).

(2) The conditions (i) and (ii) need only hold in $D \subset [a, b] \times [0, T]$.

Proof: We begin by showing that if Φ_μ is nonincreasing for every $\mu \in \mathbb{R}^+$ then (2.9) holds. For any t and τ with $0 < \tau < t < T$, $\Phi_\mu(t) \leq \Phi_\mu(\tau)$ implies

$$[\int_{D_t} \{(v_2 - v_1)^+\}^\mu dx]^{1/\mu} \leq (\frac{t}{\tau})^{2A/\mu} [\int_{D_\tau} \{(v_2 - v_1)^+\}^\mu dx]^{1/\mu} .$$

Thus, letting $\mu \rightarrow \infty$ we obtain

$$\max_{D_t} (v_2 - v_1)^+ \leq \max_{D_\tau} (v_2 - v_1)^+$$

and (2.9) follows by letting $\tau \rightarrow 0$.

For arbitrary fixed $\mu \in \mathbb{R}^+$ set $W = \{(v_2 - v_1)^+\}^\mu$. Then

$$W_t = Q W_x \quad \text{in } D'(Q) \quad (2.10)$$

where $Q = v_{1x} + v_{2x}$ satisfies

$$|Q| \leq 2Bt^{-1/2}.$$

Assume temporarily that the $v_i \in C^2(Q)$ and write (2.10) in the form

$$W_t = (QW)_x - Q_x W. \quad (2.11)$$

For $\sigma, \tau \in (0, T)$ with $\sigma < \tau$ integrate over $D_\sigma^\tau \equiv \{(x, t) \in D: \sigma \leq t \leq \tau\}$ to obtain

$$\int_{D_\tau} W dx = \int_{D_\sigma} W dx - \iint_{D_\sigma^\tau} Q_x W dx dt + \int_\sigma^\tau W(Q - y') \big|_{x=b-y} dt - \int_\sigma^\tau W(Q + y') \big|_{x=a+y} dt,$$

where $y = y(t) = 4Bt^{1/2}$. The third and fourth integrals on the right hand side are nonpositive since $|Q| \leq 2Bt^{-1/2} = y'$. On the other hand, by the semiconvexity condition (i), $-Q_x \leq 2A/t$. Therefore if we set

$$f(t) = \int_{D_t} W dx,$$

then

$$f(\tau) \leq f(\sigma) + 2A \int_\sigma^\tau \frac{1}{t} f(t) dt. \quad (2.12)$$

We conclude that

$$f(\tau) \leq \left(\frac{\tau}{\sigma}\right)^{2A} f(\sigma)$$

so that $\phi_\mu(t) = t^{-2A} f(t)$ is nonincreasing on $(0, T)$. By continuity, the same holds on $[0, T]$.

If the $v_i \in C^2(Q)$ we approximate $A = v_{1x} + v_{2x}$ by a sequence of $C^2(Q)$ - functions A^n such that $|A^n| \leq 2Bt^{-1/2}$, $A^n \geq -2t^{-1}$ and $A^n \rightarrow A$ in $L^1_{loc}(Q)$. Write (2.10) in the form

$$w_t = (A^n w)_x - A^n_x w + (A - A^n)_x w_x.$$

Since

$$|\iint_{D_\sigma^\tau} (A - A^n)_x w_x dx dt| \leq (2B\sigma^{-1/2})^\mu \|A - A^n\|_{L^1(D_\sigma^\tau)},$$

arguing as above we find

$$f(\tau) \leq f(\sigma) + 2A \int_\sigma^\tau \frac{1}{t} f(t) dt + (2B\sigma^{-1/2})^\mu \|A - A^n\|_{L^1(D_\sigma^\tau)}.$$

from which we derive (2.12) by letting $n \rightarrow \infty$.

3. CONVERGENCE OF THE INTERFACE

In this section we consider problem (P_ε) with a fixed initial datum v_0 which we assume to be bounded, nonnegative and continuous. In addition, we assume that v_0 vanishes on $\mathbb{R}^+$. Without loss of generality, we may assume that $0 = \sup \{x: v_0(x) > 0\}$. The right-hand interface of v_ε is then $x = \zeta_\varepsilon(t)$, where

$$\zeta_\varepsilon(t) = \sup\{x \in \mathbb{R}: v_\varepsilon(x, t) > 0\}, \quad 0 \leq t < \infty. \quad (3.1.a)$$

It is known that ζ_ε is a continuous, nondecreasing function in $[0, \infty)$. We shall prove that as $\varepsilon \rightarrow 0$, $\zeta_\varepsilon(t)$ converges to the right-hand interface $\zeta(t)$ of the solution v of (0.4) obtained as a limit of v_ε , i.e., to

$$\zeta(t) = \sup \{x \in \mathbb{R}: v(x, t) > 0\}, \quad 0 \leq t < \infty. \quad (3.1.b)$$

Before stating our main result we summarize the relevant properties of

$\zeta_\varepsilon(t)$ and $\zeta(t)$. For every $\varepsilon > 0$ and $\tau > 0$, $\zeta_\varepsilon \in C[0, \infty) \cap \text{Lip}[\tau, \infty)$ and there exists a waiting time $t_\varepsilon^* \in [0, \infty)$ such that $\zeta_\varepsilon(t) = 0$ for $0 \leq t \leq t_\varepsilon^*$, and $\zeta_\varepsilon \in C^1(t_\varepsilon^*, \infty)$ with $\zeta_\varepsilon'(t) > 0$ if $t > t_\varepsilon^*$. It is shown in [V3] that

$$T_\varepsilon / (B_\varepsilon)^\varepsilon \leq t_\varepsilon^* \leq \mu_\varepsilon T_\varepsilon / (B_\varepsilon)^\varepsilon, \quad (3.2)$$

where

$$B_\varepsilon = B_\varepsilon(u_0) = \sup_{x < 0} \left\{ |x|^{-\frac{\varepsilon+2}{\varepsilon}} \int_x^0 u_0(\xi) d\xi \right\},$$

$$T_\varepsilon = \frac{1}{2(\varepsilon+1)} \left(\frac{\varepsilon}{\varepsilon+2} \right)^{\varepsilon+1},$$

and μ_ε is a constant such that $\mu_\varepsilon > 1$, $\mu_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$. Note that $B_\varepsilon < \infty$ is the necessary and sufficient condition to have positive t_ε^* . The equation (0.7), $\zeta_\varepsilon'(t) = -v_{\varepsilon x}(\zeta_\varepsilon(t), t)$, is satisfied on the interface if $t \neq t_\varepsilon^*$, [CF]. Moreover we have ([V1])

$$\zeta_\varepsilon'' + \frac{\varepsilon+1}{(\varepsilon+2)t} \zeta_\varepsilon' > 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+). \quad (3.3)$$

(3.3) means that the function $\zeta_\varepsilon'(t) t^{\frac{\varepsilon+1}{\varepsilon+2}}$ is nondecreasing in $[0, \infty)$. It follows from (0.7) and (1.6) that

$$\zeta_\varepsilon'' > - \left(\frac{\varepsilon+1}{\varepsilon+2} \right) \left(\frac{2}{\varepsilon+2} \right)^{1/2} N^{1/2} t^{-3/2}. \quad (3.4)$$

As for $\zeta(t)$ it is well-known that $\zeta \in C[0, \infty) \cap \text{Lip}[\tau, \infty)$ for every $\tau > 0$, that (0.7) holds, and that ζ' is continuous except for an at most countable number of times t_i at which the one-sided derivatives $D^+\zeta(t_i)$ and $D^-\zeta(t_i)$ exist and satisfy $D^+\zeta(t_i) > D^-\zeta(t_i)$, ([V]). These properties of ζ also follow from our results below.

The following is an expanded version of Theorem 2.

THEOREM 2'. (A) As $\epsilon \downarrow 0$ the family $\{\zeta_\epsilon(t)\}$ converges uniformly on compact
subsets of $[0, \infty)$ to the interface $\zeta(t)$ of the function $v = \lim v_\epsilon$.

(B) Moreover $\zeta'_\epsilon \rightarrow \zeta'$ in $L^p_{loc}(\mathbb{R}^+)$ for every $1 < p < \infty$ and ζ satisfies

$$\zeta''(t) + \frac{1}{2t} \zeta'(t) > 0 \text{ in } \mathbb{R}^+ . \quad (3.3')$$

(C) The waiting time t^* of ζ_ϵ converges to the waiting time t^* of ζ
given by

$$t^* = 1/4B ,$$

where

$$B = \sup_{x < 0} \{|x|^{-2} v_0(x)\} .$$

Proof. (A) Let $N = \|v_0\|_{L^\infty(\mathbb{R})}$. We claim that

$$0 < \zeta_\epsilon(t) < 2\sqrt{Nt} . \quad (3.5)$$

To prove this we compare v_ϵ with the function

$$\bar{v}(x, t) = \begin{cases} N & \text{in } \mathbb{R}^- \times \mathbb{R}^+ , \\ N - x^2/4t & \text{in } [0, 2\sqrt{Nt}) \times \mathbb{R}^+ , \\ 0 & \text{in } [2\sqrt{Nt}, \infty) \times \mathbb{R}^+ . \end{cases}$$

Observe that $\bar{v}_t = (\bar{v}_x)^2$ in Q with initial values

$$v(x, 0) = \begin{cases} N & \text{for } x \in \mathbb{R}^- , \\ 0 & \text{for } x \in \mathbb{R}^+ . \end{cases}$$

Moreover, in the support $\bar{\Omega}$ of $\bar{v}$ we have

$$\bar{v}_{xx} \leq 0 \text{ in } D'(\bar{\Omega}) . \quad (3.6.a)$$

and in $\{(x,t) \in Q : 0 < x < 2\sqrt{Nt}\}$ we have precisely

$$\bar{v}_{xx} = -\frac{1}{2t} . \quad (3.6.b)$$

Therefore in $\bar{\Omega}$, $\bar{v}$ satisfies

$$\bar{v}_t - \varepsilon \bar{v} \bar{v}_{xx} - (\bar{v}_x)^2 = -\varepsilon \bar{v} \bar{v}_{xx} > 0$$

while for every $\varepsilon > 0$

$$v_{\varepsilon t} - \varepsilon v_{\varepsilon} v_{\varepsilon xx} - \varepsilon (v_{\varepsilon x})^2 = 0 \quad (3.7.b)$$

in Q . Since $\bar{v}(x,0) > v_0(x)$ in $\mathbb{R}$ it follows from the maximum principle that $\bar{v} > v_{\varepsilon}$ in Q and this implies (3.5).

Since the solutions are not smooth at the interfaces the maximum principle cannot be applied directly and an auxiliary argument is necessary. Fix $\varepsilon > 0$ and let $w(x,t) = v(x-\delta, t)$ with $\delta > 0$. We shall prove that for every t $\zeta_{\varepsilon}(t) < \bar{\zeta}(t) + \delta$ and $u \leq w$ in Q . Let $t_1 = \inf \{t > 0 : \zeta_{\varepsilon}(t) > \bar{\zeta}(t) + \delta\}$. Clearly $0 < t_1 < \infty$. We consider now the region $S = \{(x,t) \in \Omega_{\varepsilon} : 0 < t \leq t_1\}$. Since $\bar{\Omega} \supset S$ it follows from the standard maximum principle that $v_{\varepsilon} \leq w$ in S . Therefore if $t_1 < \infty$ we have $v(\cdot, t_1) \leq w(\cdot, t_1)$ in $(-\infty, \zeta(t_1))$ and hence in $\mathbb{R}$. On the other hand at the point (x_1, t_1) with $x_1 = \zeta_{\varepsilon}(t_1) = \zeta(t_1) + \delta$ we have

$$i) \quad v_{\varepsilon}(x_1, t_1) = w(x_1, t_1) ,$$

$$ii) \quad v_{\varepsilon x}(x_1^-, t_1) = -\zeta'_{\varepsilon}(t_1) < -\bar{\zeta}'(t_1) = w_x(x_1^-, t_1)$$

and

$$iii) \quad v_{\varepsilon xx}(x, t_1) = -\frac{1}{(\varepsilon+2)t} > w_{xx}(x, t_1) = -\frac{1}{2t} \text{ if } 0 < x < x_1 .$$

From i),ii),iii) it follows that $v_\epsilon(\cdot, t_1) > w(\cdot, t_1)$ for $0 < x < x_1$, in contradiction to the above result. Therefore $t_1 = \infty$ and $v_\epsilon \leq w$ in Q . Finally, let $\delta \rightarrow 0$ to obtain $v_\epsilon \leq \bar{v}$.

In view of (3.5), (0.7) and Lemma 1.3 the family $\{\zeta_\epsilon\}$ is uniformly bounded and equicontinuous on any compact subset of $\mathbb{R}^+$. Thus there exists a function $Z \in C(\mathbb{R}^+)$ and a sequence $\{\epsilon_n\}$ such that $\epsilon_n \rightarrow 0$ and $\zeta_n \equiv \zeta_{\epsilon_n} \rightarrow Z$ locally uniformly in $\mathbb{R}^+$. It is easily seen that Z satisfies (3.3') and is locally Lipschitz continuous and nondecreasing. Moreover, (3.5) implies that $Z(0) = 0$ and Z is Hölder continuous with exponent 1/2 at $t=0$. Therefore the convergence of ζ_n to Z is uniform on $[0, T]$ for every $T > 0$. We shall now show that Z is the right hand interface for v so that, in particular, the whole family ζ_ϵ converges to ζ .

Suppose that $(\bar{x}, \bar{t}) \in Q$ is such that $Z(\bar{t}) < \bar{x}$. Then $\zeta_n(\bar{t}) < \bar{x}$ for all sufficiently large n . It follows that $v_n(\bar{x}, \bar{t}) = 0$ for all sufficiently large n . Therefore $v(\bar{x}, \bar{t}) = \lim v_n(\bar{x}, \bar{t}) = 0$ so that

$$Z(t) \geq \sup \{x \in \mathbb{R} : v(x, \bar{t}) > 0\} = \zeta(\bar{t}).$$

Since $\zeta_\epsilon \geq 0$, it follows that $Z \geq 0$. If for some $t > 0$ we have $Z(\bar{t}) = 0$ then $Z=0$ on $[0, \bar{t}]$ and Z is the interface for v on $[0, \bar{t}]$. Suppose now that $Z(\bar{t}) > 0$ and consider an $\bar{x}$ such that

$$\frac{1}{3} Z(\bar{t}) < \bar{x} < Z(\bar{t}). \quad (3.8)$$

For sufficiently large n ,

$$\bar{x} < \zeta_n(\bar{t}) = \int_0^{\bar{t}} \zeta_n'(t) dt < (\epsilon_n + 2)\bar{t} \zeta_n'(\bar{t})$$

because of the nondecreasing nature of $\zeta_n'(t) t^{\frac{\epsilon+1}{\epsilon+2}}$. Therefore, in view of (0.7),

$$v_{nx}(\zeta_n(\bar{t}), \bar{t}) < -\frac{\bar{x}}{(\epsilon_n+2)t}.$$

By Taylor's theorem and Lemma 1.2 we have, with $\zeta_n = \zeta_n(t)$,

$$\begin{aligned} v_n(\bar{x}, \bar{t}) &= v_n(\zeta_n, \bar{t}) + (\bar{x} - \zeta_n) v_{nx}(\zeta_n, \bar{t}) + \frac{1}{2} (\bar{x} - \zeta_n)^2 v_{nxx}(\cdot, \bar{t}) \\ &> \frac{(\zeta_n - x)}{(\epsilon_n + 2)t} \left\{ \bar{x} - \frac{1}{2} (\zeta_n - \bar{x}) \right\} > 0. \end{aligned}$$

Thus, if $\bar{x}$ satisfies (3.8) we let $n \rightarrow \infty$ to obtain, with $\zeta = \zeta(\bar{t})$,

$$v(\bar{x}, \bar{t}) > \frac{\zeta - \bar{x}}{2t} \left\{ \bar{x} - \frac{1}{2} (\zeta - \bar{x}) \right\} > 0.$$

We conclude that $Z(t) \leq \zeta(t)$.

(B) In view of (0.7) and Lemma 3, the family $\{\zeta'_\epsilon\}$ is uniformly bounded in $[\tau, \infty)$ for any $\tau > 0$. Moreover, according to (3.4),

$$\zeta''_\epsilon > -\frac{N^{1/2}}{\tau^{3/2}}$$

in $[\tau, \infty)$ if $\epsilon \leq 1$. It follows from Lemma 3.1 of [Li] that $\{\zeta'_\epsilon\}$ is relatively compact in $L^p_{loc}(\mathbb{R}^+)$ for every $p \in [1, \infty)$. Thus, in particular, $\zeta'_\epsilon \rightarrow \zeta'$ in $L^p_{loc}(\mathbb{R}^+)$ for every $p \in [1, \infty)$ and for almost every time $t > 0$.

(C) By definition

$$\begin{aligned} \frac{(B_\epsilon)^\epsilon}{T_\epsilon} &= 2(\epsilon+1) \left(\frac{\epsilon+2}{\epsilon} \right)^{\epsilon+2} \left\{ \sup_{\mathbb{R}^+} |x|^{-\frac{\epsilon+2}{\epsilon}} \int_x^0 u_0 d\xi \right\}^\epsilon \\ &= \frac{2(\epsilon+2)^{\epsilon+1}}{\epsilon^3} \sup_{\mathbb{R}^+} \left\{ |x|^{-2} \left(\frac{1}{|x|} \int_x^0 v_0^{\frac{1}{\epsilon}} d\zeta \right)^\epsilon \right\}. \end{aligned}$$

If

$$B = \sup_{\mathbb{R}^+} \left\{ \frac{v_0(x)}{|x|^2} \right\}$$

then

$$\left(\frac{1}{|x|} \int_x^0 v_0^{\frac{1}{2}} d\zeta \right)^\epsilon \leq \frac{B|x|^2 \epsilon^\epsilon}{(2+\epsilon)^\epsilon}.$$

Therefore

$$\limsup_{\varepsilon \downarrow 0} \frac{(B_\varepsilon)^\varepsilon}{T_\varepsilon} < 4B .$$

On the other hand, for each $B_1 \in (0, B)$ there exist $x_1 = x_1(B_1) < 0$ and $\delta_1 = \delta_1(B_1) \in (0, -x_1)$ such that

$$v_0(x) > B_1 |x|^2 \quad \text{for } |x - x_1| < \delta_1 .$$

For $\delta \in (0, \delta_1]$ set $I_\delta = [x_1 - \delta, x_1 + \delta] \subset [x_1 - \delta, 0)$. If $x = x_1 - \delta$ then

$$|x|^{-2} \left(\frac{1}{|x|} \int_x^0 v_0^\varepsilon d\zeta \right)^\varepsilon > |x|^{-2} \left(\frac{1}{|x|} \int_{I_\delta} v_0^\varepsilon d\zeta \right)^\varepsilon > \left(\frac{2\delta}{|x_1 - \delta|} \right)^\varepsilon B_1 \left| \frac{x_1 + \delta}{x_1 - \delta} \right|^2 .$$

Hence

$$\frac{(B_\varepsilon)^\varepsilon}{T_\varepsilon} > \frac{2(\varepsilon+2)^{\varepsilon+1}}{\varepsilon^\varepsilon} \left(\frac{2\delta}{|x_1 - \delta|} \right)^\varepsilon B_1 \left| \frac{x_1 + \delta}{x_1 - \delta} \right|^2 ,$$

which implies

$$\liminf_{\varepsilon \downarrow 0} \frac{(B_\varepsilon)^\varepsilon}{T_\varepsilon} > 4B_1 \left| \frac{x_1 + \delta}{x_1 - \delta} \right|^2 .$$

Now let $\delta \downarrow 0$ and $B_1 \uparrow B$ to obtain

$$\liminf_{\varepsilon \downarrow 0} \frac{(B_\varepsilon)^\varepsilon}{T_\varepsilon} > 4B \quad . \quad \#$$

As a consequence of Theorem 2 we know that $\zeta'(t) t^{1/2}$ is nondecreasing and

$$\zeta'_\varepsilon \rightarrow \zeta' \tag{3.9}$$

almost everywhere in $\mathbb{R}^+$. We shall conclude this section by giving a more precise statement of (3.9). Note that it follows from the monotonicity of $\zeta'(t) t^{1/2}$, that $\zeta'(t)$ has at most jumping discontinuities with positive jumps.

THEOREM 3. For all $t > 0$ and every sequence $\varepsilon_m \downarrow 0$ such that $\zeta'_{\varepsilon_m}(t)$ converges we have

$$\lim_{\varepsilon_m} \zeta'_\varepsilon(t) \in [\zeta'(t-), \zeta'(t+)] . \quad (3.10)$$

In particular $\zeta'_\varepsilon \rightarrow \zeta'$ at every point where ζ' exists.

Proof. Fix $t_0 \in \mathbb{R}^+$ and let $P = \zeta'(t_0+)$. We claim first that

$$\limsup_{\varepsilon \downarrow 0} \zeta'_\varepsilon(t_0) \leq P . \quad (3.11)$$

Suppose for contradiction that

$$\limsup_{\varepsilon \downarrow 0} \zeta'_\varepsilon(t_0) = P + 2\lambda$$

for some $\lambda > 0$. It follows from (3.3) that

$$\zeta'_\varepsilon(t) \geq \zeta'_\varepsilon(t_0) \left(\frac{t_0}{t}\right)^{\frac{\varepsilon+1}{\varepsilon+2}} .$$

Let $\{\varepsilon_m\}$ be a sequence such that $\zeta'_{\varepsilon_m}(t_0) \rightarrow P + 2\lambda$. Then for all sufficiently large m and $t > t_0$

$$\zeta'_{\varepsilon_m}(t) \geq (P + \lambda) \left(\frac{t_0}{t}\right)^{1/2} .$$

On the other hand, by the definition of P , there is a $\delta > 0$ such that

$\zeta'(t) \leq (P + \frac{\lambda}{4})$ for all t such that $0 < t - t_0 < \delta$. Therefore we have

$$\zeta_{\varepsilon_m}(t) \geq \zeta_{\varepsilon_m}(t_0) + 2(P + \lambda)t_0^{1/2}(t^{1/2} - t_0^{1/2}) \geq \zeta(t_0) + 2(P + \lambda)t_0^{1/2}(t^{1/2} - t_0^{1/2})$$

$$> \zeta(t_0) + (P + \frac{\lambda}{2})(t - t_0) \geq \zeta(t) + \frac{\lambda}{4}(t - t_0)$$

for all $t > t_0$ such that $2(P + \lambda)t_0^{1/2} > (P + \frac{\lambda}{4})(t^{1/2} + t_0^{1/2})$. Since this contradicts the uniform convergence of $\zeta_{\varepsilon_m} \rightarrow \zeta$ we conclude that (3.11) holds. A similar argument with $t < t_0$ shows that

$$\liminf_{\varepsilon \downarrow 0} \zeta'_\varepsilon(t_0) \geq \zeta'(t_0-) .$$

4. CONCAVE SOLUTIONS

If the initial data $v_{\varepsilon 0}$ are concave in their support then the limit function v_0 also has this property. Moreover, by the results of [GJ] and [BV] the corresponding solutions v_{ε} of (P_{ε}) and v of (P_0) are concave in their supports as functions of x for each fixed $t > 0$. In particular we have

$$0 > v_{\varepsilon xx}(\cdot, t) > -\frac{1}{(2+\varepsilon)t}, \quad (4.1.a)$$

and

$$0 > v_{xx}(\cdot, t) > -\frac{1}{2t} \quad (4.1.b)$$

for $t > 0$. The presence of upper estimates allows us to obtain the following convergence result:

THEOREM 4. (A) Let $v_{\varepsilon 0}$, v_0 , v_{ε} , v be as in Theorem 1 and assume in addition that $v_{\varepsilon 0}$ is concave on its support for each $\varepsilon > 0$. Then v_{ε} , $v_{\varepsilon x}$, $v_{\varepsilon t}$ converge uniformly to v , v_x , v_t resp. on compact subsets of the closure of

$$\Omega = \{(x, t) \in Q: v(x, t) > 0\}$$

(B) The interfaces $\zeta_{\varepsilon}(t)$ and $\zeta(t)$ corresponding to v_{ε} and v respectively are C^1 concave functions of t for $0 < t < \infty$ and $\zeta'_{\varepsilon}(t) \rightarrow \zeta'(t)$ uniformly in (τ, T) for every $\tau, T > 0$ with $\tau < T$.

Proof. (A) To fix the ideas let us assume that each $v_{\varepsilon 0}$ vanishes outside a finite interval $I_{\varepsilon} = (a_{\varepsilon}, b_{\varepsilon})$. Then the subset Ω_{ε} of Q where v_{ε} is positive has the form

$$\Omega_{\varepsilon} = \{(x, t): \zeta_{\varepsilon}^{-}(t) < x < \zeta_{\varepsilon}(t)\},$$

where $x = \zeta_{\varepsilon}^{-}(t)$ is the left-hand interface for v_{ε} .

By Theorem 2 we know that $\zeta_\epsilon(t) \rightarrow \zeta(t)$ uniformly in $[0, T]$ for any $T > 0$. In the same way $\zeta_\epsilon^-(t)$ converges to the left-interface $\zeta^-(t)$ of the limit function v . It is clear that v is positive in the set

$$\Omega = \{(x, t) \in Q : \zeta^-(t) < x < \zeta(t)\}.$$

In view of Lemma 1.3, $\{v_{\epsilon x}\}$ is bounded uniformly in Q_τ for $\tau > 0$. It follows from (4.1.a) that the family $\{v_{\epsilon x}\}$ is Lipschitz continuous with respect to the x -variable locally in Ω and uniformly in ϵ . Since $w_\epsilon = v_{\epsilon x}$ satisfies the equation

$$w_t = \epsilon v_{\epsilon} w_{xx} + (\epsilon + 2) v_{\epsilon x} w_x \quad (4.2)$$

in Ω_ϵ , it follows from [G] that the family $\{w_\epsilon\}$ is Hölder continuous in t with exponent $1/2$ and the Hölder constant is locally bounded in Ω independent of ϵ . Therefore we conclude that

$$v_{\epsilon x} \rightarrow v_x$$

uniformly on compact sets of Ω . Finally the convergence of $v_{\epsilon t}$ follows from $v_{\epsilon t} - (v_{\epsilon x})^2 = \epsilon v_{\epsilon} v_{\epsilon xx}$ since

$$0 > \epsilon v_{\epsilon} v_{\epsilon xx} > \frac{\epsilon N}{(\epsilon + 2)t}. \quad (4.3)$$

(B) The uniform convergence of ζ'_ϵ to ζ' follows from the estimate ([BV])

$$0 > \zeta''_\epsilon(t) > \frac{1}{(\epsilon + 2)t} \zeta'(t). \quad \# \quad (4.4)$$

A second result that can be obtained with concave initial data concerns monotone convergence.

THEOREM 5. Let $v_{\epsilon 0}$ and v_0 be as in Theorem 4 and assume, in addition, that $v_{\epsilon 0}(x) \rightarrow v_0(x)$ for every $x \in \mathbb{R}$ as $\epsilon \rightarrow 0$. Then $v_\epsilon \rightarrow v$.

Proof. We want to prove that, given $\varepsilon' > \varepsilon > 0$, we have $v_{\varepsilon'} \leq v_{\varepsilon}$ everywhere in Q . The idea of the argument is the following. Consider the equation

$$L_{\varepsilon}(w) = w_t - \varepsilon w w_{xx} - (w_x)^2.$$

v_{ε} is a smooth solution of $L_{\varepsilon}w = 0$ in Ω_{ε} , whereas $v_{\varepsilon'}$ is a subsolution in $\Omega_{\varepsilon'}$ because $L_{\varepsilon}(v_{\varepsilon'}) = (\varepsilon' - \varepsilon)v_{\varepsilon'} v_{\varepsilon'xx} < 0$. Since $v_{\varepsilon 0} > v_{\varepsilon' 0}$ we can use the maximum principle to conclude that $v_{\varepsilon 0} > v_{\varepsilon' 0}$.

Since the domains Ω_{ε} and $\Omega_{\varepsilon'}$ of $v_{\varepsilon}, v_{\varepsilon'}$ are not necessarily the same there is a difficulty in applying the classical maximum principle which can be overcome as follows. Assume that the support of $v_{\varepsilon' 0}$ is bounded and that we replace v_{ε} by the solution $\bar{v}_{\varepsilon}$ of $L_{\varepsilon}w = 0$ with initial data $w(x, 0) = v_{\varepsilon 0} + \delta$ for some $\delta > 0$. Now since v_{ε} is positive and C^{∞} everywhere in Q and $v_{\varepsilon'}(x, t) = 0$ for large $|x|$, we easily conclude that $v_{\varepsilon'} \leq \bar{v}_{\varepsilon}$ in Q . The stated result follows by approximation since solutions of the porous medium equation depend continuously on the initial data. #

As an example consider the Barenblatt solution

$$V_{\varepsilon}(x, t) = \frac{(r_{\varepsilon}(t)^2 - x^2)_+}{2(\varepsilon+2)(t+1)} \quad (4.5)$$

where $r_{\varepsilon}(t) = K(1+t)^{1/(\varepsilon+2)}$ and K a positive constant. As $\varepsilon \rightarrow 0$ we have

$$V_{\varepsilon}(x, t) + V(x, t) = 1/4 \left(K^2 - \frac{x^2}{t+1} \right)_+,$$

while for the interfaces we get

$$r_{\varepsilon}(t) + r(t) = K(1+t)^{1/2}.$$

5. THE CASE $m < 1$

We can also consider a limit process for the solutions of porous medium equation as $m \rightarrow 1$ with $m < 1$. As noted in the Introduction, if we look at the density (i.e., if the initial data $u_{\epsilon 0}$ converge as $m \rightarrow 1$) the solutions of $u_t = (u^m)_{xx}$ converge to a solution of the heat equation.

If we look at the variable v defined as before by

$$v = \frac{m}{m-1} u^{m-1} = - \frac{m}{1-m} u^{-(1-m)} \quad (5.1)$$

we see first that $u > 0$ implies $v < 0$ and also that $u \rightarrow 0$ implies $v \rightarrow -\infty$ while $u \rightarrow \infty$ implies $v \rightarrow 0$. Moreover if we put

$$\epsilon = 1 - m$$

then v formally satisfies the equation

$$v_t = \epsilon |v| v_{xx} + (v_x)^2. \quad (5.2)$$

Now if the initial datum $v(x,0)$ is kept fixed as $\epsilon \rightarrow 0$ we want to prove that the solutions of (5.2) converge again to a solution of equation (0.4).

Following the outline of the proof of Theorem 1 we obtain a solution v_ϵ of (P_ϵ) , obtain estimates for v_ϵ and $v_{\epsilon x}$ and finally pass to the limit $\epsilon \rightarrow 0$.

To begin with, it is proved in [AB] that for every $0 < m < 1$ and non-negative $u_0 \in L^1(\mathbb{R})$ there exists a unique function $u \in C[0, \infty; L^1(\mathbb{R})) \cap C^\infty(Q)$ such that

$$\begin{aligned} u_t, \Delta u^m &\in L^1_{loc}(Q), \\ u_t &= \Delta u^m \quad \text{in } Q, \\ u(\cdot, 0) &= u_0. \end{aligned} \quad (5.3)$$

Moreover u is positive everywhere in Q (so that there is no interface) and the following estimates hold $([AB],[BC])$

$$\frac{(1-m)v}{(m+1)t} < v_t < -\frac{v}{t}, \quad (5.4)$$

and

$$v_{xx} > -\frac{1}{(m+1)t}, \quad (5.5)$$

where v is given by (5.1). Of course v satisfies equation (5.2) in Q .

Using (5.4) and (5.5) we obtain directly from (5.2) the following interesting pointwise bound for $|v_x|$:

$$|v_x|^2 < \frac{2|v|}{(m+1)t}. \quad (5.6)$$

It is worth noting that, as compared with (1.2a) and (1.6), the estimate for v_t is bilateral and the bound for $v_x(x,t)$ depends only on the value of v at (x,t) .

Solutions with general initial data $u_0 \in L^1_{loc}(\mathbb{R})$ are constructed by [HP]. Moreover they prove that uniqueness in the appropriate class holds and that when $u_0 > 0$ the estimates noted above are valid for all the solutions. Using these estimates and arguing as in Section 2 we can prove the following result. Consider a family of initial functions $v_{\varepsilon 0} \in C(\mathbb{R})$ that satisfy

$$0 > v_{\varepsilon 0}(x) > -N \quad (5.7)$$

for some $N > 0$ and every $x \in \mathbb{R}$, and let v_ε denote the solution of the problem

$$v_t = -\varepsilon v v_{xx} + (v_x)^2 \quad \text{in } Q, \quad (P_\varepsilon)$$

$$v(x,0) = v_{0\varepsilon}(x) \quad \text{in } \mathbb{R}.$$

We have

THEOREM 6. Assume that as $\epsilon \rightarrow 0$, $\{v_{\epsilon 0}\}$ converges uniformly on compact subsets of $\mathbb{R}$ to a function v_0 . Then v_{ϵ} converges uniformly on compact subsets of Q to the solution v of the problem

$$\begin{aligned} v_t &= (v_x)^2 \quad \text{in } Q, \\ v(x, 0) &= v_0(x) \quad \text{in } \mathbb{R}, \end{aligned} \quad (P_0)$$

satisfying the condition $v_{xx} > -1/(2t)$ in $\Delta'(Q)$. Moreover $v_{\epsilon x} \rightarrow v_x$ in $L^p_{loc}(Q)$ for every $p \in [1, \infty)$.

Note also that (5.4) implies in the limit the following estimate for negative solutions of (P_0)

$$0 < v_t < \frac{|v|}{t} \quad (5.8)$$

Together with (5.7) this implies that solutions of $v_t = (v_x)^2$ approach the maximum value (here $v=0$) as $t \rightarrow \infty$ and x is fixed with a rate at most $O(1/t)$. This is exactly the rate for the self-similar solutions (0.20).

Let us finally remark that the estimates (5.4)-(5.6) are true with suitable constants for the solutions of the corresponding d -dimensional problem

$$u_t = \Delta u^m \quad \text{in } Q = \mathbb{R}^d \times (0, \infty),$$

$$u(\cdot, 0) = u_0 \in L^1_{loc}(\mathbb{R}^d),$$

if $u_0 \geq 0$ and $d(1-m) < 2$. In particular the crucial estimate (5.6) becomes

$$|\nabla v|^2 < \frac{2}{2-d(1-m)} \cdot \frac{|v|}{t} \quad (5.6')$$

and the proof of Theorem 6 applies essentially unchanged in several dimensions.

Regarding the connection between (0.1) and (0.4), the transformation (0.17) is still valid if $0 < m < 1$ and transforms positive solutions of (0.1) into negative supersolutions of (0.4). In particular the Barenblatt solution

$$\bar{u}(x,t) = t^{-\frac{1}{m+1}} \left(C + \frac{1-m}{2m(m+1)} t^{\frac{x^2}{2(m+1)}} \right)^{-\frac{1}{1-m}} \quad (5.7)$$

transforms into the solution

$$V(x,t) = - \left(K + \frac{x^2}{4\tau} \right)_+ \quad (5.8)$$

with $K = Cm/(1-m) > 0$. The case $K = 0$ in (5.8) corresponds to the special solution of (0.1) given by

$$v(x,t) = - \frac{x^2}{2(m+1)t} \quad (5.9)$$

As in the case $m > 1$, these particular solutions represent the asymptotic behaviour of a large class of solutions, cf. [V2].

For $m=1$ the transformations (0.5), (0.17) should be replaced by

$$v(x,t) = \log(u(x,t)), \quad \tau=t, \quad \text{and} \quad V(x,\tau) = \log(u(x,t)t^{1/2}). \quad (5.10)$$

ACKNOWLEDGEMENTS

The authors are grateful to M.C. Crandall, P.L. Lions and P.E. Souganidis for their interest in this work and helpful suggestions. The second author wishes to thank the Institute for Mathematics and its Applications and the University of Minnesota for their hospitality while this paper was written.

REFERENCES

- [A1] Aronson, D.G. Regularity properties of flows through porous media, SIAM J. Appl. Math. 17 (1969), 461-467.
- [A2] Aronson, D.G. Regularity properties of flows through porous media: the interface, Arch. Rational Mech. Anal. 37 (1970), 1-10.
- [AB] Aronson, D.G. - Bénéilan, Ph. Régularité des solutions de l'équation des milieux poreux dans R^n , C.R. Acad. Sci. Paris 288 (1979), 103-105.
- [AC] Aronson, D.G. - Caffarelli, L.A. The initial trace of a solution of the porous medium equation, Trans. Amer. Math. Soc., 280 (1983), 351-366.
- [ACK] Aronson, D.G. - Caffarelli, L.A. - Kamin, S. How an initially stationary interface begins to move in porous medium flow, SIAM J. Math. Anal. 14 (1983), 639-658.
- [ACV] Aronson, D.G. - Caffarelli, L.A. - Vazquez, J.L. Interfaces with a corner-point in one-dimensional porous medium flow, Comm. Pure Applied Math., to appear.
- [BC1] Bénéilan, Ph. - Crandall, M.G., The continuous dependence on ϕ of the solutions of $u_t - \Delta\phi(u) = 0$, Indiana Univ. Math. J. 30, (1981), 161-177.
- [BC2] Bénéilan, Ph. - Crandall, M.G. Regularizing effects of homogeneous evolution equations, Contributions to Analysis and Geometry, Supplement to American J. Math., D.N. Clark et.al. eds., Baltimore (1981), 23-39.
- [BCP] Bénéilan, Ph. - Crandall, M.G. - Pierre, M. Solutions of the porous medium equation in R^n under optimal conditions on initial values, Indiana Univ. Math. J. 33, (1984), 51-87.
- [BV] Bénéilan, Ph. - Vazquez, J.L. Concavity of solutions of the porous medium equation, to appear.
- [B] Benton, S.H. The Hamilton-Jacobi equation: a global approach, Academic Press, New York, 1977.
- [CF] Caffarelli, L.A. - Friedman, A. Regularity of the free-boundary for the one-dimensional flow of gas in a porous medium, Amer. J. Math. 101 (1979), 1193-1218.
- [CEL] Crandall, M.G. - Evans, L.C. - Lions, P.L. Some properties of viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. Math. Soc. 282 (1984), 487-502.
- [CL] Crandall, M.G. - Lions, P.L. Viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. Math. Soc. 277 (1983), 1-42.
- [D] Dafermos, C.M. Characteristics in hyperbolic conservation laws in Nonlinear Analysis and Mechanics, Vol. I, ed. R.J. Knops, London: Pitman 1977, 1-58.

- [DK] Dahlberg, B.E. - Kenig, C.E. Nonnegative solutions of the porous medium equation, Comm. Partial Diff. Equations, 9 (1984), 409-437.
- [G] Gilding, B.H. Hölder continuity of solutions of parabolic equations, J. London Math. Soc. 13 (1976), 103-106.
- [GJ] Gravelleau, J.L. - Jamet, P. A finite-different approach to some degenerate nonlinear parabolic equations, SIAM J. Appl. Math. 20 (1971), 199-223.
- [HP] Herrero, M.A. - Pierre, M. The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$, Mathematics Research Center TS Report # , 1984
- [H] Hopf, E. The partial differential equation $u_t + uu_x = \mu u_{xx}$. Comm. Pure Applied Math. 3 (1950), 201-230.
- [K] Kamin, S. The asymptotic behaviour of the solution of the filtration equation, Israel J. Math. 14 (1973), 76-78.
- [KC] Kath, W.L. - Cohen, D.S. Waiting time behaviour in a nonlinear diffusion equation. Studies in Applied Maths. 67, (1982), 79-105.
- [Kn] Knerr, B. The porous medium equation in one dimension, Trans. Amer. Math. Soc. 234 (1977), 381-415.
- [La] Lax, P.D. Hyperbolic systems of conservation laws II, Comm. Pure Applied Math. 10 (1957), p. 537-566.
- [Li] Lions, P.L. Generalized solutions of Hamilton-Jacobi equations, Research Notes in Math. #69, Pitman, Boston, 1982.
- [LSV] Lions, P.L. - Souganidis, P.E. - Vazquez, J.L. In preparation.
- [LP] Liu, T.-P. - Pierre, M. Source-solutions and asymptotic behavior in conservation laws, J. Diff. Equations 51 (1984), 419-441.
- [O] Oleinik, O.A. Discontinuous solutions of nonlinear differential equations, Amer. Math. Soc. Translations (2) 26 (1983), p. 95-172 (Doklady Akad. Nauk SSSR , 109 (1956), 1098-1101 (in Russian))
- [OKC] Oleinik, O.A. - Kalashnikov, A.S. - Czhou, Y.L. The Cauchy problem and boundary problems for equations of the type of nonstationary filtration, Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 667-704 (in Russian).
- [V1] Vazquez, J.L. Asymptotic behaviour and propagation properties of the one-dimensional flow of a gas in a porous medium, Trans. Amer. Math. Soc. 277 (1983), 505-527.
- [V2] Vazquez, J.L. Behaviour of the velocity of one-dimensional flows in porous media, Trans. Amer. Math. Soc. 286 (1984), 787-802.
- [V3] Vazquez, J.L. The interfaces of one-dimensional flows in porous media, Trans. Amer. Math. Soc. 285 (1984), 717-737.

#	Author(s)	Title	#	Author(s)	Title
78		Abstracts for the Workshop on Bayesian Analysis in Economics and Game Theory	120	D.R.-J. Chillingworth,	Three Introductory Lectures on Differential Topology and its Applications
79	G. Chichilnitsky, G.M. Heal,	Existence of a Competitive Equilibrium in L and Sobolev Spaces	121	Giorgio Vergara Caffarelli,	Green's Formulas for Linearized Problems with Live Loads
80	Thomas P. Selman,	Time-dependent Solutions of a Nonlinear System in Semiconductivity Theory, II: Boundedness and Periodicity	122	F. Chiarenza and N. Garofalo,	Unique Continuation for Nonnegative Solutions of Schrödinger Operators
81	Yakar Kannal,	Engaging in R&D and the Emergence of Expected Non-convex Technologies	123	J.-L. Ericksen,	Constitutive Theory for some Constrained Elastic Crystals
82	Herve Moulin,	Choice Functions over a Finite Set: A Summary	124	Minoru Murata,	Positive solutions of Shrodinger Equations
83	Herve Moulin,	Choosing from a Tournament	125	John Maddocks and Gareth P. Parry,	A Model for Twinning
84	David Schmiedler,	Subjective Probability and Expected Utility Without Additivity	126	M. Kaneko and M. Wooders,	The Core of a Game with a Continuum of Players and Finite Coalitions: Nonemptiness with Bounded Sizes of Coalitions and Finite Coalitions in Production Economies with an Infinite Dimensional Commodity Space
85	I.G. Kevrekidis, R. Aris, L.D. Schmidt, and S. Pelikan,	The Numerical Computation of Invariant Circles of Maps	128	Myrna Holtz Wooders,	A Tiebout Theorem
86	F. William Lawvere,	State Categories, Closed Categories, and the Existence of Semi-Continuous Entropy Functions	129	Abstracts for the Workshop on Theory and Applications of Liquid Crystals	
87	F. William Lawvere,	Functional Remarks on the General Concept of Chaos	130	Yoshikazu Giga,	A Remark on A Priori Bounds for Global Solutions of Semilinear Heat Equations
88	Steven R. Williams,	Necessary and Sufficient Conditions for the Existence of a Locally Stable Message Process	131	M. Chipot and G. Vergara-Caffarelli,	The N-Membranes Problem
89	Steven R. Williams,	Implementing a Generic Smooth Function	132	P.-L. Lions and P.E. Souganidis,	Differential Games and Directional Derivatives of Viscosity Solutions of Isaacs' Equations II
90	Dilip Abreu,	Infinitely Repeated Games with Discounting: A General Theory	133	G. Capriz and P. Glowine,	On Virtual Effects During Diffusion of a Dispersed Medium in a Suspension
91	J.S. Jordan,	Instability in the Implementation of Walrasian Allocations	134	Fall Quarter Seminar Abstracts	
92	Myrna Holtz Wooders, William R. Zame,	Large Games: Fair and Stable Outcomes	135	Umberto Mosco,	Wiener Criterion and Potential Estimates for the Obstacle Problem
93	J.-L. Noakes,	Critical Sets and Negative Bundles	136	Chi-Sing Man,	Dynamic Admissible States, Negative Absolute Temperature, and the Entropy Maximum Principle
94	Graciela Chichilnitsky,	Von Neumann-Morgenstern Utilities and Cardinal Preferences	137	Abstracts for the Workshop on Oscillation Theory, Computation, and Methods of Compensated Compactness	
95	J.-L. Ericksen,	Twinning of Crystals	138	Arle Leizarowitz,	Tracking Nonperiodic Trajectories with the Overtaking Criterion
96	Anna Nagurney,	On Some Market Equilibrium Theory Paradoxes	139	Arle Leizarowitz,	Convex Sets in R^2 as Affine Images of some Fixed Set in R^3
97	Anna Nagurney,	Sensitivity Analysis for Market Equilibrium	140	Arle Leizarowitz,	Stochastic Tracking with the Overtaking Criterion
98	Abstracts for the Workshop on Equilibrium and Stability Questions in Continuum Physics and Partial Differential Equations		141	Abstracts from the Workshop on Amorphous Polymers and Non-Newtonian Fluids	
99	Millard Beatty,	A Lecture on Some Topics in Nonlinear Elasticity and Elastic Stability	142	Winter Quarter Seminar Abstracts	
100	Filomena Pacella,	Central Configurations of the N-Body Problem via the Equivariant Morse Theory			
101	D. Carlson and A. Hoger,	The Derivative of a Tensor-valued Function of a Tensor			
102	Kenneth Mount,	Privacy Preserving Correspondence			
103	Millard Beatty,	Finite Amplitude Vibrations of a Neo-hookean Oscillator			
104	D. Emmons and N. Yannellis,	On Perfectly Competitive Economies: Loeb Economies			
105	E. Mascolo and R. Schianchi,	Existence Theorems in the Calculus of Variations			
106	D. Kinderlehrer,	Twinning of Crystals (II)			
107	R. Chen,	Solutions of Minimax Problems Using Equivalent Differentiable Equations			
108	D. Abreu, D. Pearce, and E. Stacchetti,	Optimal Cartel Equilibria with Imperfect Monitoring			
109	R. Lauterbach,	Hopf Bifurcation from a Turning Point			
110	C. Kohn,	An Equilibrium Model of Quits under Optimal Contracting			
111	M. Kaneko and M. Wooders,	The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results			
112	Halm Brezis,	Remarks on Sublinear Equations			
113	D. Carlson and A. Hoger,	On the Derivatives of the Principal Invariants of a Second-order Tensor			
114	Raymond Deneckere and Steve Pelikan,	Competitive Chaos			
115	Abstracts for the Workshop on Homogenization and Effective Moduli of Materials and Media				
116	Abstracts for the Workshop on the Classifying Spaces of Groups				
117	Umberto Mosco,	Pointwise Potential Estimates for Elliptic Obstacle Problems			
118	J. Rodrigues,	An Evolutionary Continuous Casting Problem of Stefan Type			
119	C. Mueller and F. Weisler,	Single Point Blow-up for a General Semilinear Heat Equation			