**A FREE BOUNDARY PROBLEM WITH NONLINEAR JUMP
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A FREE BOUNDARY PROBLEM WITH NONLINEAR JUMP AND KINETICS ON THE FREE BOUNDARIES

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Abstract. In this paper we study a free boundary problem arising in modeling titanium silicide film growth. The problem involves two free boundaries, one of them lies inside the domain. The interface conditions at the interior free boundary consist of $\frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} = -k(u - v)$, the nonlinearity $v = \beta(u)$ and kinetics $v - u = V_2$, where k is a positive constant, β is a monotone increasing function and V_2 is the normal velocity of the free boundary. We establish local existence and uniqueness of a classical solution. Global existence is also investigated in some special cases.

Keywords. Free boundary problem, Stefan problem, nonlinear equations, transmission problems, a priori estimates, film growth modeling.

AMS subject classification. Primarily 35B04. Secondly 75B23.

1. Introduction. In this paper we are concerned with the following free boundary problem: For any family of curves $g_3(x, t)$ in R^2 , find two families of curves $y = g_1(x, t)$ and $y = g_2(x, t)$, for $x \in R^1$, $0 \leq t \leq T$, satisfying $g_3(x, t) < g_2(x, t) < g_1(x, t)$, and two functions $u(x, y, t)$ and $v(x, y, t)$ defined in

$$\Omega_1(t) = \{(x, y) \in R^2 : g_2(x, t) < y < g_1(x, t)\},$$

and

$$\Omega_2(t) = \{(x, y) \in R^2 : g_3(x, t) < y < g_2(x, t)\},$$

respectively such that u and v satisfy

$$(1.1) \quad \Delta u = 0 \quad \text{in} \quad \Omega_1(t),$$

$$(1.2) \quad \Delta v = 0 \quad \text{in} \quad \Omega_2(t),$$

for any $0 \leq t \leq T$, together with the boundary and interface conditions

$$(1.3) \quad \frac{\partial u}{\partial n} + u = 0 \quad \text{on} \quad \Gamma_1(t),$$

$$(1.4) \quad \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} = -k(u - v) \quad \text{on} \quad \Gamma_2(t),$$

$$(1.5) \quad u = \beta(v) \quad \text{on} \quad \Gamma_2(t),$$

$$(1.6) \quad v = \varphi \quad \text{on} \quad \Gamma_3(t),$$

$$(1.7) \quad \frac{\partial u}{\partial n} = -V_1 \quad \text{on} \quad \Gamma_1(t),$$

$$(1.8) \quad u - v = V_2 \quad \text{on} \quad \Gamma_2(t),$$

and the initial conditions

$$(1.9) \quad g_i(x, 0) = g_i^0(x), \quad i = 1, 2,$$

where k is a positive constant, β is a non-negative and monotone increasing function satisfying some extra conditions (see (2.1) in the next section), and

$$\Gamma_i(t) = \{(x, y) : y = g_i(x, t)\}, \quad i = 1, 2, 3.$$

In (1.3), (1.4) and (1.7), $\vec{n}$ is the unit vector normal to $\Gamma_i(t)$. The direction of $\vec{n}$ is chosen such that it is exterior normal on Γ_1 and Γ_2 with respect to Ω_1 and Ω_2 , respectively. In (1.7) and (1.8), V_i is the normal velocity of $\Gamma_i(t)$ defined by

$$(1.10) \quad V_i = \frac{g_{i,t}(x, t)}{\sqrt{1 + [g_{i,x}(x, t)]^2}}, \quad i = 1, 2.$$

Note that, among the three boundaries, $\Gamma_1(t)$ and $\Gamma_2(t)$ are the free boundaries.

The problem (1.1)–(1.10) arises in modeling the titanium silicide film growth [2, Chapter 8]. In the model, u is the concentration of titanium silicide in the monosilicide region, v is the concentration of titanium silicide in the disilicide region. Γ_1 is the interface between titanium and monosilicide. The conditions (1.3) and (1.4) describe the conversion rates of the concentrations across the interfaces. They are the consequence of the kinetic heating. The growth conditions (1.7) and (1.8) follow from conservation

of mass. The relation (1.5) is derived from the experiments. Note that in the actual physical model, the third boundary Γ_3 is also a free boundary along with an extra condition $\frac{\partial v}{\partial n} = V_3$, where V_3 is the normal velocity of Γ_3 .

In the special case that the boundary $\Gamma_2(t)$ coincides with $\Gamma_3(t)$ (i.e., (1.1), (1.3), (1.7) with $u = \varphi$ on $\Gamma_2(t)$,) the problem is reduced to one phase Stefan type problem. Friedman and Hu [4] studied this case. They proved local existence and uniqueness of classical solution for the general initial data and global existence for near flat initial curves.

In the present paper, we are concerned with existence and uniqueness for problem (1.1)–(1.10). Comparing to the problem in [4], there are two major difficulties here. First, the problem (1.1)–(1.10) involves a free boundary $\Gamma_2(t)$ with two phases. Secondly, the interface condition (1.5) is strongly nonlinear. If we ignore the effects of the kinetic heating, i.e., fixing Γ_1, Γ_2 and dropping conditions (1.7)–(1.8) from (1.1)–(1.10), the problem reduces to a diffraction problem. Ladyzhenskaya, Rivkind and Ural'tseva [7] and many authors studied the diffraction problems for the linear equations in the case that the connection functions β is linear. Kutev and Lions [6] considered nonlinear problems with discontinuous coefficients. They proved existence and uniqueness of the classical solutions in bounded domains. The main feature in the present problem (with fixed Γ_1 and Γ_2) is that the nonlinearity β is introduced on the boundary $\Gamma_2(t)$. We shall see that the diffraction problem (1.1)–(1.6) is essentially a quasilinear elliptic problem with discontinuous coefficients and a nonlinear jump condition on the normal derivatives across the interior boundary $\Gamma_2(t)$. It is completely different from those in [6] and [7]. Without appropriate assumptions on β , existence of solutions is not always guaranteed. In fact, there are a plenty of examples that show that there exists no solution in general even though the connection function β is smooth and monotone (see Lemma 6.1.) In addition, the authors of [6] treated the problems with discontinuous coefficients as a limit of regular elliptic equations. It occurs to us that their treatments

can not be extended to our situation.

We shall first study existence, uniqueness and regularity of the solutions for problem (1.1)–(1.6) with some suitable assumptions on β (see (2.1) below.) These assumptions are motivated by the physical phenomenon and proven rigorously unremovable in some sense. We shall then establish existence and uniqueness of a classical solution for the free boundary problem (1.1)–(1.10) for $0 \leq t \leq T$ for some $T > 0$. In the case that the initial data are axially symmetric, we shall show that there exists a global solution.

2. Preliminaries. We begin by introducing some function spaces. For any domain G (bounded or unbounded), let $W^{m,p}(G)$ and $C^{m+\mu}(G)$ be the usual Sobolev space and Hölder continuous function space with the norms $\|\cdot\|_{W^{m,p}(G)}$ and $\|\cdot\|_{C^{m+\mu}}$, respectively. Set $G_r = G \cap B_r$, where B_r is the ball centered at the origin with the radius r . We denote by $W_{loc}^{m,p}(G)$ the set of all functions f such that $f|_{G_r} \in W^{m,p}(G_r)$ for any $r > 0$, with the local norm

$$\|f\|_{W_{loc}^{m,p}(G)} = \sup\{\|f\|_{W^{m,p}(G_{r+1} \setminus G_r)} : r = 1, 2, \dots\} < \infty.$$

Let $\Gamma \in \partial G$ be a portion of the boundary G . Denote by $W_\Gamma^{m,p}(G)$ the subset of all functions f in $W^{m,p}(G)$ such that $f|_\Gamma = 0$. The space $W_{loc,\Gamma_3}^{1,p}(G)$ is defined analogously.

Throughout paper, we assume that $\beta \in C^{2+\mu}(R^1)$ satisfying

$$(2.1) \quad \beta(0) = 0, \quad 1 \geq \beta'(x) > 0,$$

and that the initial curves $y = g_i^0(x) \in C^{2+\mu}(R^1)$ for $i = 1, 2$, $g_3(\cdot, t) \in C^{2+\mu}(R^1)$, for any $0 \leq t \leq T$, and $g_{3t} \in C^0(R^1 \times [0, T])$. For simplicity, we also assume that there exist constants b_j , $j = 0, 1, 2$, with $b_0 > 0$ such that

$$(2.2) \quad g_3(x, 0) \leq b_2, \quad b_2 + 2b_0 \leq g_2^0(x) \leq b_1, \quad b_1 + 2b_0 \leq g_1^0(x).$$

Note that these assumptions simply say that the initial curves are uniformly separated. The boundary condition on Γ_3 is assumed to be

$$(2.3) \quad 0 \leq \varphi \leq 1, \quad \nabla \varphi \in L^2(\Omega), \quad \varphi \in C^{2+\mu}(\Omega).$$

The assumption (2.1) is motivated by the physical experiments [9] and the numerical results in 1- D case (see [2, Chapter 8]) that show that $\beta(x)$ is close to $\min(c_0, x)$ for some $0 < c_0 < 1$. Assumption (2.3) generalizes the ideal case that $\varphi = 1$. The smoothness assumptions are for technical reasons. We also point out that the assumption (2.1) is optimal in the sense that there exists no solution for (1.1)–(1.6) in general if (2.1) is violated. Some nonexistence results will be shown in section 6.

To study the free boundary problem, we first need to understand the nonlinear diffraction problem (1.1)–(1.6) with the fixed domains $\Omega_i = \Omega_i(0)$ and their boundaries $\Gamma_i = \Gamma_i(0)$.

Set $\bar{\Omega} = \Omega_1 \cup \Omega_2 \cup \Gamma_2$. Denote by $C_0^\infty(\bar{\Omega})$ the set of all functions in $C^\infty(\bar{\Omega})$ that vanish in $\bar{\Omega} \setminus B_r$ for some $r > 0$. For any $\xi \in C_0^\infty(\bar{\Omega})$ such that $\xi = 0$ on Γ_3 , we multiply (1.1) and (1.2) by ξ and use Green's identity. It follows that

$$(2.4) \quad \iint_{\Omega_1} \nabla u \cdot \nabla \xi \, d\sigma + \int_{\Gamma_1} u \xi \, ds = - \int_{\Gamma_2} \frac{\partial u}{\partial n} \xi \, ds,$$

$$(2.5) \quad \iint_{\Omega_2} \nabla v \cdot \nabla \xi \, d\sigma = \int_{\Gamma_2} \frac{\partial v}{\partial n} \xi \, ds.$$

Adding up (2.4) and (2.5) and applying the interface condition (1.4), we thus obtain the following weak formulation:

$$(2.6) \quad \begin{aligned} & \iint_{\Omega_1} \nabla u \cdot \nabla \xi \, d\sigma + \iint_{\Omega_2} \nabla v \cdot \nabla \xi \, d\sigma \\ & + \int_{\Gamma_1} u \xi \, ds + k \int_{\Gamma_2} (v - u) \xi \, ds = 0. \end{aligned}$$

The above observation motivates us to define the weak solutions.

DEFINITION 2.1. *A weak solution of (1.1)–(1.6) is a pair of bounded functions $(u, v) \in W_{loc}^{1,2}(\Omega_1) \times W_{loc}^{1,2}(\Omega_2)$ such that $u = \beta(v)$ on Γ_2 and $v = \varphi$ on Γ_3 , and (2.6) holds for any $\xi \in W_{\Gamma_3}^{1,2}(\bar{\Omega})$ that has the compact support in $\bar{\Omega}$.*

We have the following maximum principle.

LEMMA 2.1. *Let (u, v) be a weak solution of (1.1)–(1.6). Then*

$$(2.7) \quad 0 \leq u \leq \beta(1), \quad 0 \leq v \leq 1.$$

Proof. Let $\xi \in C_0^\infty(\bar{\Omega})$ be a cut-off function such that $\xi = 1$ on $\Omega \cap B_R$. It is easy to verify that the function defined by

$$\eta = \begin{cases} u^- \xi^2 & \text{in } \Omega_1 \\ -\beta(-v^-) \xi^2 & \text{in } \Omega_2 \end{cases}$$

is a legitimate test function. It follows from (2.6) that

$$\begin{aligned} & \iint_{\Omega_1} |\nabla u^-|^2 \xi^2 d\sigma + \iint_{\Omega_2} \beta'(-v^-) |\nabla v^-|^2 \xi^2 d\sigma \\ & - \int_{\bar{\Gamma}_1} uu^- \xi^2 ds + k \int_{\bar{\Gamma}_2} (u-v) u^- \xi^2 ds \\ & \leq \iint_{\Omega_1} 2 |\xi \nabla u^-| |u^-| |\nabla \xi| d\sigma + \iint_{\Omega_2} 2 |\beta(-v^-)| |\xi \nabla v^-| |\nabla \xi| d\sigma. \end{aligned}$$

Since $uu^- \leq 0$, and on Γ_2

$$(u-v)u^- = (\beta(v)-v)u^- \geq 0,$$

where we have applied the mean value theorem and used the fact (2.1), we thus obtain

$$\begin{aligned} & \iint_{\Omega_1} |\nabla u^-|^2 \xi^2 d\sigma + \iint_{\Omega_2} \beta'(-v^-) |\nabla v^-|^2 \xi^2 d\sigma \\ & \leq \iint_{\Omega_1} 2 |\xi \nabla u^-| |u^-| |\nabla \xi| d\sigma + \iint_{\Omega_2} 2 |\beta(-v^-)| |\xi \nabla v^-| |\nabla \xi| d\sigma. \end{aligned}$$

It follows that, since u and v are bounded,

$$(2.8) \quad \iint_{\Omega_1} |\nabla u^-|^2 \xi^2 d\sigma + \iint_{\Omega_2} |\nabla v^-|^2 \xi^2 d\sigma \leq c \iint_{\Omega} |\nabla \xi|^2 d\sigma,$$

where c is a constant depending on the bounds of ξ . We now choose ξ such that $\xi = 1$ on $\Omega \cap B_R$, $\xi = 0$ on $\Omega \setminus B_{2R}$, $0 \leq \xi \leq 1$, and $|\nabla \xi| \leq cR^{-1}$. It follows from (2.8) that

$$\iint_{\Omega_1} |\nabla u^-|^2 \xi^2 d\sigma + \iint_{\Omega_2} |\nabla v^-|^2 \xi^2 d\sigma \leq \frac{c|\Omega \cap B_{2R}|}{R^2} \leq \frac{c}{R},$$

since Ω lies in a finite strip $-\|g_3\|_{C^0} \leq y \leq \|g_1\|_{C^0}$. Let $R \rightarrow \infty$, we thus proved that $u, v \geq 0$. To show the other side of (2.7), we take the test function

$$\xi = \begin{cases} (u - \beta(1))^+ \xi^2 & \text{in } \Omega_1 \\ (\beta(v) - \beta(1))^+ \xi^2 & \text{in } \Omega_2. \end{cases}$$

We can then derive analogously that

$$\iint_{\Omega_1} |\nabla (u - \beta(1))^+|^2 \xi^2 d\sigma + \iint_{\Omega_2} |\nabla (v - 1)^+|^2 \xi^2 d\sigma \leq \frac{c}{R}.$$

The assertion follows by taking $R \rightarrow \infty$.

The Definition 2.1 is somewhat inconvenient because of the jump condition $u = \beta(v)$. To avoid it, we introduce a function $\alpha(s) = \beta^{-1}(s)$. Note that α is defined only in the range of β , and

$$(2.9) \quad \alpha(0) = 0, \quad \alpha'(s) \geq 1.$$

Set

$$(2.10) \quad w = \beta(v), \quad \text{in } \Omega_2.$$

Problem (1.1)–(1.6) is thus reduced to equation (1.1), (1.6) and

$$(2.11) \quad \nabla \cdot (\alpha'(w) \nabla w) = 0 \quad \text{in } \Omega_2,$$

$$(2.12) \quad u = w \quad \text{in } \Gamma_2,$$

$$(2.13) \quad \frac{\partial u}{\partial n} - \alpha'(w) \frac{\partial w}{\partial n} = -k(u - \alpha(w)) \quad \text{on } \Gamma_2,$$

$$(2.14) \quad w = \beta(\varphi) = \psi \quad \text{on } \Gamma_3.$$

Note that $0 \leq \psi \leq \beta(1) \leq 1$. For any $\xi \in C_0^\infty(\bar{\Omega})$ such that $\xi = 0$ on Γ_3 , we have the equivalent weak formulation

$$(2.15) \quad \begin{aligned} \iint_{\Omega_1} \nabla u \cdot \nabla \xi d\sigma + \iint_{\Omega_2} \nabla \alpha(w) \cdot \nabla \xi d\sigma \\ + \int_{\Gamma_1} u \xi ds + k \int_{\Gamma_2} (\alpha(u) - u) \xi ds = 0. \end{aligned}$$

Thanks to the maximum principle Lemma 2.1, Definition 2.1 is equivalent to the following more natural definition.

DEFINITION 2.2. *A weak solution for (1.1)–(1.6) is a function $U \in W_{loc}^{1,2}(\Omega)$ such that $0 \leq U \leq 1$, $(U - \psi) \in W_{\Gamma_3}^{1,2}(\Omega)$ and (2.15) holds for any $\xi \in W_{\Gamma_3}^{1,2}(\Omega)$ that has*

compact support in $\bar{\Omega}$, where $u = U|_{\Omega_1}$, $w = U|_{\Omega_2}$, the restrictions of U in Ω_1 and Ω_2 , respectively.

For convenience, we shall write $U = (u, w)$. Throughout the paper, we shall adopt Definition 2.2. We conclude this section by the following estimate.

LEMMA 2.2. *Let (u, w) be a weak solution. Then there exists a constant c such that*

$$\|\nabla(u, w)\|_{L^2(\Omega)} \leq c.$$

Moreover, if $\varphi \in W^{1,2}(\Omega)$, then

$$\|(u, w)\|_{W^{1,2}(\Omega)} \leq c.$$

Proof. Choose a function $\tilde{\varphi}$ that is identical to φ near Γ_3 such that $\tilde{\varphi} = 0$ near Γ_1 and Γ_2 , and $\|\nabla \tilde{\varphi}\|_{L^2(\Omega)} \leq c \|\nabla \varphi\|_{L^2(\Gamma_3)}$. For simplicity, we still denote by $\psi = \beta(\tilde{\varphi})$. Let $\xi \in C_0^\infty(\bar{\Omega})$. In (2.15) we replace ξ by $(u - \psi, w - \psi)\xi^2$. Then

$$\begin{aligned} & \iint_{\Omega_1} |\nabla u|^2 \xi^2 d\sigma + \iint_{\Omega_2} \alpha'(w) |\nabla w|^2 \xi^2 d\sigma \\ & \quad + \int_{\Gamma_1} u^2 \xi^2 ds + k \int_{\Gamma_2} (\alpha(u) - u) u \xi^2 ds \\ & \leq \iint_{\Omega_1} |\nabla u| |\nabla \psi| \xi^2 d\sigma + \iint_{\Omega_1} |\nabla u| |\nabla \xi^2| |u - \psi| d\sigma \\ & \quad + \iint_{\Omega_2} \alpha'(w) |\nabla w| |\nabla \psi| \xi^2 d\sigma + \iint_{\Omega_2} \alpha'(w) |\nabla w| |\nabla \xi^2| |w - \psi| d\sigma \end{aligned}$$

Since $1 \leq \alpha'(w) \leq c$, it follows Hölder's inequality that

$$\begin{aligned} & \iint_{\Omega_1} |\nabla u|^2 \xi^2 d\sigma + \iint_{\Omega_2} |\nabla w|^2 \xi^2 d\sigma \\ (2.16) \quad & \leq c \iint_{\Omega} (|\nabla \psi|^2 \xi^2 + |\nabla \xi|^2) \end{aligned}$$

where we used the fact that $(\alpha(u) - u)u \geq 0$ and Lemma 2.1. Note that since $\psi = \beta(\varphi)$, we have $|\nabla \psi| \leq |\nabla \tilde{\varphi}|$. Hence $\nabla \psi \in L^2$. We now choose ξ such that $\xi = 1$ in B_r , $\xi = 0$ outside B_{2r} , $0 \leq \xi \leq 1$, and $|\nabla \xi| \leq cr^{-1}$. It follows from (2.16) that

$$\iint_{\Omega_1} |\nabla u|^2 \xi^2 d\sigma + \iint_{\Omega_2} |\nabla w|^2 \xi^2 d\sigma \leq c \left(1 + \frac{1}{r}\right).$$

The first assertion follows by letting $r \rightarrow \infty$. Since $w = \psi$ on Γ_3 , by the Sobolev inequalities we deduce that $(u, w) - \psi \in W^{1,2}$. The second assertion follows.

3. Some a Priori Estimates. In this section we shall first derive some estimates on $W_{loc}^{2,2}$ - norm and then use them to obtain a priori estimates on $C^{2+\mu}$ - norm. Let $\Gamma_i = \{y = g_i(x)\}$ for $i = 1, 2, 3$. Throughout the section, we assume that $g_i \in C^{2+\mu}$ that satisfy (2.2). Since u and v solve (1.1) and (1.2) in Ω_1 and Ω_2 with the boundary condition (1.3) and (1.6) on Γ_1 and Γ_3 respectively, and $w = \beta(v)$, by Lemma 2.1, Lemma 2.2 and the standard regularity results for the linear elliptic equations [5], we know that near Γ_1 and Γ_3 the $C^{2+\mu}$ - norms of u and w are uniformly bounded. Hence we only need to take care the situation near Γ_2 .

LEMMA 3.1. *Let (u, w) be a weak solution of (1.1)–(1.6). Assume that $(u, w) \in W_{loc}^{1,4}(\Omega)$. Then $u \in W_{loc}^{2,2}(\Omega_1)$, $w \in W_{loc}^{2,2}(\Omega_2)$, and*

$$(3.1) \quad \|u\|_{W_{loc}^{2,2}(\Omega_1)} + \|w\|_{W_{loc}^{2,2}(\Omega_2)} \leq c \|(u, w)\|_{W_{loc}^{1,4}(\Omega)}$$

where c is a constant depending only on $\|g_i\|_{C^{2+\mu}}$ (for fixed β and φ .)

Proof. Since $g_2 \in C^{2+\mu}$, by a local transformation, we may assume without loss of generality that Γ_2 is x - axis. For any $(x_0, y_0) \in \Gamma_2$, let $B_r(x_0, y_0)$ be a ball contained in Ω and $B_r^\mp = B_r(x_0, y_0) \cap \{(x, y) : y \gtrless 0\}$. For any function f and h small, we write the difference quotients by

$$\begin{aligned} \Delta^h f(x, y) &= (f(x+h, y) - f(x, y)) h^{-1}, \\ \Delta^{-h} f(x, y) &= (f(x, y-h) - f(x, y)) h^{-1}. \end{aligned}$$

Then for any $\xi \in C_0^\infty(B_r)$, we have

$$\iint_{B_r^+} \nabla u \cdot \nabla (\Delta^{-h} \xi) d\sigma = - \iint_{B_r^+} \Delta^h (\nabla u) \cdot \nabla \xi d\sigma.$$

It follows from the weak formulation (2.15) that

$$\iint_{B_r^+} \Delta^h (\nabla u) \cdot \nabla \xi d\sigma + \iint_{B_r^-} \Delta^h (\nabla \alpha(w)) \cdot \nabla \xi d\sigma = k \int_{\Gamma_2} (\alpha(u) - u) \Delta^{-h} \xi ds.$$

Since

$$\Delta^h(\nabla \alpha(w)) = \alpha'(w(x+h, y)) \Delta^h(\nabla w) + \Delta^h(\alpha'(w)) \nabla w,$$

we obtain

$$\begin{aligned} (3.2) \quad & \iint_{B_r^+} \Delta^h(\nabla u) \cdot \nabla \xi \, d\sigma + \iint_{B_r^-} \alpha'(w(x+h, y)) \Delta^h(\nabla w) \cdot \nabla \xi \, d\sigma \\ &= - \iint_{B_r^-} \Delta^h(\alpha'(w)) \nabla w \cdot \nabla \xi \, d\sigma + k \int_{\Gamma_2} (\alpha(u) - u) \Delta^{-h} \xi \, ds. \end{aligned}$$

We now replace ξ by

$$\eta = \begin{cases} \left(\Delta^h u \right) \xi^2 & \text{in } \Omega_1 \\ \left(\Delta^h w \right) \xi^2 & \text{in } \Omega_2 \end{cases}$$

in (3.2). Then (3.2) becomes

$$\begin{aligned} & \iint_{B_r^+} \left| \Delta^h(\nabla u) \right|^2 \xi^2 \, d\sigma + \iint_{B_r^-} \alpha'(w(x+h, y)) \left| \Delta^h(\nabla w) \right|^2 \xi^2 \, d\sigma \\ &= - \iint_{B_r^+} 2\xi \Delta^h u \Delta^h(\nabla u) \cdot \nabla \xi \, d\sigma \\ & \quad - \iint_{B_r^-} 2\alpha'(w(x+h, y)) \xi \Delta^h w \Delta^h(\nabla w) \cdot \nabla \xi \, d\sigma \\ & \quad - \iint_{B_r^-} \Delta^h(\alpha'(w)) \nabla w \cdot \nabla (\Delta^h w) \xi^2 \, d\sigma \\ & \quad - \iint_{B_r^-} \Delta^h(\alpha'(w)) \nabla w \cdot \nabla (\xi^2) (\Delta^h w) \, d\sigma \\ & \quad + k \int_{\Gamma_2} (\alpha(u) - u) \Delta^{-h} \left((\Delta^h u) \xi^2 \right) \, ds \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

By Lemma 2.2, when h is small enough we have

$$|I_1| \leq c \left(\iint_{B_r^+} \left| \Delta^h(\nabla u) \right|^2 \xi^2 \right)^{\frac{1}{2}}, \quad |I_2| \leq c \left(\iint_{B_r^-} \left| \Delta^h(\nabla w) \right|^2 \xi^2 \right)^{\frac{1}{2}}.$$

Since

$$(3.3) \quad \left(\Delta^h \alpha \right) (w) \leq c \left| \Delta^h w \right|,$$

we obtain

$$|I_3| \leq c \left(\iint_{B_r^-} |\nabla w|^2 |\Delta^h w|^2 \right)^{\frac{1}{2}} \left(\iint_{B_r^-} |\Delta^h (\nabla w)|^2 \xi^2 \right)^{\frac{1}{2}},$$

$$|I_4| \leq c \iint_{B_r^-} |\nabla w| |\Delta^h w|^2 d\sigma.$$

It remains to estimate I_5 . Since $\Gamma_2 = \{y = 0\}$, by integration by parts,

$$\begin{aligned} I_5 &= -k \int_{\Gamma_2} (\Delta^h(\alpha(u)) - \Delta^h u) (\Delta^h u) \xi^2 ds \\ &= k \iint_{B_r^+} \frac{\partial}{\partial y} [(\Delta^h(\alpha(u)) - \Delta^h u) (\Delta^h u) \xi^2] d\sigma \\ &= k \iint_{B_r^+} (\Delta^h(\alpha'(u) u_y) - \Delta^h u_y) (\Delta^h u) \xi^2 d\sigma \\ &\quad + k \iint_{B_r^+} (\Delta^h(\alpha) - \Delta^h u) [(\Delta^h u_y) \xi^2 + 2(\Delta^h u) \xi \xi_y] d\sigma. \end{aligned}$$

Hence, by (3.3),

$$\begin{aligned} |I_5| &\leq c \left(\left(\iint_{B_r^+} |\nabla u|^2 |\Delta^h u|^2 d\sigma \right)^{\frac{1}{2}} + \|\Delta^h u\|_{L^2(B_r^+)} \right) \\ &\quad \times \left(\iint_{B_r^+} |\Delta^h (\nabla u)|^2 \xi^2 d\sigma \right)^{\frac{1}{2}}. \end{aligned}$$

Combining the above estimates for I_i , we obtain

$$\begin{aligned} &\iint_{B_r^+} |\Delta^h (\nabla u)|^2 \xi^2 d\sigma + \iint_{B_r^-} |\Delta^h (\nabla w)|^2 \xi^2 d\sigma \\ &\leq c \left(\|\Delta^h u\|_{w^{1,4}(B_r^+)}^4 + \|\Delta^h w\|_{w^{1,4}(B_r^-)}^4 \right). \end{aligned}$$

By choosing ξ such that $\xi = 1$ in $B_{r/2}$ and taking limit as $h \rightarrow 0$, we obtain

$$(3.4) \quad \|\nabla u_x\|_{L^2(B_{r/2}^+)}^2 + \|\nabla w_x\|_{L^2(B_{r/2}^-)}^2 \leq c \left(\|\Delta^h u\|_{w^{1,4}(B_r^+)}^4 + \|\Delta^h w\|_{w^{1,4}(B_r^-)}^4 \right).$$

Since the above inequality holds for any $(x_0, y_0) \in \Gamma_2$ with a fixed r and c , and u is harmonic in Ω_1 and $\alpha'(w) \Delta w + \alpha''(w) |\nabla w|^2 = 0$ in Ω_2 , the assertion (3.1) follows.

LEMMA 3.2. *Under the assumptions of Lemma 3.1, there exists a constant c depending on g_i, φ, β such that*

$$(3.5) \quad \|u\|_{W_{loc}^{2,2}(\Omega_1)} + \|w\|_{W_{loc}^{2,2}(\Omega_2)} \leq c.$$

Proof. Taking $\xi \in C_0^\infty(\Omega)$ in (2.15) and applying integration by parts, we have

$$\begin{aligned} & \iint_{\Omega_1} \nabla u \cdot \nabla \xi \, d\sigma + \iint_{\Omega_2} \alpha'(w) \nabla w \cdot \nabla \xi \, d\sigma + k \iint_{\Omega_1} (u\xi)_y \sqrt{1 + (g_{2x})^2} \, d\sigma \\ & + k \iint_{\Omega_2} (\alpha(w)\xi)_y \sqrt{1 + (g_{2x})^2} \, d\sigma = 0. \end{aligned}$$

We can rewrite this equality as

$$\iint_{\Omega} (a \nabla U \cdot \nabla \xi + b U_y \xi + f U \xi_y) \, d\sigma = 0,$$

where

$$\begin{aligned} a &= \begin{cases} 1 & \text{in } \Omega_1 \\ \alpha'(w) & \text{in } \Omega_2 \end{cases}, \quad b = \begin{cases} k \sqrt{1 + (g_{2x})^2} & \text{in } \Omega_1 \\ k \alpha'(w) \sqrt{1 + (g_{2x})^2} & \text{in } \Omega_2 \end{cases}, \\ f &= \begin{cases} k \sqrt{1 + (g_{2x})^2} & \text{in } \Omega_1 \\ \frac{k \alpha(w)}{w} \sqrt{1 + (g_{2x})^2} & \text{in } \Omega_2 \end{cases}. \end{aligned}$$

Since $|a| + |b| + |f| \leq c$, by [7, Theorem 2] (see also [8]), it follows that for small r and $(x_0, y_0) \in \Gamma_2$, we have

$$\iint_{B_{2r}(x_0, y_0) \cap \Omega} |U_x|^2 \, d\sigma \leq cr^{2\mu}$$

for some $0 < \mu < 1$. Note that although Theorem 2 in [7] states slightly different, the proof is local. It can be easily extended to the above situation. Hence we apply it directly. Since $U \in W_{loc}^{2,2}$, by [7, Lemma 1], we thus obtain, for $\xi = \eta|U_x|$, where $\eta \in C_0^\infty$, that

$$\begin{aligned} (3.6) \quad & \iint_{B_r \cap \Omega} |U_x|^4 \eta^2 \, d\sigma \leq cr^\mu \iint_{B_r^+ \cap \Omega} (|\nabla u_x|^2 \eta^2 + |u_x|^2 |\nabla \eta|^2) \, d\sigma \\ & + cr^\mu \iint_{B_r^+ \cap \Omega} (|\nabla w_x|^2 \eta^2 + |w_x|^2 |\nabla \eta|^2) \, d\sigma. \end{aligned}$$

Choosing η such that $\eta = 1$ in B_r and $\eta = 0$ outside B_{2r} and using (3.4) (after taking limit as $h \rightarrow 0$), we obtain

$$\iint_{B_r \cap \Omega} |U_x|^4 d\sigma \leq c.$$

The assertion follows from (3.4).

REMARK 3.1. *We actually can show that Lemma 3.2 is true without the assumption $U \in W_{loc}^{1,4}$. Since we would not use this result, the addition assumption is introduced to simplify our proof.*

LEMMA 3.3. *Under the assumption of Lemma 3.1, we have $u \in C^{1+\mu}(\Omega_1)$, $w \in C^{1+\mu}(\Omega_2)$ for some $0 < \mu < 1$, and there exists a constant c depending only on the initial data g_i, β, φ such that*

$$\|u\|_{C^{1+\mu}} + \|w\|_{C^{1+\mu}} \leq c.$$

Proof. Let $\xi \in C_0^\infty(\Omega)$ and Take ξ_x instead of ξ in (2.15). By integration by parts, we obtain

$$\begin{aligned} & \iint_{\Omega_1} \nabla u_x \cdot \nabla \xi d\sigma + \iint_{\Omega_2} \alpha'(w) \nabla w_x \cdot \nabla \xi d\sigma + \iint_{\Omega_2} \alpha''(w) w_x \nabla w \cdot \nabla \xi d\sigma \\ & + k \iint_{\Omega_1} \left(u_y \xi_x \sqrt{1 + (g_{2x})^2} - \left(u \sqrt{1 + (g_{2x})^2} \right)_x \xi_y \right) d\sigma \\ & + k \iint_{\Omega_2} \left(\alpha'(w) w_y \xi_x \sqrt{1 + (g_{2x})^2} - \left(\alpha(w) \sqrt{1 + (g_{2x})^2} \right)_x \xi_y \right) d\sigma = 0 \end{aligned}$$

We can rewrite the above equality as

$$\iint_{\Omega} (a \nabla U_x \cdot \nabla \xi + b_1 U_x \xi_x + b_2 U_x \xi_y + f_1 \xi_x + f_2 \xi_y) d\sigma,$$

where

$$a = \begin{cases} 1 & \text{in } \Omega_1 \\ \alpha'(w) & \text{in } \Omega_2, \end{cases} \quad b_1 = \begin{cases} 0 & \text{in } \Omega_1 \\ \alpha''(w) w_x & \text{in } \Omega_2, \end{cases}$$

$$b_2 = \begin{cases} -k \sqrt{1 + (g_{2x})^2} & \text{in } \Omega_1 \\ -k \alpha'(w) \sqrt{1 + (g_{2x})^2} + \alpha''(w) w_y & \text{in } \Omega_2, \end{cases}$$

$$f_1 = \begin{cases} ku_y \sqrt{1 + (g_{2x})^2} & \text{in } \Omega_1 \\ k\alpha'(w) w_y \sqrt{1 + (g_{2x})^2} & \text{in } \Omega_2, \end{cases}$$

$$f_2 = \begin{cases} -ku \left(\sqrt{1 + (g_{2x})^2} \right)_x & \text{in } \Omega_1 \\ -k\alpha(w) \left(\sqrt{1 + (g_{2x})^2} \right)_x & \text{in } \Omega_2. \end{cases}$$

By [7, Theorem 2] and Lemma 3.2, we obtain

$$\iint_{B_r \cap \Omega} |\nabla U_x|^2 d\sigma \leq cr^{2\mu}.$$

Since U is harmonic in $\Omega_1 \cup \Omega_2$, it follows

$$\iint_{B_r \cap \Omega_1} |\nabla^2 u|^2 d\sigma + \iint_{B_r \cap \Omega_2} |\nabla^2 w|^2 d\sigma \leq cr^{2\mu}.$$

The assertion follows from the Morrey lemma [5, Lemma 12.2].

4. Diffraction Problem (1.1)–(1.6). In this section we shall establish existence and uniqueness of a solution (u, v) for the diffraction problem (1.1)–(1.6). We shall further show that $u \in C^{2+\mu}(\Omega_1)$, $v \in C^{2+\mu}(\Omega_2)$. We need the following version of Lerray-Schauder fixed point theorem [3, §7.1 Theorem 3].

LEMMA 4.1. *Let X be a Banach space, $B_r = \{z \in X : \|z\| \leq r\}$, and T a compact mapping of $B_r \times [0, 1] \mapsto X$. Suppose that $T(x, 0) = 0$ and that any solution of $T(x, t) = x$ satisfies*

$$\|x\| \leq c.$$

Then there exists a fixed point for the mapping $T(\cdot, 1)$.

Now let $\alpha_t(s) = (1 - t)s + t\alpha(s)$. Then $\alpha_t(0) = 0$ and $\alpha'_t(s) \geq 1$. Hence α_t has the same properties as α that we need to derive the estimates in Lemma 3.1–Lemma 3.3. For any $(\tilde{u}, \tilde{w}) \in C^0(\Omega)$ such that $\tilde{u} \in C^1(\Omega_1)$, $\tilde{w} \in C^1(\Omega_2)$, and $0 \leq \tilde{u}, \tilde{w} \leq 1$, we consider the following problem: Find $(u, w) \in W_{loc}^{1,2}(\Omega)$, $w = t\psi$ on Γ_3 such that for

any $\xi \in W_{loc}^{1,2}(\Omega)$, $\xi = 0$ on Γ_3 , the equality

$$(4.1) \quad \begin{aligned} & \iint_{\Omega_1} \nabla u \cdot \nabla \xi \, d\sigma + \iint_{\Omega_2} \alpha'_t(|\tilde{w}|) \nabla w \cdot \nabla \xi \, d\sigma \\ & + \int_{\Gamma_1} u \xi \, ds + k \int_{\Gamma_2} (\hat{\alpha}_t(|\tilde{w}|) w - u) \xi \, ds = 0 \end{aligned}$$

holds, where $\hat{\alpha}_t(s) = \alpha_t(s)/s$. Note that $\hat{\alpha}_t$ is smooth and $\hat{\alpha}_t(s) \geq 1$.

LEMMA 4.2. *There exists a unique solution $U_t = (u_t, w_t)$ of (4.1). Furthermore, $U_t \in C^\mu(\Omega)$, $u_t \in C^{1+\mu}(\Omega_1)$, $w_t \in C^{1+\mu}(\Omega_2)$, and there exists a constant c depending only on g_i, φ, β such that*

$$(4.2) \quad \|u_t\|_{C^{1+\mu}} + \|w_t\|_{C^{1+\mu}} \leq c(\|\tilde{u}\|_{C^1} + \|\tilde{w}\|_{C^1}).$$

Proof. For any large integer n , consider the following truncated problem: Find $U_n = (u_n, w_n) \in W^{1,2}(\Omega \cap B_n)$, $U_n = t\psi$ on $\Gamma_n = \Gamma_3 \cap B_n$ such that

$$(4.3) \quad \begin{aligned} & \iint_{\Omega_1 \cap B_n} \nabla u_n \cdot \nabla \xi \, d\sigma + \iint_{\Omega_2 \cap B_n} \alpha'_t(|\tilde{w}|) \nabla w_n \cdot \nabla \xi \, d\sigma \\ & + \int_{\Gamma_1 \cap B_n} u_n \xi \, ds + k \int_{\Gamma_2 \cap B_n} (\hat{\alpha}_t(|\tilde{w}|) w_n - u_n) \xi \, ds = 0 \end{aligned}$$

holds for any $\xi \in W_{\Gamma_n}^{1,2}(\Omega \cap B_n)$. Since $\alpha'_t, \hat{\alpha}_t(s) \geq 1$, by the Lax-Milgram theorem, (4.3) admits a unique solution. Moreover, by the standard method, we can easily derive the maximum principle and $W^{1,2}$ -estimates. Hence

$$0 \leq u_n, w_n \leq t\psi, \quad \iint_{\Omega \cap B_n} |\nabla U_n|^2 \, d\sigma \leq c(n).$$

We can also show that, by taking a cut-off function, for any $n > r > 0$ we have

$$\iint_{\Omega \cap B_r} |\nabla U_n|^2 \, d\sigma \leq c(r).$$

Hence for any fixed $r > 0$, there exists a subsequence $U_{n(r)}$ such that $U_{n(r)}|_{\Omega \cap B_r}$ weakly converges to a function U in $W^{1,2}(\Omega \cap B_r)$. By the diagonal argument, we can further select a subsequence, denote it by U_n , and $U_t \in W_{loc}^{1,2}(\Omega)$ such that $U_n \rightarrow U_t$ weakly in

$W^{1,2}(\Omega \cap B_r)$ for any $r > 0$. By taking limit in (4.3), we obtain a weak solution for (4.1). Uniqueness follows from the maximum principle. By slightly modifying the proofs of Lemma 3.1, Lemma 3.2 and Lemma 3.3, we obtain $u_t \in C^{1+\mu}(\Omega_1)$, $w_t \in C^{1+\mu}(\Omega_2)$ and estimates (4.2).

THEOREM 4.1. *There exists a solution $U = (u, w)$ of (1.1)–(1.6). Furthermore, $u \in C^{2+\mu}(\Omega_1)$, $w \in C^{2+\mu}(\Omega_2)$.*

Proof. Let $X = \{U = (u, w) \in C^0(\Omega) : u \in C^1(\Omega_1), w \in C^1(\Omega_2)\}$ with the norm $\|U\| = \|u\|_{C^1} + \|w\|_{C^1}$, and B_1^X be the unit ball in X . We define a mapping $T : B_1^X \times [0, 1] \mapsto X$ by $T((\tilde{u}, \tilde{w}), t) = (u, w)$, where (u, w) is the solution of (4.1). By Lemma 4.2 and (4.2), T is compact, and $T((\tilde{u}, \tilde{w}), 0) = 0$. It follows from Lemma 3.3 that all the assumptions in Lemma 4.1 are satisfied. Therefore, there exists a fixed point U of $T(\cdot, 1)$, i.e., $T(U, 1) = U$, a solution of (1.1)–(1.6).

It remains to show that the solution belong to $C^{2+\mu}$. Since $g_2 \in C^{2+\mu}$, by a transformation $y \mapsto y - g_2(x)$, we may assume that $\Gamma_2 = \{y = 0\}$, and in $B_r^\pm = B_r^\pm(x_0, y_0)$, for any $(x_0, y_0) \in \Gamma_2$, u and w satisfy

$$L^+(x)u = 0 \quad \text{in } B_r^+, \quad L^+(x)\alpha(w) = 0 \quad \text{in } B_r^-,$$

where $L^+(x)$ is the uniform elliptic operators defined by

$$L^+(x)u = u_{xx} - 2g_2'(x)u_{xy} + (1 + (g_2')^2)u_{yy} - g_2''(x)u_y,$$

respectively. Also (u, w) satisfy on $\Gamma_2 \cap B_r$ the interface condition $u = w$ and

$$\begin{aligned} & g_2'(x)(u - \alpha(w))_x + (1 + (g_2'(x))^2)(u - \alpha(w))_y \\ &= k\sqrt{1 + (g_2')^2}(u - \alpha(w)). \end{aligned}$$

Let $\hat{w}(x, y) = w(x, -y)$. Then in B_r^+ , u and $\hat{w}$ solve $L^+(x)u = L^-(x)\alpha(\hat{w}) = 0$ with the boundary conditions $u = \hat{w}$ and

$$\begin{aligned} & g_2'(x)(u - \alpha(\hat{w}))_x + (1 + (g_2'(x))^2)(u + \alpha(\hat{w}))_y \\ &= k\sqrt{1 + (g_2')^2}(u - \alpha(\hat{w})), \end{aligned}$$

where

$$L^-(x)u = u_{xx} + 2g'_2(x)u_{xy} + (1 + (g'_2)^2)u_{yy} + g''_2(x)u_y.$$

Since by Lemma 4.2, $u, w \in C^{1+\mu}$, it follows from [1] that $u, \hat{w} \in C^{2+\mu}(B_r^+)$ and that their $C^{2+\mu}(B_r^+)$ -norms are bounded by their $C^{1+\mu}(B_r^+)$ -norms. Note that (x_0, y_0) is an arbitrary point on Γ_2 . The assertion follows.

REMARK 4.1. *Since from Lemma 3.3, $\|u\|_{C^{1+\mu}} + \|w\|_{C^{1+\mu}} \leq c$. It follows from the proof of the above theorem that $\|u\|_{C^{2+\mu}} + \|w\|_{C^{2+\mu}} \leq c$ with a constant c depending only on g_i, φ, β . In addition, if $g_2, \beta \in C^\infty$, then the same arguments show that u, w are of C^∞ .*

THEOREM 4.2. *The solution of the diffraction problem (1.1)–(1.6) is unique.*

Proof. Let (u_i, w_i) for $i = 1, 2$, be two solutions. Set $u = u_1 - u_2$, $w = w_1 - w_2$. By Lemma 2.2, ∇u_i and ∇w_i are of L^2 . From (2.15), we have

$$(4.4) \quad \begin{aligned} & \iint_{\Omega_1} \nabla u \cdot \nabla \xi \, d\sigma + \iint_{\Omega_2} \alpha'(w_1) \nabla w \cdot \nabla \xi \, d\sigma \\ & + \iint_{\Omega_2} (\alpha'(w_1) - \alpha'(w_2)) \nabla w_2 \cdot \nabla \xi \, d\sigma \\ & + \int_{\Gamma_1} u \xi \, ds + k \int_{\Gamma_2} (\alpha(u_1) - \alpha(u_2) - u) \xi \, ds = 0, \end{aligned}$$

for any $\xi \in W_{\Gamma_3}^{1,2}(\Omega)$ with a compact support. Choose $\eta \in C_0^\infty(R^1)$ such that $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$, $0 \leq \eta \leq 1$. Define η_r in Ω by $\eta_r(x, y) = \eta(r^{-1}x)$. It is easy to see that $0 \leq \eta_r(x, y) \leq 1$ and that as $r \rightarrow \infty$,

$$(4.5) \quad \begin{aligned} \eta_r & \longrightarrow 1 \quad \text{a.e. in } \Omega, \\ \nabla \eta_r & \longrightarrow 0 \quad \text{in } L^2(\Omega). \end{aligned}$$

We now take the following test function

$$\xi = \begin{cases} \frac{u^+ \eta_r}{u^+ + \delta} & \text{in } \Omega_1 \\ \frac{w^+ \eta_r}{w^+ + \delta} & \text{in } \Omega_2, \end{cases}$$

where $\delta > 0$. Write

$$(4.6) \quad \alpha(u_1) - \alpha(u_2) = u \int_0^1 \alpha'(su + u_2) ds = u\gamma.$$

From the fact $\alpha' \geq 1$, we know that $\gamma \geq 1$. Hence $\alpha(u_1) - \alpha(u_2) - u = u(\gamma - 1) \geq 0$ if $u \geq 0$. It follows from (4.5) that

$$\begin{aligned} & \delta \iint_{\Omega_1} \frac{|\nabla u^+|^2 \eta_r}{(u^+ + \delta)^2} d\sigma + \delta \iint_{\Omega_2} \alpha'(w_1) \frac{|\nabla w^+|^2 \eta_r}{(w^+ + \delta)^2} d\sigma \\ & \leq \delta \iint_{\Omega_2} |\alpha'(w_1) - \alpha'(w_2)| \frac{|\nabla w^+| |\nabla w_2| \eta_r}{(w^+ + \delta)^2} d\sigma + o\left(\frac{1}{r}\right), \end{aligned}$$

where

$$o\left(\frac{1}{r}\right) = \iint_{\Omega_1} |\nabla u| |\nabla \eta_r| d\sigma + c \iint_{\Omega_2} |\nabla w| |\nabla \eta_r| d\sigma.$$

Since $|\alpha'(w_1) - \alpha'(w_2)| \leq c|w| \leq c$, by letting $r \rightarrow \infty$ and using (4.5) we obtain

$$\begin{aligned} & \iint_{\Omega_1} \left| \nabla \ln \left(\frac{u^+}{\delta} + 1 \right) \right|^2 d\sigma + \iint_{\Omega_2} \left| \nabla \ln \left(\frac{w^+}{\delta} + 1 \right) \right|^2 d\sigma \\ & \leq c \iint_{\Omega_2} |\nabla w_2| |\nabla w^+| d\sigma \leq c. \end{aligned}$$

Since $\Omega = \{g_1(x) < y < g_3(x)\}$ with $0 < g_3 - g_1 \leq c$, and $w^+ = 0$ on Γ_3 , the *Poincaré* inequality

$$\begin{aligned} & \iint_{\Omega_1} \left| \ln \left(\frac{u^+}{\delta} + 1 \right) \right|^2 d\sigma + \iint_{\Omega_2} \left| \ln \left(\frac{w^+}{\delta} + 1 \right) \right|^2 d\sigma \\ & \leq c \iint_{\Omega_1} \left| \nabla \ln \left(\frac{u^+}{\delta} + 1 \right) \right|^2 d\sigma + c \iint_{\Omega_2} \left| \nabla \ln \left(\frac{w^+}{\delta} + 1 \right) \right|^2 d\sigma \leq c \end{aligned}$$

holds. Let $\delta \rightarrow 0$. It follows $u^+ = 0$, $w^+ = 0$. Analogously, we can show that $u^- = 0$, $w^- = 0$. The assertion follows.

5. Existence and Uniqueness of (1.1)–(1.10). In this section, we establish existence a uniqueness of a classical solution for (1.1)–(1.10). Recall that a solution of (1.1)–(1.10) consists of g_i , for $i = 1, 2$, defined in $R^1 \times [0, T]$, u defined in $\{(x, y, t) :$

$(x, y) \in \Omega_1(t)$, $0 \leq t \leq T$ and w defined in $\{(x, y, t) : (x, y) \in \Omega_2(t), 0 \leq t \leq T\}$. Denote it by (u, w, g_1, g_2) . For any three variable function $f(x, y, t)$, we say $f \in C_x^m$ if for any fixed y, t in its domain, $f(\cdot, y, t) \in C^m$ as a function of x . Denote the corresponding norm by $\|f\|_{C_x^m}$. Similarly we define the spaces C_y^m , C_t^m , and C_{xy}^m .

THEOREM 5.1. *Suppose that the initial conditions in (1.10) satisfy (2.2) and*

$$\|g_1^0\|_{C^{2+\mu}} + \|g_2^0\|_{C^{2+\mu}} \leq M.$$

Then there exists a unique $C^{2+\mu}$ solution of (1.1)–(1.10) for some time $T > 0$.

Proof. We consider the following set of functions Y_{M, b_i}^T

$$\begin{aligned} Y_{M, b_i}^T &= \{(g_1, g_2,) : g_i \in C_x^{2+\mu}(R^1), g_{it} \in C^0(R^1 \times [0, T]), \\ &\quad \|g_i\|_{C_x^{2+\mu}} \leq 2M, \|g_{it}\|_{C^0} \leq 4M, 0 \leq t \leq T, i = 1, 2, \\ &\quad g_3 \leq b_2, b_2 + b_0 \leq g_2 \leq b_1, b_1 + b_0 \leq g_1, 0 \leq t \leq T, \\ &\quad g_i(x, 0) = g_i^0(x) \text{ for } i = 1, 2\}. \end{aligned}$$

For any $(g_1, g_2) \in Y_{M, b_i}^T$, we set $\Gamma_i = \Gamma_i(t)$ by (1.9). By Theorem 4.1, there exists a unique solution (u, w) of (1.1)–(1.6) that satisfy $\|u\|_{C^{2+\mu}} + \|w\|_{C^{2+\mu}} \leq c(M, b_i)$ (see Remark 4.1). Let $\hat{g}_1, \hat{g}_2$ be the solutions of

$$(5.1) \quad \hat{g}_{1t} = u(x, g_1(x, t), t) \sqrt{1 + (\hat{g}_{1x})^2},$$

$$(5.2) \quad \hat{g}_{2t} = k(u(x, g_2(x, t), t) - v(x, g_2(x, t), t)) \sqrt{1 + (\hat{g}_{2x})^2},$$

respectively, with the initial conditions

$$\hat{g}_1(x, 0) = g_1(x), \hat{g}_2(x, 0) = g_2(x),$$

where $v = \alpha(w)$. We have thus defined a mapping $\mathcal{A}$ in Y_{M, b_i}^T by $\mathcal{A}(g_1, g_2) = (\hat{g}_1, \hat{g}_2)$, where $\hat{g}_1$ and $\hat{g}_2$ are defined by (5.1) and (5.2). We now show that $\mathcal{A}$ maps Y_{M, b_i}^T into itself for some $M, T > 0$.

We shall follow the ideas from [4] to estimate the solution of (5.2). Suppose at time being that $g_2, \beta \in C^\infty$. Then by Remark 4.1, u and w are also smooth along Γ_2 .

Consider, instead of (5.2) the following parabolic equation

$$(5.3) \quad g_t^\epsilon = \epsilon g_{xx}^\epsilon + k(u(x, g_2, t) - v(x, g_2, t)) \sqrt{1 + (g_x^\epsilon)^2}$$

with the same initial condition. For any $\epsilon > 0$, it is known (see, e.g., [3]) there exists a unique solution g^ϵ . By comparison principle [3, p.52], we have

$$-ct + \inf g_2^0(x) \leq g^\epsilon(x, t) \leq ct + \sup g_2^0(x),$$

where the constant c depending on

$$M_0 = \max \left\{ 1, k, b_0, \|\beta\|_{C^{2+\mu}}, \|\alpha\|_{C^{2+\mu}}, \|g_1^0\|_{C^{2+\mu}}, \|g_2^0\|_{C^{2+\mu}} \right\}.$$

Therefore for small $T > 0$ (depending on M_0)

$$b_2 + b_0 \leq g^\epsilon(x, t) \leq b_1.$$

By differentiating (5.3) with respect to x and comparing to the function $u_1 = ct + M$ with $M > M_0$ and a suitable constant c (depending on M, M_0) we derive by comparison principle that $|g_x^\epsilon(x, t)| \leq 2M$ for small T (depending on M, M_0 .) Analogously, we can obtain the estimates $|g_{xx}^\epsilon(x, t)| \leq 2M$ and $|g_t^\epsilon(x, t)| \leq 2M$, for M large enough (depending on M_0) and T small (depending on M_0, M .) Let $\epsilon \rightarrow 0$ in (5.3), we have a unique solution of (5.2) and the estimate on $|\hat{g}_{2xx}|$ and $|\hat{g}_{2t}|$. To estimate the $C^{2+\mu}$ – norm, we differentiating (5.2) to obtain

$$\begin{aligned} \frac{\partial}{\partial t} \hat{g}_{2xx} &= \frac{\hat{g}_{2x} h(x, t)}{\sqrt{1 + (\hat{g}_{2x})^2}} \hat{g}_{2xxx} + \frac{h(x, t)}{(1 + (\hat{g}_{2x})^2)^{3/2}} (\hat{g}_{2xx})^2 \\ &\quad + \frac{2\hat{g}_{2x} h(x, t)_x}{\sqrt{1 + (\hat{g}_{2x})^2}} \hat{g}_{2xx} + \sqrt{1 + (\hat{g}_{2x})^2} h(x, t)_{xx}. \end{aligned}$$

where $h(x, t) = k(u(x, g_2(x, t), t) - v(x, g_2(x, t), t))$. By integrating the above equation along the characteristics

$$\frac{\partial \Phi}{\partial t} = -\frac{\hat{g}_{2x} h(\Phi, t)}{\sqrt{1 + (\hat{g}_{1x})^2}}, \quad \Phi(0) = x,$$

we can derive that, for any $x, \hat{x}$,

$$\begin{aligned}
& |\hat{g}_{2xx}(\Phi(x, t), t) - \hat{g}_{2xx}(\Phi(\hat{x}, t), t)| \\
& \leq |g_{2xx}^0(x, t) - g_{2xx}^0(\hat{x}, t)| + ct|x - \hat{x}|^\mu \\
& \quad + c \int_0^t |\hat{g}_{2xx}(\Phi(x, t), t) - \hat{g}_{2xx}(\Phi(\hat{x}, t), t)| dt,
\end{aligned}$$

for some $c = c(M_0, M)$, where we used facts that $\|\hat{g}_2\|_{C_x^2} \leq c$ and $\|u\|_{C_x^{2+\mu}} + \|v\|_{C_x^{2+\mu}} \leq c$. It follows from the Gronwall's inequality that $\|\hat{g}_{2xx}\|_{C_x^\mu} \leq 2M$, if M is large T is small (depending on M, M_0 .) By Lemma 2.1 we also obtain from (5.2) that $\|\hat{g}_{2t}\|_{C^0} \leq 4M$ if M is also large than $2k$. Note that we have derived the estimates

$$(5.4) \quad \|\hat{g}_2\|_{C_x^{2+\mu}} \leq 2M, \quad \|\hat{g}_{2t}\|_{C^0} \leq 4M$$

under the additional assumption that g_2 and β are smooth. In general case, we only need to replace h and g_2 in (5.3) with $\xi * h(\cdot, g_2)$, $\xi * g_2$, where ξ is the δ -mollifier. From the above derivation, we obtain the corresponding estimates $\|\hat{g}_{2\delta}\|_{C_x^{2+\mu}} \leq 2M$, $\|\hat{g}_{2\delta t}\|_{C^0} \leq 4M$, for some M and T . Note that here M depends only on M_0 , $\|\xi * g_2^0\|_{C_x^{2+\mu}}$, and that T depends only on M, M_0 , and $\|\xi * h(\cdot, g_2)\|_{C_x^{2+\mu}}$. Let $\delta \rightarrow 0$, we obtain the desired the result. By the same methods, we can show that there exists a unique solution $\hat{g}_1$ of (5.1) and

$$\|\hat{g}_1\|_{C_x^{2+\mu}} \leq 2M, \quad \|\hat{g}_{1t}\|_{C^0} \leq 4M.$$

Hence for large M and small $T > 0$, the operator $\mathcal{A}$ maps Y_{M, b_i}^T into itself.

Let (g_1, g_2) and (f_1, f_2) be two elements in Y_{M, b_i}^T , and (u, v) and $(\hat{u}, \hat{v})$ are the corresponding solutions to (1.1)–(1.6), respectively. We need the following lemma.

LEMMA 5.1. *With the notion as above, the following estimates hold:*

$$\begin{aligned}
(5.5) \quad & \|u(\cdot, g_1, t) - \hat{u}(\cdot, f_1, t)\|_{C_x^{1+\mu}} + \|u(\cdot, g_2, t) - \hat{u}(\cdot, f_2, t)\|_{C_x^{1+\mu}} \\
& + \|v(\cdot, g_2, t) - \hat{v}(\cdot, f_2, t)\|_{C_x^{1+\mu}} + \|v(\cdot, g_3, t) - \hat{v}(\cdot, f_3, t)\|_{C_x^{1+\mu}} \\
& \leq c \left(\|g_1 - f_1\|_{C_x^{1+\mu}} + \|g_2 - f_2\|_{C_x^{1+\mu}} \right),
\end{aligned}$$

where c depends on T, M .

Proof. Denote by Ω_i and Γ_i the domains and the boundaries for (u, v, g_1, g_2) , and by $\hat{\Omega}_i$ and $\hat{\Gamma}_i$ the domains and the boundaries for $(\hat{u}, \hat{v}, f_1, f_2)$. Set

$$h(t) = \sup_{i=1,2} \sup_{x \in R^1} |g_i(x, t) - f_i(x, t)|.$$

For small t , by the uniform continuity we have $h(t) \leq b_0/4$ if T small. Consider the domains

$$\begin{aligned}\Omega_1^h &= \{(x, y : g_2 + h(t) < y < g_1(x, t) - h(t))\}, \\ \Omega_2^h &= \{(x, y : g_3(x, t) < y < g_2 - h(t))\},\end{aligned}$$

and denote the corresponding boundaries by $\Gamma_1^h = \{y = g_1(x, t) - h(t)\}$, $\Gamma_2^{h+} = \{y = g_2 + h(t)\}$, $\Gamma_2^{h-} = \{y = g_2 - h(t)\}$, and $\Gamma_3 = \{y = g_3(x, t)\}$. Note that $\Omega_i^h \subseteq \Omega_i \cap \hat{\Omega}_i$. By the translation $(x, y) \mapsto (x, y + 2h)$, we can move Ω_2^h towards y direction such that Γ_2^{h-} coincides with Γ_2^{h+} . Hence we may identify Γ_2^{h+} to the boundary Γ_2^{h-} . Write

$$(5.6) \quad \frac{\partial(u - \hat{u})}{\partial n} = \varphi_1 \quad \text{on } \Gamma_1^h,$$

$$(5.7) \quad \frac{\partial(u - \hat{u})}{\partial n} \Big|_{\Gamma_2^{h+}} - \frac{\partial(v - \hat{v})}{\partial n} \Big|_{\Gamma_2^{h-}} = \varphi_2 \quad \text{on } \Gamma_2^{h+},$$

$$(5.8) \quad (u - \hat{u}) \Big|_{\Gamma_2^{h+}} - (\beta(v) - \beta(\hat{v})) \Big|_{\Gamma_2^{h-}} = \varphi_3 \quad \text{on } \Gamma_2^{h+}.$$

Note that

$$(5.9) \quad \vec{n} = \frac{(-g_{2x}(x, t), 1)}{\sqrt{1 + (g_{2x}(x, t))^2}} \quad \text{on } \Gamma_2^{h\pm} \text{ and } \Gamma_2,$$

$$(5.10) \quad \vec{n} = \frac{(-f_{2x}(x, t), 1)}{\sqrt{1 + (f_{2x}(x, t))^2}} \quad \text{on } \hat{\Gamma}_2.$$

From the boundary condition (1.4) we have the expression

$$\begin{aligned}\varphi_2 &= \left(\frac{\partial u}{\partial n} - \frac{\partial \hat{u}}{\partial n} + k(u - \hat{u}) \right) \Big|_{\Gamma_2^{h+}} - \left(\frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} + k(u - v) \right) \Big|_{\Gamma_2} \\ &\quad + \left(\frac{\partial \hat{u}}{\partial n} - \frac{\partial \hat{v}}{\partial n} + k(\hat{u} - \hat{v}) \right) \Big|_{\hat{\Gamma}_2} - \left(\frac{\partial v}{\partial n} - \frac{\partial \hat{v}}{\partial n} + k(v - \hat{v}) \right) \Big|_{\Gamma_2^{h-}} \\ &\quad - k \left((u - \hat{u}) \Big|_{\Gamma_2^{h+}} - (v - \hat{v}) \Big|_{\Gamma_2^{h-}} \right)\end{aligned}$$

Since $u, \hat{u} \in C^{2+\mu}$, it follows that

$$(5.11) \quad \|\nabla u(\cdot, g_2 + h, t) - \nabla u(\cdot, g_2, t)\|_{C_x^\mu} \leq ch,$$

$$\|\nabla \hat{u}(\cdot, g_2 + h, t) - \nabla \hat{u}(\cdot, f_2, t)\|_{C_x^\mu} \leq c(h + \|g_2 - f_2\|_{C_x^{1+\mu}}).$$

The same estimates hold for v and $\hat{v}$. Hence

$$(5.12) \quad \|\varphi_2\|_{C_x^\mu} \leq c(h + \|g_2 - f_2\|_{C_x^{1+\mu}}),$$

where we have used (5.9)–(5.10). Analogously, we can show that

$$(5.13) \quad \|\varphi_1\|_{C_x^\mu} + \|\varphi_3\|_{C_x^{1+\mu}} \leq c(\|g_1 - f_1\|_{C_x^{1+\mu}} + \|g_2 - f_2\|_{C_x^{1+\mu}}).$$

By the maximum principle (that can be proved in the way similar to the proof of Lemma 2.1), we obtain, by using (5.12) and (5.13),

$$\|u - \hat{u}\|_{L^\infty(\Omega_1^h)} + \|v - \hat{v}\|_{L^\infty(\Omega_2^h)} \leq c(\|g_1 - f_1\|_{C_x^{1+\mu}} + \|g_2 - f_2\|_{C_x^{1+\mu}}).$$

Let w_1 and w_2 be the harmonic conjugates of $(u - \hat{u})$ and $(v - \hat{v})$, respectively such that

$$w_1 = (u - \hat{u}), \quad w_2 = (v - \hat{v}) \quad \text{on } \Gamma_2^{h+}.$$

Then in a neighborhood G of $\Gamma_2^{h+} = \Gamma_2^{h-}$, we have

$$(5.14) \quad \Delta w_1 = 0 \text{ on } G \cap \Omega_1^h,$$

$$(5.15) \quad \Delta w_2 = 0 \text{ on } G \cap \Omega_2^h,$$

$$(5.16) \quad \frac{\partial w_1}{\partial \tau} - \frac{\partial w_2}{\partial \tau} = \varphi_2 \text{ on } \Gamma_2^{h+},$$

$$(5.17) \quad w_1 - w_2 \gamma = \varphi_2 \text{ on } \Gamma_2^{h+},$$

where τ is the tangential unit vector and

$$\gamma = \int_0^1 \beta'(sv + (1-s)\hat{v}) \, ds.$$

We now change variable $y \mapsto y - (g_2 + h)$. System (5.14)–(5.17) becomes

$$(5.18) \quad w_{1xx} - 2g_{2x}w_{1xy} + \left(1 + (g_{2x})^2\right)w_{1yy} - g_{2xx}w_{1y} = 0 \text{ in } y > 0,$$

$$(5.19) \quad w_{2xx} - 2g_{2x}w_{2xy} + \left(1 + (g_{2x})^2\right)w_{2yy} - g_{2xx}w_{2y} = 0 \text{ in } y < 0,$$

$$(5.20) \quad \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial x} = \varphi_2 \text{ on } y = 0.$$

$$(5.21) \quad w_1 - w_2\gamma = \varphi_3 \text{ on } y = 0.$$

By the results in [1] (see also [10]) and (5.12), we deduce that near $y = 0$

$$\|w_1\|_{C_{xy}^{1+\mu}} + \|w_2\|_{C_{xy}^{1+\mu}} \leq c \left(\|g_1 - f_1\|_{C_x^{1+\mu}} + \|g_2 - f_2\|_{C_x^{1+\mu}} \right).$$

Similarly (and easilier) we can derive the above estimates near Γ_1^h and Γ_3 . In the interior of Ω_i^h , the same estimates are guaranteed by the Schauder estimates. Therefore

$$\|u - \hat{u}\|_{C_{xy}^{1+\mu}(\Omega_1^h)} + \|v - \hat{v}\|_{C_{xy}^{1+\mu}(\Omega_2^h)} \leq c \left(\|g_1 - f_1\|_{C_x^{1+\mu}} + \|g_2 - f_2\|_{C_x^{1+\mu}} \right).$$

The inequality (5.6) follows from (5.11).

Completion of the proof for Theorem 5.1:

Now for any $(g_1, g_2), (f_1, f_2) \in Y_{M, b_i}^T$, set (u, v) and $(\hat{u}, \hat{v})$ be the corresponding solutions to (1.1)–(1.6), respectively, and $(\hat{g}_1, \hat{g}_2) = \mathcal{A}(g_1, g_2)$, $(\hat{f}_1, \hat{f}_2) = \mathcal{A}(f_1, f_2)$. By differentiating in x the expressions

$$\hat{g}_{2t} = (u(x, g_2, t) - v(x, g_2, t)) \sqrt{1 + (\hat{g}_{2x})^2}$$

and

$$\hat{f}_{2t} = (\hat{u}(x, f_2, t) - \hat{v}(x, f_2, t)) \sqrt{1 + (\hat{f}_{2x})^2},$$

and using (5.6) and the same arguments as in the proof of (5.4) (i.e., integration along the characteristics), we derive, for any $t \leq T$

$$(5.22) \quad \sup_{0 \leq t \leq T} \|\hat{f}_2 - \hat{g}_2\|_{C_x^{1+\mu}} \leq cT \sup_{0 \leq t \leq T} \left(\|f_1 - g_1\|_{C_x^{1+\mu}} + \|f_2 - g_2\|_{C_x^{1+\mu}} \right).$$

Analogously, we obtain estimates

$$(5.23) \quad \sup_{0 \leq t \leq T} \|\hat{f}_1 - \hat{g}_1\|_{C_x^{1+\mu}} \leq cT \sup_{0 \leq t \leq T} (\|f_1 - g_1\|_{C_x^{1+\mu}} + \|f_2 - g_2\|_{C_x^{1+\mu}}).$$

Therefore, under the $C_x^{1+\mu}$ -norm, the mapping $\mathcal{A}$ is a contraction for T small enough. It follows that there exists a unique fixed point, say $(g_1, g_2) \in Y_{M, b_i}^T$. Hence (u, v, g_1, g_2) is a solution of (1.1)–(1.10).

Suppose that there exist two solution (u, v, g_1, g_2) and $(\hat{u}, \hat{v}, f_1, f_2)$ that belong to $C^{2+\mu}$ in the corresponding domains. Suppose also that there exists a constant $\delta > 0$ such that $g_1(x, t) - g_2(x, t) \geq \delta$, $g_2(x, t) - g_3(x, t) \geq \delta$, $f_1(x, t) - f_2(x, t) \geq \delta$, and $f_2(x, t) - f_3(x, t) \geq \delta$. Then the estimates (5.22) and (5.23) hold with $\hat{f}_i = f_i$, $\hat{g}_i = g_i$ and a constant c depending on δ and their $C^{2+\mu}$ -norms. Thus for small t , we have $g_1 = f_1$, $g_2 = f_2$, and hence, by Theorem 4.2, $u = \hat{u}$, $v = \hat{v}$. The above procedure can be extended to any time $t \leq T$.

From the proof of the Theorem 5.1 and Remark 4.1, we can extend the solution as long as $\|g_i(\cdot, T)\|_{C_x^{2+\mu}}$ remains bounded and three boundaries Γ_i do not intersect each other.

COROLLARY 5.1. *Suppose that (u, v, g_1, g_2) be a solution for $0 \leq t \leq T$ and there exists a $b_0 > 0$ such that*

$$(5.24) \quad \inf_x \{g_1(x, T) - g_2(x, T), g_2(x, T) - g_3(x, T)\} \geq b_0.$$

Then there exists a $\delta > 0$ such that the solution can be extended to $0 \leq t \leq T + \delta$. Furthermore, δ depends only on $\|g_i(\cdot, T)\|_{C_x^{2+\mu}}$ and b_0 .

6. Axially Symmetric Solutions. We consider the problem (1.1)–(1.10) in the special case that the initial data (1.9) and g_3 are axially symmetric with respect to x -axis, i.e., independent of x . We are interested in looking for solutions for all time $t \geq 0$. We observe that when $k = 1$, from (1.3) it follows that $g_{2t} = u - v \leq 0$. Hence in order to obtain a global solution, we must impose certain growth condition

on the boundary $\Gamma_3(t)$. To find the proper condition, supplement (1.1)–(1.10) with the equation

$$(6.1) \quad V_3 = \frac{\partial v}{\partial n},$$

where V_3 is the normal velocity of Γ_3 defined by (1.10) by replacing g_i with g_3 . We now seek for a global solution for problem (1.1)–(1.10), (6.1) that involves three free boundaries Γ_1 , Γ_2 , and Γ_3 . Note that this is in fact the original model for titanium silicide film growth.

Since the initial data are axially symmetric, we expect that the solutions are also axially symmetric. The problem (1.1)–(1.10), (6.1) thus reduces to

$$(6.2) \quad u_{yy} = 0 \quad \text{for } g_2 \leq y \leq g_1,$$

$$(6.3) \quad v_{yy} = 0 \quad \text{for } g_3 \leq y \leq g_2,$$

$$(6.4) \quad u_y + u = 0 \quad \text{on } y = g_1,$$

$$(6.5) \quad u_y - v_y = -k(u - v) \quad \text{on } y = g_2,$$

$$(6.6) \quad u = \beta(v) \quad \text{on } y = g_2,$$

$$(6.7) \quad v = 1 \quad \text{on } y = g_3,$$

$$(6.8) \quad g_{1t} = u \quad \text{on } y = g_1,$$

$$(6.9) \quad g_{2t} = u - v \quad \text{on } y = g_2,$$

$$(6.10) \quad g_{3t} = v_y \quad \text{on } y = g_3,$$

and the initial conditions

$$(6.11) \quad g_i(0) = g_i^0, \quad i = 1, 2, 3,$$

where g_i^0 is constant satisfying (2.2). Note that any given g_i , the diffraction problem (1.1)–(1.6) is equivalent to system (6.2)–(6.7).

Consider first the diffraction problem (6.2)–(6.7). From equation (6.2), we can write

$$u = l_1 y + l_2, \quad v = l_3 y + l_4.$$

From the boundary conditions (6.4)–(6.7), we obtain the following system for l_i :

$$l_1 (1 + g_1) + l_2 = 0,$$

$$l_1 (1 + k g_2) - l_3 (1 + k g_2) + (l_2 - l_4) k = 0,$$

$$l_1 g_2 + l_2 = \beta (l_3 g_2 + l_4),$$

$$l_3 g_3 + l_4 = 1.$$

We thus see that l_i are determined by

$$(6.12) \quad l_1 = -\frac{\beta(\theta)}{1 + g_1 - g_2},$$

$$(6.13) \quad l_2 = -l_1 (1 + g_1),$$

$$(6.14) \quad l_3 = \frac{\theta - 1}{g_2 - g_3},$$

$$(6.15) \quad l_4 = 1 - l_3 g_3,$$

where θ is a solution of

$$(6.16) \quad F(\theta) \equiv \frac{1 - k - k(g_1 - g_2)}{1 + g_1 - g_2} \beta(\theta) + \frac{1 + k(g_2 - g_3)}{g_2 - g_3} \theta - \frac{1}{g_2 - g_3} = 0.$$

It follows that the diffraction problem (6.2)–(6.7) admits a solution if and only if the algebraic equation (6.16) has at least one positive solution.

From the expression of $F(\theta)$, one may construct a plenty of examples such that $F(\theta)$ has no positive roots even though β is smooth and monotone. In particular, we have the following non-existence results to show that the assumption (2.1) can not be simply removed.

LEMMA 6.1. *Suppose that $k = 1$, $\beta(0) = 0$, $\beta'(s) \geq 1 + \delta$ for $s > 0$, where $\delta > 0$ is a constant. Then there exists no nonnegative solution for (1.1)–(1.6) for some given domains Ω_1 and Ω_1 .*

Proof. From (6.16), we have

$$\frac{\partial}{\partial \theta} F(\theta) = -\frac{g_1 - g_2}{1 + g_1 - g_2} \beta'(\theta) + \frac{1 + g_2 - g_3}{g_2 - g_3}.$$

We can choose g_i such that $g_1 - g_2$ and $g_2 - g_3$ are large enough. Then

$$\frac{\partial}{\partial \theta} F(\theta) < 0.$$

Since $F(0) < 0$, there exists no positive solution for $F(\theta) = 0$.

We now consider the free boundary problem (6.2)–(6.11). For simplicity, we may assume from now on that $k = 1$. From (6.12)–(6.15) and the motion equations (6.8)–(6.10) on the free boundaries, one sees that the system (6.2)–(6.11) reduces to the following system:

$$(6.17) \quad g_{1t} = \frac{\beta(\theta)}{1 + g_1 - g_2},$$

$$(6.18) \quad g_{2t} = \frac{\beta(\theta)}{1 + g_1 - g_2} + \frac{\theta - 1}{g_2 - g_3},$$

$$(6.19) \quad g_{3t} = \frac{\theta - 1}{g_2 - g_3},$$

together with (6.16) and the initial conditions $g_i(0) = g_i^0$ that satisfy $g_1^0 - g_2^0 \geq b_0$, $g_2^0 - g_3^0 \geq b_0 > 0$. Introducing new functions $p = g_1 - g_2 > 0$, $q = g_2 - g_3 > 0$, then system (6.16), (6.17)–(6.19) becomes

$$(6.20) \quad p_t = \frac{1 - \theta}{q}, \quad q_t = \frac{\beta(\theta)}{1 + p},$$

$$(6.21) \quad g_{3t} = \frac{\theta - 1}{q},$$

$$(6.22) \quad F(\theta, p, q) \equiv \frac{-p}{1 + p} \beta'(\theta) + \frac{1 + q}{q} \theta - \frac{1}{q} = 0.$$

It suffices to solve (6.20) and (6.22) with the initial conditions $p(0) = g_1^0 - g_2^0 \geq b_0 > 0$, $q(0) = g_2^0 - g_3^0 \geq b_0$. Since (2.1),

$$\frac{\partial}{\partial \theta} F(\theta, p, q) = \frac{-p}{1 + p} \beta'(\theta) + 1 + \frac{k}{q} > 0, \text{ if } p, q > 0.$$

In addition, for $p, q > 0$

$$F(0, p, q) = -\frac{1}{q} < 0, \quad F(1, p, q) = \frac{\beta(1)}{1+p} + 1 - \beta(1) > 0.$$

It follows that for any $p, q > 0$, there exists a unique solution $0 < \theta(p, q) < 1$ that solves (6.22). By differentiating (6.22) with respect to p, q , we have

$$\begin{aligned} \theta_p \left(\frac{\beta'(\theta)}{1+p} - \beta'(\theta) + 1 + \frac{1}{q} \right) - \frac{\beta(\theta)}{(1+p)^2} &= 0, \\ \theta_q \left(\frac{\beta'(\theta)}{1+p} - \beta'(\theta) + 1 + \frac{1}{q} \right) - \frac{\theta-1}{q^2} &= 0. \end{aligned}$$

Therefore it follows, since $0 < \theta(p, q) < 1$ and $0 < \beta'(s) \leq 1$,

$$(6.23) \quad 0 \leq \theta_p \leq \frac{\beta(\theta)}{(1+p)\beta'(\theta)} \leq c,$$

$$(6.24) \quad -\frac{1}{q} \leq \frac{\theta-1}{q} \leq \theta_q \leq 0.$$

We now show that there exists a unique solution (p, q) for (6.20) and (6.22). Since $p(0), q(0) \geq b_0 > 0$, and from (6.23) and (6.24) θ_p and θ_q are bounded in the region $p, q \geq b_0/2$. It follows from the standard ODE that (6.20) admits a unique solution in the region $p, q \geq b_0/2$ for small time interval $0 \leq t \leq T$. From (6.20) and the fact that $0 < \theta(p, q) < 1$, we know that $p(t)$ and $q(t)$ increase in time. Hence $p(t), q(t) \geq b_0$. Since T depends only on the initial conditions and the bounds in (6.23) and (6.24), It follows that T actually is independent. Therefore, the solution can be extended to any time t . We thus obtain a global solution. Note that $p(t)$ and $q(t)$ are increasing function, and $p'(t)$ and $q'(t)$ are bounded. We have just proved

THEOREM 6.1. *There exists a unique global solution to the problem (1.1)–(1.10), (6.1) if the initial data are axially symmetric.*

We next study the asymptotic behavior of the global solutions.

THEOREM 6.2. *The solution of (6.20)–(6.22) has the following properties:*

$$(6.25) \quad c_1 \sqrt{t+1} \leq p \leq t + c_2, \quad c_1 \ln(t+1) \leq q \leq c_2 \sqrt{t+1},$$

$$g_1 - q = p + g_3 = g_1(0) - g_2(0) + g_3(0),$$

where c_i are some positive constants.

Proof. For convenience, in what follows, c is always a constant that may be different in different occasions. We have known from the previous discussion that $p(t) \geq g_1(0) - g_2(0) > 0$, $q(t) \geq g_2(0) - g_3(0) > 0$. Since $0 \leq \theta \leq 1$ and $0 < \beta' \leq 1$, there exists a constant $1 > \eta > 0$ such that $\eta\theta \leq \beta(\theta) \leq \theta$. It follows from (6.22) that

$$\frac{1}{q} + \frac{\eta\theta p}{1+p} \leq \theta \left(\frac{1+q}{q} \right) \leq \frac{1}{q} + \frac{\theta p}{1+p}.$$

Hence

$$\begin{aligned} \frac{1}{1+q} &\leq \frac{1+p}{1+p+q+(1-\eta)pq} \\ (6.26) \quad &\leq \theta \leq \frac{1+p}{1+p+q}. \end{aligned}$$

Substituting this inequality into the second equation in (6.20), we obtain

$$q_t \leq \frac{\theta}{1+p} \leq \frac{1}{1+p+q} \leq \frac{1}{1+q}.$$

It follows that

$$(6.27) \quad q^2 \leq q^2 + 2q \leq 2t + c.$$

Next, from (6.20), we have

$$q_t + (pq)_t = 1 - \theta + \beta(\theta).$$

Applying (6.26), we derive

$$\eta \leq q_t + (pq)_t \leq 1.$$

Therefore

$$(6.28) \quad \eta t + c \leq q(1+p) \leq t + c.$$

Combining (6.27) and (6.28) we obtain

$$(6.29) \quad q \leq c\sqrt{t+1}, \quad p \geq c\sqrt{t+1}.$$

From (6.20) and (6.26) we know that $p_t \leq 1/(q+1) \leq 1$. Hence $p \leq t+c$. It follows from (6.26) and (6.28) that

$$q_t = \frac{\beta(\theta)}{1+p} \geq \frac{\eta\theta}{1+p} \geq \frac{\eta}{(1+p)(1+q)} \geq \frac{\eta}{2t+c}.$$

By integration,

$$q \geq \frac{\eta}{2} \ln(2t+c) + c.$$

LEMMA 6.2. *Let g_i^0 and $\bar{g}_i^0$ be two sets of initial data, and (p, q) and $(\bar{p}, \bar{q})$ be the corresponding solutions for (6.20) and (6.22).*

(i) *If $g_1^0 \geq \bar{g}_1^0 \geq \bar{g}_2^0 \geq g_2^0$, $g_3^0 \geq \bar{g}_3^0$, then*

$$p(t) \geq \bar{p}(t), \quad q(t) \leq \bar{q}(t);$$

(ii) *If $\bar{g}_1^0 \geq g_1^0 \geq g_2^0 \geq \bar{g}_2^0$, $\bar{g}_3^0 \geq g_3^0$, then*

$$p(t) \leq \bar{p}(t), \quad q(t) \geq \bar{q}(t).$$

Proof. Write

$$F_1(p, q) = \frac{1-\theta}{q}, \quad F_2(p, q) = \frac{\beta(\theta)}{1+p}.$$

Then from (6.23) and (6.24), we know that

$$\frac{\partial F_i}{\partial p} \leq 0, \quad \frac{\partial F_i}{\partial q} \leq 0.$$

The assertion follows from the standard comparison theorem from ODE.

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