**STABLE NUCLEATION FOR THE GINZBURG-LANDAU
SYSTEM WITH AN APPLIED MAGNETIC FIELD**

By

P. Bauman

D. Phillips

and

Q. Tang

IMA Preprint Series # 1416

July 1996

**INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA**

**514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455**

Stable Nucleation for the Ginzburg-Landau System with an Applied Magnetic Field

P. Bauman*   D. Phillips†   Q. Tang‡

Abstract. We consider the Ginzburg-Landau system with an applied magnetic field and analyze the behavior of solutions when the domain is a cylinder (of radius $\bar{r}$) and the applied field is parallel to the axis. It is shown that there is an upper critical value $\bar{h}$ such that if the modulus of the applied field is greater than $\bar{h}$, the normal (nonsuperconducting) state (in which the order parameter is identically zero) is stable and if the modulus of the applied field is slightly below $\bar{h}$, the normal state is unstable. In addition, it is shown that there is a positive lower critical value $\underline{h} \leq \bar{h}$ such that the normal state is unstable if the modulus of the applied field is less than $\underline{h}$ and stable if the modulus is slightly above $\underline{h}$.

In the case of type II materials for which the Ginzburg-Landau constant κ is large, it is shown that there is a discrete set of radii $\mathcal{B}(\kappa)$ such that if $\bar{r} \notin \mathcal{B}(\kappa)$ and $\kappa\bar{r}$ is sufficiently large, then for each applied field of modulus slightly less than $\bar{h}$ (or slightly more than $\underline{h}$) there is precisely one small superconducting solution (up to a gauge transformation) which is stable. Moreover for this solution, the complex-valued order parameter ψ is zero only on the axis of the cylinder, and its winding number is proportional to the product of κ^2 and the cross-sectional area of the cylinder. In addition, the solution exhibits “surface superconductivity” as predicted by the physicists, de Gennes and St. James.

---

*This research sponsored in part by NSF Grant #DMS 9310258 and by the Institute for Mathematics and its Applications with funds provided by the National Science Department. Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395.

†This research sponsored in part by NSF Grant #DMS9102574 and by the Institute for Mathematics and its Applications with funds provided by the National Science Department. Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395

‡School of Mathematical and Physical Sciences, University of Sussex, Brighton BN1 9QH, UK.

1 Introduction

In this paper we consider the nucleation of solutions to the mathematical problem describing a cylindrical superconducting material placed in a vacuum subject to an applied magnetic field parallel to its generator. We assume that the applied magnetic field is of the form $h\mathbf{e}_3$, where h is a positive constant. In this case, an electric current and magnetic field are induced to flow in the material according to the Ginzburg Landau theory. In this theory, the above phenomenon is described in terms of a complex order parameter ψ (where $|\psi|^2$ represents the density of superconducting electron pairs) and a two-dimensional vector field $\mathbf{A}$, called the magnetic potential, such that $\text{curl } \mathbf{A}$ represents the induced magnetic field. Let the cross section of the material be given by a bounded, simply connected domain $\Omega \subset \mathbf{R}^2$ where $\partial\Omega$ is regular. According to this theory, the pair $(\psi, \mathbf{A})$ is an equilibrium state for the free energy

$$G(\psi, \mathbf{A}) = \int_{\Omega} \left(\frac{1}{2} \left| \left(\frac{i}{\kappa} \nabla + \mathbf{A} \right) \psi \right|^2 + \frac{1}{4} (1 - |\psi|^2)^2 + \frac{1}{2} |\text{curl } \mathbf{A} - h\mathbf{e}_3|^2 \right) dx$$

where $\text{curl } \mathbf{A} = (\partial_{x_1} A_2 - \partial_{x_2} A_1)\mathbf{e}_3$ and $\kappa > 0$ is the Ginzburg-Landau constant which is determined from the material. Thus $(\psi, \mathbf{A})$ satisfies the boundary value problem

$$\left\{ \begin{array}{l} \left(\frac{i}{\kappa} \nabla + \mathbf{A} \right)^2 \psi - \psi + |\psi|^2 \psi = 0, \\ (\text{curl})^2 \mathbf{A} + \frac{i}{2\kappa} (\psi^* \nabla \psi - \psi \nabla \psi^*) + |\psi|^2 \mathbf{A} = 0, \end{array} \right\} \quad \text{in } \Omega, \quad (1.1)$$

$$\left\{ \begin{array}{l} \mathbf{n} \cdot \left(\frac{i}{\kappa} \nabla + \mathbf{A} \right) \psi = 0, \\ \text{curl } \mathbf{A} = h\mathbf{e}_3 \end{array} \right\} \quad \text{on } \partial\Omega,$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. The solution is called normal (nonsuperconducting) if it satisfies $\psi \equiv 0$ and $\text{curl } \mathbf{A} \equiv h\mathbf{e}_3$ in Ω .

The Ginzburg-Landau free energy is invariant under the gauge transformation $(\psi, \mathbf{A}) \rightarrow (\psi', \mathbf{A}')$ where

$$\psi' \equiv \psi e^{i\kappa\chi}, \quad \mathbf{A}' \equiv \mathbf{A} + \nabla\chi$$

for an arbitrary real-valued function χ . By choosing χ appropriately, we may distinguish a regular solution which satisfies the additional conditions

$$\begin{aligned} \text{div } \mathbf{A} &= 0 \quad \text{in } \Omega, \\ \mathbf{A} \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

From elliptic regularity theory, for this particular choice of gauge (known as the Coulomb gauge), an equilibrium solution is classical. Thus we consider solutions of (1.1) in the space

$$\mathcal{M} = \{(\psi, \mathbf{A}) : \psi \in C^2(\bar{\Omega}; \mathbf{C}), \mathbf{A} \in C^2(\bar{\Omega}; \mathbf{R}^2), \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega, \mathbf{A} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

This gauge determines χ up to an arbitrary constant. Thus, given $(\psi, \mathbf{A})$ the orbit $\{e^{i\tau}\psi, \mathbf{A}\} : \tau \in \mathbf{R}\}$ is its gauge equivalent family in $\mathcal{M}$.

A solution $(\psi, \mathbf{A})$ is called *stable* if the second variation of G is nonnegative at $(\psi, \mathbf{A})$, that is,

$$\frac{d^2}{d\varepsilon^2} G(\psi + \varepsilon\zeta, \mathbf{A} + \varepsilon\mathbf{B})|_{\varepsilon=0} \geq 0$$

for all $(\zeta, \mathbf{B}) \in \hat{\mathcal{M}}$ where $\hat{\mathcal{M}}$ is the closure of $\mathcal{M}$ in $W^{1,2}(\Omega; \mathbf{C} \times \mathbf{R}^2)$. A stable solution is said to be *strictly stable* if equality holds iff $(\zeta, \mathbf{B})$ is tangent to the gauge equivalent orbit at $(\psi, \mathbf{A})$; i.e.,

$$(\zeta, \mathbf{B}) = c \frac{d}{d\tau} (e^{i\tau}\psi, \mathbf{A})|_{\tau=0} = c(i\psi, 0) \text{ for some } c \in \mathbf{R}.$$

Let κ be fixed. Physicists have predicted that when h is sufficiently large, the normal state should be strictly stable. As h decreases it is conjectured that there is a value $\bar{h} > \kappa$ where strict stability is lost and bifurcation occurs generating a nontrivial branch $(\psi(\varepsilon), \mathbf{A}(\varepsilon), h(\varepsilon))$ for $|\varepsilon| < \delta$, for some $\delta > 0$, with $(\psi(0), \mathbf{A}(0), h(0)) = (0, \mathbf{A}(0), \bar{h})$. For $\varepsilon \neq 0$ it is expected that $|\psi(x, \varepsilon)| > 0$ except for isolated points called vortices. On this branch it is also conjectured that ψ develops as a superconducting sheath around $\partial\Omega$; that is $|\psi|$ becomes uniformly positive in a strip near $\partial\Omega$ and then rapidly decreases as one moves towards the interior of Ω . The above quasi-static process is called nucleation, and the development of a superconducting region near the boundary of Ω is called surface nucleation. (See [9].) It is these phenomena that we analyze here.

We consider the case of a disk, $\Omega = B_{\bar{r}}$. The normal state is then given by

$$\psi \equiv 0 \quad \text{and} \quad \mathbf{A} = h\mathbf{a}_N \tag{1.2}$$

where $\mathbf{a}_N = \frac{1}{2}(-x_2, x_1)$. The second variation of G at the normal state gives

$$\kappa^2 \frac{d^2}{d\varepsilon^2} G(\varepsilon\zeta, h\mathbf{a}_N + \varepsilon\mathbf{B})|_{\varepsilon=0} = \int_{B_{\bar{r}}} (|(i\nabla + \kappa h\mathbf{a}_N)\zeta|^2 - \kappa^2|\zeta|^2 + \kappa^2|\operatorname{curl} \mathbf{B}|^2) dx. \tag{1.3}$$

Thus setting

$$\lambda = \lambda(\omega, \bar{r}) \equiv \inf_{\substack{\|\zeta\|_{L^2}=1 \\ \zeta \in W^{1,2}(B_{\bar{r}}; \mathbb{C})}} \int_{B_{\bar{r}}} |(i\nabla + \omega^2 \mathbf{a}_N)\zeta|^2 dx \quad (1.4)$$

we see that the normal state is strictly stable iff $\lambda = \lambda((\kappa h)^{\frac{1}{2}}, \bar{r}) > \kappa^2$ and unstable when $\lambda < \kappa^2$. If $\lambda = \kappa^2$, then *degenerate stability* occurs, that is,

$$\frac{d^2}{d\varepsilon^2} G(\varepsilon\zeta, h\mathbf{a}_N + \varepsilon\mathbf{B})|_{\varepsilon=0} \geq 0 \text{ for all } (\zeta, \mathbf{B}) \in \mathcal{M} \quad (1.5)$$

and there exists $\psi_0 \neq 0$ such that we have equality for $(\zeta, \mathbf{B}) = (\psi_0, 0)$. This is the situation in which stable bifurcation is possible. We note that if a smooth bifurcation curve $(\psi(\varepsilon), \mathbf{A}(\varepsilon), h(\varepsilon))$ exists near $(0, h_0\mathbf{a}_N, h_0)$, with $(\psi(0), \mathbf{A}(0), h(0)) = (0, h_0\mathbf{a}_N, h_0)$, then $\frac{d}{d\varepsilon}(\psi(\varepsilon), \mathbf{A}(\varepsilon))(0) = (\psi_0, 0)$ for such a ψ_0 . In particular the structure of $\psi(\varepsilon)$ will correspond to that of $\varepsilon\psi_0$ for ε small.

Let $\mathcal{H}(\bar{r}, \kappa) = \{h > 0 : \lambda((\kappa h)^{\frac{1}{2}}, \bar{r}) = \kappa^2\}$. We show that $\mathcal{H}(\bar{r}, \kappa)$ is a nonempty finite set for any $\kappa, \bar{r} > 0$. Moreover $\kappa < h_0$ for each $h_0 \in \mathcal{H}$ as conjectured. We also prove the following asymptotic estimate: For $h_0 \in \mathcal{H}(\bar{r}, \kappa)$,

$$\frac{\kappa}{h_0} = \underline{\mu}^2 + \frac{\ell\mu}{\kappa\bar{r}} + O((\kappa\bar{r})^{-2}) \text{ as } \kappa\bar{r} \rightarrow \infty \quad (1.6)$$

where $\underline{\mu}$ and ℓ are specific constants with $0 < \underline{\mu} < 1$ and $\ell < 0$. (See Theorem 7.1.) In particular, this holds for $\bar{h} = \bar{h}(\bar{r}, \kappa) \equiv \max\{h : h \in \mathcal{H}(\bar{r}, \kappa)\}$ and $\underline{h} = \underline{h}(\bar{r}, \kappa) \equiv \min\{h : h \in \mathcal{H}(\bar{r}, \kappa)\}$.

If we recenter our coordinates at the edge of the disk and let $\bar{r} \rightarrow \infty$, this corresponds to the cross section of Ω_0 tending to the halfspace, $\Omega = \{(x_1, x_2) : x_1 < 0\}$. The problem on the half space was studied by de Gennes and St. James in [9]. They showed that in this case, degenerate stability occurs at $h = \kappa\underline{\mu}^{-2}$, the limit as $\bar{r} \rightarrow \infty$ in (1.6).

For $h_0 \in \mathcal{H}(\bar{r}, \kappa)$ one obtains from (1.5) that ψ_0 solves the linearized problem

$$\begin{aligned} \mathcal{L}\psi_0 &\equiv (i\nabla + \kappa h_0 \mathbf{a}_N)^2 \psi_0 = \kappa^2 \psi_0 & \text{in } B_{\bar{r}}, \\ \partial_r \psi_0 &= 0 & \text{on } \partial B_{\bar{r}}. \end{aligned} \quad (1.7)$$

We show that the solution set to this problem has a basis of the form

$$\mathcal{S} = \{\varphi_{m_1}(r)e^{im_1\theta}, \dots, \varphi_{m_k}(r)e^{im_k\theta}\}$$

where $0 \leq m_1 < m_2 < \dots < m_k$ are integers, $\varphi_{m_j} \geq 0$ and $\varphi_{m_j}(r) = O(r^{m_j})$ as $r \rightarrow 0$. For κ fixed the set of $\bar{r}$ for which $k > 1$ is a discrete set, $\{\bar{r}_i\}$, with $\lim_{i \rightarrow \infty} \bar{r}_i = \infty$.

To be more specific, let us assume that $\bar{r} \in \mathbf{R} - \{\bar{r}_i\}$. Then for $h_0 \in \mathcal{H}(\bar{r}, \kappa)$ we prove that $\mathcal{S} = \{\varphi_m(r)e^{im\theta}\}$ for some $m = m(\bar{r}, \kappa, h_0)$. (See Theorem 7.3 and Lemma 7.5.)

If $\kappa\bar{r}$ is sufficiently small, we show that $m = 0$ and $\varphi_0 > 0$ on $[0, \bar{r}]$; in particular the linearized problem has no vortices. For $\kappa\bar{r}$ large, we prove an asymptotic estimate:

$$m = \frac{1}{2} \left(\frac{\kappa\bar{r}}{\underline{\mu}} - \left(\frac{\ell}{2\underline{\mu}^2} + \underline{\mu} \right) \right)^2 + O(1) \text{ as } \kappa\bar{r} \rightarrow \infty. \quad (1.8)$$

(See Theorems 7.1 and 7.6.) In particular, $m \rightarrow \infty$ as $\kappa\bar{r} \rightarrow \infty$.

Assume that κ and $\bar{r}$ are such that $m \geq 1$. Normalizing φ_m so that $\varphi_m(\bar{r}) = 1$ we show that $\{\varphi_m\}$ are uniformly positive in a boundary layer $0 \leq \bar{r} - r \leq \frac{C}{\kappa}$ and that φ_m decays rapidly to zero as r decreases from $\bar{r}$ to zero. This is the phenomenon of a superconducting sheath described above. Moreover we find that

$$\lim_{\bar{r} \rightarrow \infty} \varphi_m(\bar{r} + \tau) \equiv \varphi_\infty(\underline{\mu}^{-1}\kappa\tau + \underline{\mu}) \text{ for } \tau < 0$$

exists and gives the boundary layer profile for the half-plane found in [9] where $m = m(\bar{r}, \kappa, h_0)$. (See Theorem 7.4.)

We show that except in a discrete set of radii $\mathcal{B}(\kappa)$, a simple bifurcation occurs at each $h_0 \in \mathcal{H}(\bar{r}, \kappa)$. Thus, if $\bar{r} \in \mathbf{R}^+ - \mathcal{B}(\kappa)$, $h_0 \in \mathcal{H}(\bar{r}, \kappa)$ and $\mathcal{S} = \{\varphi_m(r)e^{im\theta}\}$ then there is a $\delta > 0$ and a C^∞ curve of solutions to (1.1), namely

$$(\psi(\varepsilon), \mathbf{A}(\varepsilon), h(\varepsilon)) : (-\delta, \delta) \rightarrow \mathcal{M} \times \mathbf{R}^+$$

such that

$$\text{i) } (\psi(\varepsilon), \mathbf{A}(\varepsilon), h(\varepsilon)) = (\varepsilon\varphi_m(r)e^{im\theta}, h_0\mathbf{a}_N, h_0) + O(\varepsilon^2) \text{ as } \varepsilon \rightarrow 0, \\ \text{and}$$

$$\text{ii) } (\psi(\varepsilon), \mathbf{A}(\varepsilon)) = (|\psi|(\varepsilon, r)e^{im\theta}, A(\varepsilon, r)\mathbf{a}_N)$$

$$\text{where } \begin{array}{lll} |\psi|(\varepsilon, r) > 0 & \text{for } 0 < r \leq \bar{r} & \text{and} \\ |\psi|(\varepsilon, r) = O(r^m) & \text{as } r \rightarrow 0. \end{array}$$

Moreover (up to gauge invariance) these are the only nontrivial solutions of (1.1) near $(0, h_0\mathbf{a}_N, h_0)$ in $\mathcal{M} \times \mathbf{R}$. (See Theorems 8.6 and 8.8.) Thus due to symmetry and

the strong magnetic field $\psi(\varepsilon)$ retains the structure of $\frac{d}{d\varepsilon}(\psi(\varepsilon))(0)$ for ε small, that is, $\psi(\varepsilon)$ has one vortex with winding number m .

We also analyze the stability of the above branch of solutions. Setting

$$\sigma_0 = \left(2 \int_0^{\bar{r}} \varphi_m^4(r) r dr - 4 \int_0^{\bar{r}} \left(\int_r^{\bar{r}} \left(\frac{h_0 s}{2} - \frac{m}{\kappa s} \right) \varphi_m^2(s) ds \right)^2 r dr \right) / \int_0^{\bar{r}} \varphi_m^2(r) r dr$$

we show that $(\psi(\varepsilon), \mathbf{A}(\varepsilon))$ is strictly stable for all sufficiently small $\varepsilon \neq 0$ if $\sigma_0 > 0$ and unstable if $\sigma_0 < 0$. Furthermore we show that there are constants $c_1, c_2 > 0$ so that if $\kappa > c_1$ and $\kappa \bar{r} > c_2$, then $\sigma_0 > 0$. Thus for κ and $\kappa \bar{r}$ large the transition from normal to a superconducting state is accomplished by the continuous stable formation of a superconducting sheath at the boundary. (See Theorem 8.12.) This agrees with what is observed in experiments.

It is conjectured for type II superconductors ($\kappa \gg 1$) that continuing along the branch of solutions, $(\psi(\varepsilon), \mathbf{A}(\varepsilon), h(\varepsilon))$, at some point fission occurs where the vortex splits into m vortices each with winding number one. Assuming this scenario, then (1.8) is significant since it gives a sharp estimate for the number of vortices in terms of κ and $\bar{r}$.

Finally we comment on past work, which has been for semi-infinite domains. In [9], de Gennes and St. Jmes discuss “surface state solutions” in an infinite slab, $-d < x_1 < d$, $-\infty < x_2 < \infty$, in $\mathbf{R}^2$ as well as half plane solutions. In the case of the slab they consider solutions for the linearized problem of the form $g(x_1)e^{icx_2}$ for some $c \in \mathbf{R}$. For d and κ fixed they give an ansatz for finding a degenerately stable state. This involves a principal eigenvalue problem for $g(x_1)$ with c fixed and then a minimization over the continuous parameter c . Assuming this ansatz, Odeh [8] showed existence of solutions to the nonlinear problem in the slab of width d based on bifurcation theory. In this case the bifurcation was done in a family of functions not permitting vortices. In [1] and [2] Bolley and Bolley and Helffer made the ansatz for the problem on the slab rigorous and proved an asymptotic formula analogous to (1.6) as $d \rightarrow \infty$. In the present work, due to the fact that the domain is a disk, one is led by Fourier series to solutions for the linearized problem of the form $\varphi(r)e^{im\theta}$ where $m \in \mathbf{Z}$ with $\varphi(0) = 0$ for $m \neq 0$. Since the parameter m is discrete here, in contrast to the case of the infinite slab, a different analysis is required.

With regard to stability, an analysis for the half-plane in terms of formal asymptotics was given by Chapman in [3] and [4]. There Chapman predicted stability (in the context of gradient flow) for κ large based on a formula similar to ours for σ_0 .

An important feature of the results we prove here is that they rigorously establish the existence, uniqueness, and stability of a branch of solutions to (1.1) (containing

vortices) which nucleate from the normal state. This is done within the general class, $\mathcal{M}$, which does not impose any special form due to symmetry or other considerations.

2 Stability of Normal Solutions in a Disk

Fix h and $\kappa > 0$ and let $\omega = (h\kappa)^{\frac{1}{2}}$. Set

$$\mathcal{G}(\zeta) = \int_{B_{\bar{r}}} |i\nabla + \omega^2 \mathbf{a}_N) \zeta|^2 dx \text{ for } \zeta \in W^{1,2}(B_{\bar{r}}; \mathbf{C})$$

where $\mathbf{a}_N = \mathbf{a}_N(x) \equiv \frac{1}{2}(-x_2, x_1)$ and

$$\lambda = \lambda(\omega, \bar{r}) = \inf \left\{ \frac{\mathcal{G}(\zeta)}{\|\zeta\|_{L^2(B_{\bar{r}})}^2} : \zeta \in W^{1,2}(B_{\bar{r}}; \mathbf{C}) \text{ and } \|\zeta\|_{L^2(B_{\bar{r}})} > 0 \right\}. \quad (2.1)$$

In this section we make several elementary observations concerning stability of normal solutions and the eigenvalue problem (2.1).

From formula (1.3) for the second variation of the Ginzburg-Landau energy at $\psi \equiv 0$, $\mathbf{A} = h\mathbf{a}_N$, we have:

Lemma 2.1 *The normal state (1.2) at $(\bar{r}, \kappa, h)$ is*

$$\begin{aligned} & \text{strictly stable} && \text{iff } \lambda(\omega, \bar{r}) > \kappa^2, \\ & \text{unstable} && \text{iff } \lambda(\omega, \bar{r}) < \kappa^2, \\ \text{and } & \text{degenerately stable} && \text{iff } \lambda(\omega, \bar{r}) = \kappa^2. \end{aligned}$$

where $\omega = (h\kappa)^{\frac{1}{2}}$.

If $\zeta \in W^{1,2}(B_{\bar{r}}; \mathbf{C})$ and we represent ζ by its Fourier series,

$$\zeta(r, \theta) = \sum_{n=-\infty}^{\infty} \zeta_n(r) e^{in\theta}$$

where the ζ_n are complex valued, then

$$\begin{aligned} & \int_0^{\bar{r}} \left(\sum_{n=-\infty}^{\infty} (|\zeta'_n|^2 + \frac{n^2 |\zeta_n|^2}{r^2}) \right) r dr < \infty \text{ and} \\ \mathcal{G}(\zeta) &= 2\pi \int_0^{\bar{r}} \left(\sum_{n=-\infty}^{\infty} \left(|\zeta'_n|^2 + \left(\frac{\omega^2 r}{2} - \frac{n}{r} \right)^2 |\zeta_n|^2 \right) \right) r dr. \end{aligned}$$

Setting

$$\lambda_n = \lambda_n(\omega, \bar{r}) = \inf \left\{ \frac{\int_0^{\bar{r}} (|\xi'(r)|^2 + (\frac{\omega^2 r}{2} - \frac{n}{r})^2 \cdot |\xi(r)|^2) r dr}{\int_0^{\bar{r}} |\xi(r)|^2 r dr} : \right. \\ \left. \xi \in W^{1,2}([0, \bar{r}]; r dr) \text{ and } \int_0^{\bar{r}} |\xi(r)|^2 r dr > 0 \right\}, \quad (2.2)$$

we have from Sturm-Liouville theory:

Lemma 2.2 *For each n the infimum in (2.2) is taken on by a real-valued function $\varphi_n(r)$ such that $\varphi_n \in W^{1,2}([0, \bar{r}]; r dr)$, $\varphi_n(r) > 0$ for $0 < r \leq \bar{r}$, and φ_n satisfies:*

$$\left\{ \begin{array}{l} \varphi_n'' + \frac{\varphi_n'}{r} - \left(\frac{\omega^2 r}{2} - \frac{n}{r} \right)^2 \varphi_n = -\lambda_n \varphi_n \text{ in } (0, \bar{r}), \\ \varphi_n \text{ is bounded, and } \varphi_n'(\bar{r}) = 0. \end{array} \right\} \quad (2.3)$$

Furthermore, any solution of (2.3) is of the form $\alpha \varphi_n$ for some constant $\alpha \in \mathbb{C}$.

It follows from (2.1), (2.2), and Lemma 2.2 that

$$\begin{aligned} \mathcal{G}(\zeta) &\geq 2\pi \cdot \sum_{n=-\infty}^{\infty} \lambda_n \cdot \int_0^{\bar{r}} |\zeta_n|^2 r dr \\ &\geq 2\pi \cdot \sum_{n=-\infty}^{\infty} \lambda \cdot \int_0^{\bar{r}} |\zeta_n|^2 r dr \\ &= \lambda \cdot \|\zeta\|_{L^2(B_{\bar{r}})}^2 \end{aligned}$$

for all $\zeta \in W^{1,2}(B_{\bar{r}}; \mathbb{C})$. In particular, we see that $\lambda(\omega, \bar{r}) = \inf_{n \in \mathbb{Z}} \lambda_n(\omega, \bar{r})$. Now, it is clear from the definition of λ_n that $\lim_{|n| \rightarrow \infty} \lambda_n(\omega, \bar{r}) = \infty$. Thus $\lambda(\omega, \bar{r}) = \lambda_n(\omega, \bar{r})$ for a finite number of n and we have the following lemma:

Lemma 2.3 *The set of minimizers ζ of (2.1) spans a nontrivial finite dimensional subspace with a basis given by*

$$S = \{\varphi_{n_1}(r)e^{in_1\theta}, \dots, \varphi_{n_k}(r)e^{in_k\theta}\}$$

where $n_1 < n_2 < \dots < n_k$, φ_{n_j} has the properties described in Lemma 2.2, and $\lambda(\omega, \bar{r}) = \lambda_{n_j}(\omega, \bar{r})$ for $j = 1, \dots, k$.

The first step in studying nucleation on a cylinder is given $\bar{r}$ and κ to determine the stability of the normal solution for different values of h . This is characterized by the function $\lambda = \lambda(\omega, \bar{r}) = \lambda((h\kappa)^{\frac{1}{2}}, \bar{r})$ and the result of Lemma 2.1. The second step is to study the bifurcation and stability of solutions to (1.1) near values of h where the normal state is degenerately stable, i.e., for values of h near h_0 where $h_0 \in \mathcal{H}(\bar{r}, \kappa) \equiv \{h : \lambda((h\kappa)^{\frac{1}{2}}, \bar{r}) = \kappa^2\}$. We shall accomplish this step by determining those n for which $\lambda(\omega, \bar{r}) = \lambda_n(\omega, \bar{r})$. We first analyze the one-dimensional variational problems (2.2) in the next section.

3 The One-Dimensional Variational Problems

Fix $n \in \mathbf{Z}$ and $\omega = (h\kappa)^{\frac{1}{2}} > 0$. It will be convenient to consider (2.2) and (2.3) under a change of variables which eliminates ω from the left side of the differential equation in (2.3). To this end, let $x = \omega r$ and $\bar{x} = \omega \bar{r}$. Setting

$$\xi(x) = \xi_n(x) \equiv \varphi_n\left(\frac{x}{\omega}\right) \text{ and } \mu = \lambda_n^{\frac{1}{2}}/\omega = (\lambda_n(\omega, \bar{r}))^{\frac{1}{2}}/\omega,$$

we obtain from (2.3) and Lemma 2.2 the problem:

$$\begin{cases} \mathcal{L}_n \xi \equiv \xi'' + \frac{\xi'}{x} - \left(\frac{x}{2} - \frac{n}{x}\right)^2 \xi = -\mu^2 \xi \text{ for } 0 < x < \bar{x} \\ \text{where } \xi \text{ is bounded, } \xi \geq 0, \quad \xi'(\bar{x}) = 0 \\ \text{and } \xi \not\equiv 0. \end{cases} \quad (3.1)$$

By standard (Frobenius) theory, the solution ξ_n is regular near $x = 0$ and

$$\xi_n(x) = a_n x^{|n|} + o(|x|^{|n|}) \text{ as } x \rightarrow 0 \quad (3.2)$$

for some $a_n > 0$. Moreover, $\xi_n > 0$ for $0 < x \leq \bar{x}$.

The above change of variables in the minimization problem (2.2) yields:

$$\begin{aligned} \text{minimize } \frac{\mathcal{J}(\xi, n, \bar{x})}{\int_0^{\bar{x}} \xi^2 x \, dx} &\equiv \frac{\int_0^{\bar{x}} [(\xi')^2 + (\frac{x}{2} - \frac{n}{x})^2 \xi^2] x \, dx}{\int_0^{\bar{x}} \xi^2 x \, dx} \\ \text{in the class } &\left\{ \xi \in W^{1,2}([0, \bar{x}]; x \, dx) : \int_0^{\bar{x}} \xi^2 x \, dx > 0 \right\}. \end{aligned} \quad (3.3)$$

By Lemma 2.2 and Sturm Liouville theory, the relation between problems (3.1) and (3.3) may be summarized as follows:

Lemma 3.1 *For each $\bar{x} > 0$ there exists a unique $\mu = \mu(\bar{x}, n) > 0$ for which (3.1) is solvable. The solution $\xi_n(x) \equiv \xi_n(x, \bar{x})$ is unique under the normalization $\xi_n(\bar{x}) = 1$. It follows that under this normalization ξ_n is the unique nonnegative minimizer for (3.3) and $\mu^2(\bar{x}, n) = \mathcal{J}(\xi_n, n, \bar{x}) / \int_0^{\bar{x}} \xi_n^2 x dx$.*

Proof: Consider (3.1). From Sturm-Liouville theory an eigenvalue has a nonnegative eigenfunction iff the eigenvalue is minimal. Thus μ^2 is uniquely determined. Moreover the eigenspace for this problem is one-dimensional. Thus the solution is determined uniquely up to a multiplicative constant. ■

The remainder of this section is devoted to establishing properties of the eigenvalues $\mu^2(\bar{x}, n)$ and their corresponding eigenfunctions $\xi_n(x, \bar{x})$ which we shall need later.

We first analyze the case $n = 0$.

Lemma 3.2 *We have*

$$0 < \mu^2(\bar{x}, 0) < \min\left(1, \frac{\bar{x}^2}{4}\right) \text{ for } \bar{x} > 0.$$

Proof: Let $v(x) = e^{-\frac{x^2}{4}}$. Then v satisfies $\mathcal{L}_0 v = -v$. Multiplying this equation by $-xv$ and integrating from 0 to $\bar{x}$ gives

$$\int_0^{\bar{x}} v^2 x dx + \bar{x} v'(\bar{x}) v(\bar{x}) = \mathcal{J}(v) \geq \mu^2(\bar{x}, 0) \int_0^{\bar{x}} v^2 x dx.$$

Since $v' < 0$ we deduce that $\mu^2(\bar{x}, 0) < 1$. On the other hand since $\int_0^{\bar{x}} v^2 x dx = 1 - e^{-\frac{\bar{x}^2}{2}}$ we get

$$\mu^2(\bar{x}, 0) \leq \left(1 - e^{-\frac{\bar{x}^2}{2}} - \frac{\bar{x}^2}{2} e^{-\frac{\bar{x}^2}{2}}\right) / (1 - e^{-\frac{\bar{x}^2}{2}}) < \frac{\bar{x}^2}{4}.$$

To see that the last inequality is valid, we note that it is equivalent to showing $1 - e^s + se^s < -\frac{s}{2}(1 - e^s)$ for $s < 0$. If we set $j(s) = 1 + \frac{s}{2} - e^s + \frac{s}{2}e^s$ then it follows that $j(0) = j'(0) = 0$ and $j''(s) < 0$ for $s \leq 0$. Thus $j < 0$ for $s < 0$. ■

Lemma 3.3 *If $n = 0$ then $\xi'_n < 0$ on $(0, \bar{x})$ and $\xi_n > 0$ on $[0, \bar{x}]$.*

Proof: The second inequality follows from (3.2) for $n = 0$. From equation (3.1) with $n = 0$, we have $\xi'_0(0) = 0$ and $\xi''_0(0) < 0$. Let $\tilde{x}$ be the first positive value such that $\xi'_0(\tilde{x}) = 0$. Then $\xi''_0(\tilde{x}) \geq 0$, whence from the equation $\frac{\tilde{x}^2}{4} \geq \mu^2$. Thus for $x \geq \tilde{x}$,

$$\frac{1}{x} \frac{d}{dx}(x \xi'_0(x)) = \left(\frac{x^2}{4} - \mu^2\right) \xi_0 \geq \left(\frac{\tilde{x}^2}{4} - \mu^2\right) \xi_0 \geq 0$$

and $x\xi'_0(x)$ is increasing on $(\tilde{x}, \bar{x})$. Since $\xi'_0(\bar{x}) = 0 = \xi'_0(\tilde{x})$ we must have $\tilde{x} = \bar{x}$. Thus $\xi'_0 < 0$ on $(0, \bar{x})$. ■

Given $\bar{r}$, κ , and h , we wish to identify those n for which $\lambda_n((h\kappa)^{\frac{1}{2}}, \bar{r}) = \lambda((h\kappa)^{\frac{1}{2}}, \bar{r}) = \inf_{n' \in \mathbb{Z}} \lambda_{n'}((h\kappa)^{\frac{1}{2}}, \bar{r})$. This is equivalent to being given $\bar{x}$, κ , and h and finding those n such that $\mu^2(\bar{x}, n) = \inf_{n' \in \mathbb{Z}} \mu^2(\bar{x}, n')$. The following lemma implies that $\mu^2(\bar{x}, n) > 1$ for $n < 0$. Since $\mu^2(\bar{x}, 0) < 1$ by Lemma 3.2, it follows that $n \geq 0$ are the only integers of interest in this issue.

Lemma 3.4 *If $n < 0$ then $\mu^2(\bar{x}, n) \geq 2|n|$.*

Proof: We have $(\frac{x}{2} - \frac{n}{x})^2 = (\frac{x}{2} + \frac{|n|}{x})^2 \geq 2|n|$. Thus,

$$\mu^2(\bar{x}, n) \int_0^{\bar{x}} \xi_n^2 x dx = \mathcal{J}(\xi_n) \geq 2|n| \int_0^{\bar{x}} \xi_n^2 x dx.$$

Next we analyze the behavior of ξ_n for $n > 0$. We show that ξ_n either has one interior critical point (an absolute maximum) or ξ_n is strictly increasing on $(0, \bar{x})$.

Lemma 3.5 *Fix $n > 0$ and let $\mu = \mu(\bar{x}, n)$. If $\bar{x} \leq \mu + \sqrt{\mu^2 + 2n}$ then $\xi'_n > 0$ on $(0, \bar{x})$. If $\bar{x} > \mu + \sqrt{\mu^2 + 2n}$ then there exists $\tilde{x}_n \in (\sqrt{\mu^2 + 2n} - \mu, \sqrt{\mu^2 + 2n} + \mu)$ such that $\xi'_n > 0$ on $(0, \tilde{x}_n)$ and $\xi'_n < 0$ on $(\tilde{x}_n, \bar{x})$.*

Proof: From (3.1) we have $\xi'_n > 0$ on a maximal interval $(0, \tilde{x}_n)$ with $\xi'_n(\tilde{x}_n) = 0$ and $\xi''_n(\tilde{x}_n) \leq 0$. This implies that $((\frac{\tilde{x}_n}{2} - \frac{n}{\tilde{x}_n})^2 - \mu^2) \leq 0$ which is equivalent to $\tilde{x}_n \in [\sqrt{\mu^2 + 2n} - \mu, \sqrt{\mu^2 + 2n} + \mu]$. Now from (3.1), $x\xi'_n(x)$ is increasing on $[0, \sqrt{\mu^2 + 2n} - \mu]$. Thus in fact $\tilde{x}_n > \sqrt{\mu^2 + 2n} - \mu$.

If $\tilde{x}_n < \bar{x} \leq \sqrt{\mu^2 + 2n} + \mu$ then $x\xi'_n(x)$ decreases on $[\tilde{x}_n, \bar{x}]$ and $\xi'_n(\bar{x}) = 0$ is not possible. Thus if $\bar{x} \leq \sqrt{\mu^2 + 2n} + \mu$, then $\tilde{x}_n = \bar{x}$ and ξ_n is increasing on $[0, \bar{x}]$.

Assume $\bar{x} > \sqrt{\mu^2 + 2n} + \mu$. We have that $x\xi'_n(x)$ decreases on $[\sqrt{\mu^2 + 2n} - \mu, \sqrt{\mu^2 + 2n} + \mu]$ and increases on the intervals $[0, \sqrt{\mu^2 + 2n} - \mu]$ and $[\sqrt{\mu^2 + 2n} + \mu, \bar{x}]$. Since $\xi'_n(\bar{x}) = 0$ we must have $\tilde{x}_n \in (\sqrt{\mu^2 + 2n} - \mu, \sqrt{\mu^2 + 2n} + \mu)$ and $\tilde{x}_n$ must be the unique critical point for ξ_n in $(0, \bar{x})$. ■

Definition 3.1 Let $n \in \{0, 1, 2, \dots\}$, $\mu \in \mathbf{R}$, and let $g_n(x) = g_n(x, \mu)$ be the solution to

$$\mathcal{L}_n g_n \equiv g_n'' + \frac{g_n'}{x} - \left(\frac{x}{2} - \frac{n}{x}\right)^2 g_n = -\mu^2 g_n \text{ for } 0 < x < \infty \quad (3.4)$$

with $g_n(x) = x^n + o(x^n)$ as $x \rightarrow 0^+$. By standard theory of ordinary differential equations, g_n has an extension which is a real analytic function of $(x, \mu) \in \mathbf{R}^2$ for each n .

Remark 1 The function $v_n(x) = x^n e^{-x^2/4}$ is a solution of (3.4) on $(0, \infty)$ with $\mu = 1$. Thus $g_n(x, 1) = x^n e^{-x^2/4}$. Furthermore $v_n'(x) = 0$ if and only if $x = 0$ or $x = \sqrt{2n}$.

Remark 2 Let $n \in \{0, 1, 2, \dots\}$ and $\bar{x} > 0$ and set $\mu = \mu(\bar{x}, n)$. Let $\xi_n(x) \equiv \xi_n(x, \bar{x})$ be the solution of (3.1) with $\xi_n(\bar{x}) = 1$ and assume without loss of generality that $\xi_n(x)$ has been extended to $(0, \infty)$ so that ξ_n satisfies (3.4). Recall that $\xi_n(x) = a_n x^n + o(x^n)$ as $x \rightarrow 0^+$ for some $a_n > 0$ (depending on $\bar{x}$). Thus

$$g_n(x, \mu(\bar{x}, n)) = a_n^{-1} \cdot \xi_n(x, \bar{x}).$$

We shall need the following properties of $\mu(\bar{x}, n)$.

Lemma 3.6 For each $n \in \{0, 1, 2, \dots\}$, the functions $\mu(\cdot, n)$ are analytic on $(0, \infty)$ and

$$\frac{\partial}{\partial x} \mu^2(x, n) = \frac{[(\frac{x}{2} - \frac{n}{x})^2 - \mu^2(x, n)] x g_n^2(x, \mu(x, n))}{\int_0^x g_n^2(s, \mu(x, n)) s ds}. \quad (3.5)$$

Proof: For ease of notation, fix n and set $\mu(x) \equiv \mu(x, n)$ and $g(x, \mu) \equiv g_n(x, \mu)$. From Remark 2 we see that $\mu(x)$ satisfies the equation $g_x(\bar{x}, \mu(\bar{x})) = 0$ for $\bar{x} > 0$. Thus $\mu(x)$ is analytic provided $g_{x\mu}(\bar{x}, \mu(\bar{x})) \neq 0$ for all $\bar{x} > 0$. To verify this we differentiate (3.4) with respect to μ to obtain

$$g_{\mu xx} + \frac{g_{\mu x}}{x} - \left(\frac{x}{2} - \frac{n}{x}\right)^2 g_\mu + \mu^2 g_\mu = -2\mu g.$$

Multiplying the equation by $-gx$, evaluating at $(x, \mu(\bar{x}))$, and integrating on $(0, \bar{x})$ we find that

$$\begin{aligned} 2\mu(\bar{x}) \int_0^{\bar{x}} g^2 x dx &= - \int_0^{\bar{x}} gx(\mathcal{L}_n + \mu^2) g_\mu \cdot dx \\ &= - \int_0^{\bar{x}} g_\mu x(\mathcal{L}_n + \mu^2) g dx - \bar{x} g_{\mu x}(\bar{x}, \mu(\bar{x})) g(\bar{x}, \mu(\bar{x})) \\ &= -\bar{x} g_{\mu x}(\bar{x}, \mu(\bar{x})) g(\bar{x}, \mu(\bar{x})), \end{aligned}$$

where $g = g(x) \equiv g(x, \mu(\bar{x}))$. Thus

$$g_{x\mu}(\bar{x}, \mu(\bar{x})) = -\frac{2\mu(\bar{x}) \int_0^{\bar{x}} g^2 x dx}{\bar{x}g(\bar{x}, \mu(\bar{x}))} < 0.$$

Hence $\mu(x)$ is analytic and satisfies

$$\frac{d}{dx}\mu(x) = -g_{xx}/g_{x\mu} = \left[\left(\frac{x}{2} - \frac{n}{x} \right)^2 - \mu^2 \right] x g^2(x, \mu(x)) / 2\mu(x) \int_0^x g^2(s, \mu(x)) s ds.$$

■

Lemma 3.7 *For each n in $\{0, 1, 2, \dots\}$, we have*

- i) $\mu(x, n) > 1$ if $x < \sqrt{2n}$,
- ii) $\mu(x, n) < 1$ if $x > \sqrt{2n}$,
- iii) $\lim_{x \rightarrow \infty} \mu(x, n) = 1$.

Proof: Fix n and $\bar{x} > 0$, and set $\mu(x) = \mu(x, n)$ and $g(x) = g_n(x, \mu(\bar{x}))$. Let $v(x) = x^n e^{-\frac{x^2}{4}}$ and recall that $\mathcal{L}_n v = -v$. Multiplying (3.4) by xv and integrating by parts twice on the interval $(0, \bar{x})$, we get

$$-\bar{x}g(\bar{x})v'(\bar{x}) = (1 - \mu^2(\bar{x})) \int_0^{\bar{x}} g v x dx.$$

Since $xv' = (n - \frac{x^2}{2})v$ we get

$$-g(\bar{x})v(\bar{x}) \left(n - \frac{\bar{x}^2}{2} \right) = (1 - \mu^2(\bar{x})) \int_0^{\bar{x}} g v x dx.$$

Thus i) and ii) follow directly.

To show iii) let $k = \frac{g}{v}$. From (3.2) we have $0 < \lim_{x \rightarrow 0} k(x) < \infty$ and

$$vk'' + vk'/x + 2k'v' = (1 - \mu^2(\bar{x}))kv.$$

We rewrite this as

$$(xv^2k')' = (1 - \mu^2)kxv^2.$$

Multiplying this equation by k and integrating yields

$$\bar{x}v^2(\bar{x})k'(\bar{x})k(\bar{x}) - \int_0^{\bar{x}} v^2k'^2 x dx = (1 - \mu^2(\bar{x})) \int_0^{\bar{x}} g^2 x dx.$$

Now $0 = g'(\bar{x}) = (k'v + kv')(\bar{x})$. Thus at $\bar{x}$:

$$\bar{x}v^2k'k = -\bar{x}vk^2v' = g^2(\bar{x})\left(\frac{\bar{x}^2}{2} - n\right).$$

Hence,

$$\left(\frac{\bar{x}^2}{2} - n\right)g^2(\bar{x}) \geq (1 - \mu^2(\bar{x})) \int_0^{\bar{x}} g^2x \, dx. \quad (3.6)$$

From ii) we have $0 < \mu(\bar{x}) < 1$ for $\bar{x} > \sqrt{2n}$. From (3.5) then $\mu'(\bar{x}) > 0$ for all large $\bar{x}$. Thus $\lim_{x \rightarrow \infty} \mu(\bar{x})$ exists and moreover $\lim_{i \rightarrow \infty} \mu'(\bar{x}_i) = 0$ for some sequence $\{\bar{x}_i\}$ approaching $+\infty$. Equation (3.5) then gives

$$\lim_{i \rightarrow \infty} \bar{x}_i^2 g^2(\bar{x}_i) \Big/ \int_0^{\bar{x}_i} g^2x \, dx = 0.$$

Hence from (3.6) we have

$$\lim_{\bar{x} \rightarrow \infty} \mu(\bar{x}) = \lim_{i \rightarrow \infty} \mu(\bar{x}_i) = 1. \quad \blacksquare$$

The next step is to use (3.5) to analyze $\mu'(x, n) \equiv \frac{\partial \mu}{\partial x}(x, n)$.

Lemma 3.8 *For each $x > 0$ we have $\mu'(x, 0) > 0$.*

Proof: From Lemma 3.2 we have $\mu^2(x, 0) < \frac{x^2}{4}$. Hence from (3.5) we have $\mu'(x, 0) > 0$. $\blacksquare$

Lemma 3.9 *For each $n \in \{1, 2, 3, \dots\}$ there exists $x_n > \sqrt{2n}$ such that*

- i) $\mu'(x, n) < 0$ for $0 < x < x_n$,
- ii) $\mu'(x, n) > 0$ for $x > x_n$.

Moreover, setting $\mu_n = \mu(x_n, n)$ we have

$$\text{iii) } x_n = \mu_n + \sqrt{\mu_n^2 + 2n}.$$

Proof: From Remark 1 following Lemma 3.5 we have $\mu(\sqrt{2n}, n) = 1$. Then from (3.5), $\mu'(\sqrt{2n}, n) < 0$. Fix n and choose a_n in $[0, \sqrt{2n})$ such that $(a_n, \sqrt{2n})$ is the maximal such interval on which $\mu'(\cdot, n) < 0$. Assume $a_n > 0$. Then $\mu'(a_n, n) = 0$. We will show that this is impossible.

From (3.5) it follows that $p_n(a_n) \equiv (\frac{a_n}{2} - \frac{n}{a_n})^2 - \mu^2(a_n, n) = 0$. Hence either

$$\begin{aligned} a_n &= x_n^1 = \sqrt{\mu^2(a_n, n) + 2n} - \mu(a_n, n) \\ \text{or } a_n &= x_n^2 \equiv \sqrt{\mu^2(a_n, n) + 2n} + \mu(a_n, n). \end{aligned} \quad (3.7)$$

Since $a_n < \sqrt{2n}$, we must have $a_n = x_n^1$. This implies that

$$\left(\frac{x}{2} - \frac{n}{x}\right)^2 - \mu^2(a_n, n) > 0 \text{ for } 0 < x < a_n.$$

From (3.1) then $x \cdot \frac{\partial g_n}{\partial x}(x, \mu(a_n, n))$ is increasing on $[0, a_n]$. It follows that $\frac{\partial g_n}{\partial x}(a_n, \mu(a_n, n)) > 0$ which is a contradiction. This proves that $\mu'(x, n) < 0$ for $x < \sqrt{2n}$.

We next show that there exists a unique critical point x_n for $\mu(\cdot, n)$. Note that from (3.5), $\mu^2(\cdot, n)$ decreases as long as $p_n(x) \equiv [(\frac{x}{2} - \frac{n}{x})^2 - \mu^2(x, n)] < 0$. Furthermore $(\frac{x}{2} - \frac{n}{x})^2$ is increasing for $x > \sqrt{2n}$ and $p_n(\sqrt{2n}) = -1$. Thus there exists $b_n > \sqrt{2n}$ such that $p_n(x)$ increases to 0 on $[\sqrt{2n}, b_n]$. Now $\mu'(x, n) < 0$ for $x < b_n$ and $\mu'(b_n, n) = 0$. Moreover, from (3.5), $\mu''(b_n, n) > 0$. Hence there is a maximal interval (b_n, c_n) on which $\mu'(\cdot, n) > 0$. If $c_n < \infty$ we have $\mu'(c_n, n) = 0$ and it follows from (3.5) that $p_n(c_n) = 0$. Since $c_n > \sqrt{2n}$ we have $p'_n(c_n) > 0$. This implies that $p_n(x) < 0$ for $x < c_n$ with $|x - c_n|$ sufficiently small. Thus $\mu'(x, n) < 0$ for such x which is impossible. This proves that $c_n = \infty$ and we have shown that μ has a unique critical point $x_n = b_n$ for which i) and ii) hold.

To prove iii) we note that $\mu'(x_n, n) = 0$ implies that $p_n(x_n) = 0$. Since $x_n > \sqrt{2n}$ we see as in (3.7) that iii) gives an implicit formula for x_n . ■

A sketch of the eigenvalues, $\mu^2(x, n)$ for $n = 0, 1, 2, \dots$ is given in Figure 1. We remark that nonmonotone behavior of eigenvalues for the Neumann problem and regular Sturm-Liouville operators was studied by Dauge and Helffer in [6].

Figure 1: The eigenvalues $\mu^2(x, n)$

4 Limits of Eigenvalues and Eigenvectors as n Approaches Infinity

A key step of our work in this paper is a detailed analysis of the lower envelope,

$$\mu_e(x) \equiv \inf_n \mu(x, n),$$

since the values of h such that $\frac{\kappa}{h} = \mu_e^2((h\kappa)^{\frac{1}{2}}\bar{r})$ correspond to normal states which are degenerately stable. In this section, we prove that for large n , the $\mu(x, n)$ are approximately translates of one another. In addition, after appropriate dilation and translation, the eigenfunctions $g_n(x, \mu(\bar{x}, n))$ (and $\xi_n(x, \bar{x})$) converge to a solution of a limiting boundary value problem as $n \rightarrow \infty$. To this end, fix $n \geq 1$ and set

$$y \equiv x - \sqrt{2n}, \quad \bar{y} \equiv \bar{x} - \sqrt{2n}, \text{ and } \nu(\bar{y}, n) \equiv \mu(\bar{x}, n).$$

We are interested only in $\bar{x}$ for which $\mu(\bar{x}, n) \leq 1$, thus for $\bar{x} \geq \sqrt{2n}$. Thus we shall assume that $\bar{y} \geq 0$ and hence (by Lemma 3.7) $0 < \nu(\bar{y}, n) \leq 1$. Set

$$f_n(y) \equiv f_n(y, \bar{y}) = \xi_n(x, \bar{x}) = \frac{g_n(x, \mu(\bar{x}, n))}{g_n(\bar{x}, \mu(\bar{x}, n))} \text{ for } -\sqrt{2n} \leq y \leq \bar{y}.$$

In terms of y and f_n , (3.1) and (3.4) become:

$$\begin{cases} f_n'' + \frac{f_n'}{y+\sqrt{2n}} - y^2 \left[1 - \frac{1}{2} \left(\frac{y}{y+\sqrt{2n}} \right) \right]^2 f_n = -\nu^2(\bar{y}, n) f_n \text{ for } -\sqrt{2n} < y, \\ \text{where } f_n \geq 0, f_n \text{ is bounded for } -\sqrt{2n} \leq y \leq \bar{y}, \\ \text{and } f_n'(\bar{y}) = 0. \end{cases} \quad (4.1)$$

The above problem has been normalized by assuming in addition that

$$f_n(\bar{y}) = 1. \quad (4.2)$$

Lemma 4.1 *Let f_n satisfy (4.1) and (4.2). Given $0 < L < \infty$ there is a constant c independent of n and $\bar{y}$ such that if $0 \leq \bar{y} \leq L$ then*

$$\begin{aligned} c^{-1} \leq f_n(y) \leq c \quad \text{for } -1 \leq y \leq \bar{y}, \\ |f_n'(y)| \leq c \quad \text{for } -1 \leq y \leq \bar{y}. \end{aligned}$$

Proof: The coefficients in (4.1) are uniformly bounded in n on $[-1, L]$. Thus the lemma follows from standard estimates and the fact that $f_n \geq 0$ and $f_n(\bar{y}) = 1$. ■

Next we use a comparison function to estimate f_n uniformly in n .

Lemma 4.2 *There is a constant $c = c(L)$ so that if $0 \leq \bar{y} \leq L$ and $f_n(y) \equiv f_n(y, \bar{y})$, then for $-\sqrt{2n} \leq y \leq \bar{y}$ we have*

$$\begin{aligned} \text{i)} \quad & f_n(y) \leq ce^{-y^2/4} \quad \text{for } n \geq 1, \\ \text{ii)} \quad & f_n(y) \leq c \left(1 + \frac{y}{\sqrt{2n}}\right)^4 e^{-y^2/4} \quad \text{for } n \geq 4, \\ \text{iii)} \quad & |f'_n(y)| \leq c \left(1 + \frac{y}{\sqrt{2n}}\right) e^{-y^2/8} \quad \text{for } n \geq 4. \end{aligned}$$

Proof: Set $w_n(y) = (1 + \frac{y}{\sqrt{2n}})^n e^{-\frac{y^2}{4} - \frac{\sqrt{2n}y}{2}}$. Then w_n is a solution to (4.1) and (4.2) with $\bar{y} = 0$ and $\nu = 1$. Write

$$w_n(y) = \exp \left(n \ln \left(1 + \frac{y}{\sqrt{2n}} \right) - \frac{y^2}{4} - \frac{\sqrt{2n}}{2} y \right).$$

Using the fact that $\ln(1+x) \leq x$ for $x > -1$, it follows that

$$(n-m) \ln \left(1 + \frac{y}{\sqrt{2n}} \right) - \frac{y^2}{4} - \frac{\sqrt{2n}}{2} y \leq -\frac{y^2}{4} - \frac{my}{\sqrt{2n}} \leq m - \frac{y^2}{4}$$

for $m \leq n$ and $-\sqrt{2n} \leq y \leq 0$. Thus setting $m = 0$ and 4 we see that

$$\begin{cases} w_n(y) \leq ce^{-y^2/4} & \text{for } n \geq 1, \\ \text{and } w_n(y) \leq c \left(1 + \frac{y}{\sqrt{2n}}\right)^4 e^{-y^2/4} & \text{for } n \geq 4, \end{cases} \quad (4.3)$$

where c is independent of n .

We now use the maximum principle and w_n as a comparison function to bound f_n . We can express (4.1) as

$$f''_n + a_1 f'_n + a_2 f_n = 0 \quad \text{for } -\sqrt{2n} \leq y \leq \bar{y}.$$

Since $0 < \nu(\bar{y}, n) \leq 1$ we have $a_2 \leq 0$ for $-\sqrt{2n} \leq y \leq -1$. Moreover, w_n satisfies

$$w''_n + a_1 w'_n + a_2 w_n = (\nu^2 - 1)w_n \leq 0.$$

Thus, w_n is a supersolution on $(-\sqrt{2n}, -1)$. Applying Lemma 4.1 to both w_n and f_n we see that $f_n(-1) \leq c \leq c^2 w_n(-1)$. Moreover, $f_n(-\sqrt{2n}) = w_n(-\sqrt{2n}) = 0$. Thus, by the maximum principle

$$f_n \leq c^2 w_n \quad \text{for } -\sqrt{2n} \leq y \leq -1.$$

Properties i) and ii) follow from this, (4.3), and Lemma 4.1. Property iii) follows by integrating (4.1) from $-\sqrt{2n}$ to $y \leq \bar{y}$ and using i) and ii). ■

We now show that $\nu^2(\bar{y}, n)$ and $f_n(y, \bar{y})$ converge as $n \rightarrow \infty$.

Lemma 4.3 *The function $\bar{y} \rightarrow \nu^2(\bar{y}, n)$ converges in $C^1([0, L])$ and the function $\bar{y} \rightarrow f_n(y, \bar{y})$ converges in $C([0, L]; C^2([-M, M]))$ as $n \rightarrow \infty$ for any $0 < L, M < \infty$.*

Proof: We first show that $\bar{y} \rightarrow \nu^2(\bar{y}, n)$ converges in $C([0, L])$. Arguing by contradiction, one sees that it suffices to prove that there is a function $\nu_\infty^2(\bar{y})$ so that for any $\bar{y} \in [0, L]$ and sequences $\{n_j\}$, $\{\bar{y}_j\}$ for which $\lim_{j \rightarrow \infty} n_j = \infty$, $\lim_{j \rightarrow \infty} \bar{y}_j = \bar{y}$, and $\{\nu^2(\bar{y}_j, n_j)\}$ is Cauchy, we have $\lim_{j \rightarrow \infty} \nu^2(\bar{y}_j, n_j) = \nu_\infty^2(\bar{y})$.

Set $a^2 = \lim_{j \rightarrow \infty} \nu^2(\bar{y}_j, n_j)$. It follows from standard ordinary differential equation estimates that $\{f_n(y, \bar{y}_j)\}$ converge in $C^2([-M, M])$ for each $M < \infty$ to a solution $f_\infty(y) \equiv f_\infty(y, \bar{y})$ of

$$\begin{cases} f_\infty'' - y^2 f_\infty = -a^2 f_\infty & \text{for } y \in \mathbf{R}, \\ 0 \leq f_\infty(y) \leq ce^{-y^2/4} & \text{for } y \leq \bar{y}, \\ f_\infty(\bar{y}) = 1, \quad f_\infty'(\bar{y}) = 0. \end{cases} \quad (4.4)$$

We claim that a^2 is uniquely determined by $\bar{y}$ so that $\nu_\infty(\bar{y}) \equiv [\nu_\infty^2(\bar{y})]^{1/2}$ is well defined. To see this let k_1 and k_2 be two solutions of (4.4) with $a = a_1$ and a_2 , respectively. Then

$$-a_1^2 \int_{-\infty}^{\bar{y}} k_2 k_1 dy = \int_{-\infty}^{\bar{y}} k_2 (k_1'' - y^2 k_1) dy = -a_2^2 \int_{-\infty}^{\bar{y}} k_2 k_1 dy.$$

Hence $a_1 = a_2$. This proves that $y \rightarrow \nu^2(\bar{y}, n)$ converges in $C([0, L])$.

It follows that $\bar{y} \rightarrow f_n(y, \bar{y})$ converges in $C([0, L]; C^2([-M, M]))$ as $n \rightarrow \infty$ since the coefficients in (4.1) converge uniformly. Using this and Lemma 4.2 we see that the right hand side of

$$\frac{d}{d\bar{y}} \nu^2(\bar{y}, n) = \left[\bar{y}^2 \left(1 - \frac{1}{2} \left(\frac{\bar{y}}{\bar{y} + \sqrt{2n}} \right) \right)^2 - \nu^2 \right] \cdot \left(1 + \frac{\bar{y}}{\sqrt{2n}} \right) / \int_{-\sqrt{2n}}^{\bar{y}} \left(1 + \frac{y}{\sqrt{2n}} \right) f_n^2(y, \bar{y}) dy \quad (4.5)$$

converges uniformly on $[0, L]$. Hence, $\{\nu^2(\bar{y}, n)\}$ converges in $C^1([0, L])$ as $n \rightarrow \infty$. ■

Remark 3 The limit problem satisfied by $f_\infty(y) \equiv f_\infty(y, \bar{y}) = \lim_{n \rightarrow \infty} f_n(y, \bar{y})$ and $\nu_\infty = \nu_\infty(\bar{y}) = \lim_{n \rightarrow \infty} \nu(\bar{y}, n)$ is:

$$\begin{aligned} f_\infty'' - y^2 f_\infty &= -\nu_\infty^2(\bar{y}) \cdot f_\infty & \text{for } y \leq \bar{y}, \\ 0 \leq f_\infty(y) &\leq ce^{-y^2/4} & \text{for } y \leq \bar{y}, \\ f_\infty(\bar{y}) &= 1 \text{ and } f_\infty'(\bar{y}) = 0, \end{aligned} \tag{4.6}$$

and

$$\frac{d}{d\bar{y}} \nu_\infty^2(\bar{y}) = [\bar{y}^2 - \nu_\infty^2(\bar{y})] / \int_{-\infty}^{\bar{y}} f_\infty^2(s, \bar{y}) ds \tag{4.7}$$

where $\bar{y} \geq 0$, $\nu_\infty(0) = 1$.

We next show that the critical points of $\{\nu(\bar{y}, n)\}$ converge to the critical point of $\nu_\infty(\bar{y})$ as $n \rightarrow \infty$.

Lemma 4.4 Let $y_n \equiv x_n - \sqrt{2n}$ where x_n is defined by Lemma 3.9. Then $\lim_{n \rightarrow \infty} y_n = \underline{y}$ exists and $\nu_\infty(\underline{y}) = \inf_{y \geq 0} \nu_\infty(y) = \underline{y}$.

Proof: First from Lemmas 3.7 and 3.9 we have $0 \leq y_n \leq 2$. Next since $\nu(y, n)$ minimizes at y_n it follows that $\nu_\infty(y)$ minimizes on the entire interval $[\lim_{n \rightarrow \infty} y_n, \lim_{n \rightarrow \infty} y_n]$. It follows from (4.7) that ν_∞ cannot be constant on an interval of positive length. Thus $\lim_{n \rightarrow \infty} y_n \equiv \underline{y}$ exists and ν_∞ minimizes at $\underline{y}$. Finally from (4.7) we see that $\underline{y} = \nu_\infty(\underline{y})$ since $\nu_\infty'(\underline{y}) = 0$. ■

Corollary 4.5 Let $\mu_n = \mu(x_n, n) = \nu(y_n, n)$. Then $\lim_{n \rightarrow \infty} \mu_n = \underline{\mu}$ exists and $\underline{y} = \underline{\mu}$. Moreover, $0 < \underline{\mu} < 1$.

Proof: From Lemma 3.9 we see that if $\lim_{n \rightarrow \infty} y_n$ exists then $\lim_{n \rightarrow \infty} \mu_n$ exists and the two limits agree. Note that $\underline{\mu} = \inf_{y \geq 0} \nu_\infty(y)$. Since $0 \leq \nu_\infty \leq 1$ we see that if $\underline{\mu} = 1$ then $\nu_\infty \equiv 1$ and this is not consistent with (4.7). If $\underline{\mu} = 0$ then $\underline{y} = 0$. We have seen that $\nu_\infty(0) = \lim_{n \rightarrow \infty} \nu(0, n) = 1$. This would imply $0 = \nu_\infty(0) = 1$. ■

Remark 4 The constant $\underline{\mu}^2$ is estimated by 0.59 (see [9]).

The last estimate of this section shows that the functions $\{\mu^2(x, n)\}$ have uniform quadratic growth near their minimum points.

Lemma 4.6 Let $n \geq 1$. There are positive constants δ and θ independent of n so that

$$\theta^{-1}|x - x_n|^2 \leq \mu^2(x, n) - \mu_n^2 \leq \theta|x - x_n|^2 \text{ for } |x - x_n| < \delta.$$

Proof: Without loss of generality we work with y and $\nu^2(y, n)$ to show that

$$\theta^{-1}|y - y_n|^2 \leq \nu^2(y, n) - \nu^2(y_n, n) \leq \theta|y - y_n|^2. \quad (4.8)$$

By factoring the term in brackets in (4.5) and using (3.5) we get

$$\frac{d}{dy}\nu^2(y, n) = \left(y - (\nu + \sqrt{\nu^2 + 2n} - \sqrt{2n})\right) k(y, n).$$

Moreover, since $\nu(\cdot, n) \rightarrow \nu_\infty(\cdot)$ and $y_n \rightarrow \underline{y}$ as $n \rightarrow \infty$ there are constants k_0 and $\bar{\delta}$ so that $0 < k_0^{-1} \leq k(y, n) \leq k_0$ for $|y - y_n| < \bar{\delta}$.

In particular $|\frac{d}{dy}\nu^2(y, n)|$ is uniformly bounded for $|y - y_n| < \bar{\delta}$ and we get

$$|\nu^2(y, n) - \nu^2(y_n, n)| \leq C|y - y_n|$$

where C is independent of n . Now since $y_n = \nu(y_n, n) + \sqrt{\nu^2(y_n, n) + 2n} - \sqrt{2n}$ we have

$$\left|\frac{d}{dy}\nu^2(y, n)\right| \leq C(|y - y_n| + |\nu(y, n) - \nu(y_n, n)|) \leq \bar{C}|y - y_n|.$$

Hence

$$|\nu^2(y, n) - \nu^2(y_n, n)| \leq \bar{C}|y - y_n|^2.$$

Thus,

$$\frac{d}{dy}\nu^2(y, n) = (y - y_n)k + M|y - y_n|^2$$

where $M = M(y, n)$ is bounded uniformly in n . We see (4.8) follows directly from this for δ sufficiently small. ■

5 Asymptotic Estimates

In this section we use the uniform quadratic growth of the $\mu(x, n)$ near x_n given in Lemma 4.6 to derive rigorous asymptotic estimates of $\mu_n \equiv \mu(x_n, n)$. These estimates will be used in Section 6 to study the eigenvalues $\mu_e(\bar{x}) \equiv \inf_{n \in \mathbb{Z}} \mu(\bar{x}, n)$. To do this, we compare the variational problem (3.3) for the disk to one on a half-line.

Consider the minimization problem for

$$J(v, \bar{y}) = \int_{-\infty}^{\bar{y}} (v^2 + y^2 v^2) dy \quad (5.1)$$

in the class $\{v \in W^{1,2}((-\infty, \bar{y})) : \int_{-\infty}^{\bar{y}} v^2 dy = 1\}$. Due to the potential y^2 a minimizer exists, f , in spite of the fact that the domain is unbounded. It follows as in Lemma 3.1 that the minimum is $\nu_\infty^2(\bar{y})$ and that the minimizer is a multiple of $f_\infty(y, \bar{y})$. (See (4.6) and (4.7).) Moreover, it follows as in Lemma 3.7 and 3.8 that $0 < \nu_\infty(\bar{y}) < 1$ for $\bar{y} > 0$, $\nu_\infty(0) = 1$, ν_∞ has a single critical point $\underline{\mu}$ (from Corollary 4.5) for which $0 < \underline{\mu} < 1$ at which ν_∞ minimizes, and $\underline{\mu} = \nu_\infty(\underline{\mu})$.

Lemma 5.1 *There exists a constant $K < \infty$ such that $|\underline{\mu}^2 - \mu_n^2| \leq Kn^{-\frac{1}{2}}$ and hence $|\underline{\mu} - \mu_n| \leq 2Kn^{-\frac{1}{2}}$.*

Proof: We compare the variational problems (3.3) and (5.1).

First we show that there is a constant K for which

$$\underline{\mu}^2 \leq \mu_n^2 + Kn^{-\frac{1}{2}}. \quad (5.2)$$

Let $f_n(y) = f_n(y, y_n)$ solve (4.1) and (4.2) where $\nu(\bar{y}, n)$ minimizes at $\bar{y} = y_n$ (and thus $\nu(y_n, n) = \mu(x_n, n) \equiv \mu_n$). Extend f_n to $(-\infty, y_n]$ by setting $f_n = 0$ in $(-\infty, -\sqrt{2n})$. Note that

$$\frac{y}{y + \sqrt{2n}} = \frac{y}{\sqrt{2n}} \left(1 + \frac{y}{\sqrt{2n}}\right)^{-1}$$

and $\{y_n\}$ is bounded in $(0, \infty)$. From this, (4.1), and Lemma 4.2, we have

$$-(f_n'' - y^2 f_n + \mu_n^2 f_n) \leq Cn^{-\frac{1}{2}} e^{-\frac{y^2}{16}} \text{ for } y \leq y_n.$$

Using $f_n'(y_n) = 0 = f_n(-\sqrt{2n})$, we multiply this by f_n and integrate to get

$$J(f_n, y_n) = \int_{-\infty}^{y_n} (f_n'^2 + y^2 f_n^2) dy \leq \mu_n^2 \int_{-\infty}^{y_n} f_n^2 dy + Cn^{-\frac{1}{2}}. \quad (5.3)$$

Since $\underline{\mu} = \nu_\infty(\underline{\mu}) \leq \nu_\infty(y_n)$ we get

$$\underline{\mu}^2 \int_{-\infty}^{y_n} f_n^2 dy \leq \nu_\infty^2(y_n) \int_{-\infty}^{y_n} f_n^2 dy \leq J(f_n, y_n). \quad (5.4)$$

From Lemmas 4.1 and 4.2 there exists $c > 0$ so that $c^{-1} \leq \int_{-\infty}^{y_n} f_n^2 dy \leq c$. Hence from (5.3) and (5.4) we obtain (5.2).

Next we show that $\mu_n^2 \leq \underline{\mu}^2 + Kn^{-\frac{1}{2}}$. Let $\chi_n \in C^\infty(\mathbf{R})$ such that $|\chi'_n|, |\chi''_n| \leq 4$ and

$$\chi_n = 1 \text{ if } y > \frac{-\sqrt{2n}}{2} + 1, \quad \chi_n = 0 \text{ if } y < \frac{-\sqrt{2n}}{2}.$$

Set $\eta_n(y) = \chi_n(y) \cdot f_\infty(y, y_n)$, where $f_\infty(y, y_n)$ is the solution of (4.6) with $\bar{y} = y_n$. Note that for $y \geq -\frac{\sqrt{2n}}{2}$:

$$\left| \frac{1}{y + \sqrt{2n}} \right| \leq (2n)^{-\frac{1}{2}} \text{ and } \left| \frac{y}{y + \sqrt{2n}} \right| \leq 2^{\frac{1}{2}} |y| n^{-\frac{1}{2}}.$$

Then

$$-\left(\eta_n'' + \frac{\eta_n'}{y + \sqrt{2n}} - y^2 \left(1 - \frac{1}{2} \left(\frac{y}{y + \sqrt{2n}} \right) \right)^2 \eta_n + \nu_\infty^2(y_n) \eta_n \right) \leq Cn^{-\frac{1}{2}} e^{-\frac{y^2}{16}} \quad (5.5)$$

for $y \leq y_n$.

Set $\tilde{\eta}_n(x) = \eta_n(x - \sqrt{2n})$. Multiplying (5.5) by $\eta_n(y) \cdot (y + \sqrt{2n})$ and integrating we obtain

$$\begin{aligned} \mu_n^2 \int_{-\sqrt{2n}}^{y_n} \eta_n^2(y) \cdot (y + \sqrt{2n}) dy &\leq \mathcal{J}(\tilde{\eta}_n, n, x_n) \\ &= \int_{-\sqrt{2n}}^{y_n} \left\{ (\eta_n'(y))^2 + y^2 \left(1 - \frac{1}{2} \left(\frac{y}{y + \sqrt{2n}} \right) \right)^2 \eta_n^2(y) \right. \\ &\quad \left. (y + \sqrt{2n}) \right\} dy \leq \nu_\infty^2(y_n) \int_{-\sqrt{2n}}^{y_n} \eta_n^2(y) \cdot (y + \sqrt{2n}) dy + C. \end{aligned}$$

Using Lemmas 4.1 and 4.2 there exists $c > 0$ so that

$$c^{-1} n^{\frac{1}{2}} \leq \int_{-\sqrt{2n}}^{y_n} \eta_n^2(y) \cdot (y + \sqrt{2n}) dy \leq cn^{\frac{1}{2}}.$$

Hence $\mu_n^2 \leq \nu_\infty^2(y_n) + Cn^{-\frac{1}{2}}$. Passing to the limit as $n \rightarrow \infty$ in (4.8) of Lemma 4.6 gives $\nu_\infty^2(y) - \underline{\mu}^2 \leq M(y - \underline{\mu})^2$ for $y \geq 0$ and some $M < \infty$. Hence

$$\mu_n^2 - \underline{\mu}^2 \leq M(y_n - \underline{\mu})^2 + Cn^{-\frac{1}{2}} \leq 2M(\mu_n - \underline{\mu})^2 + 2M(y_n - \mu_n)^2 + Cn^{-\frac{1}{2}}.$$

Recall that $y_n = \mu_n + \sqrt{\mu_n^2 + 2n} - \sqrt{2n}$. Thus $|y_n - \mu_n| \leq \bar{C}n^{-\frac{1}{2}}$ and we obtain

$$\mu_n^2 - \underline{\mu}^2 \leq 2M(\mu_n - \underline{\mu})^2 + \bar{C} n^{-\frac{1}{2}}.$$

From Corollary 4.5 $\mu_n \rightarrow \underline{\mu} > 0$. Thus we have

$$\mu_n^2 \leq \underline{\mu}^2 + Kn^{-\frac{1}{2}} \text{ for some } K > 0.$$

Combining this with (5.2) proves the lemma. $\blacksquare$

We next refine the asymptotic estimate for $\{\mu_n\}$. From Lemma 5.1 we have $|\mu_n - \underline{\mu}| \leq 4K(2n)^{-\frac{1}{2}}$. Also, since $y_n = x_n - \sqrt{2n} = \mu_n + \sqrt{\mu_n^2 + 2n} - \sqrt{2n}$ for $n \geq 1$, it follows that

$$|y_n - \underline{\mu}| \leq \tilde{K}(2n)^{-\frac{1}{2}}. \quad (5.6)$$

From (4.1) and (4.2), $f_n(y) \equiv f_n(y, y_n)$ satisfies

$$f_n'' + \frac{f_n'}{\sqrt{2n}(1 + \frac{y}{\sqrt{2n}})} - y^2 \left(1 - \frac{y}{2\sqrt{2n}(1 + \frac{y}{\sqrt{2n}})}\right)^2 f_n + \mu_n^2 f_n = 0 \quad (5.7)$$

for $-\sqrt{2n} < y \leq y_n$ with $f_n(y_n) = 1$ and $f_n'(y_n) = 0$. From Lemma 4.2 we have

$$0 \leq f_n(y) \leq c \left(1 + \frac{y}{\sqrt{2n}}\right)^4 e^{-\frac{y^2}{4}}, \quad |f_n'(y)| \leq c \left(1 + \frac{y}{\sqrt{2n}}\right) e^{-\frac{y^2}{8}} \text{ for } -\sqrt{2n} \leq y \leq \bar{y}. \quad (5.8)$$

Set $f_0(y) = f_\infty(y, \underline{\mu})$. From Lemmas 4.3, 4.4, and Corollary 4.5, we have $f_n(y) \equiv f_n(y, y_n) \rightarrow f_0(y)$ in $C^2([-M, M])$ as $n \rightarrow \infty$ for each $M < \infty$. Thus f_0 satisfies

$$\begin{aligned} f_0'' - y^2 f_0 + \underline{\mu}^2 f_0 &= 0, \\ f_0(\underline{\mu}) &= 1, \quad f_0'(\underline{\mu}) = 0, \end{aligned} \quad (5.9)$$

and

$$f_0(y), |f_0'(y)| \leq ce^{-\frac{y^2}{8}} \text{ for } y \leq 2. \quad (5.10)$$

Set

$$d_n(y) = \sqrt{2n} \cdot [f_n(y) - f_0(y)] \equiv \sqrt{2n} \cdot [f_n(y, y_n) - f_\infty(y, \underline{\mu})]$$

for $-\sqrt{2n} \leq y \leq y_n$. From (5.7) and (5.9), we see that d_n satisfies

$$\begin{aligned} d_n'' - y^2 \cdot d_n + \underline{\mu}^2 \cdot d_n &= \frac{-f_n'}{(1 + \frac{y}{\sqrt{2n}})} - \frac{y^3 f_n}{(1 + \frac{y}{\sqrt{2n}})} \\ &+ \frac{y^4 f_n}{4\sqrt{2n}(1 + \frac{y}{\sqrt{2n}})^2} - \sqrt{2n}(\mu_n^2 - \underline{\mu}^2) f_n \end{aligned} \quad (5.11)$$

for $-\sqrt{2n} < y \leq y_n$. We shall need the following estimates for $\{d_n\}$:

Lemma 5.2 *The functions $\{d_n\}$ satisfy $d_n(y_n) = O(n^{-1})$ and $d'_n(y_n) = O(n^{-\frac{1}{2}})$ as $n \rightarrow \infty$.*

Proof: Since $f_n(y_n) = f_0(\underline{\mu}) = 1$, we have

$$d_n(y_n) = \frac{f_0(\underline{\mu}) - f_0(y_n)}{(\frac{1}{\sqrt{2n}})}.$$

Now from (5.9), $f''_0(\underline{\mu}) = f'_0(\underline{\mu}) = 0$. From this and (5.6), we have

$$|d_n(y_n)| \leq C|y_n - \underline{\mu}|^3 \cdot \sqrt{2n} \leq \bar{C}n^{-1}.$$

Next consider

$$d'_n(y_n) = \frac{f'_n(y_n) - f'_0(y_n)}{(\frac{1}{\sqrt{2n}})} = -\sqrt{2n} \cdot f'_0(y_n).$$

We have

$$|d'_n(y_n)| \leq C\sqrt{2n} \cdot |y_n - \underline{\mu}|^2 \leq \bar{C}n^{-\frac{1}{2}}.$$

■

Lemma 5.3 *There is a constant $C < \infty$ independent of n such that*

$$\int_{-\sqrt{2n}}^2 (d_n'^2 + d_n^2) dy \leq C.$$

Proof: Using (5.8), Lemma 5.2 and equation (5.11) we can apply standard estimates for the initial value problem to conclude that d_n and d'_n are uniformly bounded on $[-2, 2]$ for $n \geq 5$. Next we multiply (5.11) by d_n and integrate from $-\sqrt{2n}$ to -2 . Then (5.8) and (5.10) allow us to estimate $\int_{-\sqrt{2n}}^{-2} (d_n'^2 + y^2 d_n^2) dy \leq C$ where C depends on $d_n(-2)$ and $d'_n(-2)$. Since these terms are uniformly bounded it follows that C is independent of n . The lemma follows directly. ■

We are now in a position to prove the following asymptotic estimate of μ_n^2 :

Theorem 5.4 *There is a constant ℓ (independent of n) such that*

$$\mu_n^2 = \underline{\mu}^2 + \frac{\ell}{\sqrt{2n}} + O(n^{-1}) \text{ as } n \rightarrow \infty.$$

Proof: Multiplying (5.11) by f_0 , integrating by parts, and using (5.9) we obtain

$$\begin{aligned} (d'_n f_0 - d_n f'_0) \Big|_{y=-\sqrt{2n}}^{y=y_n} &= - \int_{-\sqrt{2n}}^{y_n} \left[\frac{(f'_n f_0 + y^3 f_n f_0)}{(1 + \frac{y}{\sqrt{2n}})} \right. \\ &\quad \left. - \frac{y^4 f_n f_0}{4\sqrt{2n}(1 + \frac{y}{\sqrt{2n}})^2} \right] dy - \sqrt{2n} \cdot (\mu_n^2 - \underline{\mu}^2) \cdot \int_{-\sqrt{2n}}^{y_n} f_n f_0 dy. \end{aligned} \quad (5.12)$$

We write this as $I = II + III$. From Lemma 5.2, (5.8) and (5.10) we have

$$I = O(n^{-\frac{1}{2}}) \text{ as } n \rightarrow \infty.$$

Next we rewrite II . Since $f_n = f_0 + (2n)^{-\frac{1}{2}} \cdot d_n$ and $\frac{1}{(1 + \frac{y}{\sqrt{2n}})} = 1 - \frac{y}{\sqrt{2n}(1 + \frac{y}{\sqrt{2n}})}$ we have

$$\begin{aligned} II &= - \int_{-\sqrt{2n}}^{y_n} (f'_0 f_0 + y^3 f_0^2) dy \\ &\quad + \frac{1}{\sqrt{2n}} \int_{-\sqrt{2n}}^{y_n} \left[(-d'_n f_0 - y^3 d_n f_0) + \frac{(f'_n f_0 + y^3 f_n f_0)y}{(1 + \frac{y}{\sqrt{2n}})} + \frac{y^4 f_n f_0}{4(1 + \frac{y}{\sqrt{2n}})^2} \right] dy. \end{aligned}$$

Using (5.6), (5.8), (5.10), and Lemma 5.3 we see that the second term on the right is bounded in absolute value by $\frac{C}{\sqrt{2n}}$. Thus

$$II = - \int_{-\infty}^{\underline{\mu}} (f'_0 f_0 + y^3 f_0^2) dy + O(n^{-\frac{1}{2}}) \text{ as } n \rightarrow \infty.$$

Finally estimating III similarly, we have

$$III = -\sqrt{2n}(\mu_n^2 - \underline{\mu}^2) \cdot \int_{-\infty}^{\underline{\mu}} f_0^2 dy + O(n^{-\frac{1}{2}}) \text{ as } n \rightarrow \infty.$$

It follows that Theorem 5.4 is valid with

$$\ell \equiv - \int_{-\infty}^{\underline{\mu}} (f_0 f'_0 + y^3 f_0^2) dy \Big/ \int_{-\infty}^{\underline{\mu}} f_0^2 dy. \quad (5.13)$$

■

Using (5.9) we may determine ℓ as follows:

Lemma 5.5 *We have*

$$\ell = -1 \Big/ \left(3 \int_{-\infty}^{\underline{\mu}} f_0^2 dy \right) < 0.$$

Proof: From (5.13) we see that

$$\ell \int_{-\infty}^{\underline{\mu}} f_0^2 dy = - \int_{-\infty}^{\underline{\mu}} \left(\frac{(f_0^2)'}{2} + y^3 f_0^2 \right) dy = -\frac{1}{2} - \int_{-\infty}^{\underline{\mu}} y^3 f_0^2 dy. \quad (5.14)$$

Multiplying (5.9) by $2f_0'$ and integrating by parts, we get $\int_{-\infty}^{\underline{\mu}} y f_0^2 dy = 0$.

Now multiplying (5.9) by $2y^2 f_0'$ and integrating we get

$$-2 \int_{-\infty}^{\underline{\mu}} y (f_0')^2 dy + 4 \int_{-\infty}^{\underline{\mu}} y^3 f_0^2 dy = 2\underline{\mu}^2 \int_{-\infty}^{\underline{\mu}} y f_0^2 dy = 0. \quad (5.15)$$

Finally, multiplying (5.9) by $y f_0$ and integrating we get

$$-\frac{1}{2} - \int_{-\infty}^{\underline{\mu}} y (f_0')^2 dy - \int_{-\infty}^{\underline{\mu}} y^3 f_0^2 dy = -\underline{\mu}^2 \int_{-\infty}^{\underline{\mu}} y f_0^2 dy = 0. \quad (5.16)$$

From (5.15) and (5.16) we have

$$\int_{-\infty}^{\underline{\mu}} y^3 f_0^2 dy = -\frac{1}{6}.$$

Thus from (5.14) we obtain

$$\ell \int_{-\infty}^{\underline{\mu}} f_0^2 dy = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}.$$

■

Corollary 5.6 *For ℓ as defined above, we have*

$$\mu_n = \underline{\mu} + \frac{\ell}{2\underline{\mu}\sqrt{2n}} + O(n^{-1})$$

and

$$y_n = \underline{\mu} + \left(\frac{\ell}{2\underline{\mu}} + \frac{\underline{\mu}^2}{2} \right) \cdot \frac{1}{\sqrt{2n}} + O(n^{-1})$$

as $n \rightarrow \infty$.

Proof: From Taylor's Theorem if $c > 0$, $x \geq c$, and $x_0 \geq c$, we have

$$\sqrt{x} = \sqrt{x_0} + \frac{(x - x_0)}{2\sqrt{x_0}} + |x - x_0|^2 \cdot O(c^{-3/2}).$$

Thus (setting $x = \mu_n^2$ and $x_0 = \underline{\mu}^2$),

$$\mu_n = \underline{\mu} + \frac{1}{2\underline{\mu}}(\mu_n^2 - \underline{\mu}^2) + O(|\mu_n^2 - \underline{\mu}^2|^2).$$

From Theorem 5.4 and the above estimate, we have

$$\mu_n = \underline{\mu} + \frac{\ell}{2\underline{\mu}\sqrt{2n}} + O(n^{-1}). \quad (5.17)$$

Now recall that

$$y_n = \mu_n + \sqrt{\mu_n^2 + 2n} - \sqrt{2n}.$$

Setting $x = \mu_n^2 + 2n$, $x_0 = 2n$, and $c = 2n$ in the formula for $\sqrt{x}$ we have

$$\sqrt{\mu_n^2 + 2n} = \sqrt{2n} + \frac{\underline{\mu}^2}{2\sqrt{2n}} + O(n^{-3/2}).$$

Thus

$$y_n = \mu_n + \frac{\underline{\mu}^2}{2\sqrt{2n}} + O(n^{-3/2}). \quad (5.18)$$

The corollary now follows from (5.17) and (5.18). ■

6 Analysis of the Eigenvalues

In this section we conclude our analysis of the lower envelope $\mu_e(x) = \inf_{n \in \mathbb{Z}} \mu(x, n)$. We note that since the eigenvalues $\lambda(\omega, \bar{r})$ of the linearized problem defined by (1.4) satisfy $\lambda = \inf_{n \in \mathbb{Z}} \lambda_n(\omega, \bar{r})$ and $\lambda_n(\omega, \bar{r}) = \omega^2 \cdot \mu^2(\omega \bar{r}, n)$, it follows that $\lambda(\omega, \bar{r}) = \omega^2 \cdot \mu_e^2(\omega \bar{r})$. (See Sections 2 and 3.)

Recall that $\mu(x, 0) < 1$ and $\mu(x, n) \geq 1$ for $n \leq -1$. Moreover if $n \geq 1$ and $x \leq \sqrt{2n}$ then $\mu(x, n) \geq 1$. Hence

$$\mu_e(x) = \min_{0 \leq n \leq n_0} \mu(x, n) \text{ for } 0 \leq x \leq \sqrt{2n_0}.$$

Since each $\mu(x, n)$ is analytic and no two are identical the intersection points of any finite number cannot accumulate in $(0, \infty)$. Thus $\mu_e(x)$ is a continuous piecewise analytic function such that $0 < \mu_e < 1$. Moreover there are nonnegative sequences $\{m_j : j = 0, 1, 2, \dots\}$ in $\mathbf{Z}$ and $\{\beta_j : j = 0, 1, 2\}$ in $\mathbf{R}$ with $m_0 = 0 = \beta_0$, $\beta_j < \beta_{j+1}$ and $\lim_{j \rightarrow \infty} \beta_j = \infty$ so that

$$\mu_e(x) = \mu(x, m_j) \text{ for all } x \text{ in } [\beta_j, \beta_{j+1}].$$

The following result provides estimates of $\mu_e^2(x)$ and $m_j(x)$ for large x .

Theorem 6.1 *There are constants c_0 and c_1 so that if $x \geq c_0$ and if m satisfies $\mu_e(x) = \mu(x, m)$, then*

$$|2m - (x - \underline{\mu})^2| \leq c_1 \quad (6.1)$$

and

$$\mu_e^2(x) = \underline{\mu}^2 + \ell x^{-1} + O(x^{-2}) \text{ as } x \rightarrow \infty. \quad (6.2)$$

Proof: Consider the sequence of critical points $\{x_n\}$ for $\{\mu(x, n)\}$ where $n \geq 1$. From Corollary 5.6 and the fact that $y_n = x_n - \sqrt{2n}$, there are constants $N_1 \geq 1$ and $c > 0$ such that

$$c^{-1}n^{-\frac{1}{2}} \leq x_{n+1} - x_n \leq cn^{-\frac{1}{2}} \leq \delta \text{ for } n \geq N_1. \quad (6.3)$$

where δ is the constant in Lemma 4.6. Let c_0 be so large that if $x_{n+1} \geq c_0$ then $n \geq N_1$.

Let $x \geq c_0$ and let n satisfy $x_n \leq x < x_{n+1}$. Then $0 \leq |x - x_n| \leq cn^{-\frac{1}{2}} \leq \delta$. Thus from Lemma 4.6 and Theorem 5.4,

$$\mu_e^2(x) \leq \mu^2(x_n, n) + (\mu^2(x, n) - \mu^2(x_n, n)) \leq \underline{\mu}^2 + \ell(2n)^{-\frac{1}{2}} + cn^{-1}. \quad (6.4)$$

Since $\ell < 0$ we can choose $N_2 \geq N_1$ so large that

$$\underline{\mu}^2 + \ell(2n)^{-\frac{1}{2}} + cn^{-1} < \underline{\mu}^2 \text{ for } n \geq N_2. \quad (6.5)$$

Since $\lim_{n \rightarrow \infty} \mu_n = \underline{\mu}$ we can also assume that

$$\underline{\mu}^2 < \mu_n^2 + \theta^{-1}\delta^2 \text{ for } n \geq N_2 \quad (6.6)$$

where δ and θ are from Lemma 4.6.

From Lemma 3.7 we have $\lim_{x \rightarrow \infty} \mu(x, j) = 1$ for $j = 0, 1, 2, \dots$. Thus we can assume c_0 is so large that

$$\mu(x, j) \geq \underline{\mu} \text{ for } 0 \leq j \leq N_2 \text{ and } x \geq c_0. \quad (6.7)$$

(If not, we replace c_0 by the maximum of c_0 and $\bar{c}_0$ where $\bar{c}_0$ satisfies (6.7).)

Now let m be such that $\mu_e(x) = \mu(x, m)$. From (6.4) and (6.5) we see that $\mu_e(x) < \underline{\mu}$. Thus from (6.7) then $m \geq N_2$.

Next we claim that $|x - x_m| < \delta$. If not from Lemma 3.9 it follows that $\mu(x, m) \geq \min\{\mu(x_m + \delta, m), \mu(x_m - \delta, m)\}$. Then from Lemma 4.6 we get $\mu^2(x, m) \geq \mu_m^2 + \theta^{-1}\delta^2$. Whence from (6.6) we have $\mu^2(x, m) > \underline{\mu}^2$ which is impossible.

Thus

$$|x_n - x_m| \leq |x - x_m| + |x - x_n| \leq \delta + cn^{-\frac{1}{2}} \leq \bar{c}$$

where $\bar{c}$ is independent of x, m and n . Since $m, n \geq N_1$, we use (6.3) to deduce that

$$|\sqrt{m} - \sqrt{n}| \equiv |m - n|/(\sqrt{m} + \sqrt{n}) \leq c|x_m - x_n| \leq c\bar{c}. \quad (6.8)$$

If either $m \geq 4n$ or $n \geq 4m$, then $|\sqrt{m} - \sqrt{n}| \geq (\sqrt{m} + \sqrt{n})/4$. From this and (6.8), it follows that $\sqrt{m} + \sqrt{n} \leq 4c\bar{c}$ independent of x, m and n . For c_0 sufficiently large, this is impossible. Thus $4^{-1}n \leq m \leq 4n$ and from (6.8), $|m - n| \leq c\sqrt{n}$. From Theorem 5.4 then $|\mu_m^2 - \mu_n^2| \leq \bar{c}/n$. Thus using Lemma 4.6 and $\mu_e(x) = \mu(x, m)$ we have

$$\begin{aligned} (4\theta)^{-1}|x_m - x_n|^2 &\leq \theta^{-1}|x - x_m|^2 + \theta^{-1}|x - x_n|^2 \\ &\leq \mu_e^2(x) - \mu_m^2 + \theta^{-1}c^2n^{-1} \\ &\leq (\mu^2(x, n) - \mu_n^2) + (\mu_n^2 - \mu_m^2) + \bar{c}n^{-1} \\ &\leq \theta|x - x_n|^2 + \bar{c}n^{-1} \\ &\leq \bar{c}n^{-1}. \end{aligned}$$

Therefore $|x_m - x_n| \leq \bar{c}n^{-\frac{1}{2}}$. It follows from (6.8) that $|m - n| \leq \bar{c}$ where $\bar{c}$ is independent of x, m and n .

Now since $x_n = \sqrt{2n} + \underline{\mu} + O(n^{-\frac{1}{2}})$ and $|x - x_n| \leq cn^{-\frac{1}{2}}$, we deduce that

$$\begin{aligned} |2m - (x - \underline{\mu})^2| &\leq |2m - 2n| + |2n - (x_n - \underline{\mu})^2| \\ &\quad + |(x_n - \underline{\mu})^2 - (x - \underline{\mu})^2| \leq c_1 \end{aligned}$$

where c_1 is independent of x and m . This proves (6.1).

To prove (6.2) assume that $x \geq c_0$, $\mu_e(x) = \mu(x, m)$, and $x_n \leq x < x_{n+1}$. From (6.1) it follows that there exists $c_2 \geq c_0$ and $c_3 > 0$ independent of x and m such that

$$c_3^{-1} \cdot x^2 \leq 2m \leq c_3 x^2 \text{ for } x \geq c_2. \quad (6.9)$$

Thus

$$\begin{aligned} |\mu_e^2(x) - \underline{\mu}^2 - \frac{\ell}{x}| &= |\mu(x, m)^2 - \underline{\mu}^2 - \frac{\ell}{x}| \\ &\leq |\mu(x, m)^2 - \mu_m^2| + |\mu_m^2 - \underline{\mu}^2 - \frac{\ell}{\sqrt{2m}}| \\ &\quad + |\frac{\ell}{\sqrt{2m}} - \frac{\ell}{x}|. \end{aligned}$$

From Lemma 4.6 and our previous estimates, we have

$$\begin{aligned} |\mu(x, m)^2 - \mu_m^2| &\leq \theta |x - x_m|^2 \\ &\leq 2\theta(|x - x_n|^2 + |x_n - x_m|^2) \\ &\leq \bar{c}/m \end{aligned}$$

where $\bar{c}$ is independent of x and m . Also by Theorem 5.4, we have

$$|\mu_m^2 - \underline{\mu}^2 - \frac{\ell}{\sqrt{2m}}| \leq \bar{c}/m.$$

Finally by (6.9) if $x \geq c_2$ then

$$\begin{aligned} |\frac{\ell}{\sqrt{2m}} - \frac{\ell}{x}| &= |\frac{\ell(x - \sqrt{2m})}{x\sqrt{2m}}| \\ &\leq \bar{c}|x - \sqrt{2m}|/m \end{aligned}$$

where $\bar{c}$ is independent of $\bar{x}$ and m . But

$$\begin{aligned} |x - \sqrt{2m}| &\leq |x - x_n| + |x_n - \sqrt{2n}| + |\sqrt{2n} - \sqrt{2m}| \\ &\leq \bar{c} \end{aligned}$$

by our previous estimates. Thus

$$|\mu_e^2(x) - \underline{\mu}^2 - \frac{\ell}{x}| \leq \bar{c}/m \leq \bar{c}/x^2$$

for $x \geq c_2$ where $\bar{c}$ is independent of x and m . ■

7 Reformulation of the Asymptotic Estimates

In this section we reformulate our asymptotic estimates in terms of $\bar{r}$, κ , and h . Given $\bar{r}$ and κ let

$$h_0 \in \mathcal{H}(\bar{r}, \kappa) \equiv \{h : \kappa^2 = \lambda((h\kappa)^{\frac{1}{2}}, \bar{r})\} = \left\{h : \frac{\kappa}{h} = \mu_e^2((h\kappa)^{\frac{1}{2}} \bar{r})\right\}.$$

Theorem 7.1 Let $h_0 \in \mathcal{H}(\bar{\tau}, \kappa)$. Then

$$(7.1) \quad \frac{\kappa}{h_0} = \bar{\mu}_2^2 + \bar{\mu} \ell (\kappa \bar{\tau})^{-1} + O((\kappa \bar{\tau})^{-2})$$

and

$$(7.2) \quad h_0 \bar{\tau} = \bar{\mu}^{-2} \kappa \bar{\tau} - \bar{\mu}^{-3} \ell + O((\kappa \bar{\tau})^{-1}) \text{ as } \kappa \bar{\tau} \rightarrow \infty.$$

Moreover, if m is such that $\mu_e((h_0 \kappa)^{\frac{1}{2} \bar{\tau}}) = \mu((h_0 \kappa)^{\frac{1}{2} \bar{\tau}}, m)$ then

$$(7.3) \quad 2m = (\bar{\mu}^{-1} \kappa \bar{\tau})^2 - \kappa \bar{\tau} (2 + \bar{\mu}^{-3} \ell) + O(1) \text{ as } \kappa \bar{\tau} \rightarrow \infty.$$

Proof: Let $\kappa \bar{\tau} \geq M \gg 1$. Since $\mu_e < 1$ we see that if $\frac{h_0}{\kappa} = \mu_2^e((h_0 \kappa)^{\frac{1}{2} \bar{\tau}})$, then $h_0 \bar{\tau} > \kappa \bar{\tau} \geq M$. Using (6.2) with $x = (\kappa h_0)^{\frac{1}{2} \bar{\tau}}$ we have

$$(7.4) \quad \frac{\kappa}{h_0} = \bar{\mu}_2^2 + \ell \kappa^{-\frac{1}{2}} h_0^{-\frac{1}{2}} \bar{\tau}^{-1} + O((\kappa \bar{\tau})^{-2}) \text{ as } \kappa \bar{\tau} \rightarrow \infty.$$

In particular $\frac{\kappa}{h_0} - \bar{\mu}_2^2 = O((\kappa \bar{\tau})^{-1})$. Hence $\bar{\mu} = \kappa^{\frac{1}{2}} h_0^{-\frac{1}{2}} + O((\kappa \bar{\tau})^{-1})$ and $\bar{\mu}^{-1} = \kappa^{-\frac{1}{2}} h_0^{\frac{1}{2}} + O((\kappa \bar{\tau})^{-1})$. Thus

$$\bar{\mu} (\kappa \bar{\tau})^{-1} = \kappa^{-\frac{1}{2}} h_0^{-\frac{1}{2}} \bar{\tau}^{-1} + O((\kappa \bar{\tau})^{-2})$$

and (7.1) follows from this and (7.4). Similarly multiplying (7.4) by $h_0 \bar{\tau} \bar{\mu}^{-2}$ we obtain

$$\kappa \bar{\tau} \bar{\mu}^{-2} = h_0 \bar{\tau} + \ell \kappa^{-\frac{1}{2}} h_0^{\frac{1}{2}} \bar{\mu}^{-2} + O((\kappa \bar{\tau})^{-1})$$

$$= h_0 \bar{\tau} + \ell \bar{\mu}^{-3} + O((\kappa \bar{\tau})^{-1}).$$

This proves (7.2). Now to prove (7.3) we note that from (7.1) we have

$$\frac{\kappa}{h_0} = (\bar{\mu} + (\kappa \bar{\tau})^{-1} \ell / 2)^2 + O((\kappa \bar{\tau})^{-2}).$$

Since $\bar{x} = (h_0 \kappa)^{\frac{1}{2} \bar{\tau}} = (\kappa / h_0)^{\frac{1}{2}} \cdot h_0 \bar{\tau}$, it follows from this and (7.2) that

$$(7.5) \quad \bar{x} = [\bar{\mu} + (\kappa \bar{\tau})^{-1} \ell / 2 + O((\kappa \bar{\tau})^{-2})] h_0 \bar{\tau}$$

$$= \bar{\mu}^{-1} \kappa \bar{\tau} - \bar{\mu}^{-2} \ell / 2 + O((\kappa \bar{\tau})^{-1}).$$

Using this in (6.1) gives (7.3). ■

Theorem 7.2 Fix $\bar{\tau}$ and κ . Then $\mathcal{H}(\bar{\tau}, \kappa)$ is a finite nonempty set of positive numbers. Set $\underline{h} = \underline{h}(\bar{\tau}, \kappa) \equiv \min\{h : h \in \mathcal{H}(\bar{\tau}, \kappa)\}$ and $\bar{h} = \bar{h}(\bar{\tau}, \kappa) \equiv \max\{h : h \in \mathcal{H}(\bar{\tau}, \kappa)\}$. The normal state corresponding to $(\bar{\tau}, \kappa, h)$ is unstable for $h < \underline{h}$ and strictly stable for $h > \bar{h}$. For each $h_0 \in \mathcal{H}(\bar{\tau}, \kappa)$, the normal states corresponding to $(\bar{\tau}, \kappa, h_0)$ are states of degenerate stability which are estimated by (7.2). Thus there is a constant C (independent of $\bar{\tau}$ and κ) such that

$$|\bar{h} - \underline{h}| \leq C\kappa^{-1}(\bar{\tau})^{-2}.$$

In particular for $\bar{\tau}$ large $\mathcal{H}(\bar{\tau}, \kappa)$ is contained in a small interval of magnetic strength.

Proof: Given $\bar{\tau}$ and κ set $j(h) = h\mu_e^2((h\kappa)^{\frac{1}{2}}\bar{\tau})$ for $h \geq 0$. Then $j \in C([0, \infty))$, piecewise analytic and nonconstant on any subinterval. Moreover $j(0) = 0$ and using (6.2) we see that $\lim_{h \rightarrow \infty} j(h) = \infty$. Thus $\mathcal{H}(\bar{\tau}, \kappa) = \{h : \kappa = j(h)\}$ is a nonempty finite set of positive numbers. If $h < \underline{h}$ then $j(h) < \kappa$. This implies that $\kappa^2 > \kappa h\mu_e^2((h\kappa)^{\frac{1}{2}}\bar{\tau}) = \lambda((h\kappa)^{\frac{1}{2}}, \bar{\tau})$. Thus from Lemma 2.1 we see that the normal state corresponding to $(\bar{\tau}, \kappa, h)$ is unstable. Similarly, the normal state corresponding to $(\bar{\tau}, \kappa, h)$ is strictly stable if $h > \bar{h}$. ■

Next we describe the solutions to (1.7) at $(\bar{\tau}, \kappa, h_0)$ for $h_0 \in \mathcal{H}(\bar{\tau}, \kappa)$.

Theorem 7.3 Fix $\bar{\tau}, \kappa$ and $h_0 \in \mathcal{H}(\bar{\tau}, \kappa)$. Then the linearized problem (1.7) with $h = h_0$ has its solution set spanned by

$$\mathcal{S}(\bar{\tau}, \kappa, h_0) = \{\varphi_{m_1}(r)e^{im_1\theta}, \dots, \varphi_{m_k}(r)e^{im_k\theta}\}$$

where m_ℓ is a nonnegative integer, $m_i \neq m_\ell$ for $i \neq \ell$, and $\varphi_n(r)$ is a positive multiple of $\xi_n((h_0\kappa)^{\frac{1}{2}}r, (h_0\kappa)^{\frac{1}{2}}\bar{\tau})$.

The m_ℓ satisfy the asymptotic estimate (7.3). In particular we see that there is a constant C independent of $\bar{\tau}$ and κ such that $k \leq C$.

Proof: The structure of $\mathcal{S}$ follows from §2. ■

For $\kappa \geq \kappa_0 > 0$ fixed we now point out that the boundary layer profiles converge to that found by de Gennes and St. James for the half plane as $\bar{\tau} \rightarrow \infty$.

Theorem 7.4 Fix $\kappa_0 > 0$. For $\kappa \geq \kappa_0$ and $\bar{\tau}$ and $h_0 \in \mathcal{H}(\bar{\tau}, \kappa)$ given, let $\varphi_{m_\ell}(r)$ be as in Theorem 7.3 with $\varphi_{m_\ell}(\bar{\tau}) = 1$. Recall that $f_0(y) \equiv f_\infty(y, \underline{\mu})$ satisfies (4.6) with $\bar{y} = \nu_\infty(\bar{y}) = \underline{\mu}$. Then

$$\lim_{\bar{\tau} \rightarrow \infty} \sup_{\substack{-\bar{\tau} \leq \tau \leq 0, \\ 1 \leq \ell \leq k(\bar{\tau}, h_0), \\ h_0 \in \mathcal{H}(\bar{\tau}, \kappa)}} |\varphi_\ell(\bar{\tau} + \tau) - f_0(\underline{\mu}^{-1}\kappa\tau + \underline{\mu})| = 0. \quad (7.6)$$

Proof: Let $h_0 \in \mathcal{H}(\bar{\tau}, \kappa)$, $-\bar{\tau} \leq \tau \leq 0$, $1 \leq \ell \leq k(\bar{\tau}, h_0)$, and set

$$\begin{aligned} y = y(\tau) &\equiv (h_0 \kappa)^{\frac{1}{2}}(\bar{\tau} + \tau) - (2m_\ell)^{\frac{1}{2}} \\ &= (h_0 \kappa)^{\frac{1}{2}}\tau + ((h_0 \kappa)^{\frac{1}{2}}\bar{\tau} - (2m_\ell)^{\frac{1}{2}}). \end{aligned} \quad (7.7)$$

This is the change of variables used to obtain (4.1). From (7.5) we have

$$|(h_0 \kappa)^{\frac{1}{2}} - \underline{\mu}^{-1}\kappa| \leq \frac{c}{\bar{\tau}} \quad (7.8)$$

where c depends on κ_0 . From (6.1) we have

$$|(2m)^{\frac{1}{2}} - (x - \underline{\mu})| \leq \frac{c}{x} \text{ for } x \geq c_0$$

if m satisfies $\mu_e(x) = \mu(x, m)$. Setting $m = m_\ell$ and $x = (h_0 \kappa)^{\frac{1}{2}}\bar{\tau}$ we obtain

$$|(2m_\ell)^{\frac{1}{2}} - ((h_0 \kappa)^{\frac{1}{2}}\bar{\tau} - \underline{\mu})| \leq c(\kappa\bar{\tau})^{-1}. \quad (7.9)$$

From (7.7)-(7.9) we obtain

$$|y(\tau) - (\underline{\mu} + \underline{\mu}^{-1}\kappa\tau)| \leq C(|\tau| + \kappa^{-1})\bar{\tau}^{-1}.$$

Hence using the exponential decay of f_0 and the fact that $\kappa \geq \kappa_0$ we see that $f_0(y(\tau))$ converges uniformly to $f_0(\underline{\mu} + \underline{\mu}^{-1}\kappa\tau)$ on $-\infty < \tau \leq 0$ as $\bar{\tau} \rightarrow \infty$. Furthermore from Lemmas 4.2 and 4.3 we have that $\varphi_{m_\ell}(\bar{\tau} + \tau)$ converges uniformly to $f_0(y(\tau))$ on $-\bar{\tau} \leq \tau \leq 0$ as $\kappa\bar{\tau} \rightarrow \infty$. Thus the lemma follows. $\blacksquare$

We now show that except for a discrete set of radii $\mathcal{S}$ has just one element.

Lemma 7.5 *For each $\kappa > 0$ there is a discrete set $\mathcal{A}_1 \equiv \{\bar{\tau}_k : k = 1, 2, \dots\}$ with $\lim_{k \rightarrow \infty} \bar{\tau}_k = \infty$ so that for any $\bar{\tau} \in \mathbb{R}^+ - \mathcal{A}_1$ and $h_0 \in \mathcal{H}(\bar{\tau}, \kappa)$, $\mathcal{S}(\bar{\tau}, \kappa, h_0)$ has one element, $\mathcal{S} = \{\varphi_{m_1}(r)e^{im_1\theta}\}$.*

Proof: Fix κ , $\bar{\tau}$, and $h_0 \in \mathcal{H}(\bar{\tau}, \kappa)$. The set $\mathcal{S}$ will have more than one element iff $(h_0 \kappa)^{\frac{1}{2}}\bar{\tau} = \beta_k$ for some k where the sequence $\{\beta_k\}$ was introduced at the beginning of §6. For each such k , h_0 is uniquely determined by the equation $h_0 = \kappa/\mu_e^2(\beta_k)$. Hence, $\bar{\tau}$ is uniquely determined by $\bar{\tau} = \beta_k(h_0 \kappa)^{-\frac{1}{2}} = \beta_k \mu_e(\beta_k)/\kappa \equiv \bar{\tau}_k$. $\blacksquare$

We now describe solutions to the linearized problem (1.7) for $\kappa\bar{\tau}$ small.

Theorem 7.6 *There is a constant $C < \infty$ such that if $\kappa\bar{\tau} < C$ then $\mathcal{H}(\kappa, \bar{\tau}) = \{\bar{h}\}$ and $\mathcal{S}(\bar{\tau}, \kappa, \bar{h}) = \{\varphi_0(r)\}$ where $\varphi_0 > 0$ in $[0, \bar{\tau}]$ and $\varphi'_0 \leq 0$.*

Proof: Set $\gamma = \inf_{x \geq \sqrt{2}} \mu_e(x) > 0$. Suppose $\kappa\bar{r} < 2^{\frac{1}{2}}\gamma \equiv C$. If $h_0 \in \mathcal{H}(\kappa, \bar{r})$ we claim that $h_0\kappa\bar{r}^2 < 2$. Indeed if not then $(h_0\kappa)^{\frac{1}{2}}\bar{r} \geq \sqrt{2}$. Hence $\mu_e((h_0\kappa)^{\frac{1}{2}}\bar{r}) \geq \gamma$. Since $h_0 \in \mathcal{H}(\bar{r}, \kappa)$ we have $\kappa\bar{r} = h_0\bar{r}\mu_e^2((h_0\kappa)^{\frac{1}{2}}\bar{r}) \geq \frac{2\gamma^2}{\kappa\bar{r}}$ which is a contradiction.

Thus setting $x_0 = (h_0\kappa)^{\frac{1}{2}}\bar{r}$ we have $x_0 < \sqrt{2}$. From §6 we see that $\mu_e(x_0) = \mu(x_0, 0)$. Thus $h_0 \in \mathcal{H}(\bar{r}, \kappa) = \{h : \kappa = h\mu((h\kappa)^{\frac{1}{2}}\bar{r}, 0)\}$. From Lemma 3.8 the function $h \rightarrow h\mu((h\kappa)^{\frac{1}{2}}, 0)$ is increasing. Thus $\mathcal{H}(\bar{r}, \kappa)$ consists of one element, $h = \bar{h}$. Moreover by construction $0 = \beta_0 < (\bar{h}\kappa)^{\frac{1}{2}}\bar{r} < \sqrt{2} < \beta_1$. Thus from Lemma 7.5, $\mathcal{S}(\bar{r}, \kappa, \bar{h})$ consists of one element. The structure of φ_0 follows from Lemma 3.3. ■

Remark 5 For $\kappa\bar{r}$ sufficiently small then we have shown that the linearized problem has no vortices.

8 Analysis of Nucleation

In this section we study the bifurcation of solutions in a neighborhood of a degenerately stable normal state. For each κ , and all $\bar{r}$ outside of a discrete set of radii, we show that there exists a unique branch of superconducting solutions nucleating away from each of the degenerately stable normal states. Moreover if κ and $\kappa\bar{r}$ are sufficiently large we prove that the solutions along this branch are stable. These results are summarized in Theorem 8.12. We shall need the following lemmas.

Lemma 8.1 Let $\varphi = \varphi(r, \omega, \lambda_m)$ satisfy

$$\varphi_{rr} + \frac{\varphi_r}{r} - \left(\frac{\omega^2 r}{2} - \frac{m}{r} \right)^2 \varphi = -\lambda_m \varphi \text{ for } r > 0 \quad (8.1)$$

and

$$\varphi(r, \omega, \lambda_m) = r^m + o(r^m) \text{ as } r \rightarrow 0 \quad (8.2)$$

where $m = 0, 1, 2, \dots$ and $\omega, \lambda_m \in \mathbf{R}$. Assume that

$$\varphi_r(\bar{r}, \omega^0, \lambda_m^0) = 0 \text{ for some } \bar{r} > 0. \quad (8.3)$$

Then the solutions of $\varphi_r(\bar{r}, \omega, \lambda_m) = 0$ for (ω, λ_m) near (ω^0, λ_m^0) are given as $(\omega, \lambda_m(\omega, \bar{r}))$ where

$$\frac{\partial \lambda_m}{\partial \omega} \cdot \int_0^{\bar{r}} \varphi^2 r \, dr = 2\omega \int_0^{\bar{r}} \left(\frac{\omega^2 r}{2} - \frac{m}{r} \right) \varphi^2 r^2 \, dr. \quad (8.4)$$

Proof: If we apply the Implicit Function Theorem to (8.3) we see that $\frac{\partial \lambda_m}{\partial \omega} \varphi_{r\lambda_m} = -\varphi_{r\omega}$. To compute $\varphi_{r\lambda_m}$ and $\varphi_{r\omega}$ at $(\bar{r}, \omega, \lambda_m(\omega, \bar{r}))$ we differentiate (8.1) with respect to λ_m and ω , integrate these equations by parts against $r\varphi$ and use the fact that φ is a solution to (8.1)-(8.3). We get

$$r \varphi_{r\lambda_m} \varphi \Big|_{(\bar{r}, \omega, \lambda_m(\omega, \bar{r}))} = - \int_0^{\bar{r}} \varphi^2 r \, dr$$

and

$$r \varphi_{r\omega} \varphi \Big|_{(\bar{r}, \omega, \lambda_m(\omega, \bar{r}))} = 2\omega \int_0^{\bar{r}} \left(\frac{\omega^2 r}{2} - \frac{m}{r} \right) \varphi^2 r^2 \, dr.$$

The lemma follows. ■

Recall that $\beta_0 = 0$ and $\{\beta_j : j = 1, 2, \dots\}$ is the set where $\mu_e(\bar{x})$ and at least two of the $\{\mu(\bar{x}, j)\}$ intersect.

Lemma 8.2 *Let $\mathcal{C} \equiv \{\bar{x} > 0 : \bar{x} \notin \{\beta_j\} \text{ and } \frac{d}{d\bar{x}}(\bar{x}\mu_e(\bar{x})) \neq 0\}$. Then $\mathcal{C}$ is a discrete set, $\mathcal{C} = \{c_k\}$. Moreover $\lim_{k \rightarrow \infty} c_k = \infty$ if $\mathcal{C}$ is infinite.*

Proof: Set $\mathcal{C}_j = \mathcal{C} \cap (\beta_j, \beta_{j+1})$ for $j = 0, 1, 2, \dots$. Now $\mu_e(\bar{x}) = \mu(\bar{x}, m_j)$ on (β_j, β_{j+1}) for some integer m_j and $m_0 = 0$. Also we have shown that $\bar{x}\mu(\bar{x}, m_j)$ is analytic and nonconstant. Thus $\mathcal{C}_0 = \emptyset$ and $\mathcal{C}_j$ is at most a finite set for each $j \geq 1$. ■

Lemma 8.3 *Fix $\kappa > 0$. There is a discrete set $\mathcal{B}(\kappa) = \{r_k : k = 1, 2, \dots\}$ with $\lim_{k \rightarrow \infty} r_k = \infty$ such that for any $\bar{r} \in \mathbf{R}^+ - \mathcal{B}(\kappa)$ and $h_0 \in \mathcal{H}(\bar{r}, \kappa)$ we have*

$$\left\{ \begin{array}{ll} \text{i)} & \mathcal{S}(\bar{r}, \kappa, h_0) = \{\varphi(r)e^{im\theta}\} \text{ for some } m, \\ \text{and} & \\ \text{ii)} & \int_0^{\bar{r}} \left(\frac{h_0 r}{2} - \frac{m}{\kappa r} \right) \varphi^2 r^2 \, dr \neq 0. \end{array} \right. \quad (8.5)$$

Proof: Consider $\bar{r} \in \mathbf{R}^+ - \mathcal{A}_1$ where $\mathcal{A}_1$ is as defined in Lemma 7.5. Then $\beta_j < (h_0 \kappa)^{\frac{1}{2}} \bar{r} < \beta_{j+1}$ for some j and $\mu_e((h_0 \kappa)^{\frac{1}{2}} \bar{r}) = \mu((h_0 \kappa)^{\frac{1}{2}} \bar{r}, m_j)$ for a uniquely determined integer m_j . Thus i) is satisfied.

Now $\lambda_m(\omega, \bar{r}) = \omega^2 \mu^2(\omega \bar{r}, m)$ for $\omega, \bar{r} > 0$. Thus from Lemmas 8.1 and 8.2 we see that ii) is equivalent to $\omega \bar{r} \notin \mathcal{C}_j$ where $\mathcal{C}_j$ is defined in Lemma 8.2 such that $m = m_j$. Let $\mathcal{C} = \{c_\ell\}$. The conditions $(h\kappa)^{\frac{1}{2}} \bar{r} = c_\ell$ and $\mu_e^2(c_\ell) = \frac{\kappa}{h}$ imply that $\frac{\kappa}{\mu_e^2(c_\ell)} = \frac{c_\ell^2}{\kappa \bar{r}^2}$ which uniquely determines $\bar{r} \equiv r_\ell$. Set $\mathcal{A}_2 = \{r_\ell\}$. Then for $\bar{r} \notin \mathcal{B}(\kappa) \equiv \mathcal{A}_1 \cup \mathcal{A}_2$ both i) and ii) hold. ■

Given $\kappa > 0$ and $\bar{r} \in \mathbf{R}^+ - \mathcal{B}(\kappa)$ we shall prove in this section that bifurcation occurs for each $h_0 \in \mathcal{H}(\bar{r}, \kappa)$. Thus there is a $\delta > 0$ and a smooth curve $(\psi(\varepsilon), \mathbf{A}(\varepsilon), h(\varepsilon))$ of solutions to (1.1) for $|\varepsilon| < \delta$ such that $(\psi(0), \mathbf{A}(0), h(0)) = (0, h_0 \mathbf{a}_N, h_0)$ and $\psi(\varepsilon) \not\equiv 0$ for $\varepsilon \neq 0$.

Let $\psi = u + iv$, $\mathbf{a} = (a_1, a_2)$ and $\mathbf{z} = (u, v, a_1, a_2)$. Define

$$\begin{aligned} \mathbf{Z} &= \{ \mathbf{z} : u, v \in C^{2,\alpha}(\bar{B}_{\bar{r}}), a_1, a_2 \in C^{3,\alpha}(\bar{B}_{\bar{r}}), \\ &\quad \operatorname{div} \mathbf{a} = 0 \text{ in } B_{\bar{r}}, \mathbf{a} \cdot \mathbf{n} = 0 \text{ on } \partial B_{\bar{r}}, \operatorname{curl} \mathbf{a} = 0 \text{ on } \partial B_{\bar{r}}, \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} v = 0 \text{ on } \partial B_{\bar{r}} \}, \\ \mathbf{Y} &= \{ \mathbf{y} = (y_1, y_2, y_3, y_4) : y_1, y_2 \in C^\alpha(\bar{B}_{\bar{r}}), y_3, y_4 \in C^{1,\alpha}(\bar{B}_{\bar{r}}), (y_3, y_4) \cdot \mathbf{n} = 0 \text{ on } \partial B_{\bar{r}} \} \\ &\quad \text{for some constant } 0 < \alpha < 1. \end{aligned}$$

The equations (1.1) in terms of $\mathbf{z}$ and h (with $\mathbf{z} \in \mathbf{Z}$ and $\mathbf{A} = h\mathbf{a}_N + \mathbf{a}$) are $0 = F_1 = F_2 = F_3 = F_4$, where

$$\begin{aligned} F_1 &= F_1(\mathbf{z}, h) = -\kappa^{-1} \operatorname{div}[\kappa^{-1} \nabla u + (\mathbf{a} + h\mathbf{a}_N)v] - \\ &\quad (\kappa^{-1} \nabla v - (\mathbf{a} + h\mathbf{a}_N)u) \cdot (\mathbf{a} + h\mathbf{a}_N) - (1 - u^2 - v^2)u, \\ F_2 &= F_2(\mathbf{z}, h) = -\kappa^{-1} \operatorname{div}[\kappa^{-1} \nabla v - (\mathbf{a} + h\mathbf{a}_N)u] \\ &\quad + (\kappa^{-1} \nabla u + (\mathbf{a} + h\mathbf{a}_N)v) \cdot (\mathbf{a} + h\mathbf{a}_N) - (1 - u^2 - v^2)v, \\ (F_3, F_4) &= (F_3(\mathbf{z}, h), F_4(\mathbf{z}, h)) = \operatorname{curl}^2 \mathbf{a} - \kappa^{-1}(u \nabla v - v \nabla u) + (u^2 + v^2)(\mathbf{a} + h\mathbf{a}_N). \end{aligned} \tag{8.6}$$

Let $\mathbf{F} = (F_1, \dots, F_4)$.

Lemma 8.4 *The mapping $\mathbf{F} = \mathbf{F}(\mathbf{z}, h)$ defined by (8.6) takes $\mathbf{Z} \times \mathbf{R}$ into $\mathbf{Y}$.*

Proof: Let $\mathbf{a}$ satisfy the conditions in the definition of $\mathbf{Z}$. Then $\mathbf{a} = (-\eta_{x_2}, \eta_{x_1})$ for some $\eta \in C_0^{4,\alpha}(\bar{B}_{\bar{r}})$. Moreover, $\operatorname{curl} \mathbf{a} = \Delta \eta \mathbf{e}_3 = 0$ on $\partial B_{\bar{r}}$. Now $\operatorname{curl}^2 \mathbf{a} = ((\Delta \eta)_{x_2}, -(\Delta \eta)_{x_1})$. Thus $\operatorname{curl}^2 \mathbf{a} \cdot \mathbf{n} = 0$ on $\partial B_{\bar{r}}$. It follows that $(F_3, F_4) \cdot \mathbf{n} = 0$ on $\partial B_{\bar{r}}$ and we have $\mathbf{F} : \mathbf{Z} \times \mathbf{R} \rightarrow \mathbf{Y}$. $\blacksquare$

Due to the gauge invariance of the equations we need the following projection. Let $u_0, v_0 \in C^\alpha(\bar{B}_{\bar{r}})$. Set

$$\mathbf{W} = \left\{ \mathbf{y} \in \mathbf{Y} : \int_{B_{\bar{r}}} (-y_1 v_0 + y_2 u_0) dx = 0, \operatorname{div}(y_3, y_4) = 0 \right\}.$$

Let $P : \mathbf{Y} \rightarrow \mathbf{W}$ be the L^2 projection operator onto $\mathbf{W}$ and set $\mathbf{g} = P\mathbf{F}$. Then $\mathbf{g} : \mathbf{Z} \times \mathbf{R} \rightarrow \mathbf{W}$ and we have the following relation.

Lemma 8.5 *Let $(\mathbf{z}(\varepsilon), h(\varepsilon))$ be a smooth curve in $\mathbf{Z} \times \mathbf{R}$ such that $\mathbf{z}(0) = 0$, $(z'_1(0), z'_2(0)) = (u_0, v_0) \not\equiv 0$ and $\mathbf{g}(\mathbf{z}(\varepsilon), h(\varepsilon)) \equiv 0$. Then $\mathbf{F}(\mathbf{z}(\varepsilon), h(\varepsilon)) \equiv 0$ for all ε sufficiently small.*

Proof: Let $\mathbf{F}(\varepsilon)$ denote $\mathbf{F}(\mathbf{z}(\varepsilon), h(\varepsilon))$. Since $\mathbf{g}(\mathbf{z}(\varepsilon), h(\varepsilon)) = 0$ we have $(F_1(\varepsilon), F_2(\varepsilon)) = \phi(\varepsilon)(-v_0, u_0)$ for some $\phi(\varepsilon) \in \mathbf{R}$. Now for any $\mathbf{z} \in \mathbf{Z}$ and $h \in \mathbf{R}$ using (8.6) and the boundary conditions for $\mathbf{z}$ it follows that $\int_{B_{\bar{r}}} (-vF_1 + uF_2) dx \equiv 0$ where $F_i = F_i(\mathbf{z}, h)$. Hence

$$0 = \phi(\varepsilon) \int_{B_{\bar{r}}} (v_0 v(\varepsilon) + u_0 u(\varepsilon)) / \varepsilon dx \text{ for } \varepsilon \neq 0.$$

Since

$$\int_{B_{\bar{r}}} (v_0 v(\varepsilon) + u_0 u(\varepsilon)) / \varepsilon dx \rightarrow \int_{B_{\bar{r}}} (v_0^2 + u_0^2) dx \neq 0$$

by our hypotheses as $\varepsilon \rightarrow 0$ we see that $\phi(\varepsilon) \equiv 0$ for all ε small enough. Thus, $F_1(\varepsilon) = F_2(\varepsilon) = 0$ for such ε . Next we note from (8.6) that

$$\operatorname{div}(F_3, F_4) = \kappa^{-1}(v\Delta u - u\Delta v) + 2(\mathbf{a} + h\mathbf{a}_N) \cdot (u\nabla u + v\nabla v) = \kappa(uF_2 - vF_1).$$

Hence $\mathbf{g} = P\mathbf{F} = \mathbf{F}$ for ε sufficiently small. ■

Set $\hat{\mathbf{Z}} = \{\mathbf{z} \in \mathbf{Z} : \int_{B_{\bar{r}}} (-v_0 u + u_0 v) dx = 0\}$. An important fact related to gauge invariance is that for each $(u, v, \mathbf{a}) \in \mathbf{Z}$ there is a fixed rotation $(\tilde{u}, \tilde{v})$ of (u, v) such that $(\tilde{u}, \tilde{v}, \mathbf{a}) \in \hat{\mathbf{Z}}$; this follows from the intermediate value theorem since if $(u, v, \mathbf{a}) \notin \hat{\mathbf{Z}}$, we have

$$\begin{aligned} I(s) &\equiv \int_{B_{\bar{r}}} [-(u \cos s - v \sin s)v_0 + (v \cos s + u \sin s)u_0] dx \\ &\equiv \int_{B_{\bar{r}}} [-\tilde{u}(x, s)v_0(x) + \tilde{v}(x, s)u_0(x)] dx \end{aligned}$$

satisfies $I(0) \neq 0$ and $I(\Pi) = -I(0)$ and thus for some s_0 in $(0, \Pi)$ we have $I(s_0) = 0$.

Theorem 8.6 *Given $\kappa > 0$, $\bar{r} \in \mathbf{R}^+ - \mathcal{B}(\kappa)$, and $h_0 \in \mathcal{H}(\bar{r}, \kappa)$ define (u_0, v_0) by $u_0 + iv_0 = \varphi(r)e^{im\theta}$ where $\mathcal{S}(\bar{r}, \kappa, h_0) = \{\varphi(r)e^{im\theta}\}$. Then there is a $\delta > 0$ and a C^∞ curve $(\mathbf{z}(\varepsilon), h(\varepsilon)) : (-\delta, \delta) \rightarrow \hat{\mathbf{Z}} \times \mathbf{R}$ such that*

- i) $\mathbf{F}(\mathbf{z}(\varepsilon), h(\varepsilon)) = 0$
- ii) $(\mathbf{z}(0), h(0)) = (0, h_0)$
- iii) $\mathbf{z}'(0) = (u_0, v_0, 0)$
- iv) $(\hat{\mathbf{Z}} \times \mathbf{R}) \cap \mathbf{F}^{-1}(\{0\})$ near $(0, h_0)$ consists precisely of the curves $\mathbf{z} = 0$ and $(\mathbf{z}(\varepsilon), h(\varepsilon))$ for $|\varepsilon| < \delta$.

In particular, $(\psi, \mathbf{A}, h) \equiv (0, \tau \mathbf{a}_N, \tau)$ (with τ arbitrary) and $(\psi, \mathbf{A}, h) \equiv (u(\varepsilon) + iv(\varepsilon), h(\varepsilon) \mathbf{a}_N + \mathbf{a}(\varepsilon), h(\varepsilon))$ for $|\varepsilon| < \delta$ satisfy problem (1.1) and there are no other solutions (up to a gauge transformation) in a sufficiently small neighborhood of $(0, h_0 \mathbf{a}_N, h_0)$ in $\mathcal{M} \times \mathbf{R}$.

Proof: To prove i) through (iv), we see from Lemma 8.5 that it suffices to prove i) through iv) with $\mathbf{F}$ replaced by $\hat{\mathbf{g}} = \mathbf{g} \mid_{\hat{\mathbf{Z}} \times \mathbf{R}}$. We will show that $\hat{\mathbf{g}} : \hat{\mathbf{Z}} \times \mathbf{R} \rightarrow \mathbf{W}$ satisfies

$$\begin{cases} 1) & \text{Ker } \hat{\mathbf{g}}_{\mathbf{z}}(0, h_0) = \text{span}\{(u_0, v_0, 0)\}, \\ 2) & \text{codim Rng } \hat{\mathbf{g}}_{\mathbf{z}}(0, h_0) = 1, \\ 3) & \hat{\mathbf{g}}_{\mathbf{z}h}(0, h_0)(u_0, v_0, 0) \notin \text{Rng } \hat{\mathbf{g}}_{\mathbf{z}}(0, h_0), \end{cases} \quad (8.7)$$

assuming $\bar{r} \in \mathbf{R}^+ - \mathcal{B}(\kappa)$ and $h_0 \in \mathcal{H}(\bar{r}, \kappa)$. Under these conditions, it follows from standard results of bifurcation theory at simple eigenvalues that the C^∞ curve $(\mathbf{z}(\varepsilon), h(\varepsilon))$ exists satisfying i)-iv) with $\mathbf{F}$ replaced by $\hat{\mathbf{g}}$. See [5, Theorem 1.7].

Step 1: Let $(u, v, \mathbf{a}) \in \hat{\mathbf{Z}}$. Then $\hat{\mathbf{g}}_{\mathbf{z}}(0, h_0)(u, v, \mathbf{a}) = P(\mathbf{F}_{\mathbf{z}}(0, h_0)(u, v, \mathbf{a}))$. From the definition of $\mathbf{F}$, this equals zero iff

$$[\mathcal{L} - \kappa^2](u, v) = c(-v_0, u_0) \text{ and } \text{curl}^2 \mathbf{a} = \nabla p$$

for some $c \in \mathbf{R}$ and some function p . Here we have identified $\mathcal{L}$ in equation (1.7) for $u + iv$ with the real system for (u, v) , that is, we have identified

$$\mathcal{L}(u + iv) = (i\nabla + h_0 \kappa \mathbf{a}_N)^2(u + iv)$$

with the real system $(\mathcal{L}_1(u, v), \mathcal{L}_2(u, v))$ where

$$\mathcal{L}_1(u, v) = \Re(i\nabla + h_0 \kappa \mathbf{a}_N)^2(u + iv)$$

and

$$\mathcal{L}_2(u, v) = \Im(i\nabla + h_0 \kappa \mathbf{a}_N)^2(u + iv).$$

Now if $\hat{\mathbf{g}}_{\mathbf{z}}(0, h_0)(u, v, \mathbf{a}) = 0$, then

$$0 = \int_{B_{\bar{r}}} (u, v) \cdot [\mathcal{L} - \kappa^2](-v_0, u_0) dx = \int_{B_{\bar{r}}} [\mathcal{L} - \kappa^2](u, v) \cdot (-v_0, u_0) dx = c \int_{B_{\bar{r}}} (u_0^2 + v_0^2) dx$$

and

$$0 = \int_{B_{\bar{r}}} \text{curl } \mathbf{a} \cdot \text{curl } \nabla p dx = \int_{B_{\bar{r}}} \text{curl}^2 \mathbf{a} \cdot \nabla p = \int_{B_{\bar{r}}} |\nabla p|^2 dx$$

and hence $c = 0$ and $\nabla p = 0$.

Thus, $\hat{\mathbf{g}}_{\mathbf{z}}(0, h_0)(u, v, \mathbf{a}) = 0$ iff $\mathbf{F}_{\mathbf{z}}(0, h_0)(u, v, \mathbf{a}) = 0$, since $\mathbf{F}_{\mathbf{z}}(0, h_0)(u, v, \mathbf{a}) = 0$ is equivalent to

$$[\mathcal{L} - \kappa^2](u, v) = 0 \text{ and } \text{curl}^2 \mathbf{a} = 0 \text{ in } B_{\bar{r}}. \quad (8.8)$$

By Theorem 7.3 and Lemma 7.5 the Neumann problem for $\text{Ker}[\mathcal{L} - \kappa^2]$ is spanned by one complex solution, $\varphi(r)e^{im\theta} \equiv u_0 + iv_0$. Thus, in terms of real solutions, we have $\text{Ker}[\mathcal{L} - \kappa^2] \equiv \text{span}\{(u_0, v_0), (-v_0, u_0)\}$. For $(u, v, \mathbf{a}) \in \mathbf{Z}$, the observations on $\mathbf{a}$ from Lemma 8.4 are such that $\mathbf{a} = (-\eta_{x_2}, \eta_{x_1})$ for some $\eta \in C_0^{4,\alpha}(\bar{B}_{\bar{r}})$ with $\Delta\eta = 0$ on $\partial B_{\bar{r}}$. Now $\text{curl}^2 \mathbf{a} = -\Delta \mathbf{a}$ since $\text{div} \mathbf{a} = 0$. From (8.8) then if $(u, v, \mathbf{a}) \in \text{Ker} \hat{\mathbf{g}}_{\mathbf{z}}(0, h)$, we have $\nabla(\Delta\eta) = 0$ in $B_{\bar{r}}$. Whence $\Delta\eta \equiv 0$ in $B_{\bar{r}}$. Since $\eta = 0$ on $\partial B_{\bar{r}}$ this implies $\eta \equiv 0$. Thus $\mathbf{a} = 0$. This proves that

$$\text{Ker} \hat{\mathbf{g}}_{\mathbf{z}}(0, h_0) = \text{span}\{(u_0, v_0, 0), (-v_0, u_0, 0)\} \cap \hat{\mathbf{Z}} = \text{span}\{(u_0, v_0, 0)\}.$$

Step 2: We now determine $\text{Rng} \hat{\mathbf{g}}_{\mathbf{z}}(0, h_0)$. Since $\mathcal{L}$ is self adjoint, $[\mathcal{L} - \kappa^2](u, v) = (y_1, y_2)$ is solvable for u, v as in the definition of $\mathbf{Z}$ iff

$$0 = \int_{B_{\bar{r}}} (y_1 u_0 + y_2 v_0) dx = \int_{B_{\bar{r}}} (-v_0 y_1 + u_0 y_2) dx.$$

Next we claim that $\text{curl}^2 \mathbf{a} = -\Delta \mathbf{a} = \mathbf{y}$ is solvable for $\mathbf{a}$ satisfying the conditions in the definition of $\mathbf{Z}$ for any $\mathbf{y} = (y_3, y_4)$ satisfying the conditions in the definition of $\mathbf{W}$. To see this write $\mathbf{y} = (-\eta_{x_2}, \eta_{x_1})$ for some $\eta \in C_0^{2,\alpha}(\bar{B}_{\bar{r}})$. Let $k \in C_0^{4,\alpha}(\bar{B}_{\bar{r}})$ such that $\Delta k = \eta$. Then let $\mathbf{a} = (k_{x_2}, -k_{x_1})$. Thus $\text{Rng} \hat{\mathbf{g}}_{\mathbf{z}}(0, h_0) = \{\mathbf{y} \in \mathbf{W} : \int_{B_{\bar{r}}} (y_1 u_0 + y_2 v_0) dx = 0\}$. In particular $\text{codim} \text{Rng} \hat{\mathbf{g}}_{\mathbf{z}}(0, h_0) = 1$.

Step 3: We consider $\hat{\mathbf{g}}_{\mathbf{zh}}(0, h_0)(u_0, v_0, 0)$. From (8.6) and the fact that $u_0 + iv_0 = \varphi(r)e^{im\theta}$, we have

$$\begin{aligned} \mathbf{F}_{\mathbf{zh}}(0, h_0)(u_0, v_0, 0) &= \hat{\mathbf{g}}_{\mathbf{zh}}(0, h_0)(u_0, v_0, 0) \\ &= \left(\frac{-2}{\kappa} \mathbf{a}_N \cdot \nabla v_0 + 2h_0 |\mathbf{a}_N|^2 u_0, \frac{2}{\kappa} \mathbf{a}_N \cdot \nabla u_0 + 2h_0 |\mathbf{a}_N|^2 v_0, 0 \right) \\ &= \left(\frac{-1}{\kappa} \frac{\partial v_0}{\partial \theta} + \frac{h_0 r^2}{2} u_0, \frac{1}{\kappa} \frac{\partial u_0}{\partial \theta} + \frac{h_0 r^2}{2} v_0, 0 \right) \\ &= \left(\frac{-m}{\kappa} + \frac{h_0 r^2}{2} \right) (u_0, v_0, 0). \end{aligned}$$

This is not in $\text{Rng } \hat{\mathbf{g}}_{\mathbf{z}}(0, h_0)$ iff

$$\int_{B_{\bar{r}}} \left(\frac{h_0}{2} r^2 - \frac{m}{\kappa} \right) (u_0^2 + v_0^2) dx = 2\pi \int_0^{\bar{r}} \left(\frac{h_0}{2} r^2 - \frac{m}{\kappa} \right) \varphi^2(r) r dr \neq 0.$$

Since $\bar{r} \notin \mathcal{B}(\kappa)$, we see from Lemma 8.3 that this is indeed the case.

Thus (8.7) is verified and the curve $(\mathbf{z}(\varepsilon), h(\varepsilon)) \subset \hat{\mathbf{Z}} \times \mathbf{R}$ satisfying i)-iv) exists. The rest of the theorem now follows, since solutions of $\mathbf{F}(\mathbf{z}, h) = 0$ in $\mathbf{Z}$ are gauge equivalent (by a rotation of (z_1, z_2)) to solutions in $\hat{\mathbf{Z}}$, and elliptic regularity ensures that all solutions $(\psi, \mathbf{A})$ of (1.1) in $\mathcal{M}$ (with $\psi = u + iv$, $\mathbf{A} = h\mathbf{a}_N + \mathbf{a}$) are necessarily smooth and satisfy $(u, v, \mathbf{a}) \in \mathbf{Z}$. $\blacksquare$

Remark 6 We assume without loss of generality from here on that $\mathbf{z}$ and h are parameterized such that $\mathbf{z}(\varepsilon) = \mathbf{z}'(0)\varepsilon + \tilde{\mathbf{z}}(\varepsilon)$ where $\int_{B_R} \tilde{\mathbf{z}}(\varepsilon) \cdot \mathbf{z}'(0) dx = 0$. Also note that if $\mathbf{F}(u, v, \mathbf{a}, h) = \mathbf{0}$ then $\mathbf{F}(-u, -v, \mathbf{a}, h) = \mathbf{0}$. Property iv) in Theorem 8.6 and this choice of parameterization then imply the following result:

Corollary 8.7 The function $\psi(\varepsilon)$ is odd and $\mathbf{a}(\varepsilon)$ and $h(\varepsilon)$ are even functions in ε for ε small. In particular

$$\begin{aligned} \psi(\varepsilon) &= \varepsilon\psi_0 + O(\varepsilon^3), \\ \mathbf{a}(\varepsilon) &= \varepsilon^2\mathbf{a}_0 + O(\varepsilon^4), \\ h(\varepsilon) &= h_0 + h_1\varepsilon^2 + O(\varepsilon^4) \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \tag{8.9}$$

Proof: Given ε , since $(-u(\varepsilon), -v(\varepsilon), \mathbf{a}(\varepsilon))$ is a solution with $h = h(\varepsilon)$, then from Theorem 8.6 iv) there is an s so that $(u(s), v(s), \mathbf{a}(s)) = (-u(\varepsilon), -v(\varepsilon), \mathbf{a}(\varepsilon))$ with $h(s) = h(\varepsilon)$. Now $\mathbf{z}'(0) = (u_0, v_0, 0)$. Thus

$$s \int_{B_{\bar{r}}} (u_0^2 + v_0^2) dx = \int_{B_{\bar{r}}} \mathbf{z}(s) \cdot \mathbf{z}'(0) dx = - \int_{B_{\bar{r}}} \mathbf{z}(\varepsilon) \cdot \mathbf{z}'(0) dx = -\varepsilon \int_{B_{\bar{r}}} (u_0^2 + v_0^2) dx.$$

Whence $s = -\varepsilon$ and the corollary is proved. $\blacksquare$

We now show that the bifurcating branch has the same structure as $\psi'(0)$.

Theorem 8.8 Given $\kappa > 0$ let $\bar{r} \in \mathbf{R}^+ - \mathcal{A}_2$, $h_0 \in \mathcal{H}(\bar{r}, \kappa)$ and $\varphi(r)e^{im\theta} \in \mathcal{S}(\bar{r}, \kappa, h_0)$. Then there is a C^∞ branch of solutions to (1.1), namely

$$\left\{ \begin{array}{l} (\psi(\varepsilon), A(\varepsilon), h(\varepsilon)) \in \mathcal{M} \times \mathbf{R} \text{ for } |\varepsilon| < \delta \text{ such that} \\ (\psi(0), \mathbf{A}(0), h(0)) = (0, h_0 \mathbf{a}_N, h_0), \\ \frac{d}{d\varepsilon}(\psi, \mathbf{A}, h)(0) = (\varphi(r)e^{im\theta}, 0, 0), \\ \text{and} \\ \psi(\varepsilon) = |\psi|(\varepsilon, r)e^{im\theta}, \quad \mathbf{A}(\varepsilon) = \left(a(\varepsilon, r) + \frac{h(\varepsilon)r}{2} \right) \Theta \\ \text{where } \Theta = (-x_2, x_1)/r. \end{array} \right. \quad (8.10)$$

Moreover, for each ε in $(-\delta, \delta)$ we have:

$$\left\{ \begin{array}{l} |\psi|(\varepsilon, r) > 0 \text{ for } 0 < r \leq \bar{r}, \\ \text{and} \\ |\psi|(\varepsilon, r) = \varepsilon[\beta(\varepsilon)r^m + o(r^m)] \text{ as } r \rightarrow 0 \\ \text{where } \beta(\varepsilon) \geq \beta_0 > 0. \end{array} \right. \quad (8.11)$$

Proof: We use an analysis similar to that used in Theorem 8.6. Set

$$\begin{aligned} \mathbf{Z}_m &= \{(u, v, \mathbf{a}) \in \mathbf{Z} : u + iv = \zeta(r)e^{im\theta}, \quad \mathbf{a} = \phi(r)\Theta \\ &\quad \text{for some real valued functions } \zeta(r) \text{ and } \phi(r)\}, \\ \mathbf{W}_m &= \{(w_1, w_2, w_3, w_4) \in W : w_1 + iw_2 = \zeta(r)e^{im\theta} \\ &\quad \text{and } (w_3, w_4) = \phi(r)\Theta \text{ for some real valued functions } \zeta(r) \text{ and } \phi(r)\}. \end{aligned}$$

Then $\mathbf{F}$ from (8.6) is such that

$$\mathbf{F} : (\mathbf{z}, h) \in \mathbf{Z}_m \times \mathbf{R} \rightarrow \mathbf{W}_m.$$

The argument used in Theorem 8.6 can be applied directly to $\mathbf{F}$ and (8.10) is proved. Note that $h'(0) = 0$ follows from Corollary 8.7.

For each ε the function $|\psi|(\varepsilon, r)$ is a smooth solution to the equation

$$|\psi|'' + \frac{|\psi|'}{r} + \kappa^2 \left[- \left(a + \frac{hr}{2} - \frac{m}{\kappa r} \right)^2 + 1 - |\psi|^2 \right] |\psi| = 0 \text{ for } 0 < r < \bar{r}.$$

From elliptic regularity theory $|\psi|(\cdot, r)$ is in fact a smooth mapping valued in C^k for any integer k . Thus, $\beta(\varepsilon) \equiv \frac{1}{\varepsilon} \frac{d^m}{dr^m} |\psi|(\varepsilon, 0)$ for $\varepsilon \neq 0$ is continuous with $\beta(0) = \lim_{\varepsilon \rightarrow 0} \beta(\varepsilon) = \frac{d}{d\varepsilon} \left(\frac{d^m}{dr^m} |\psi|(\varepsilon, 0) \right) \Big|_{\varepsilon=0} = \frac{d^m}{dr^m} \varphi(0) > 0$. It follows that $|\psi|(\varepsilon, r) = \varepsilon(\beta(\varepsilon)r^m + o(r^m))$ as $r \rightarrow 0$. Furthermore we have $\frac{d}{d\varepsilon} |\psi|(0, r) \equiv \varphi(r) > 0$ for $0 < r \leq \bar{r}$. It follows from these two facts that $|\psi|(\varepsilon, r) > 0$ for $0 < r \leq \bar{r}$ and ε small. Thus (8.11) is proved. $\blacksquare$

Remark 7 While Theorem 8.8 gives a detailed description of the bifurcating branch, Theorem 8.6 guarantees that it is the only nontrivial branch (up to a gauge transformation), regardless of symmetry consideration, near $(0, h_0 \mathbf{a}_N, h_0)$ in $\mathcal{M} \times \mathbb{R}$.

Given a solution $(\mathbf{z}(\varepsilon), h(\varepsilon))$ as in (8.9), its formal asymptotic expansion is derived in [3], [4] and [7]. From this and Corollary 8.7, writing $\psi(\varepsilon) = u(\varepsilon) + iv(\varepsilon)$, we have that ψ_0 satisfies (1.7), $(0, 0, \mathbf{a}_0) \in \mathbf{Z}$ with $\mathbf{a}_0 = a_0(r)\Theta$,

$$-\operatorname{curl}^2 \mathbf{a}_0 = \frac{i}{2\kappa}(\psi_0^* \nabla \psi_0 - \psi_0 \nabla \psi_0^*) + h_0 \mathbf{a}_N |\psi_0|^2 \text{ in } B_{\bar{r}} \quad (8.12)$$

and

$$h_1 = \left(\frac{1}{2} \int_{B_{\bar{r}}} |\psi_0|^4 dx - \int_{B_{\bar{r}}} |\operatorname{curl} \mathbf{a}_0|^2 dx \right) / \int_{B_{\bar{r}}} \mathbf{a}_N \cdot \operatorname{curl}^2 \mathbf{a}_0 dx. \quad (8.13)$$

With $\psi_0 = \varphi(r)e^{im\theta}$, (8.12) becomes

$$(r^{-1}(ra_0)')'\Theta = \left(\frac{h_0 r}{2} - \frac{m}{\kappa r} \right) \varphi^2(r)\Theta.$$

In this case, since $\operatorname{curl} \mathbf{a}_0 = 0$ on $\partial B_{\bar{r}}$ it follows that

$$\operatorname{curl} \mathbf{a}_0 = r^{-1}(ra_0)'\mathbf{e}_3 = - \int_r^{\bar{r}} \left(\frac{h_0 r}{2} - \frac{m}{\kappa r} \right) \varphi^2(r) dr \mathbf{e}_3$$

and (8.13) becomes

$$h_1 = \frac{-(\int_0^{\bar{r}} \varphi^4 r dr - 2 \int_0^{\bar{r}} (\int_r^{\bar{r}} (\frac{h_0 s}{2} - \frac{m}{\kappa s}) \varphi^2(s) ds)^2 r dr)}{\int_0^{\bar{r}} (\frac{h_0 r}{2} - \frac{m}{\kappa r}) \varphi^2 r^2 dr} \quad (8.14)$$

We now analyze the stability of the bifurcating branch. Let

$$\hat{\mathcal{M}} = \{(\rho, \eta, b_1, b_2) \in W^{1,2}(B_{\bar{r}}); \mathbf{b} = (b_1, b_2), \operatorname{div} \mathbf{b} = 0 \text{ in } B_{\bar{r}} \text{ and } \mathbf{b} \cdot \mathbf{n} = 0 \text{ on } \partial B_{\bar{r}}\}.$$

For $\mathbf{z} = (u, v, \mathbf{a}) \in \hat{\mathcal{M}}$, $\psi = u + iv$, and $\mathbf{A} = \mathbf{a} + h\mathbf{a}_N$, we have

$$\begin{aligned} G(\psi, \mathbf{A}) &= \int_{B_{\bar{r}}} \left(\frac{1}{2} (|\kappa^{-1} \nabla u + (\mathbf{a} + h\mathbf{a}_N)v|^2 + |\kappa^{-1} \nabla v - (\mathbf{a} + h\mathbf{a}_N)u|^2) \right. \\ &\quad \left. + \frac{1}{4} (1 - u^2 - v^2)^2 + \frac{1}{2} |\operatorname{curl} \mathbf{a}|^2 \right) dx \equiv \tilde{G}(\mathbf{z}, h). \end{aligned}$$

For $\mathbf{z}, \hat{\mathbf{z}} \in \hat{\mathcal{M}}$ set

$$D\tilde{G}(\mathbf{z}, h; \hat{\mathbf{z}}) = \frac{d}{d\tau} G(\mathbf{z} + \tau\hat{\mathbf{z}}, h) \big|_{\tau=0}.$$

For $\mathbf{z} \in \mathbf{Z}$ and $\hat{\mathbf{z}} \in \hat{\mathcal{M}}$ we have

$$D\tilde{G}(\mathbf{z}, h; \hat{\mathbf{z}}) = \int_{B_R} \mathbf{F}(\mathbf{z}, h) \cdot \hat{\mathbf{z}} \, dx. \quad (8.15)$$

For $\mathbf{z}, \hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2 \in \hat{\mathcal{M}}$ set

$$D^2\tilde{G}(\mathbf{z}, h; \hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2) = \frac{\partial^2}{\partial\tau_1\partial\tau_2} \tilde{G}(\mathbf{z} + \tau_1\hat{\mathbf{z}}_1 + \tau_2\hat{\mathbf{z}}_2, h) \big|_{\tau_1=\tau_2=0} = D^2\tilde{G}(\mathbf{z}, h; \hat{\mathbf{z}}_2, \hat{\mathbf{z}}_1). \quad (8.16)$$

From (8.15) for $\mathbf{z}, \hat{\mathbf{z}}_1 \in \mathbf{Z}$ and $\hat{\mathbf{z}}_2 \in \hat{\mathcal{M}}$ we see that

$$D^2\tilde{G}(\mathbf{z}, h; \hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2) = \int_{B_{\bar{r}}} \mathbf{F}_{\mathbf{z}}(\mathbf{z}, h) \hat{\mathbf{z}}_1 \cdot \hat{\mathbf{z}}_2 \, dx. \quad (8.17)$$

The stability of a solution $\mathbf{z}$ is analyzed by considering the second variation of $\tilde{G}(\mathbf{z}, h)$, namely $D^2\tilde{G}(\mathbf{z}, h; \hat{\mathbf{z}}, \hat{\mathbf{z}})$, for $\hat{\mathbf{z}} \in \hat{\mathcal{M}}$. The conditions, $\operatorname{div} \mathbf{A} = 0$ in $B_{\bar{r}}$, $\mathbf{A} \cdot \mathbf{n} = 0$ on $\partial B_{\bar{r}}$ distinguish $\mathbf{z}$ up to a constant; that is, if $\mathbf{z} = (u, v, \mathbf{a})$, $\psi = u + iv$ and $\mathbf{q}(\tau) \equiv (u(\tau), v(\tau), \mathbf{a})$ with $\psi(\tau) = e^{i\tau}\psi$ then, $\mathbf{q}(\tau)$ is also a solution, $0 = \mathbf{F}(\mathbf{q}(\tau), h)$ for all τ . In particular,

$$0 = \frac{d}{d\tau} \mathbf{F}(\mathbf{q}(\tau), h) \big|_{\tau=0} = \mathbf{F}_{\mathbf{z}}(\mathbf{z}, h) \mathbf{q}_{\tau}(0) = \mathbf{F}_{\mathbf{z}}(\mathbf{z}, h) (-v, u, 0), \quad (8.18)$$

and it follows from (8.17) that $D^2\tilde{G}(\mathbf{z}, h; \mathbf{q}_{\tau}(0), \hat{\mathbf{z}}) = 0$ for all $\hat{\mathbf{z}} \in \hat{\mathcal{M}}$. Set

$$\mathcal{M}_{\perp}(\mathbf{z}) \equiv \left\{ (\rho, \eta, \mathbf{b}) \in \hat{\mathcal{M}} : \int_{B_{\bar{r}}} (-v\rho + u\eta) \, dx = 0 \right\}.$$

By considering $D^2\tilde{G}(\mathbf{z}, h; \hat{\mathbf{z}}, \hat{\mathbf{z}})$ only for $\hat{\mathbf{z}} \in \mathcal{M}_{\perp}(\mathbf{z})$ we measure the stability of G in directions perpendicular to the subset of $\mathbf{Z}$ that is gauge invariant to $\mathbf{z}$. In fact for any $\hat{\mathbf{z}} \in \hat{\mathcal{M}}$, if we write it as $\hat{\mathbf{z}} = \hat{\mathbf{z}}_{\perp} + c\mathbf{q}_{\tau}(0)$ where $\hat{\mathbf{z}}_{\perp} \in \mathcal{M}_{\perp}$ and $\mathbf{q}_{\tau}(0) = (-v, u, 0)$, then from (8.16), (8.17) and (8.18) we see that

$$D^2\tilde{G}(\mathbf{z}, h; \hat{\mathbf{z}}, \hat{\mathbf{z}}) = D^2\tilde{G}(\mathbf{z}, h; \hat{\mathbf{z}}_{\perp}, \hat{\mathbf{z}}_{\perp}).$$

Theorem 8.9 Consider the curve $(\mathbf{z}(\varepsilon), h(\varepsilon))$ from Theorem 8.6. There is a $\delta_0 > 0$ and continuous functions $\sigma(\varepsilon) : (-\delta_0, \delta_0) \rightarrow \mathbf{R}$ and $\mathbf{w}(\varepsilon) : (-\delta_0, \delta_0) \rightarrow \hat{\mathcal{M}}$ with $\mathbf{w}(\varepsilon) \in \mathcal{M}_\perp(\mathbf{z}(\varepsilon))$ and $\mathbf{w}(0) = \mathbf{z}'(0)/\|\mathbf{z}(0)\|_{L^2}$ for which

$$\inf_{\substack{\hat{\mathbf{z}} \in \mathcal{M}_\perp(\mathbf{z}(\varepsilon)) \\ \|\hat{\mathbf{z}}\|_{L^2} = 1}} D^2\tilde{G}(\mathbf{z}(\varepsilon), h(\varepsilon); \hat{\mathbf{z}}, \hat{\mathbf{z}}) = D^2\tilde{G}(\mathbf{z}(\varepsilon), h(\varepsilon); \mathbf{w}(\varepsilon), \mathbf{w}(\varepsilon)) = \sigma(\varepsilon).$$

Proof: Set $\mathbf{z}(\varepsilon) = (u(\varepsilon), v(\varepsilon), \mathbf{a}(\varepsilon))$, $\hat{\mathbf{z}} = (\rho, \eta, \mathbf{b})$, $D^2\tilde{G}(\mathbf{z}(\varepsilon), h(\varepsilon), \hat{\mathbf{z}}, \hat{\mathbf{z}}) \equiv D^2\tilde{G}(\varepsilon; \hat{\mathbf{z}}, \hat{\mathbf{z}})$ and

$$\begin{aligned} \mathcal{M}(\varepsilon) &\equiv \mathcal{M}_\perp(\mathbf{z}(\varepsilon)) \text{ for } \varepsilon \neq 0, \\ \mathcal{M}(0) &\equiv \left\{ \hat{\mathbf{z}} \in \hat{\mathcal{M}} : \int_{B_{\bar{r}}} (-v'(0)\rho + u'(0)\eta) dx = 0 \right\}. \end{aligned}$$

Note that the normal direction varies continuously at $\varepsilon = 0$, since $(-\frac{v(\varepsilon)}{\varepsilon}, \frac{u(\varepsilon)}{\varepsilon}) \rightarrow (-v'(0), u'(0))$ as $\varepsilon \rightarrow 0$. Now

$$\begin{aligned} D^2\tilde{G}(\varepsilon; \hat{\mathbf{z}}, \hat{\mathbf{z}}) &= \int_{B_{\bar{r}}} \left\{ \kappa^{-2}(|\nabla\rho|^2 + |\nabla\eta|^2) - (1 - u^2 - v^2)(\rho^2 + \eta^2) \right. \\ &\quad + 2(u\rho + v\eta)^2 + |\mathbf{a} + h(\varepsilon)\mathbf{a}_N|^2(\rho^2 + \eta^2) - 2\kappa^{-1}(\rho\nabla\eta \cdot (\mathbf{a} + h(\varepsilon)\mathbf{a}_N) - \eta\nabla\rho \cdot \mathbf{b}) \\ &\quad \left. + |\operatorname{curl} \mathbf{b}|^2 + (u^2 + v^2)|\mathbf{b}|^2 + 4(\mathbf{a} + h(\varepsilon)\mathbf{a}_N) \cdot \mathbf{b}(u\rho + v\eta) - \kappa^{-1}4(u\nabla\eta \cdot \mathbf{b} - v\nabla\rho \cdot \mathbf{b}) \right\} dx. \end{aligned}$$

The assumptions on $\mathbf{b}$ imply that $\mathbf{b} = (-k_{x_2}, k_{x_1})$ for some $k \in W_0^{1,2}(B_{\bar{r}}) \cap W^{2,2}(B_{\bar{r}})$ and $\operatorname{curl} \mathbf{b} = \Delta k \mathbf{e}_3$. Hence $\|D\mathbf{b}\|_{L^2(B_{\bar{r}})} \leq \|D^2k\|_{L^2(B_{\bar{r}})} \leq C\|\operatorname{curl} \mathbf{b}\|_{L^2(B_{\bar{r}})}$. Standard elliptic theory then implies that there exists $\hat{\mathbf{w}} \in \mathcal{M}(\varepsilon)$ with $\|\hat{\mathbf{w}}\|_{L^2} = 1$ such that

$$D^2\tilde{G}(\varepsilon; \hat{\mathbf{w}}, \hat{\mathbf{w}}) = \inf_{\substack{\hat{\mathbf{z}} \in \mathcal{M}(\varepsilon) \\ \|\hat{\mathbf{z}}\|_{L^2} = 1}} D^2\tilde{G}(\varepsilon; \hat{\mathbf{z}}, \hat{\mathbf{z}}) \equiv \sigma(\varepsilon). \quad (8.19)$$

Moreover, $\hat{\mathbf{w}}$ satisfies

$$D^2\tilde{G}(\varepsilon; \hat{\mathbf{w}}, \hat{\mathbf{z}}) = \sigma(\varepsilon) \int_{B_{\bar{r}}} \hat{\mathbf{w}} \cdot \hat{\mathbf{z}} dx \text{ for all } \hat{\mathbf{z}} \in \hat{\mathcal{M}}. \quad (8.20)$$

Furthermore if $\lim_{n \rightarrow \infty} \varepsilon_n = \varepsilon_0$ and $\{\hat{\mathbf{w}}(\varepsilon_n)\}$ satisfy (8.19) then for a subsequence $\lim_{k \rightarrow \infty} \hat{\mathbf{w}}(\varepsilon_{n_k}) = \hat{\mathbf{w}}$ where $\hat{\mathbf{w}} \in \mathcal{M}(\varepsilon_0)$ and satisfies (8.19) for $\varepsilon = \varepsilon_0$. In particular at $\varepsilon_0 = 0$, $\hat{\mathbf{w}} \in \operatorname{span}\{\mathbf{z}'(0)\}$ and $\sigma(0) = 0$. This implies that $\dim\{\hat{\mathbf{w}} \in \mathcal{M}(\varepsilon) : \hat{\mathbf{w}} \text{ solves (8.20)}\} = 1$ for all ε sufficiently small. It follows that there is a $\delta_0 > 0$ such that for $|\varepsilon| < \delta_0$ we can define $\mathbf{w}(\varepsilon)$ as the unique element in $\mathcal{M}(\varepsilon)$ with $\|\mathbf{w}(\varepsilon)\|_{L^2} = 1$ satisfying (8.19) and such that $\int_{B_{\bar{r}}} \mathbf{w}(\varepsilon) \cdot \mathbf{z}'(0) dx > 0$. By uniqueness and compactness $\mathbf{w}(\varepsilon)$ is continuous. $\blacksquare$

Theorem 8.10 Let $\mathbf{z}'(0) = (u_0, v_0, 0)$ with $u_0 + iv_0 = \varphi(r)e^{im\theta}$ and $h_0 \in \mathcal{H}(\bar{r}, \kappa)$. Then

$$\lim_{\varepsilon \rightarrow 0} \sigma(\varepsilon)/\varepsilon^2 = \left\{ 2 \int_0^{\bar{r}} \varphi^4 r \, dr - 4 \int_0^{\bar{r}} \left(\int_r^{\bar{r}} \left(\frac{h_0 s}{2} - \frac{m}{\kappa s} \right) \varphi^2 ds \right)^2 r \, dr \right\} / \int_0^{\bar{r}} \varphi^2 r \, dr \equiv \sigma_0.$$

Proof: We differentiate $\mathbf{F}(\mathbf{z}(\varepsilon), h(\varepsilon)) = 0$ to get

$$\mathbf{F}_{\mathbf{z}}(\mathbf{z}(\varepsilon), h(\varepsilon))\mathbf{z}'(\varepsilon) + \mathbf{F}_h(\mathbf{z}(\varepsilon), h(\varepsilon)) \cdot h'(\varepsilon) = 0.$$

Consider $\mathbf{w}(\varepsilon)$ from the previous theorem. We see using (8.16), (8.17) and (8.20) that

$$\sigma(\varepsilon) \int_{B_{\bar{r}}} \mathbf{w}(\varepsilon) \cdot \mathbf{z}'(\varepsilon) dx = \int_{B_{\bar{r}}} \mathbf{w}(\varepsilon) \cdot \mathbf{F}_{\mathbf{z}}(\mathbf{z}(\varepsilon), h(\varepsilon))\mathbf{z}'(\varepsilon) dx = -h'(\varepsilon) \int_{B_{\bar{r}}} \mathbf{w}(\varepsilon) \cdot \mathbf{F}_h(\mathbf{z}(\varepsilon), h(\varepsilon)) dx. \quad (8.21)$$

Now since $\mathbf{F}(0, h) \equiv 0$ we see using (8.9) that $\mathbf{F}_h(\mathbf{z}(\varepsilon), h(\varepsilon)) = \varepsilon \mathbf{F}_{\mathbf{zh}}(0, h_0)\mathbf{z}'(0) + O(\varepsilon^2)$ as $\varepsilon \rightarrow 0$ in $\mathbf{Z}$. Furthermore

$$h'(\varepsilon) = 2h_1\varepsilon + O(\varepsilon^3) \text{ as } \varepsilon \rightarrow 0$$

and

$$\mathbf{w}(\varepsilon) \rightarrow \mathbf{z}'(0)/\|\mathbf{z}'(0)\|_{L^2} \text{ in } L^2(B_{\bar{r}}).$$

Thus from (8.21) we see

$$\lim_{\varepsilon \rightarrow 0} \sigma(\varepsilon)/\varepsilon^2 = -2h_1 \frac{\int_{B_{\bar{r}}} \mathbf{z}'(0) \cdot \mathbf{F}_{\mathbf{zh}}(0, h_0)\mathbf{z}'(0) dx}{\|\mathbf{z}'(0)\|_{L^2}^2}.$$

From the proof of Theorem 8.6 we have

$$\int_{B_{\bar{r}}} \mathbf{z}'(0) \cdot \mathbf{F}_{\mathbf{zh}}(0, h_0)\mathbf{z}'(0) dx = 2\pi \int_0^{\bar{r}} \left(\frac{h_0 r}{2} - \frac{m}{\kappa r} \right) \varphi^2 r^2 dr.$$

The theorem follows from this and (8.14). ■

Theorem 8.10 ensures that $\sigma(\varepsilon) = \sigma_0\varepsilon^2 + o(\varepsilon^2)$ as $\varepsilon \rightarrow 0$. Hence for $\varepsilon \neq 0$ and small the stability of $\mathbf{z}(\varepsilon)$ is determined by the sign of σ_0 . If $\sigma_0 < 0$ then $\mathbf{z}(\varepsilon)$ is unstable and if $\sigma_0 > 0$, $\mathbf{z}(\varepsilon)$ is strictly stable. In the following theorem we show that $\sigma_0 > 0$ for κ and $\kappa\bar{r}$ sufficiently large.

Theorem 8.11 *There are constants c_1, c_2 so that if $\kappa > c_1$, $\kappa\bar{r} > c_2$ and $h_0 \in \mathcal{H}(\bar{r}, \kappa)$ then $\sigma_0 > 0$.*

Proof: We see from Theorem 8.10 that $\sigma_0 > 0$ iff

$$\int_0^{\bar{r}} \varphi^4 r \, dr > 2 \int_0^{\bar{r}} \left(\int_r^{\bar{r}} \left(\frac{h_0 s}{2} - \frac{m}{\kappa s} \right) \varphi^2 ds \right)^2 r \, dr. \quad (8.22)$$

Assume $\varphi(\bar{r}) = 1$. Let $y = y_m(r) = (h_0 \kappa)^{\frac{1}{2}} r - \sqrt{2m}$ and set $\varphi_m(y) = \varphi(r(y)) = \varphi((y + \sqrt{2m})/(h_0 \kappa)^{\frac{1}{2}})$.

Step 1: From (7.8) and (7.9) we get $c^{-1}m^{\frac{1}{2}} \leq \kappa\bar{r} \leq cm^{\frac{1}{2}}$ and

$$|y_m(\bar{r}) - \underline{\mu}| \leq c(\kappa\bar{r})^{-1}. \quad (8.23)$$

Thus we can apply the estimates from Lemmas 4.1 and 4.2 to φ_m uniformly in m . We get

$$\int_0^{\bar{r}} \varphi^4 r \, dr = (h_0 \kappa)^{-1} \int_{-\sqrt{2m}}^{y_m(\bar{r})} \varphi_m^4(y) (y + \sqrt{2m}) dy \geq cm^{\frac{1}{2}} (h_0 \kappa)^{-1} \geq \bar{c} \frac{\bar{r}}{\kappa}$$

for some $\bar{c}(c_2) > 0$ provided c_2 is sufficiently large.

Step 2: To estimate the right side of (8.22) we consider

$$\int_r^{\bar{r}} \left(\frac{h_0 s}{2} - \frac{m}{\kappa s} \right) \varphi^2 ds = \sqrt{\frac{h_0}{\kappa}} \int_r^{\bar{r}} \left(\frac{(h_0 \kappa)^{\frac{1}{2}} s}{2} - \frac{m}{(h_0 \kappa)^{\frac{1}{2}} s} \right) \varphi^2 ds.$$

This equals

$$\kappa^{-1} \int_{y_m(r)}^{y_m(\bar{r})} \left(\frac{y + \sqrt{2m}}{2} - \frac{m}{y + \sqrt{2m}} \right) \varphi_m^2 dy = \kappa^{-1} \int_{y_m(r)}^{y_m(\bar{r})} y \left(1 - \frac{1}{2} \left(\frac{y}{y + \sqrt{2m}} \right) \right) \varphi_m^2 dy.$$

Let $f_0(y)$ solve (5.9). We write this as

$$\kappa^{-1} \left(\int_{y_m(r)}^{y_m(\bar{r})} \left(y f_0^2 + y(\varphi_m + f_0)(\varphi_m - f_0) + \frac{1}{2} \left(\frac{-y^2}{y + \sqrt{2m}} \right) \varphi_m^2 \right) dy \right) = \kappa^{-1} (I + II + III).$$

Step 3: From the proof of Lemma 5.5 it follows that $\int_{-\infty}^{\underline{\mu}} y f_0^2 dy = 0$. Thus

$$|I| = \left| \int_{y_m(\bar{r})}^{\underline{\mu}} y f_0^2 dy + \int_{-\infty}^{y_m(r)} y f_0^2 dy \right|.$$

Using (8.23) and (5.10) we have

$$|I| \leq c((\kappa\bar{r})^{-1} + e^{-y_m^2(r)/8}).$$

From (7.9) we have

$$y_m(r) = (h_0\kappa)^{\frac{1}{2}}r - (h_0\kappa)^{\frac{1}{2}}\bar{r} + \underline{\mu} + O((\kappa\bar{r})^{-1}).$$

Thus by (7.1) there are constants $a, b > 0$ depending on c_2 so that $y_m(r)^2 > a(\kappa(r - \bar{r}))^2 - b$. Hence there is a constant $\bar{a} > 0$, independent of κ , such that

$$|I| \leq c((\kappa\bar{r})^{-1} + e^{-\bar{a}(\kappa(r-\bar{r}))^2}). \quad (8.24)$$

Step 4: Applying Hölder's inequality to II we obtain

$$|II| \leq \left(\int_{-\infty}^{y_m(\bar{r})} y^2(f_0^2 + \varphi_m^2) dy \right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{y_m(\bar{r})} (f_0 - \varphi_m)^2 dy \right)^{\frac{1}{2}}.$$

From Lemma 4.2 and (4.6) we see that the first term on the right is uniformly bounded. Next using the same argument as in Lemma 5.3 we see that the second term is bounded by $cm^{-\frac{1}{2}}$. Thus,

$$|II| \leq c(\kappa\bar{r})^{-1}. \quad (8.25)$$

Applying Lemma 4.2 and (8.23) to III gives

$$|III| \leq c(\kappa\bar{r})^{-1}. \quad (8.26)$$

Step 5: From Step 2, (8.24), (8.25) and (8.26) we have

$$d(r) \equiv \left| \int_r^{\bar{r}} \left(\frac{h_0 s}{2} - \frac{m}{\kappa s} \right) \varphi^2 ds \right| \leq c\kappa^{-1}((\kappa\bar{r})^{-1} + e^{-\bar{a}(\kappa(r-\bar{r}))^2}).$$

Thus,

$$\int_0^{\bar{r}} d^2(r)r dr \leq \bar{c}_1\kappa^{-4} + \bar{c}_2\kappa^{-2}\bar{r} \int_0^{\bar{r}} e^{-2\bar{a}(\kappa(r-\bar{r}))^2} dr \leq \bar{c}_1\kappa^{-4} + \bar{c}_3\kappa^{-3}\bar{r}.$$

Step 6: From Steps 1 and 5 we see that if $\kappa\bar{r} \geq c_2$ for c_2 sufficiently large there are positive constants b_1, b_2 , and b_3 for which

$$\sigma_0 \geq b_1\kappa^{-1}\bar{r} - b_2\kappa^{-4} - b_3\kappa^{-3}\bar{r} = \kappa\bar{r}(b_1\kappa^{-2} - b_3\kappa^{-4}) - b_2\kappa^{-4}.$$

Thus,

$$\sigma_0 \geq c_2(b_1\kappa^{-2} - b_3\kappa^{-4}) - b_2\kappa^{-4} > 0$$

for $\kappa \geq c_1$ sufficiently large. ■

The following theorem summarizes our results for κ and $\kappa\bar{r}$ sufficiently large.

Theorem 8.12 *For κ and $\kappa\bar{\tau}$ sufficiently large there is a constant $C > 0$ so that*

- i) *There is precisely a finite set $\mathcal{H}(\bar{\tau}, \kappa) = \{\underline{h} \leq \dots \leq \bar{h}\}$ contained in the interval $(\underline{\mu}^{-2}\kappa - \underline{\mu}^{-3}\ell\bar{\tau}^{-1} - C\kappa^{-1}\bar{\tau}^{-2}, \underline{\mu}^{-2}\kappa - \underline{\mu}^{-3}\ell\bar{\tau}^{-1} + C\kappa^{-1}\bar{\tau}^{-2})$ such that the normal state $\psi = 0$, $\mathbf{A} = h_0\mathbf{a}_N$ is degenerately stable iff $h_0 \in \mathcal{H}(\bar{\tau}, \kappa)$.*
- ii) *If the applied field $h\mathbf{e}_3$ satisfies $h < \underline{h} \equiv \underline{h}(\bar{\tau}, \kappa)$ then the normal state is unstable.*
- iii) *If $h > \bar{h} \equiv \bar{h}(\bar{\tau}, \kappa)$, the normal state is strictly stable.*
- iv) *There is a discrete set $\mathcal{B}(\kappa)$ such that if $\bar{\tau} \in \mathbf{R}^+ - \mathcal{B}(\kappa)$ then for each h_0 in $\mathcal{H}(\bar{\tau}, \kappa)$ there is a nontrivial branch $(\psi(\varepsilon), \mathbf{A}(\varepsilon), h(\varepsilon))$ satisfying Theorems 8.6 and 8.8 with $(\psi(0), \mathbf{A}(0), h(0)) = (0, h_0\mathbf{a}_N, h_0)$ for which $(\psi(\varepsilon), \mathbf{A}(\varepsilon))$ is strictly stable for $\varepsilon \neq 0$. The family $\{(\psi(\varepsilon), \mathbf{A}(\varepsilon), h(\varepsilon))\}$ are the only non-normal solutions of (1.1) (up to a gauge transformation) in a C^2 neighborhood of $(0, h_0\mathbf{a}_N, h_0)$. Moreover if $h_0 = \underline{h}$ or $h_0 = \bar{h}$ then $\frac{dh}{d\varepsilon}(0) = 0$ and $\frac{d^2h}{d\varepsilon^2}(0) < 0$ so that $h(\varepsilon) < h_0$ for $0 \neq \varepsilon$ sufficiently small.*

Proof: Except for the last assertion in iv) these results follow directly from Theorems 7.1, 7.2, 8.6, 8.8, 8.10, and 8.11.

Consider then $h(\varepsilon)$ such that $h(0) = \underline{h}$ or $\bar{h}$. From Corollary 8.7 we see that $\frac{dh}{d\varepsilon}(0) = 0$ and that the sign of $\frac{d^2h}{d\varepsilon^2}(0)$ is determined by the quotient (8.14). From Theorem 8.11 the numerator is negative provided κ and $\kappa\bar{\tau}$ are sufficiently large. From Lemmas 8.1 and 8.3 the denominator has the same sign as

$$\frac{d}{d\omega}(\omega^2\mu^2(\omega\bar{\tau}, m)) \text{ at } \omega = (h(0)\kappa)^{\frac{1}{2}} \quad (8.27)$$

Moreover this term is nonzero since $\bar{\tau} \notin \mathcal{B}(\kappa)$. From the proof of ii) given above in Theorem 7.2, it follows that $\omega^2\mu^2(\omega\bar{\tau}, m)$ is increasing just to the left of $(\underline{h}\kappa)^{\frac{1}{2}}$ and just to the right of $(\bar{h}\kappa)^{\frac{1}{2}}$. Thus the derivative in (8.27) is positive and hence $\frac{d^2h}{d\varepsilon^2}(0) < 0$. ■

Remark 8 *It follows that if $\bar{\tau} \in \mathbf{R}^+ - \mathcal{B}(\kappa)$ and $\bar{h} = \bar{h}(\bar{\tau}, \kappa)$ with κ and $\kappa\bar{\tau}$ sufficiently large, then in a neighborhood of the normal state $(0, \bar{h}\mathbf{a}_N)$ all solutions to (1.1) (up to a gauge transformation) lie on two curves:*

i) *A curve of normal solutions*

$$\tau \in (-\delta, \delta) \rightarrow (0, (\bar{h} + \tau)\mathbf{a}_N, \bar{h} + \tau)$$

that are unstable for $-\delta < \tau < 0$ and strictly stable for $0 < \tau < \delta$.

ii) *A curve of nucleating superconducting solutions*

$$\varepsilon \in (-\delta, \delta) \rightarrow (\psi(\varepsilon), \mathbf{A}(\varepsilon), h(\varepsilon))$$

such that $(\psi(0), A(0), h(0)) = (0, \bar{h}\mathbf{a}_N, \bar{h})$ satisfying Theorem 8.12 which are strictly stable for $\varepsilon \neq 0$ and such that $h(\varepsilon) < \bar{h}$ for $\varepsilon \neq 0$.

This proves that stable nucleation is brought on by decreasing h through $\bar{h}$ as is observed in experiment [9].

References

- [1] Bolley, C.: *Modélisation de champ de retard à la condensation d'un supraconducteur par un problème de bifurcation*. M²AN 26:2, 235–287 (1992).
- [2] Bolley, C., and Helffer, B.: *An application of semi-classical analysis to the asymptotic study of the supercooling field of a superconducting material*. Am. Inst. Henri Poincaré, 58:2 (1993) 189–233.
- [3] Chapman, S. J.: *Nucleation of superconductivity in decreasing fields, I*. Euro. Jnl. of Applied Mathematics 5 (1994), 449–468.
- [4] Chapman, S. J.: *Nucleation of superconductivity in decreasing fields, I*. Euro. Jnl. of Applied Mathematics 5 (1994), 469–494.
- [5] Crandall, M. G., and Rabinowitz, P. H.: *Bifurcation for simple eigenvalues*. J. Func. Analysis 8 (1971), 321–340.
- [6] Dauge, M. and Helffer, B.: *Eigenvalues variation, I, Neumann problem for Sturm-Liouville operators*. J. Differential Equations 104 (1993), 243–262.
- [7] Millman, M. H., and Keller, J. B.: *Perturbation theory of nonlinear boundary-value problems*. J. Math. Phys. 10 (1969) 342.

- [8] Odeh, F.: *Existence and bifurcation theorems for the Ginzburg-Landau equations*. J. Math Phys. 8:12 (1967), 2351–2357.
- [9] Saint-James, D., de Gennes, P. G.: *Onset of superconductivity in decreasing fields*. Phys. Letters 7:5 (1963), 306–308.

Recent IMA Preprints

#	Author/s	Title
1325	M. Luskin,	Approximation of a laminated microstructure for a rotationally invariant, double well energy density
1326	Rakesh & P. Sacks,	Impedance inversion from transmission data for the wave equation
1327	O. Lafitte,	Diffraction for a Neumann boundary condition
1328	E. Sobel, K. Lange, J.R. O'Connell & D.E. Weeks,	Haplotyping algorithms
1329	B. Cockburn, D.A. Jones & E.S. Titi,	Estimating the number of asymptotic degrees of freedom for nonlinear dissipative systems
1330	T. Aktosun,	Inverse Schrödinger scattering on the line with partial knowledge of the potential
1331	T. Aktosun & C. van der Mee,	Partition of the potential of the one-dimensional Schrödinger equation
1332	B. Engquist & E. Luo,	Convergence of the multigrid method with a wavelet coarse grid operator
1333	V. Jakšić & C.-A. Pillet,	Ergodic properties of the Spin-Boson system
1334	S.K. Patch,	Recursive solution for diffuse tomographic systems of arbitrary size
1335	J.C. Bronski,	Semiclassical eigenvalue distribution of the non self-adjoint Zakharov-Shabat eigenvalue problem
1336	J.C. Cockburn,	Bitangential structured interpolation theory
1337	S. Kichenassamy,	The blow-up problem for exponential nonlinearities
1338	F.A. Grünbaum & S.K. Patch,	How many parameters can one solve for in diffuse tomography?
1339	R. Lipton,	Reciprocal relations, bounds and size effects for composites with highly conducting interface
1340	H.A. Levine & J. Serrin,	A global nonexistence theorem for quasilinear evolution equations with dissipation
1341	A. Boutet de Monvel & R. Purice,	The conjugate operator method: Application to DIRAC operators and to stratified media
1342	G. Michele Graf,	Stability of matter through an electrostatic inequality
1343	G. Avalos,	Sharp regularity estimates for solutions of the wave equation and their traces with prescribed Neumann data
1344	G. Avalos,	The exponential stability of a coupled hyperbolic/parabolic system arising in structural acoustics
1345	G. Avalos & I. Lasiecka,	A differential Riccati equation for the active control of a problem in structural acoustics
1346	G. Avalos,	Well-posedness for a coupled hyperbolic/parabolic system seen in structural acoustics
1347	G. Avalos & I. Lasiecka,	The strong stability of a semigroup arising from a coupled hyperbolic/parabolic system
1348	A.V. Fursikov,	Certain optimal control problems for Navier-Stokes system with distributed control function
1349	F. Gesztesy, R. Nowell & W. Pötz,	One-dimensional scattering theory for quantum systems with nontrivial spatial asymptotics
1350	F. Gesztesy & H. Holden,	On trace formulas for Schrödinger-type operators
1351	X. Chen,	Global asymptotic limit of solutions of the Cahn-Hilliard equation
1352	X. Chen,	Lorenz equations, Part I: Existence and nonexistence of homoclinic orbits
1353	X. Chen,	Lorenz equations Part II: "Randomly" rotated homoclinic orbits and chaotic trajectories
1354	X. Chen,	Lorenz equations, Part III: Existence of hyperbolic sets
1355	R. Abeyaratne, C. Chu & R.D. James,	Kinetics of materials with wiggly energies: Theory and application to the evolution of twinning microstructures in a Cu-Al-Ni shape memory alloy
1356	C. Liu,	The Helmholtz equation on Lipschitz domains
1357	G. Avalos & I. Lasiecka,	Exponential stability of a thermoelastic system without mechanical dissipation
1358	R. Lipton,	Heat conduction in fine scale mixtures with interfacial contact resistance
1359	V. Odisharia & J. Peradze,	Solvability of a nonlinear problem of Kirchhoff shell
1360	P.J. Olver, G. Sapiro & A. Tannenbaum,	Affine invariant edge maps and active contours
1361	R.D. James,	Hysteresis in phase transformations
1362	A. Sei & W. Symes,	A note on consistency and adjointness for numerical schemes
1363	A. Friedman & B. Hu,	Head-media interaction in magnetic recording
1364	A. Friedman & J.J.L. Velázquez,	Time-dependent coating flows in a strip, part I: The linearized problem
1365	X. Ren & M. Winter,	Young measures in a nonlocal phase transition problem
1366	K. Bhattacharya & R.V. Kohn,	Elastic energy minimization and the recoverable strains of polycrystalline shape-memory materials
1367	G.A. Chechkin,	Operator pencil and homogenization in the problem of vibration of fluid in a vessel with a fine net on the surface
1368	M. Carme Calderer & B. Mukherjee,	On Poiseuille flow of liquid crystals
1369	M.A. Pinsky & M.E. Taylor,	Pointwise Fourier inversion: A wave equation approach
1370	D. Brandon & R.C. Rogers,	Order parameter models of elastic bars and precursor oscillations
1371	H.A. Levine & B.D. Sleeman,	A system of reaction diffusion equations arising in the theory of reinforced random walks
1372	B. Cockburn & P.-A. Gremaud,	A priori error estimates for numerical methods for scalar conservation laws. Part II: Flux-splitting monotone schemes on irregular Cartesian grids

- 1373 B. Li & M. Luskin, Finite element analysis of microstructure for the cubic to tetragonal transformation
- 1374 M. Luskin, On the computation of crystalline microstructure
- 1375 J.P. Matos, On gradient young measures supported on a point and a well
- 1376 M. Nitsche, Scaling properties of vortex ring formation at a circular tube opening
- 1377 J.L. Bona & Y.A. Li, Decay and analyticity of solitary waves
- 1378 V. Isakov, On uniqueness in a lateral Cauchy problem with multiple characteristics
- 1379 M.A. Kouritzin, Averaging for fundamental solutions of parabolic equations
- 1380 T. Aktosun, M. Klaus & C. van der Mee, Integral equation methods for the inverse problem with discontinuous wavespeed
- 1381 P. Morin & R.D. Spies, Convergent spectral approximations for the thermomechanical processes in shape memory alloys
- 1382 D.N. Arnold & X. Liu, Interior estimates for a low order finite element method for the Reissner-Mindlin plate model
- 1383 D.N. Arnold & R.S. Falk, Analysis of a linear-linear finite element for the Reissner-Mindlin plate model
- 1384 D.N. Arnold, R.S. Falk & R. Winther, Preconditioning in $H(\text{div})$ and applications
- 1385 M. Lavrentiev, Nonlinear parabolic problems possessing solutions with unbounded gradients
- 1386 O.P. Bruno & P. Laurence, Existence of three-dimensional toroidal MHD equilibria with nonconstant pressure
- 1387 O.P. Bruno, F. Reitich, & P.H. Leo, The overall elastic energy of polycrystalline martensitic solids
- 1388 M. Fila & H.A. Levine, On critical exponents for a semilinear parabolic system coupled in an equation and a boundary condition
- 1389 J.M. Berg, W.G. Frazier, A. Chaudhary, & S.S. Banda, Optimal open-loop ram velocity profiles for isothermal forging: A variational approach
- 1390 J.M. Berg & H.G. Kwatny, Unfolding the zero structure of a linear control system
- 1391 A. Sei, High order finite-difference approximations of the wave equation with absorbing boundary conditions: A stability analysis
- 1392 A.V. Coward & Y.Y. Renardy, Small amplitude oscillatory forcing on two-layer plane channel flow
- 1393 V.A. Pliss & G.R. Sell, Approximation dynamics and the stability of invariant sets
- 1394 J.G. Cao & P. Roblin, A new computational model for heterojunction resonant tunneling diode
- 1395 C. Liu, Inverse obstacle problem: Local uniqueness for rougher obstacles and the identification of a ball
- 1396 K.A. Pericak-Spector & S.J. Spector, Dynamic cavitation with shocks in nonlinear elasticity
- 1397 G. Avalos & I. Lasiecka, Exponential stability of a thermoelastic system without mechanical dissipation II: The case of simply supported boundary conditions
- 1398 B. Brighi & M. Chipot, Approximation of infima in the calculus of variations
- 1399 G. Avalos, Concerning the well-posedness of a nonlinearly coupled semilinear wave and beam-like equation
- 1400 R. Lipton, Variational methods, bounds and size effects for composites with highly conducting interface
- 1401 B.T. Hayes & P.G. LeFloch, Non-classical shock waves in scalar conservation laws
- 1402 K.T. Joseph & P.G. LeFloch, Boundary layers in weak solutions to hyperbolic conservation laws
- 1403 Y. Diao, C. Ernst, & E.J.J. Van Rensburg, Energies of knots
- 1404 Xiaofeng Ren, Multi-layer local minimum solutions of the bistable equation in an infinite tube
- 1405 Vlastimil Pták, Krylov sequences and orthogonal polynomials
- 1406 T. Aktosun, M. Klaus, & C. van der Mee, Factorization of scattering matrices due to apportioning of potentials in one-dimensional Schrödinger-type equations
- 1407 C.-S. Man & R. Paroni, On the separation of stress-induced and texture-induced birefringence in acoustoelasticity
- 1408 D.N. Arnold, R.S. Falk, & R. Winther, Preconditioning discrete approximations of the Reissner-Mindlin plate model
- 1409 M.A. Kouritzin, On exact filters for continuous signals with discrete observations
- 1410 R. Lipton, The second Stekloff eigenvalue and energy dissipation inequalities for functionals with surface energy
- 1411 R. Lipton, The second Stekloff eigenvalue of an inclusion and new size effects for composites with imperfect interface
- 1412 W. Littman & B. Liu, The regularity and singularity of solutions of certain elliptic problems on polygonal domains
- 1413 C.R. Collins, Spurious oscillations are not fatal in computing microstructures
- 1414 M.A. Horn, Sharp trace regularity for the solutions of the equations of dynamic elasticity
- 1415 A. Friedman, B. Hu & Y. Liu, A boundary value problem for the Poisson equation with multi-scale oscillating boundary
- 1416 P. Baumann, D. Phillips & Q. Tang, Stable nucleation for the Ginzburg-Landau system with an applied magnetic field
- 1417 J.M. Berg, A strain profile for robust control of microstructure using dynamic recrystallization
- 1418 P. Klouček, Toward the computational modeling of nonequilibrium thermodynamics of the Martensitic transformations
- 1419 S. Chawla & S.M. Lenhart, Application of optimal control theory to in Situ bioremediation
- 1420 B. Li & M. Luskin, Nonconforming finite element approximation of crystalline microstructure
- 1421 H. Kang & J.K. Seo, Inverse conductivity problem with one measurement: Uniqueness of balls in $\mathbb{R}^n$