**DYNAMIC CAVITATION WITH SHOCKS
IN NONLINEAR ELASTICITY**

By

K.A. Pericak-Spector

and

Scott J. Spector

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**INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA**

**514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455**

DYNAMIC CAVITATION WITH SHOCKS IN NONLINEAR ELASTICITY

K. A. PERICAK-SPECTOR & SCOTT J. SPECTOR
DEPARTMENT OF MATHEMATICS
SOUTHERN ILLINOIS UNIVERSITY
CARBONDALE, IL 62901, USA

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ABSTRACT. The hyperbolic system of conservation laws that governs the motion of a homogeneous, isotropic, nonlinearly elastic body is shown to have a discontinuous solution for a class of stored energy functions of slow growth. This solution is admissible, by the usual entropy criterion, and in fact preferred, by the entropy rate criterion, over the smooth equilibrium solution to the same problem. The existence of such a dissipative solution shows that the equilibrium solution is dynamically unstable. This instability cannot be ascertained by linearization.

1. Introduction.

We let $\Omega \subset \mathbb{R}^n, n \geq 2$, and consider the problem: Find $\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ that satisfies

$$\begin{aligned} \operatorname{div} \mathbf{S}(\nabla \mathbf{u}) &= \mathbf{u}_{tt} \text{ in } \Omega \times [0, T], \\ \mathbf{u}(\mathbf{x}, t) &= \lambda \mathbf{x} \text{ for } \mathbf{x} \in \partial\Omega \text{ and } t \in [0, T], \\ \mathbf{u}(\mathbf{x}, 0) &= \lambda \mathbf{x} \text{ for } \mathbf{x} \in \Omega, \\ \mathbf{u}_t(\mathbf{x}, 0) &= \mathbf{0} \text{ for } \mathbf{x} \in \Omega, \end{aligned} \tag{1.1}$$

for certain homogeneous, isotropic constitutive functions $\mathbf{S}$ that grow sufficiently slowly at infinity so that weak solutions of the partial differential equations need not be continuous.

We prove that, for all λ sufficiently large, the equilibrium solution $\mathbf{u}(\mathbf{x}, t) \equiv \lambda \mathbf{x}$ is not dynamically stable due to the existence of a dissipative similarity solution

$$\mathbf{u}(\mathbf{x}, t) := \frac{\varphi(s)}{s}(\mathbf{x} - \mathbf{x}_0), \quad s := \frac{|\mathbf{x} - \mathbf{x}_0|}{t}, \tag{1.2}$$

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where φ is smooth except for a single shock (a jump in $\dot{\varphi}$) and satisfies $\varphi(0) > 0$. Thus, in particular, this solution is discontinuous at $\mathbf{x} = \mathbf{x}_0$, opening a hole in the region Ω .

These results generalize our previous work [PS 88]. There we showed that if

$$\mathbf{S}(\mathbf{F}) = \mathbf{F} + (\det \mathbf{F})h'(\det \mathbf{F})\mathbf{F}^{-T} \quad (1.3)$$

then there were certain special values of λ where (1.1) had solutions that form such cavities. The current work extends the work in [PS 88] in three respects. First, the class of constitutive relations that we herein consider is much more general than (1.3). Second, instead of showing that there are certain special values of the loading parameter λ at which solutions of the form (1.2) exist, we show that such solutions exist for all λ sufficiently large. Finally, we show in this paper that the energy is strictly decreasing, rather than merely nonincreasing, along solutions of the form (1.2). Such solutions are therefore preferred over the equilibrium solution according to the entropy-rate criterion (see Dafermos [Da 73]) as well as being admissible by the usual entropy criterion (see, e.g., Lax [La 57, La 76]).

A related interesting problem that we do not address is the admissibility of our solutions according to more general criteria such as the viscosity criterion (see, e.g., [Da 83]) or the viscosity-capillarity criterion (see, e.g., [Sl 81] or [Sh 86]).

The physical motivation for the construction of such solutions is the observation in experiments on certain elastomers (see, e.g., [GL 58]) that an initially “void free” sample will exhibit “new” holes after the material has undergone a loading cycle. One explanation for the appearance of such cavities is that microscopic preexisting holes have grown in size until they have become visible. However, since precursor holes have yet to be observed in such materials (see [Ge 90, p. G52]) and since our intuition of the underlying physics does not rule out modeling this phenomenon as the spontaneous creation of new voids (see [MS 95, p. 62]), we believe that the construction of singular solutions to the equations of elastodynamics is of interest.

The fundamental work which viewed cavitation as the creation of new holes was done by Ball. In [Ba 82] he considered the equilibrium problem corresponding to (1.1) and showed, for a certain class of constitutive relations of slow growth, that the equilibrium solution $\mathbf{u}(\mathbf{x}, t) := \lambda \mathbf{x}$ is not the global minimizer of the stored energy for large λ . He also showed, using the direct method of the calculus of variations, that among radial deformations a global minimizer exists and exhibits cavitation.

Following Ball’s work there have been a number of results on cavitation in elastostatics (see [HP 95] and the references therein). However, as far as we know, the only other

results on *dynamic* hole formation in *nonlinearly elastic solids* (there is an extensive literature on fluids) are contained in [Ba 82], [CH 89], and [Ch 95]. In [Ba 82] Ball presented a thorough analysis of the dynamic stability of the equilibrium state in isotropic, incompressible, elastic materials of slow growth. In particular he showed that for sufficiently large dead loads the equilibrium state is dynamically unstable due to the existence of a dynamic cavitory solution with the same energy. Moreover, Ball [Ba 82, p. 582] noted that “*This instability cannot be detected by formal linearization of the n -dimensional equations about the trivial solution.*” (This is also true for the instability in our problem.) In [CH 89] Chou-Wang and Horgan considered the dynamic traction problem for an incompressible, elastic ball composed of a neo-Hookean material. In their analysis a sudden dead-load traction was applied at time zero. They showed that the critical load beyond which cavitation occurs in statics is also the critical load for dynamic cavitation. No such result has been obtained for compressible materials, although it is clear that the critical value of λ beyond which dynamic cavitation occurs in compressible materials is greater than or equal to the critical value of λ at which cavitation occurs in statics and that holes will not form in compression (see section 5). In [Ch 95] Choski considered the solutions constructed in [PS 88] for a one-parameter family of almost incompressible materials and showed that the solutions converge to the isochoric solution as the energy approaches that of an incompressible material. Dynamic cavitory solutions have not yet been constructed for solids that change phase, however, for elastic fluids this possibility has been considered by Yan [Ya 95]. (See [St 93] for static solutions that might involve a change of phase.)

Finally, we wish to point out that, although we believe that our results are new from the viewpoint of both nonlinear elastodynamics and nonlinear hyperbolic systems, the approach we have used in solving the second-order, nonlinear, ordinary differential equation, which results when (1.2) is substituted into (1.1)₁, is due to Stuart ([St 85] and [St 93]). Stuart considered the radial equilibrium equation, which had been derived by Ball, and used a clever shooting argument, involving the radial component of the Cauchy stress, to deduce the existence of solutions that exhibit cavitation for a larger class of materials than was considered in [Ba 82]. Our analysis is essentially the same shooting method applied to the slightly modified ordinary differential equation that results when dynamics are introduced.

2. Preliminaries.

In the following, Ω will denote a nonempty, bounded, open subset of $\mathbb{R}^n$, $n \geq 2$. By $L^p(\Omega)$ and $W^{1,p}(\Omega)$ we denote the usual spaces of p -summable and Sobolev functions, respectively. We use the notation $L^p(\Omega; \mathbb{R}^n)$, etc., for vector-valued maps.

We write Lin for the set of all linear maps from $\mathbb{R}^n$ into $\mathbb{R}^n$ and Lin^+ for those linear maps that have positive determinant. We write ∇ and div for the gradient and divergence operators in $\mathbb{R}^n$: for a vector field $\mathbf{u}$, $\nabla \mathbf{u}$ is the tensor field with components $(\nabla \mathbf{u})_{ij} = \partial u_i / \partial x_j$; for a tensor field $\mathbf{S}$, $\text{div } \mathbf{S}$ is the vector field with components $\sum_j \partial S_{ij} / \partial x_j$. Given any function $\Psi(\mathbf{a}, \mathbf{b}, \dots, \mathbf{c})$ with vector or tensor arguments, we write, e.g., $\partial \Psi / \partial \mathbf{a}$ for the partial Fréchet derivative with respect to $\mathbf{a}$ holding the remaining arguments fixed.

We consider a body that, for convenience, we identify with the region Ω that it occupies in a fixed reference configuration. A **deformation** $\mathbf{f}$ of the body is a member of the space

$$\text{Def} := \{\mathbf{f} \in W^{1,p}(\Omega; \mathbb{R}^n) : \mathbf{f} \text{ is one-to-one a.e. with } \det \nabla \mathbf{f} > 0 \text{ a.e.}\}.$$

Given $T > 0$ a function $\mathbf{u} : [0, T] \rightarrow \text{Def}$ is called a **motion** provided that

$$\mathbf{u} \in C^0([0, T]; \text{Def}) \cap C^1([0, T]; L^p(\Omega; \mathbb{R}^n)).$$

We now suppose that $\Omega = \mathcal{B}$ is the unit ball in $\mathbb{R}^n$ and consider a function $\mathbf{f} : \mathcal{B} \setminus \{0\} \rightarrow \mathbb{R}^n$ that is **radial**, i.e.,

$$\mathbf{f}(\mathbf{x}) := \frac{r(R)}{R} \mathbf{x}, \quad R := |\mathbf{x}|, \quad (2.1)$$

where $r : [0, 1) \rightarrow [0, \infty)$.

Proposition 2.1 (Ball [Ba 82]). *Let $\mathbf{f}$ be defined by (2.1). Then $\mathbf{f} \in \text{Def}$ if and only if r is absolutely continuous on $(0, 1)$, $r'(r/R)^{n-1} > 0$ almost everywhere, and*

$$\int_0^1 \left[|r'(R)|^p + \left| \frac{r(R)}{R} \right|^p \right] R^{n-1} dR < +\infty.$$

In this case the weak derivatives of $\mathbf{f}$ are given by

$$\nabla \mathbf{f}(\mathbf{x}) = r'(R) \left(\frac{\mathbf{x}}{R} \otimes \frac{\mathbf{x}}{R} \right) + \frac{r(R)}{R} \left(\text{Id} - \frac{\mathbf{x}}{R} \otimes \frac{\mathbf{x}}{R} \right), \quad (2.2)$$

where $\text{Id} \in \text{Lin}$ is the identity mapping.

We will consider motions that, in addition to being radial, only depend on R/t . With this in mind, let $\lambda > 0$ and suppose that $\varphi : [0, \infty) \rightarrow [0, \infty)$ satisfies $\varphi(s) = \lambda s$ for all $s \geq \sigma$. Define $\mathbf{u} : \mathcal{B} \setminus \{0\} \times [0, \infty) \rightarrow \mathbb{R}^n$ by

$$\mathbf{u}(\mathbf{x}, t) := \begin{cases} \frac{\varphi(s)}{s} \mathbf{x}, & t > 0 \\ \lambda \mathbf{x}, & t = 0 \end{cases}, \quad s := |\mathbf{x}|/t. \quad (2.3)$$

Proposition 2.2 ([PS 88, p. 297]). *Let $\mathbf{u}$ be defined by (2.3) and suppose that $p < n$. If φ is continuous and piecewise smooth and $\dot{\varphi} > 0$ a.e. then $\mathbf{u}$ is a motion.*

We assume that the body is hyperelastic with smooth response function $W : \Omega \times \text{Lin}^+ \rightarrow \mathbb{R}$. W gives the stored energy $W(\mathbf{x}, \nabla \mathbf{f}(\mathbf{x}))$ at $\mathbf{x} \in \Omega$ when the body is deformed by a smooth deformation $\mathbf{f}$. The **Piola-Kirchhoff stress** $\mathbf{S} : \Omega \times \text{Lin}^+ \rightarrow \text{Lin}$ is given by

$$\mathbf{S}(\mathbf{x}, \mathbf{F}) := \frac{\partial}{\partial \mathbf{F}} W(\mathbf{x}, \mathbf{F}),$$

while the **Cauchy Stress** is defined by

$$\mathbf{T}(\mathbf{x}, \mathbf{F}) := \mathbf{S}(\mathbf{x}, \mathbf{F}) \mathbf{F}^T / \det \mathbf{F}.$$

We suppose that the body in its reference configuration is homogeneous and isotropic; thus there is a symmetric function $\Phi : (0, \infty)^n \rightarrow \mathbb{R}$ that satisfies

$$W(\mathbf{x}, \mathbf{F}) = \Phi(\lambda_1, \lambda_2, \dots, \lambda_n)$$

for all $\mathbf{F} \in \text{Lin}^+$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of $(\mathbf{F} \mathbf{F}^T)^{1/2}$. In addition the density of the body in the reference configuration will be a constant that, for convenience, we assume is one.

We further assume that

$$\Phi \in C^3((0, \infty); \mathbb{R}). \quad (\text{H1})$$

By (2.2), the eigenvalues of $(\mathbf{F} \mathbf{F}^T)^{1/2}$ are given by

$$\lambda_1 = r'(R), \quad \lambda_2 = \lambda_3 = \dots = \lambda_n = \frac{r(R)}{R} \quad (2.4)$$

and therefore to simplify notation we define $\hat{\Phi}, \hat{\Phi}_i, \hat{\Phi}_{ij}, \hat{\Phi}_{ijk} : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \hat{\Phi}(\alpha, \lambda) &:= \Phi(\alpha, \lambda, \lambda, \dots, \lambda), & \hat{\Phi}_i(\alpha, \lambda) &:= \Phi_{,i}(\alpha, \lambda, \lambda, \dots, \lambda), \\ \hat{\Phi}_{ij}(\alpha, \lambda) &:= \Phi_{,ij}(\alpha, \lambda, \lambda, \dots, \lambda), & \hat{\Phi}_{ijk}(\alpha, \lambda) &:= \Phi_{,ijk}(\alpha, \lambda, \lambda, \dots, \lambda). \end{aligned}$$

For motions $\mathbf{u}$ of the form (2.3) it now follows from (2.2), (2.4), the definition of the Piola-Kirchhoff stress, and the isotropy of the material that (see, e.g., [Ba 82, p. 568])

$$\mathbf{S}(\nabla \mathbf{u}(\mathbf{x}, t)) = \Phi_1(s) \left(\frac{\mathbf{x}}{R} \otimes \frac{\mathbf{x}}{R} \right) + \Phi_2(s) \left(\mathbf{Id} - \frac{\mathbf{x}}{R} \otimes \frac{\mathbf{x}}{R} \right),$$

where $\Phi_i(s) = \hat{\Phi}_i(\dot{\varphi}(s), \varphi(s)/s)$ and $s := |\mathbf{x}|/t$.

By a *weak solution* of (1.1) we mean a motion $\mathbf{u}$ that satisfies (1.1)_{3,4} for a.e. $\mathbf{x} \in \Omega$, (1.1)₂ (in the sense of trace) for every $t \in [0, T]$, $\mathbf{S}(\nabla \mathbf{u}(\cdot, t)) \in L^1(\Omega, \text{Lin})$ for every $t \in [0, T]$, and

$$\int_0^T \int_{\Omega} [\nabla \mathbf{v} \cdot \mathbf{S}(\nabla \mathbf{u}) - \mathbf{v}_t \cdot \mathbf{u}_t] d\mathbf{x} dt = 0 \quad (2.5)$$

for every $\mathbf{v} \in C_0^\infty(\Omega \times (0, T), \mathbb{R}^n)$.

If we now substitute (2.5) into (1.1)₁ we find that the system of partial differential equations reduces to a single second-order ordinary differential equation in the variable s .

Proposition 2.3 ([PS 88, p. 302]). *Let $\varphi \in C^0((0, \infty); (0, \infty))$ be C^2 on each of the intervals $(0, \sigma)$ and (σ, ∞) for some $\sigma > 0$. Suppose that φ satisfies the following:*

- (i) $\varphi(s) = \lambda s$ for all $s \geq \sigma$;
- (ii) $\dot{\varphi}(s) > 0$ for $s \in (0, \sigma)$;
- (iii) $\frac{d}{ds}[s^{n-1}\hat{\Phi}_1(s)] = s^{n+1}\ddot{\varphi}(s) + (n-1)s^{n-2}\hat{\Phi}_2(s)$ for $s \in (0, \sigma)$;
- (iv) $\hat{\Phi}_1(\lambda, \lambda) - \hat{\Phi}_1(\alpha, \lambda) = \sigma^2(\lambda - \alpha)$;
- (v) $T(0) := \lim_{s \rightarrow 0^+} \left(\left[\frac{s}{\varphi(s)} \right]^{n-1} \hat{\Phi}_1(\dot{\varphi}(s), \frac{\varphi(s)}{s}) \right) = 0$;
- (vi) $s^{n-1}\hat{\Phi}_2(\dot{\varphi}(s), \frac{\varphi(s)}{s}) \in L^1((0, \sigma))$;
- (vii) $\varphi(0) := \lim_{s \rightarrow 0^+} \varphi(s) > 0$.

Then $\mathbf{u}$, given by (2.3), is a weak solution of (1.1) in $\mathcal{B} \times [0, \sigma^{-1}]$. Here

$$\alpha := \lim_{s \rightarrow \sigma^-} \dot{\varphi}(s), \quad \Phi_i(s) := \hat{\Phi}_i(\dot{\varphi}(s), \frac{\varphi(s)}{s}).$$

Remarks. 1. Condition (vii) necessitates the creation of a new cavity in the material. The quantity $\varphi(0)$ is the (constant) velocity of the surface of the cavity.

2. Condition (iv) is the Rankine-Hugoniot jump condition.

3. Condition (v) is the requirement that the surface of the newly created cavity be traction free. In order to allow for a cavity with contents (see, e.g., [GT 69]) or surface energy we will replace this condition with the assumption that there is a given function $g : \mathbb{R} \rightarrow \mathbb{R}$ which satisfies

$$T(0) = g(\varphi(0)).$$

Thus the normal force on the surface of the cavity will depend on the velocity of the surface. However, in this paper we will neither examine the mechanics that would lead to such a condition nor the energetics of the resulting solution.

4. Condition (vi) ensures that the stress is integrable which is necessary for the motion to be a weak solution.

3. Existence for the Ordinary Differential Equation.

In this section we will construct functions φ that satisfy (i)-(iv) and (vii) of Proposition 2.3. In particular, we will show that the radial similarity equation that was derived in the last section has solutions on $(0, \sigma]$ that can be joined by a shock at $s = \sigma$ to a constant solution. We will not yet concern ourselves with the boundary condition at the surface of the cavity ($s = 0$). We thus consider the ordinary differential equation

$$\frac{d}{ds}[s^{n-1}\Phi_1(s)] = s^{n+1}\ddot{\varphi}(s) + (n-1)s^{n-2}\Phi_2(s) \quad (3.1)$$

or, equivalently,

$$\begin{aligned} \ddot{\varphi}(s)(\Phi_{11}(s) - s^2) &= -\left(\frac{n-1}{s}\right) \left[\Phi_{12}(s)(\dot{\varphi}(s) - \frac{\varphi(s)}{s}) + \Phi_1(s) - \Phi_2(s) \right] \\ &= \left(\frac{n-1}{s}\right) \left(\frac{\varphi(s)}{s} - \dot{\varphi}(s) \right) P(\varphi(s), \frac{\dot{\varphi}(s)}{s}) \end{aligned} \quad (\text{ODE})$$

where

$$P(q, r) := \hat{\Phi}_{12}(q, r) + \frac{\hat{\Phi}_1(q, r) - \hat{\Phi}_2(q, r)}{q - r}. \quad (3.2)$$

We suppose that there is a constant $\lambda^* > 0$ such that

$$\hat{\Phi}_{11}(q, r) > 0, \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H2})$$

For the remainder of this section we fix $\lambda \geq \lambda^*$, $\alpha \in (0, \lambda)$, and $\sigma \in (0, [\hat{\Phi}_{11}(\alpha, \lambda)]^{1/2})$. In particular we will eventually choose ((H2) implies $\tilde{\sigma}$ is well-defined)

$$\sigma^2 = \tilde{\sigma}^2(\alpha, \lambda) := \frac{\hat{\Phi}_1(\lambda, \lambda) - \hat{\Phi}_1(\alpha, \lambda)}{\lambda - \alpha} \quad (3.3)$$

in order to satisfy the Rankine-Hugoniot condition. Before proceeding with our analysis of the (ODE) we first make a hypothesis that ensures that $\tilde{\sigma}(\alpha, \lambda)$ is in the interval that we have chosen for σ . This condition will be used to guarantee that the solution of the (ODE) can be continued to $s = 0$. It will also be needed, in section 5, to prove that the shock dissipates energy.

$$\hat{\Phi}_{111}(q, r) < 0, \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H3})$$

Remark. Due to the restriction $q < r$ hypothesis (H3) need **not** imply that the stored energy grows at most quadratically at infinity. In section 6 we give an example of a stored-energy function which satisfies all of our hypotheses and which has superquadratic growth.

Lemma 3.1. *Assume (H1)-(H3). Then $\hat{\Phi}_{11}(\alpha, \lambda) - \tilde{\sigma}^2(\alpha, \lambda) > 0$.*

Proof. The mean-value theorem applied to (3.3) yields a $c \in (\alpha, \lambda)$ such that $\tilde{\sigma}^2(\alpha, \lambda) = \hat{\Phi}_{11}(c, \lambda)$. However, (H3) implies that $\hat{\Phi}_{11}(c, \lambda) < \hat{\Phi}_{11}(\alpha, \lambda)$, which together with the previous equation yields the desired result. $\square$

We now establish the local existence of a solution to the (ODE).

Proposition 3.2. *Assume (H1)-(H3). Then there is an $\epsilon = \tilde{\epsilon}(\alpha, \lambda, \sigma) > 0$ and a unique $\varphi \in C^3((\sigma - \epsilon, \sigma + \epsilon))$ that satisfies the (ODE) together with the initial conditions*

$$\varphi(\sigma) = \lambda\sigma, \quad \dot{\varphi}(\sigma) = \alpha. \quad (3.4)$$

Moreover, for all $s \in (\sigma - \epsilon, \sigma + \epsilon)$

$$\varphi(s) > 0, \quad \dot{\varphi}(s) > 0, \quad \Phi_{11}(s) - s^2 > 0, \quad (3.5)$$

$$\left(\frac{\varphi(s)}{s}\right)' = \left(\dot{\varphi}(s) - \frac{\varphi(s)}{s}\right)s^{-1} < 0. \quad (3.6)$$

Proof. By hypothesis $\hat{\Phi}_{11}(q, r) - s^2 > 0$ for all (q, r, s) in a sufficiently small neighborhood of $(\alpha, \lambda, \sigma)$. Thus by (H1) the right-hand side of the (ODE) is C^1 in a neighborhood of $(\alpha, \lambda, \sigma)$. The existence of a unique C^3 function φ is then standard. Finally, we note that each of the inequalities in (3.5) and (3.6) is clear at $s = \sigma$ and hence will be satisfied in a sufficiently small neighborhood since φ and $\dot{\varphi}$ are smooth. $\square$

In order to continue the solution (backwards) to $s = 0$ we will need to, in particular, ensure that the denominator on the right-hand side of the (ODE) is bounded away from zero. The derivative of the significant term in the denominator along the solution is

$$\frac{d}{ds} \hat{\Phi}_{11}(\dot{\varphi}(s), \frac{\varphi(s)}{s}) = \frac{n-1}{s} \frac{\dot{\varphi} - \frac{\varphi}{s}}{\Phi_{11} - s^2} (\Phi_{112}[\Phi_{11} - s^2] - P\Phi_{111}), \quad (3.7)$$

where $P = P(\dot{\varphi}(s), \varphi(s)/s)$ is given by (3.2) and we have made use of the (ODE) in deriving (3.7). If this derivative were negative then the denominator would increase as s decreases. In view of (H3) (and (3.6)) sufficient conditions for the denominator of the (ODE) to be bounded away from zero will therefore be given by

$$P(q, r) := \hat{\Phi}_{12}(q, r) + \frac{\hat{\Phi}_1(q, r) - \hat{\Phi}_2(q, r)}{q - r} > 0, \quad \lambda^* \leq r, 0 < q < r < \infty, \quad (H4)$$

and for each $r \in [\lambda^*, \infty)$ and $q \in (r, \infty)$ either

$$\hat{\Phi}_{112}(q, r) \geq 0 \quad \text{or} \quad \hat{\Phi}_{112}(q, r)\hat{\Phi}_{11}(q, r) - P(q, r)\hat{\Phi}_{111}(q, r) \geq 0. \quad (\text{H5})$$

Other hypotheses that will be used to continue the solution to zero are:

$$\lim_{\substack{q \rightarrow 0^+ \\ r \rightarrow b > 0}} \hat{\Phi}_1(q, r) = -\infty; \quad (\text{H6})$$

and, there are constants $B > 0$, and $\beta \in [0, n-1)$ such that

$$0 \leq R(q, r) \leq Br^\beta, \quad \lambda^* \leq r, \quad 0 < q < r < \infty, \quad (\text{H7})$$

where

$$R(q, r) := \frac{q\hat{\Phi}_1(q, r) - r\hat{\Phi}_2(q, r)}{q - r}.$$

Theorem 3.3. *Assume (H1)-(H7). Then there exists a unique $\varphi \in C^3((0, \sigma + \epsilon))$ that satisfies the (ODE) together with the initial conditions (3.4). Moreover, for all $s \in (0, \sigma + \epsilon)$, φ satisfies (3.5), (3.6), and*

$$\ddot{\varphi}(s) > 0, \quad \varphi(0) := \lim_{s \rightarrow 0^+} \varphi(s) \geq \lambda - \alpha > 0. \quad (3.8)$$

Proof. The standard continuation theorem for ordinary differential equations implies that the solution constructed in the previous proposition can be uniquely continued as long as the right-hand side of the (ODE) remains Lipschitz-continuous. Thus φ can be continued uniquely to an interval $(\sigma_m, \sigma + \epsilon)$; $\varphi \in C^3((\sigma_m, \sigma + \epsilon))$; and either $\sigma_m \leq 0$ (in which case we set $\sigma_m = 0$) or at least one of the following occurs

$$\lim_{s \rightarrow \sigma_m^+} \varphi(s) \text{ does not exist}, \quad (3.9)$$

$$\lim_{s \rightarrow \sigma_m^+} \dot{\varphi}(s) \text{ does not exist}, \quad (3.10)$$

$$\lim_{s \rightarrow \sigma_m^+} \varphi(s) = 0, \quad \lim_{s \rightarrow \sigma_m^+} \dot{\varphi}(s) = 0, \quad (3.11)$$

$$\lim_{s \rightarrow \sigma_m^+} \varphi(s) = +\infty, \quad \lim_{s \rightarrow \sigma_m^+} \dot{\varphi}(s) = +\infty, \quad (3.12)$$

$$\lim_{s \rightarrow \sigma_m^+} (\Phi_{11}(s) - s^2) = 0. \quad (3.13)$$

Moreover, we may assume that each of the inequalities in (3.5) is satisfied for all $s \in (\sigma_m, \sigma + \epsilon)$.

To show that (3.6) is satisfied on (σ_m, σ) suppose for the sake of contradiction that $\omega := \dot{\varphi}(s_0) = \varphi(s_0)/s_0$ for some $s_0 \in (\sigma_m, \sigma)$. Then both φ and $\varphi^*(s) := \omega s$ are solutions of the (ODE) with $\varphi^*(s_0) = \varphi(s_0) = \omega s_0$ and $\dot{\varphi}^*(s_0) = \dot{\varphi}(s_0) = \omega$. A standard uniqueness theorem yields $\varphi^*(s) = \varphi(s)$ for $s \in (s_0, \sigma]$ and hence, in view of (3.4), $\alpha = \omega = \lambda$, a contradiction.

Next, we note that (3.6), (3.7), (H3), (H4), and (H5) imply that $\frac{d}{ds}(\Phi_{11}(s) - s^2) < 0$ and hence, since $\Phi_{11}(\sigma) - \sigma^2 > 0$, we see that (3.13) is not possible. Also (3.5)₂ implies that φ is monotone while the (ODE), (3.5)₃, (3.6), and (H4) imply that

$$\ddot{\varphi}(s) > 0, \quad s \in (\sigma_m, \sigma + \epsilon) \quad (3.14)$$

and so $\dot{\varphi}$ is monotone. Thus (3.9) and (3.10) are not satisfied. In addition, in view of the inequalities (3.5)₂ and (3.14) we conclude that (3.12) can't occur. Moreover, once we have shown that σ_m is zero, (3.8)₁ will follow from (3.14).

In order to show that (3.11)₁ is not possible we note that, by (3.14), φ is convex on $(\sigma_m, \sigma + \epsilon)$. Therefore for $s \in (\sigma_m, \sigma)$

$$\begin{aligned} \varphi(s) &\geq \varphi(\sigma) + \dot{\varphi}(\sigma)(s - \sigma) \\ &= \lambda\sigma + \alpha(s - \sigma) \\ &= (\lambda - \alpha)\sigma + \alpha s > 0. \end{aligned}$$

Also, if $\sigma_m = 0$ this shows that (3.8)₂ is satisfied.

Finally, we consider (3.11)₂. Define

$$T(s) := \left(\frac{\varphi(s)}{s}\right)^{1-n} \hat{\Phi}_1\left(\dot{\varphi}(s), \frac{\varphi(s)}{s}\right), \quad (3.15)$$

the radial component of the Cauchy stress. If (3.11)₂ were satisfied then (H6) would yield

$$\lim_{s \rightarrow \sigma_m^+} T(s) = -\infty$$

However, by Lemma 3.4 T is bounded from below on the interval $(\sigma_m, \sigma]$ and so (3.11)₂ is not possible. Therefore $\sigma_m = 0$. $\square$

Lemma 3.4. Assume (H1)-(H7) and let $\sigma_* \geq 0$. Suppose that $\varphi \in C^3((\sigma_*, \sigma + \epsilon))$ satisfies the (ODE) and (3.4)-(3.6) on the interval $(\sigma_*, \sigma]$. Define $T(s)$ by (3.15). Then

$$0 \leq \dot{T}(s) \leq s^2 \left(\frac{\varphi(s)}{s} \right)^{1-n} \ddot{\varphi} + B(n-1) \left(\frac{-\varphi}{s} \right) \cdot \left(\frac{\varphi}{s} \right)^{\beta-n} \quad (3.16)$$

and hence there is a constant $C > 0$, which does not depend on our choice of $\lambda \geq \lambda^*$, $\alpha \in (0, \lambda)$, and $\sigma \in (0, [\hat{\Phi}(\alpha, \lambda)]^{1/2})$ such that

$$\lambda^{1-n} \hat{\Phi}_1(\alpha, \lambda) - C \lambda^{1-n+\beta} - \alpha \lambda^{1-n} \sigma^2 \leq T(t) \quad (3.17)$$

for every $t \in (\sigma_*, \sigma]$.

Proof. We rewrite (3.15) as

$$T(s) = \varphi(s)^{1-n} \left(s^{n-1} \hat{\Phi}_1(\dot{\varphi}(s), \frac{\varphi(s)}{s}) \right)$$

and then differentiate with respect to s to conclude, with the aid of (3.1), that

$$\begin{aligned} \dot{T}(s) &= (1-n) \varphi^{-n} \dot{\varphi} s^{n-1} \hat{\Phi}_1 + \varphi^{1-n} [s^{n+1} \ddot{\varphi} + (n-1) s^{n-2} \Phi_2] \\ &= -\frac{n-1}{s} \left(\frac{\varphi}{s} \right)^{-n} \left(\dot{\varphi} - \frac{\varphi}{s} \right) R\left(\dot{\varphi}, \frac{\varphi}{s}\right) + s^2 \left(\frac{\varphi}{s} \right)^{1-n} \ddot{\varphi} \\ &= \frac{n-1}{s} \left(\frac{\varphi}{s} \right)^{-n} \left(\frac{\varphi}{s} - \dot{\varphi} \right) \left[R + \frac{s \varphi P}{\Phi_{11} - s^2} \right], \end{aligned} \quad (3.18)$$

where R is given in (H7) and we have used the (ODE) to obtain (3.18)₃.

We note that by (3.5)_{1,3}, (3.6), (3.18)₃, (H4), and (H7) $\dot{T}$ is positive. To obtain the upper bound on $\dot{T}$ we consider the term in (3.18)₂ that contains the function R . In view of (3.5)₁, (3.6), and (H7) we find that

$$-\frac{n-1}{s} \left(\frac{\varphi}{s} \right)^{-n} \left(\dot{\varphi} - \frac{\varphi}{s} \right) R\left(\dot{\varphi}, \frac{\varphi}{s}\right) \leq B(n-1) \left(\frac{-\varphi}{s} \right) \cdot \left(\frac{\varphi}{s} \right)^{\beta-n},$$

which together with (3.18)₂ yields (3.16).

In order to show that (3.17) is satisfied we note that, by (3.4)₁ and (3.6),

$$s^2 \left(\frac{\varphi}{s} \right)^{1-n} \leq \sigma^2 \lambda^{1-n}, \quad s \in (\sigma_*, \sigma]. \quad (3.19)$$

If we bound the first term on the right-hand side of (3.16) from above using (3.19) and then integrate the resulting inequality with respect to s over the interval (t, σ) we find, with the aid of (3.4), (3.5)_{1,2}, (3.15), and the fact that $\beta < n-1$, that (3.17) is satisfied with $C = B(n-1)/(n-1-\beta)$. $\square$

4. The Cauchy Stress at the Cavity.

In this section we identify solutions φ of the (ODE) that satisfy (v) of Proposition 2.3. In order to accomplish this we investigate the manner in which the radial component of the Cauchy stress at the cavity depends upon the initial data. We now denote the solution of the (ODE) with initial conditions (3.4) by $\varphi(s, \alpha, \lambda, \sigma)$ and we write $T(s, \alpha, \lambda, \sigma)$ for the radial component of the Cauchy stress as given by (3.15). Since (3.5)₂ and (3.16) imply that φ and T are monotone in s the limits

$$\varphi(0, \alpha, \lambda, \sigma) := \lim_{s \rightarrow 0^+} \varphi(s, \alpha, \lambda, \sigma), \quad T(0, \alpha, \lambda, \sigma) := \lim_{s \rightarrow 0^+} T(s, \alpha, \lambda, \sigma) \quad (4.1)$$

are well-defined.

Our first result shows that the radial component of the Cauchy stress at the cavity can be made as negative as we want by choosing α near zero.

Lemma 4.1. *Assume (H1)-(H7) and let $\tilde{\sigma}$ be given by (3.3). Then*

$$\lim_{\alpha \rightarrow 0^+} T(0, \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda)) = -\infty.$$

Proof. The definition of T , (3.4), and (H6) imply that

$$\lim_{\alpha \rightarrow 0^+} T(\tilde{\sigma}(\alpha, \lambda), \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda)) = \lim_{\alpha \rightarrow 0^+} \lambda^{1-n} \hat{\Phi}_1(\alpha, \lambda) = -\infty,$$

which together with Lemma 3.4 ($\dot{T} \geq 0$) gives the desired result. $\square$

In order to satisfy the boundary condition $T(0, \alpha, \lambda, \sigma) = g(\varphi(0, \alpha, \lambda, \sigma))$ we now assume that

$$g \text{ is nonnegative, monotone increasing, and continuous,} \quad (\text{H8})$$

$$\liminf_{r \rightarrow +\infty} \left(r^{1-n} (\hat{\Phi}_1(r, r) - r \hat{\Phi}_{11}(r, r)) - g(r [\hat{\Phi}_{11}(r, r)]^{\frac{1}{2}}) \right) > 0. \quad (\text{H9})$$

We first show that T can be made greater than g by choosing λ suitably large and α close to λ .

Lemma 4.2. *Assume (H1)-(H9). Then there exists $\lambda_{cav} \geq \lambda^*$ such that for every $\lambda > \lambda_{cav}$ there is a $\delta > 0$ such that for any $\alpha \in (\lambda - \delta, \lambda)$*

$$T(0, \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda)) > g(\varphi(0, \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda))).$$

Proof. By Lemma 3.1, Lemma 3.4 ((3.17)), (4.1)₂, and the fact that $0 < \alpha < \lambda$

$$T(0, \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda)) \geq \lambda^{1-n}(\hat{\Phi}_1(\alpha, \lambda) - \alpha \hat{\Phi}_{11}(\alpha, \lambda)) - C\lambda^{1-n+\beta}.$$

We need not concern ourselves with the last term on the right-hand side of the above inequality since it goes to zero as λ goes to infinity.

We note that by Lemma 3.1 and (3.5)₂

$$\varphi(0, \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda)) \leq \varphi(\tilde{\sigma}(\alpha, \lambda), \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda)) = \lambda \tilde{\sigma}(\alpha, \lambda) \leq \lambda \hat{\Phi}_{11}(\alpha, \lambda)^{\frac{1}{2}}.$$

Hypotheses (H1), (H8), and (H9) then give the desired result. $\square$

In view of the previous two lemmas the traction boundary condition at the surface of the cavity will be satisfied, for all sufficiently large λ , by an appropriate choice of $\alpha = \alpha(\lambda)$ provided we can show that $T(0, \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda))$ and $\varphi(0, \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda))$ depend continuously upon α . In order to prove this we will need to assume that either

there are constants $N > 0$ and $\nu \in [0, n)$ such that

$$P(q, r) \leq Nr^\nu, \quad \lambda^* \leq r, \quad 0 < q < r < \infty, \quad (\text{H10}\dagger)$$

or

$$P(q, r) \leq Nr^\nu \hat{\Phi}_{11}(q, r), \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H10})$$

Lemma 4.3. *Assume (H1)-(H9) and either (H10\dagger) or (H10). Then for each $\lambda \geq \lambda^*$ the mappings*

$$(\alpha, \sigma) \mapsto \varphi(0, \alpha, \lambda, \sigma), \quad (\alpha, \sigma) \mapsto T(0, \alpha, \lambda, \sigma)$$

are continuous.

The previous three lemmas together with Lemma 3.1 and the continuity of the map $\alpha \mapsto \tilde{\sigma}(\alpha, \lambda)$ immediately give

Theorem 4.4. *Assume (H1)-(H9) and either (H10_†) or (H10). Then there exists a $\lambda_{cav} \geq \lambda^*$ such that for every $\lambda > \lambda_{cav}$ there is an $\alpha \in (0, \lambda)$ such that the (ODE) and initial conditions (3.4) have a solution $\varphi \in C^3((0, \tilde{\sigma}(\alpha, \lambda)))$ that satisfies*

$$T(0, \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda)) = g(\varphi(0, \alpha, \lambda, \tilde{\sigma}(\alpha, \lambda))).$$

Proof of Lemma 4.3. Let $\lambda \geq \lambda^*$. By a standard continuous dependence result for ordinary differential equations the mappings

$$\begin{aligned} (\alpha, \sigma) &\mapsto \varphi(s, \alpha, \lambda, \sigma), \quad (\alpha, \sigma) \mapsto \dot{\varphi} := \frac{d}{ds}\varphi(s, \alpha, \lambda, \sigma) \\ (\alpha, \sigma) &\mapsto T(s, \alpha, \lambda, \sigma), \quad (\alpha, \sigma) \mapsto \dot{T} := \frac{d}{ds}T(s, \alpha, \lambda, \sigma) \end{aligned} \quad \text{are continuous} \quad (4.3)$$

for each $s \in (0, \sigma]$. In addition, since $\dot{\varphi}$ and $\dot{T}$ are nonnegative the monotone convergence theorem implies that

$$\begin{aligned} \varphi(0, \alpha, \lambda, \sigma) &= \varphi(\sigma, \alpha, \lambda, \sigma) - \int_0^\sigma \dot{\varphi}(s, \alpha, \lambda, \sigma) ds, \\ T(0, \alpha, \lambda, \sigma) &= T(\sigma, \alpha, \lambda, \sigma) - \int_0^\sigma \dot{T}(s, \alpha, \lambda, \sigma) ds. \end{aligned} \quad (4.4)$$

Thus we fix $\alpha_0 \in (0, \lambda)$ and let $\alpha_k \rightarrow \alpha_0$ with $\alpha_k \in (0, \lambda)$ for $k = 1, 2, 3, \dots$. Similarly, fix $\sigma_0 \in (0, [\hat{\Phi}_{11}(\alpha_0, \lambda)]^{1/2})$ and let $\sigma_k \rightarrow \sigma_0$ with $\sigma_k \in (0, [\hat{\Phi}_{11}(\alpha_k, \lambda)]^{1/2})$ for $k = 1, 2, 3, \dots$. To finish the proof we will show that the integrands in (4.4) are bounded, independently of $k = 1, 2, 3, \dots$, by an integrable function. The desired result will then follow from (4.3) and the Lebesgue dominated convergence theorem.

The uniform bound on (4.4)₁ follows from the fact that (see (3.4)₂, (3.5)₂, and (3.8)₁) $0 \leq \dot{\varphi}(s, \alpha_k, \lambda, \sigma) \leq \alpha_k \leq \lambda$. In order to obtain the uniform bound on (4.4)₂ we first note that, by equation (3.16) in Lemma 3.4,

$$|\dot{T}_k| \leq s^2 \left(\frac{\varphi_k}{s} \right)^{1-n} \ddot{\varphi}_k + B(n-1) \left(\frac{-\varphi_k}{s} \right) \cdot \left(\frac{\varphi_k}{s} \right)^{\beta-n} \quad (4.5)$$

for $k = 1, 2, 3, \dots$, where we have denoted $T(s, \alpha_k, \lambda, \sigma_k)$ and $\varphi(s, \alpha_k, \lambda, \sigma_k)$ by T_k and φ_k , respectively. After differentiating and noting that $\dot{\varphi}_k > 0$ we find that the second term on the right-hand side of (4.5) is bounded by

$$B(n-1) \frac{1}{s} \left(\frac{s}{\varphi_k(s)} \right)^{n-1-\beta} \leq B(n-1) \frac{s^{n-2-\beta}}{(\lambda - \alpha_k)^{n-1-\beta}},$$

where we have used (3.5)₂, (3.8)₂, and the fact that $\beta < n-1$ to obtain the lower bound on $\varphi_k(s)$. Since $\alpha_k \rightarrow \alpha_0 < \lambda$ and $\beta < n-1$ it follows that the second term on the right-hand

side of (4.5) is bounded by an integrable function, independently of k , for k sufficiently large.

We next consider the first term on the right-hand side of (4.5). The (ODE) and the fact that $\dot{\varphi}_k > 0$ imply that this term is bounded by

$$(n-1)s\left(\frac{\varphi_k}{s}\right)^{2-n} \frac{P(\dot{\varphi}_k, \frac{\varphi_k}{s})}{\hat{\Phi}_{11}(\dot{\varphi}_k, \frac{\varphi_k}{s}) - s^2}. \quad (4.6)$$

By (3.7), (H3), (H4), and (H5) the denominator in (4.6) is bounded below by $\hat{\Phi}_{11}(\alpha_k, \lambda) - \sigma_k^2$, which converges to $\hat{\Phi}_{11}(\alpha_0, \lambda) - \sigma_0^2 > 0$ as $k \rightarrow \infty$. Therefore, by (H10 $\dagger$) there is an $L > 0$ such that, for all sufficiently large k , (4.6) is bounded by

$$sL\left(\frac{s}{\varphi_k(s)}\right)^{n-2-\nu} \leq \begin{cases} L \frac{s^{n-1-\nu}}{(\lambda-\alpha_k)^{n-2-\nu}} & \text{if } \nu < n-2 \\ L \frac{s^{n-1-\nu}}{(\sigma_k \lambda)^{n-2-\nu}} & \text{if } \nu \geq n-2 \end{cases},$$

where we have used (3.5)₂, (3.8)₂, and the fact that $\nu < n-2$ to obtain the first bound on $\varphi_k(s)$ and (3.5)₂, (3.4)₁, and the fact that $\nu \geq n-2$ to obtain the second one. As before $\alpha_k \rightarrow \alpha_0 < \lambda$, $\sigma_k \rightarrow \sigma_0$, and $\nu < n$. Thus the first term on the right-hand side of (4.5) is bounded by an integrable function, independently of k , for k sufficiently large.

If we assume (H10) rather than (H10 $\dagger$) then we rewrite (4.6) as

$$(n-1)s\left(\frac{\varphi_k}{s}\right)^{2-n} \frac{\frac{P}{\hat{\Phi}_{11}}}{1 - \frac{s^2}{\hat{\Phi}_{11}}}. \quad (4.7)$$

It is clear that the previous argument together with (H10) will yield the required bound provided that the denominator in (4.7) is bounded away from zero, uniformly in k . In order to get such a bound we note that $s \leq \sigma_k$ and hence (3.7), (H3), (H4), and (H5) yield

$$\hat{\Phi}_{11}\left(\dot{\varphi}_k(s), \frac{\varphi_k(s)}{s}\right) \geq \hat{\Phi}_{11}(\alpha_k, \lambda).$$

Therefore

$$1 - \frac{s^2}{\hat{\Phi}_{11}(\dot{\varphi}_k(s), \frac{\varphi_k(s)}{s})} \geq 1 - \frac{\sigma_k^2}{\hat{\Phi}_{11}(\alpha_k, \lambda)} \rightarrow 1 - \frac{\sigma_0^2}{\hat{\Phi}_{11}(\alpha_0, \lambda)} > 0$$

as $k \rightarrow \infty$, which yields the required bound. $\square$

5. The Energy. Admissibility.

In this section we will show that the radial similarity solutions φ , constructed in the previous sections, dissipate energy and satisfy (vi) of Proposition 2.3.

The **total energy** at time t of a weak solution $\mathbf{u}$ of (1.1) is given by

$$E(t, \mathbf{u}, \Omega) := \int_{\Omega} [W(\nabla \mathbf{u}(\mathbf{x}, t)) + \frac{1}{2} |\mathbf{u}_t(\mathbf{x}, t)|^2] d\mathbf{x}.$$

The total energy of the equilibrium solution $\hat{\mathbf{u}} = \lambda \mathbf{x}$ is therefore equal to $W(\lambda \mathbf{I}) \text{vol}(\Omega)$. In this section we show that the total energy of the cavitary solution constructed in sections 3 and 4 is *strictly* less than the energy of the equilibrium solution. Thus, since the total energy is the appropriate entropy in this context (see, e.g., Dafermos [Da 83]), the cavitary solutions are admissible by the usual entropy criterion (see, e.g., Lax [La 57, La 76]) and preferred by the entropy-rate criterion (see Dafermos [Da 73]).

We will make use of the following

Proposition 5.1 [PS 88]. *Let $\mathbf{u}$ be given by (2.3) where φ satisfies (i)-(vii) of Proposition 2.3 and*

$$s^{n-2} \hat{\Phi}_2(\dot{\varphi}(s), \frac{\varphi(s)}{s}) \in L^1((0, \sigma)), \quad \lim_{s \rightarrow 0^+} s^n \hat{\Phi}(\dot{\varphi}(s), \frac{\varphi(s)}{s}) = 0. \quad (5.1)$$

Then

$$E(t, \mathbf{u}, \mathcal{B}) = E(t, \lambda \mathbf{x}, \mathcal{B}) + t^n \sigma^n S_n J / n,$$

where

$$J := \hat{\Phi}(\alpha, \lambda) - \hat{\Phi}(\lambda, \lambda) + \frac{1}{2} [\hat{\Phi}_1(\alpha, \lambda) + \hat{\Phi}_1(\lambda, \lambda)] (\lambda - \alpha).$$

Moreover, if constitutive hypothesis (H3) is satisfied then $J < 0$ and hence for every $t > 0$

$$E(t, \mathbf{u}, \mathcal{B}) < E(t, \lambda \mathbf{x}, \mathcal{B}).$$

Remarks. 1. The elastostatics (see, e.g., [Ba 82] or [Si 86]) of constitutive relations of slow growth yields a λ_{cr} such that for all $\lambda > \lambda_{cr}$ the radial minimizer exhibits a cavity, while for all $\lambda \leq \lambda_{cr}$ the deformation $\lambda \mathbf{x}$ is the radial minimizer. One consequence of the above result is that λ_{cr} is less than or equal to the infimum of λ at which dynamic cavitation can occur (which may be smaller than λ_{cav}) since at each fixed (positive) time the dynamic cavitary solution is a radial deformation whose *elastic* energy is smaller than that of the initial deformation $\lambda \mathbf{x}$.

2. [Sp 94] gives constitutive hypotheses, which are compatible with cavitation, under which the deformation $\lambda \mathbf{x}$ is the global minimizer of the elastic energy for all sufficiently

small λ . Thus, in view of the previous remark, dynamic cavitation cannot occur when such materials are compressed.

We now assume that

$$\hat{\Phi}(q, r) \geq 0, \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H11})$$

The following lemma then both completes the proof that our solutions of the (ODE) yield motions that satisfy (1.1) and also shows that these motions are admissible.

Lemma 5.2. *Assume (H1)-(H11). Let $\varphi \in C^3((0, \sigma])$ satisfy the (ODE) and the initial conditions (3.4), where $\sigma = \tilde{\sigma}(\alpha, \lambda)$ is given by the Rankine-Hugoniot jump condition (3.3). If the boundary condition $T(0) = g(\varphi(0))$ (see (3.15) and (4.1)) is satisfied on the surface of the cavity then (5.1) and condition (vi) of Proposition 2.3 are satisfied.*

Thus, by Proposition 2.3, Theorem 4.4, Proposition 5.1, and Lemma 5.2 we have our

Main Theorem. *Assume (H1)-(H11) with $g \equiv 0$. Then there is a $\lambda_{cav} \geq \lambda^*$ such that, for all $\lambda > \lambda_{cav}$, equations (1.1) have a discontinuous weak solution. Moreover, each such solution dissipates energy and consequently the equilibrium solution $\mathbf{u}(\mathbf{x}, t) \equiv \lambda \mathbf{x}$ is not dynamically stable for such values of λ .*

Proof of Lemma 5.2. Let $\varphi \in C^3((0, \sigma])$ satisfy the (ODE). By (3.15)

$$s^{n-1} \Phi_1(s) = \varphi(s)^{n-1} T(s). \quad (5.2)$$

Then, by (3.5)_{1,2}, (3.16), and the fact that $T(0) = g(\varphi(0)) \geq 0$ (see (H8)),

$$\frac{d}{ds} (s^{n-1} \Phi_1(s)) = (n-1) \varphi(s)^{n-2} \dot{\varphi}(s) T(s) + \varphi(s)^{n-1} \dot{T}(s) \geq 0. \quad (5.3)$$

Therefore, by (3.1) and (3.8)₁, we find that for $s \in (0, \sigma)$

$$(n-1)s^{n-2} |\Phi_2(s)| \leq \frac{d}{ds} (s^{n-1} \Phi_1(s)) + \sigma^{n+1} \ddot{\varphi}(s). \quad (5.4)$$

Next, the monotone convergence theorem, (3.5)₂, and (3.8)₁ yield $\ddot{\varphi} \in L^1((0, \sigma))$. Also, since φ and T are nonnegative, we find, with the aid of (5.2) and (5.3) that

$$\lim_{s \rightarrow 0^+} s^{n-1} \Phi_1(s) \text{ exists and is finite}$$

and hence, using (5.3) and the monotone convergence theorem, that $s^{n-1}\Phi_1 \in L^1((0, \sigma))$. Therefore, since both terms on the right-hand side of (5.4) are positive we conclude that (5.1)₁ and consequently condition (vi) of Proposition 2.3 are satisfied.

A straightforward computation, which makes use of the (ODE), (see [PS 88, p. 312]) shows that φ satisfies

$$s^n \frac{d}{ds} [\Phi(s) + \frac{1}{2}(\varphi - s\dot{\varphi})^2] = \frac{d}{ds} [s^{n-1}(s\dot{\varphi} - \varphi)\Phi_1(s)]$$

and hence that φ satisfies

$$\frac{d}{ds} \left(s^n \Phi(s) - s^{n-1}(s\dot{\varphi} - \varphi)\Phi_1(s) + \frac{1}{2}s^n(\varphi - s\dot{\varphi})^2 \right) = ns^{n-1}\Phi(s) + \frac{n}{2}s^{n-1}(\varphi - s\dot{\varphi})^2.$$

If we integrate the last equation over the interval (τ, σ) and take the limit as $\tau \rightarrow 0^+$ we conclude, with the aid of (3.5)_{1,2}, (3.8)₁, (5.2), (H8), and the boundary condition $T(0) = g(\varphi(0))$, that

$$\lim_{\tau \rightarrow 0^+} \left(\tau^n \Phi(\tau) + \int_{\tau}^{\sigma} s^{n-1} \Phi(s) ds \right) \text{ exists and is finite.} \quad (5.5)$$

Since (H11) gives $\Phi \geq 0$ we conclude from (5.5) and the monotone convergence theorem that

$$s^{n-1}\Phi(s) \in L^1((0, \sigma)). \quad (5.6)$$

In particular the limit of the second term in (5.5) exists and consequently so does the limit of the first term. Finally, (5.6) implies that the limit of the first term in (5.5) must be zero, i.e., (5.1)₂ is satisfied. $\square$

6. The Jacobian. Examples.

In [St 93] Stuart considers the hypothesis

$$0 \leq \frac{\partial R(q, r)}{\partial q} \text{ or, equivalently, } q\hat{\Phi}_{11}(q, r) \leq rP(q, r), \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H12})$$

He shows that (H12), along with his other hypotheses, implies that the Jacobian

$$J(s) := \det \nabla \mathbf{u}(s) = \dot{\varphi}(s) \left(\frac{\varphi(s)}{s} \right)^{n-1},$$

is increasing along the solutions of the radial equilibrium equation. It turns out that this hypothesis does the same for our (ODE). We will make use of this property when we verify that a certain class of examples of stored energy functions satisfy our other hypotheses.

Proposition 6.1. *Assume (H1) - (H7) and (H12). Then*

$$\frac{d}{ds} \left(\dot{\varphi}(s) \left(\frac{\varphi(s)}{s} \right)^{n-1} \right) \geq 0.$$

Proof. If we take the indicated derivative and use the (ODE) we find that

$$j(s) = \frac{n-1}{s(\Phi_{11}(s) - s^2)} \left(\frac{\varphi}{s} - \dot{\varphi} \right) \left(\frac{\varphi}{s} \right)^{n-2} \left(\frac{\varphi}{s} P - \dot{\varphi}(\Phi_{11}(s) - s^2) \right),$$

which together with (3.5), (3.6), and (H12) gives the desired result. $\square$

We now restrict our attention to three dimensions and consider materials whose constitutive relation is of the form

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^3 \phi(\lambda_i) + \psi(\lambda_1 \lambda_2) + \psi(\lambda_2 \lambda_3) + \psi(\lambda_1 \lambda_3) + h(\lambda_1 \lambda_2 \lambda_3), \quad (6.1)$$

in order to illustrate the form that hypotheses (H1) - (H12) take for a well-analyzed class of materials. (Such constitutive relations were used by Ogden [Og 72] to match theory with experiments.)

For such materials we define $\tilde{\phi}(t) := t\phi'(t)$ and $\tilde{\psi}(t) := t\psi'(t)$ and make the following assumptions (cf. [Ba 82], [St 85], [Si 86], [JS 91], and especially [St 93]):

$$\phi, \psi, h \in C^3((0, \infty); [0, \infty)); \quad (O1)$$

$$\phi'' \geq 0, \psi'' \geq 0, h'' > 0 \text{ on } (0, \infty); \quad (O2)$$

$$\lim_{t \rightarrow 0^+} h'(t) = -\infty; \quad (O3)$$

$$\phi \geq 0, \psi \geq 0, h \geq 0 \text{ on } (0, \infty); \quad (O4)$$

there exist $t_0 > 0, \beta \in [1, 2)$, and $B > 0$ such that

$$0 \leq \tilde{\phi}'(t), 0 \leq \tilde{\psi}'(t) \leq Bt^{(\beta-1)/2}, \quad t \geq t_0; \quad (O5)$$

$$\tilde{\phi}'' \geq 0, \tilde{\psi}'' \geq 0 \text{ on } (0, \infty); \quad (O6)$$

$$\lim_{t \rightarrow +\infty} (h'(t) - th''(t)) > 0; \quad (O7)$$

$$\phi''' \leq 0, \psi''' \leq 0, h''' < 0 \text{ on } (0, \infty). \quad (O8)$$

Instead of (O8) one may assume that there exist $q_0 > 0$, $\delta > 0$, $A > 0$, and $\alpha \in [-1, 2)$ such that

$$\begin{aligned} \psi''' &\leq 0 \quad \text{on } (0, \infty), \\ \phi'''(t) &\leq 0, \quad h'''(t^3) < 0, \quad t \in (0, q_0], \\ \phi'''(t) &\leq At^{\alpha-2}, \quad h'''(t^3) < -At^{\alpha-8}, \quad h''(t^3) \geq \frac{A}{2}t^{\alpha-5}, \quad t \in (q_0, \infty), \quad (\text{O8}\dagger) \\ &\text{and either} \\ \phi''(t) &\geq \delta, \quad \psi''(t^2) \geq \delta, \quad \text{or} \quad h''(t^3) \geq \delta, \quad t \in (q_0, \infty). \end{aligned}$$

Before we show that for constitutive relations of the form (6.1) assumptions (O1) - (O8) imply (H1)-(H7) and (H9)-(H12) we make a couple of observations. First, if (ϕ_1, ψ_1, h_1) and (ϕ_2, ψ_2, h_2) are any triples of functions that satisfy the above hypotheses and $c_1 > 0$ and $c_2 \geq 0$ are any constants then the functions $(\phi, \psi, h) := c_1(\phi_1, \psi_1, h_1) + c_2(\phi_2, \psi_2, h_2)$ will satisfy the same assumptions. Next, some simple computations show that the functions

$$\phi(t) := a_1 t^j, \quad \psi(t) := a_2 t^k, \quad h(t) := a_3 t^m + a_4 t^{-n}$$

satisfy (O1)-(O8) provided

$$1 \leq j \leq 2, \quad 1 \leq k < 3/2, \quad 1 \leq m < 2, \quad 0 < n, \quad 0 \leq a_1, \quad 0 \leq a_2, \quad 0 < a_3, \quad 0 < a_4.$$

However, the energy for such materials only satisfies an estimate of the form ($c_1 > 0$)

$$W(\mathbf{F}) \geq c_1 |\mathbf{F}|^p + c_2$$

when $p \leq 2$. Although energies that grow so slowly may be of physical interest (see, e.g., [GL 58] and [JS 91]) we observe that (O1)-(O7) together with (O8 $\dagger$) will allow for such a growth condition with $2 < p < 3$. For example, if ψ, h , and their associated constants are as given above then we can choose

$$\phi(t) := \begin{cases} a_5(t-1)^2 + a_6(t-1) + a_7, & \text{if } t \leq 1, \\ a_8 t^p + a_9 t^q, & \text{if } t > 1, \end{cases}$$

provided $2 < p < 3$ (which, with $q_0 = 1$, provides the required growth), $1 < q < 2$,

$$\begin{aligned} a_7 &= a_8 + a_9, \quad a_5 = \frac{1}{2} [p(p-1)a_8 + q(q-1)a_9], \\ a_6 &= pa_8 + qa_9, \quad a_9 = \frac{p(p-1)(p-2)}{q(q-1)(2-q)} a_8 \end{aligned}$$

(all of which ensure that ϕ is C^3), and

$$0 < a_8, \quad 1 < m, \quad \frac{p(p-1)(p-2)}{m(m-1)(2-m)} a_8 < a_3$$

(which gives the required relation between the third derivative of ϕ and the second and third derivatives of h).

We now show that (O1)-(O7) together with either (O8) or (O8 †) satisfy (H1)-(H7) and (H9) -(H12). Note first that (O1) implies (H1) and (H11). If we differentiate (6.1) with respect to q we get

$$\begin{aligned} \hat{\Phi}_1(q, r) &= \phi'(q) + 2r\psi'(qr) + r^2 h'(qr^2), \\ \hat{\Phi}_{11}(q, r) &= \phi''(q) + 2r^2 \psi''(qr) + r^4 h''(qr^2), \\ \hat{\Phi}_{111}(q, r) &= \phi'''(q) + 2r^3 \psi'''(qr) + r^6 h'''(qr^2). \end{aligned} \tag{6.2}$$

Clearly (O2) and (6.2)₂ imply (H2), while (O3), (6.2)₁, and the convexity of ϕ and ψ (i.e., (O2)) give (H6). In view of (6.2)₃ (H3) follows from (O8). If instead (O8 †) is satisfied and $q \leq q_0$, then (H3) is clear. For $q > q_0$ (6.2)₃ and (O8 †)_{1,4} imply that a sufficient condition for (H3) to be satisfied is

$$r^6 h'''(qr^2) < -Aq^{\alpha-2}, \quad q_0 \leq q \leq r < \infty.$$

If we multiply the last inequality by q^3 and let $x = qr^2$ we arrive at an equivalent inequality

$$x^3 h'''(x) < -Aq^{\alpha+1}, \quad q_0 \leq q \leq x^{\frac{1}{3}} < \infty.$$

Since $\alpha \geq -1$ the right-hand side of the above inequality is minimized when $q = x^{1/3}$. Thus the last inequality is a consequence of (O8 †)₅.

Next, straightforward computations using (6.1) show that

$$\begin{aligned} P(q, r) &= \frac{\phi'(r) - \phi'(q)}{r - q} + r^2 \frac{\psi'(r^2) - \psi'(qr)}{r^2 - qr} + qr\psi''(qr) + qr^3 h''(qr^2), \\ R(q, r) &= \frac{\tilde{\phi}(r) - \tilde{\phi}(q)}{r - q} + \frac{\tilde{\psi}(r^2) - \tilde{\psi}(qr)}{r - q} \end{aligned} \tag{6.3}$$

and hence

$$\frac{\partial R}{\partial q} = \frac{-\tilde{\phi}'(q)(r - q) + \tilde{\phi}(r) - \tilde{\phi}(q)}{(r - q)^2} + \frac{-r\tilde{\psi}'(qr)(r - q) + \tilde{\psi}(r^2) - \tilde{\psi}(qr)}{(r - q)^2}. \tag{6.4}$$

The mean-value theorem applied to the difference quotients in $(6.3)_1$ together with (O2) yields (H4). Hypothesis (H12) is a consequence of (O6) together with Taylor's theorem applied to the numerators in (6.4).

In order to show that (H7) is satisfied we note that the mean-value theorem applied to $(6.3)_2$ yields the existence of $c^* \in (q, r)$ and $\hat{c} \in (qr, r^2)$ such that

$$R(q, r) = \tilde{\phi}'(c^*) + r\tilde{\psi}'(\hat{c}), \quad (6.5)$$

which together with (O5) yields the nonnegativity of R . Next, we note that by (O5) and the fact that $\hat{c} < r^2$

$$\tilde{\psi}'(\hat{c}) \leq B\hat{c}^{(\beta-1)/2} \leq Br^{\beta-1},$$

and thus we have obtained the appropriate growth on the second term in the right-hand side of (6.5). In order to derive a growth condition on $\tilde{\phi}'$ we note that if we integrate either $(O8)_1$ or $(O8^\dagger)_4$ we find that there are constants $K_i \geq 0$ such that for $t \geq q_0$

$$\begin{aligned} \phi''(t) &\leq K_1 t^{(\alpha-1)} + K_2, \\ \phi'(t) &\leq K_3 t^\alpha + K_2 t + K_5, \end{aligned} \quad (6.6)$$

and hence

$$\tilde{\phi}'(t) = t\phi''(t) + \phi'(t) \leq (K_1 + K_3)t^\alpha + 2K_2 t + K_5.$$

The last equation together with (6.5) then give the required upper bound on R .

In order to verify that (H9) is satisfied (when g is zero) we first note that by $(6.2)_{1,2}$

$$\begin{aligned} r^{-2}(\hat{\Phi}_1(r, r) - r\hat{\Phi}_{11}(r, r)) &= r^{-2}(\phi'(r) - r\phi''(r)) \\ &\quad + 2r^{-1}(\psi'(r^2) - r^2\psi''(r^2)) \\ &\quad + (h'(r^3) - r^3h''(r^3)). \end{aligned} \quad (6.7)$$

Now either (O8) or $(O8^\dagger)$ implies that $(6.6)_1$ is satisfied, while $(O2)_1$ implies that ϕ' is increasing. Therefore if $0 < s < r$

$$r^{-2}(\phi'(r) - r\phi''(r)) \geq r^{-2}\phi'(s) - K_1 r^{\alpha-2} - K_2 r^{-1} \rightarrow 0 \quad \text{as } r \rightarrow +\infty, \quad (6.8)$$

since $\alpha < 2$. Next, by $(O5)_2$

$$\tilde{\psi}'(t) = \psi'(t) + t\psi''(t) \leq Bt^{(\beta-1)/2},$$

which together with $(O2)_2$ yields

$$r^{-1}(\psi'(r^2) - r^2\psi''(r^2)) \geq 2r^{-1}\psi'(s^2) - Br^{\beta-2} \rightarrow 0 \quad \text{as } r \rightarrow +\infty, \quad (6.9)$$

since $\beta < 2$. Hypothesis (H9) now follows from (6.7)-(6.9) and (O7).

In order to show that (H10) is satisfied we note that the mean-value theorem applied to the difference quotients in (6.3)₁ yield $c^* \in (q, r)$ and $\hat{c} \in (qr, r^2)$ such that

$$P(q, r) = \phi''(c^*) + r^2 \psi''(\hat{c}) + qr \psi''(qr) + qr^3 h''(qr^2). \quad (6.10)$$

If (O8) is satisfied then it is clear that each term on the right-hand side of (6.10) is bounded from above by a corresponding term on the right-hand side of (6.2)₂. Thus, in this case, (H10) will be satisfied with $N = 1$ and $\nu = 0$. If instead (O8[†]) is satisfied and it happens that $c^* \leq q_0$ then (H10) follows by the same reasoning. When $q_0 < c^*$ we integrate (O8[†])₄ to get a constant K such that

$$\phi''(c^*) \leq \phi''(q) + K((c^*)^{\alpha-1} - q^{\alpha-1}) \leq \phi''(q) + |K|r^{\alpha-1}. \quad (6.11)$$

(H10) now follows from (6.2)₂, (6.10), (6.11), and either (O8[†])₇, (O8[†])₈, or (O8[†])₉.

Finally we consider (H5). We first note that (H12) is satisfied and hence that a sufficient condition for (H5) to be satisfied is that

$$q \hat{\Phi}_{111}(q, r) \leq r \hat{\Phi}_{112}(q, r), \quad 0 < q < r < \infty. \quad (6.12)$$

If we differentiate (6.1) we find that

$$r \hat{\Phi}_{112}(q, r) = 2r^2 \psi''(qr) + qr^3 \psi'''(qr) + 2r^4 h''(qr^2) + qr^6 h'''(qr^2).$$

If (O8) is satisfied then (6.12) follows from (O2) and (O8). If instead (O8[†]) is satisfied and $q \leq q_0$ then the same reasoning gives (6.12). When $q > q_0$ we conclude from (O8[†])_{1,4} that a sufficient condition for (6.12) to be satisfied is

$$Aq^{\alpha-1} \leq 2r^4 h''(qr^2), \quad q_0 < q < r < \infty,$$

We multiply both sides of this inequality by q^2 and let $x = qr^2$ to arrive at an equivalent inequality

$$Aq^{\alpha+1} \leq 2x^2 h''(x), \quad q_0 < q < x^{\frac{1}{3}} < \infty.$$

Since $\alpha \geq -1$ the left-hand side of the above inequality is maximized when $q = x^{1/3}$. Thus this inequality is a consequence of (O8[†])₆.

Appendix. List of hypotheses used in this paper. We assume that there is a $\lambda^* > 0$ such that the following hypotheses are satisfied.

$$\Phi \in C^3((0, \infty); \mathbb{R}). \quad (\text{H1})$$

$$\hat{\Phi}_{11}(q, r) > 0, \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H2})$$

$$\hat{\Phi}_{111}(q, r) < 0, \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H3})$$

$$P(q, r) := \hat{\Phi}_{12}(q, r) + \frac{\hat{\Phi}_1(q, r) - \hat{\Phi}_2(q, r)}{q - r} > 0, \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H4})$$

For each $r \in [\lambda^*, \infty)$ and $q \in (r, \infty]$ either

$$\hat{\Phi}_{112}(q, r) \geq 0 \quad \text{or} \quad \hat{\Phi}_{112}(q, r)\hat{\Phi}_{11}(q, r) - P(q, r)\hat{\Phi}_{111}(q, r) \geq 0. \quad (\text{H5})$$

$$\lim_{\substack{q \rightarrow 0^+ \\ r \rightarrow b > 0}} \hat{\Phi}_1(q, r) = -\infty. \quad (\text{H6})$$

There are constants $B > 0$, and $\beta \in [0, n - 1)$ such that

$$0 \leq R(q, r) := \frac{q\hat{\Phi}_1(q, r) - r\hat{\Phi}_2(q, r)}{q - r} \leq Br^\beta, \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H7})$$

$$g \text{ is nonnegative, monotone increasing, and continuous.} \quad (\text{H8})$$

$$\liminf_{r \rightarrow +\infty} \left(r^{1-n} (\hat{\Phi}_1(r, r) - r\hat{\Phi}_{11}(r, r)) - g(r[\hat{\Phi}_{11}(r, r)]^{\frac{1}{2}}) \right) > 0. \quad (\text{H9})$$

There are constants $N > 0$, and $\nu \in [0, n)$ such that

$$P(q, r) \leq Nr^\nu, \quad \lambda^* \leq r, \quad 0 < q < r < \infty \quad (\text{H10}^\dagger)$$

or

$$P(q, r) \leq Nr^\nu \hat{\Phi}_{11}(q, r), \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H10})$$

$$\hat{\Phi}(q, r) \geq 0, \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H11})$$

$$0 \leq \frac{\partial R(q, r)}{\partial q} \quad \text{or, equivalently,} \quad q\hat{\Phi}_{11}(q, r) \leq rP(q, r), \quad \lambda^* \leq r, \quad 0 < q < r < \infty. \quad (\text{H12})$$

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