**EXISTENCE OF THREE-DIMENSIONAL TOROIDAL MHD
EQUILIBRIA WITH NONCONSTANT PRESSURE**

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Existence of three-dimensional toroidal MHD equilibria with nonconstant pressure

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Abstract

We establish an existence result for the three-dimensional MHD equations

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla p$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{B} \cdot \mathbf{n}|_{\partial T} = 0$$

with $p \neq \text{const}$ in tori T *without symmetry*. More precisely, our theorems insure the existence of sharp boundary solutions for tori whose departure from axisymmetry is sufficiently small; they allow for construction of solutions with an arbitrary number of pressure-jumps.

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Introduction

The problem of establishing existence of solutions of the fundamental boundary value problem of ideal magnetohydrodynamic equilibrium

$$\begin{aligned}(\nabla \times \mathbf{B}) \times \mathbf{B} &= \nabla p \\ \nabla \cdot \mathbf{B} &= 0 \\ \mathbf{B} \cdot \mathbf{n}|_{\partial T} &= 0\end{aligned}\tag{1}$$

in a three dimensional torus T without symmetry and with $p \neq \text{const}$, has remained open since the early work on fusion research by H. Grad, M. Kruskal and others. The boundary value problem (1) models the equilibrium of a plasma confined in a toroidal magnetic bottle. It dates back at least to 1958 when nuclear fusion research was declassified and previous work was made public in the first International Congress on the Peaceful Use of Thermonuclear Energy. Early on Grad [21] suggested that the simplest setting in which the existence of three dimensional solutions could be established is the sharp boundary problem described below, in which p is a step function. He also conjectured that only very pathological solutions with continuous pressure profiles could exist [22]. This conjecture is unsettled to this day, and it is rather controversial, see e.g. [19, 39]. In this paper we address the first of these remarks by Grad. We show that there indeed exist non-axisymmetric three dimensional tori T for which solutions of the sharp boundary problem exist, and we thereby establish for the first time existence of non-constant-pressure solutions of the MHD equations (1) in tori without symmetry.

Besides modeling plasma equilibria, the problem (1) and its special case the sharp boundary model also arise in fluid dynamics. There, these equations describe the equilibrium of an incompressible and inviscid fluid in a vessel T , and the analogue of the sharp boundary problem is that of two toroidal jets of fluid separated by a toroidal vortex sheet. The analogy between the MHD and fluid problems is given by $B \approx u$ and $p \approx -(\frac{1}{2}|u|^2 + p)$; with these substitutions our equations are identical with the Euler equations

$$\begin{aligned}u \cdot \nabla u &= -\nabla p \text{ in } T \\ u \cdot \mathbf{n} &= 0 \text{ in } \partial T.\end{aligned}\tag{2}$$

Our result can therefore be interpreted as the first rigorous result for the existence of truly nonlinear 3-D solutions with non-zero vorticity in a toroidal domain. Our problem was discussed in the MHD and fluids settings by Arnold [2], Moffatt [34] and others. These authors observed that results of existence of solutions with non-trivial topological structure would follow provided corresponding asymptotic limits as $t \rightarrow \infty$ of solutions of a related Navier-Stokes like system could be shown to exist. A proof of existence of such asymptotic limits has not been given, however.

We will construct solutions of the three dimensional MHD equations by means of *analytic* perturbations from certain axisymmetric configurations for which solutions are known to exist [24] (see also [30, 16] for considerations on more general axisymmetric equilibria). Our solutions reflect the nonlinearity of the MHD equations in their behavior on a certain toroidal surface S , nested with and interior to ∂T , on which a jump in the pressure occurs. The aforementioned perturbations from axisymmetry are introduced through this discontinuity surface: for a perturbed surface S we find the new boundary ∂T and fields $\mathbf{B}$ in the regions interior and exterior to S . Inside the surface S the magnetic field $\mathbf{B}$ will be taken to be the gradient of a harmonic function; outside S , $\mathbf{B}$ will also be given as a harmonic field, or, more generally, as a solution of the Beltrami equations $\nabla \times \mathbf{B} = \alpha \mathbf{B}$. Thus, the pressure is constant both inside and outside S . As a simple extension, we will also present solutions with a finite number k of values of the pressure.

The solution inside the perturbed surface S can be obtained without regards to the solution in the outside region, since the former is simply defined by a homogeneous Neumann problem. To determine the solution outside S we must first determine the values of the solution *on the outside of the surface S* from the values of the inner solution. We do this through an application of the twist map theorem to a certain Hamiltonian system associated with a geometrical-optics equation on S . The idea of solving a geometrical-optics type equation through proper choice of initial values and the twist map theorem is known, and can be traced back to ideas of Grad and Moser. Some details on the solution of our surface equation were given in [5, 25]; the literature is otherwise devoted to the case of geodesics which do not completely cover the torus [3]. (The latter situation corresponds to the generic case in which our equation cannot be solved on the full surface of the torus, whereas we deal with cases which are close to integrable, and for which a solution on the entire surface can be found. A complete construction in this case is given in §3). Once the surface values of the solution have been found, the outside field can be obtained as a solution of a Cauchy problem for the Beltrami system mentioned above. The existence of an outermost toroidal magnetic surface ∂T follows from yet another application of the KAM theorem—one which leads us to certain geometrical restrictions associated with the nondegeneracy of the twist map.

Notice that in order to satisfy the hypothesis of the KAM theorem in the latter application we need the perturbation in the twist map induced by the perturbed field lines to be rather smooth and small. This cannot generally be guaranteed from a corresponding smoothness of the variations of the Cauchy initial values, since the Cauchy problem is ill posed for our equations. We are thus led to considering a setup which insures the analytic dependence of the Cauchy data on the perturbations as well as the spatial variables. In order to obtain jointly analytic Cauchy data we use jointly analytic variations of the surface S . We then show that the corresponding boundary values of the solution of the inner Neumann problem inherit the property of joint

analyticity, see §3. This analytic dependence carries over to the outer field through the Cauchy-Kovalevsky theorem, and the existence of an outer toroidal surface can be established. (The analyticity of the boundary values of the solution of the Neumann problem with respect to the spatial variables is, of course, classical [35, 32]. A theorem of analyticity with respect to boundary perturbations in the two dimensional case was given in [9] by means of holomorphic surface potentials. Our extension of that construction to the three dimensional Neumann problem is not direct, due to the nature of the singular behavior of fundamental solutions in three dimensional space and to the lack of uniqueness of solution for our Neumann problem. Thus, while we use the Fredholm theory in an analytic framework –as in [9]– our actual constructions differ substantially from those given before, see §3.

Our approach extends Grad’s original sketch [21] —which assumes field-excluded solutions, that is, solutions for which the inner field vanishes. A new element in our proof is the use of analytic setting mentioned above and announced in [8]. This analytic framework permitted us to construct an outer field satisfying the boundary condition on ∂T and thus, to establish existence of solutions with non-constant pressure. Other discussions, without rigorous proofs, which incorporate some aspects of Grad’s ideas in connection with three dimensional MHD equilibria have been given in the physics literature [5, 25]. In [5], for example, a harmonic inner field like ours was considered, the related general equation of geometrical optics was put forward, and its relation to the KAM theory was discussed. In [25], on the other hand, the authors returned to the field excluded case in which the equation of geometrical optics is identical with that of the geodesic on the torus, and they showed that the associated twist map satisfies the KAM monotonicity hypothesis.

This paper is organized as follows. In §1 we discuss the geometry and some preliminaries. In §2 we present our main result; the proof given there is based on the results of the following three sections. In §3 we discuss the properties of analyticity of solutions of the Neumann problem with respect to boundary variations. In §4 we establish existence and regularity of solutions of the surface equation. Finally, the existence of solutions for the outer shell problem is discussed in §5.

1 The ideal MHD equations

1.1 Weak solutions

Using the Maxwell stress tensor

$$T = \mathbf{B} \otimes \mathbf{B} - \frac{|\mathbf{B}|^2}{2} I, \quad (3)$$

the MHD equations (1) can be rearranged into the following system in divergence-form [18]

$$\begin{aligned}\nabla \cdot (T - pI) &= 0 \\ \nabla \cdot \mathbf{B} &= 0.\end{aligned}$$

The corresponding jump conditions at a surface S of discontinuity are then

$$[\mathbf{B} \cdot \mathbf{n}] = 0 \tag{4}$$

and

$$\left[\frac{|\mathbf{B}|^2}{2} + p \right] - \mathbf{B} \cdot \mathbf{n} [\mathbf{B}] = 0. \tag{5}$$

Here the square brackets indicate the jump across the surface S and $\mathbf{n}$ denotes the normal to the surface.

In what follows we shall only consider configurations for which, in addition,

$$\mathbf{B} \cdot \mathbf{n} = 0 \tag{6}$$

on the discontinuity surfaces. In such cases, equation (5) reduces to

$$[|\mathbf{B}|^2 + 2p] = 0. \tag{7}$$

We thus remark that a piecewise smooth solution with a discontinuity surface S is a weak solution if it satisfies the conditions (6) and (7) above. The jump

$$[\mathbf{n} \times \mathbf{B}]$$

gives the surface current on the discontinuity surface.

1.2 The geometry

We shall deal with a family of toroidal surfaces S^ϵ obtained by means of normal perturbations from an axisymmetric torus S^0 . We assume the surface S^0 is defined by a parametrization of the form

$$\mathbf{x} = (x_1, x_2, x_3) = G_0(u_1, u_2), \tag{8}$$

where $G_0 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^3$ is given by

$$G_0(u_1, u_2) = (R_0 + \xi(u_1)) \cos(2\pi u_2) \mathbf{e}_{x_1} + (R_0 + \xi(u_1)) \sin(2\pi u_2) \mathbf{e}_{x_2} + x_3(u_1) \mathbf{e}_{x_3}. \tag{9}$$

Here e_{x_1} , e_{x_2} and e_{x_3} are orthonormal unit vectors. The cross-section $(\xi(u_1), x_3(u_1))$ is a curve which contains the origin in its interior and such that $(\xi^2 + x_3^2) \leq R_0^2$. For definiteness we take the parametrization (ξ, x_3) in such a way that

$$\nabla u_1, \nabla u_2 \text{ and the outer normal } n \text{ form a left handed triple.} \quad (10)$$

Further we assume that this curve is *parametrized by normalized arclength*, that is, denoting by l_0 the length of the curve, we have

$$(\xi')^2 + (x_3')^2 = l_0^2. \quad (11)$$

Note that the coordinate u_2 , which changes by one with each wind the long way around the torus, is just a multiple of the azimuthal angle. The coordinate u_1 , on the other hand, may or may not be given by the poloidal angle; our setup allows for cross-sections which are not starshaped, such as those related to the bean-shaped plasmas; see §5.2. In the following sections we will occasionally use the cylindrical coordinates (r, u_2, z) ; note the relations

$$z = x_3$$

and

$$r = R_0 + \xi = \sqrt{x_1^2 + x_2^2}.$$

The perturbed surfaces S^ϵ , on the other hand, are given by the parametrizations

$$x = G(u_1, u_2; \epsilon) = G_0(u_1, u_2) + \epsilon Q(u_1, u_2)n \quad (12)$$

for some function $Q : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$.

The tangential gradient on the surface S^ϵ will be denoted by

$$\nabla_s = \nabla - n \frac{\partial}{\partial n}$$

where n is the unit normal to S^ϵ . (The dependence on ϵ will not be explicitly indicated when the meaning of symbols such as n or ∇_s is clear from the context). Alternatively, calling g_{ij} and g^{ij} the metric tensor and its inverse,

$$g_{ij} = \frac{\partial G}{\partial u_i} \cdot \frac{\partial G}{\partial u_j}, \quad (13)$$

$$g^{ij} g_{jk} = \delta^i_k,$$

the surface gradient is given by

$$\nabla_s \Phi = \left(g^{11} \frac{\partial G}{\partial u_1} + g^{12} \frac{\partial G}{\partial u_2} \right) \frac{\partial \Phi}{\partial u_1} + \left(g^{12} \frac{\partial G}{\partial u_1} + g^{22} \frac{\partial G}{\partial u_2} \right) \frac{\partial \Phi}{\partial u_2}.$$

Denoting by D the determinant of (g_{ij})

$$D = g_{11}g_{22} - g_{12}^2,$$

we have, of course,

$$g^{11} = g_{22}/D, \quad g^{22} = g_{11}/D, \quad g^{12} = g^{21} = -g_{12}/D.$$

1.3 Sharp boundary solutions

As we mentioned, we will consider sharp boundary solutions of the system (1), that is, solutions for which the pressure is a step function and the magnetic field is piecewise smooth—and locally equal to either a harmonic fields or a constant- α force-free field.

More precisely, we will deal with *weak solutions* $(\mathbf{B}, p)$ of the system (1) in a three dimensional toroidal volume T which verify points 1) through 4) below. In what follows $\alpha_2, \dots, \alpha_k$ and $p_1, \dots, p_k$ denote two sequences of constants, $p_{i+1} < p_i$, while T_i ($1 \leq i \leq k$) denotes a sequence of nested toroidal volumes, with $T_k = T$ and T_i contained in the interior of T_{i+1} . Further, S_i denotes the boundary of T_i , and $\mathbf{B}_i^+$ (resp. $\mathbf{B}_i^-$) denotes the limiting values of the magnetic field on S_i from outside (resp. from inside).

We thus seek solutions of equations (1) such that

1. $p = p_{i+1}$ on $T_{i+1} \setminus T_i$.
2. $\mathbf{B}_i^+$ and $\mathbf{B}_i^-$ are tangent to S_i
3. The jump condition

$$|\mathbf{B}_i^+|^2 = 2(p_i - p_{i+1}) + |\mathbf{B}_i^-|^2. \quad (14)$$

is satisfied and

4. $\mathbf{B}$ is the gradient of a (multivalued) harmonic function θ inside T_1 , and it satisfies the equation

$$\nabla \times \mathbf{B} = \alpha_i \mathbf{B} \quad (15)$$

in the region $T_i \setminus T_{i-1}$ for $2 \leq i \leq k$.

The positive quantity on the right hand member of equation (14) will be denoted by a_i

$$a_i = 2(p_i - p_{i+1}) + |\mathbf{B}_i^-|^2. \quad (16)$$

Note that a_i can be computed from the sole knowledge of the pressure jump and the field inside S_i . The assumption

$$p_{i+1} < p_i,$$

which is included above to guarantee the positivity of a_i , can be relaxed to $2(p_i - p_{i+1}) + K^2 > 0$ if $|\mathbf{B}_i^-| > K > 0$.

From points 2) and 4) above we see that $\nabla \times \mathbf{B}$ is tangent to the discontinuity surfaces. It follows from Stokes' theorem that the quantity $\mathbf{B}_i^+$ must be given by the surface gradient of a (multivalued) potential

$$\mathbf{B}_i^+ = \nabla_s \Phi_i.$$

We can thus write the geometrical optics equation

$$|\nabla_s \Phi_i|^2 = a_i, \quad (17)$$

which can be viewed as a means for computing, via the multivalued functions Φ_i , the limiting values of the field outside S_i from the corresponding inside values and pressure jump. After some manipulation (and dropping the subindex i), equation (17) can be made to read

$$g_{22}\left(\frac{\partial \Phi}{\partial u_1}\right)^2 + g_{11}\left(\frac{\partial \Phi}{\partial u_2}\right)^2 - 2g_{12}\frac{\partial \Phi}{\partial u_1}\frac{\partial \Phi}{\partial u_2} - Da(x) = 0,$$

or, further,

$$\frac{\partial \Phi^\pm}{\partial u_1} = \frac{g_{12}\frac{\partial \Phi}{\partial u_2} \pm \sqrt{D\left\{g_{22}a - \left(\frac{\partial \Phi^\pm}{\partial u_2}\right)^2\right\}}}{g_{22}}. \quad (18)$$

It is easy to convince oneself that equation (17) does not determine $\mathbf{B}_i^+$ from a_i , even after a choice is made of the arbitrary sign in equation (18). In many cases of interest, however, the additional requirement that the field has a given winding number β suffices to determine a solution Φ_i , up to an additive constant, and thus the boundary values of the magnetic field $\mathbf{B}_i^+$. (The winding number β associated with the solution of an ODE in a torus is defined as the number of times a field line winds the long way around a torus S divided the number of times it winds the short way around in the limit of an infinite number of winds [13, 4].)

For example, we will show in §1.5 that, for axisymmetric configurations, the prescription of the rotation number within certain intervals does indeed determine $\mathbf{B}_i^+$. In the non-axisymmetric case the same is true provided the departure from axisymmetry is sufficiently smooth and small, and that the rotation numbers are restricted to certain classes of Diophantine irrationals. A central element in the proof of this result is the KAM theory —which we review briefly in the following section.

1.4 KAM theory

As mentioned above, the KAM theory [27, 1, 36, 38, 41] and the associated Diophantine irrationals defined below play an important role in the solution of our surface problems.

Definition 1 *The real number β is said to be a Diophantine of type τ if there exists a positive constant γ such that there holds*

$$\left| \frac{\beta}{2\pi} - \frac{p}{q} \right| \geq \gamma q^{-\tau} \text{ for all rational numbers } \frac{p}{q}. \quad (19)$$

In what follows $C^n(\Omega)$ ($n \in \mathbb{N}$) (resp. $C^\omega(\Omega)$) denotes the class of all n -times continuously differentiable functions (resp. real analytic functions) defined in a domain $\Omega \subseteq \mathbb{R}^d$. It is to be understood that, for $k \in \mathbb{N}$,

$$C^{\eta \pm k}(\Omega) = \begin{cases} C^{n \pm k}(\Omega) & \text{if } \eta = n \in \mathbb{N} \\ C^\omega(\Omega) & \text{if } \eta = \omega. \end{cases} \quad (20)$$

Using polar coordinates (F, u) , we will call $A \subseteq \mathbb{R}^2$ the annulus $A = \{a \leq F \leq b\}$, $0 < a < b$. Also, for $h \in C^n(A)$ and for $l \leq n$ we will write

$$|h|_l = \sup_{\substack{a \leq F \leq b \\ 0 \leq i+j \leq l}} \left| \frac{\partial^{i+j} h}{\partial F^i \partial u^j} \right|.$$

Consider a twist map defined in A by the equations

$$\begin{aligned} F' &= F \\ u' &= u + \beta(F). \end{aligned} \quad (21)$$

Moser's theorem [36] relates to smooth perturbations of the map (21) of the form

$$\begin{aligned} u' &= u + \beta(F) + f(F, u) \\ F' &= F + g(F, u). \end{aligned} \tag{22}$$

It states that the perturbed map (22) admits invariant curves under certain conditions. The following version of the twist map theorem, which requires limited smoothness on the perturbations, is due to Rüssman [40].

Theorem 1 *Let $\beta(F) \in C^5[a, b]$ and let $f(F, u; \epsilon)$ and $g(F, u; \epsilon)$ be of class C^5 in the annulus A . Assume there is a constant $k > 0$ such that $k^{-1} \leq |\beta'(F)| \leq k$ and such that the estimate*

$$|f|_5 + |g|_5 \leq k$$

holds for $F \in [a, b]$ and for all u . Further, let γ be sufficiently small so that there are Diophantine irrationals β satisfying both equation (19) with $\tau = \frac{9}{4}$ and $\beta(a) + \gamma \leq \beta \leq \beta(b) - \gamma$, and let β_0 be one such irrational. Finally, assume the perturbed map (22) satisfies the self-intersection property, that is, the property that each twice-differentiable curve near a circle $F = \text{const.}$ (in norm $|\cdot|_2$) intersects its own image under the map. Under these assumptions it follows that there exists a number $\delta_0 = \delta_0(\gamma, k) \in \mathbb{R}$ such that, if $\delta < \delta_0$ and if

$$|f| + |g| < \delta$$

in A , then the twist map (22) admits a closed invariant curve $\mathcal{Q} \subseteq A$ within which the induced map has rotation number β_0 . The curve $\mathcal{Q}$ can be represented by a parametrization of the form

$$F = c + p(\eta), \quad u = \eta + q(\eta) \tag{23}$$

for some $c \in [a, b]$ and certain periodic functions p and $q \in C^2$ in such a way the induced mapping within $\mathcal{Q}$ is given by

$$\eta \mapsto \eta + \beta_0.$$

The twist maps we will consider (in §3) arise from certain Hamiltonian systems associated with our MHD equations. As is well known, denoting by ϵ a perturbation parameter, Hamiltonians $h(t, u, F; \epsilon)$ which are periodic of period 1 in both the time t and u , and jointly analytic in the time and the spatial variables u and F lead to (spatially) analytic invariant curves [27, 1, 41] for every sufficiently small fixed value of ϵ . It is also known that if a Hamiltonian $h(t, u, F; \epsilon)$ is jointly analytic in the variables $(t, u, F; \epsilon)$ (as the Hamiltonian we will consider in §3.2), then the parametrization of the invariant curve is jointly analytic in the spatial and perturbation variables. That is, under the hypothesis of joint analyticity of the Hamiltonian, the functions $p(\eta; \epsilon)$

and $q(\eta; \epsilon)$ associated with the invariant curve (23) belong to the class C^ω in η and ϵ for $0 \leq \eta \leq 2\pi$ and $|\epsilon|$ small enough; see [11, p. 124], [38, p.527], [7, p. 124].

The results given in these references establish the joint analyticity of the invariant curve in η and ϵ in case the Hamiltonian is independent of the time t for $\epsilon = 0$. An extension to the case discussed above, in which h does depend on t even for $\epsilon = 0$, follows easily from these results. Indeed, a change of variables $u \rightarrow u + \phi(t, F)$ (with a suitably chosen function $\phi = \phi(t, F)$ which is periodic in t) reduces the more general case to that in which the unperturbed Hamiltonian is time independent.

1.5 The axisymmetric case

In this section we present a construction of sharp boundary solutions $(\mathbf{B}, p)$ in the axisymmetric case. While not strictly necessary for our proof of §2, the present construction of axisymmetric solutions will permit us to introduce some basic concepts and notations.

According to §1.3, the sharp boundary field $\mathbf{B}$ in the region T_1 should be a harmonic field $\nabla\theta$ ($\Delta\theta = 0$) satisfying homogeneous Neumann conditions $\frac{\partial\theta}{\partial n} = 0$ on ∂T_1 . It is easy to see that in the present axisymmetric case the function θ must necessarily equal a constant multiple of the azimuthal angle u_2 :

$$\theta = cu_2. \quad (24)$$

Indeed, we see from Stokes theorem that for any two points $(r_1, 0, z_1)$ and $(r_2, 0, z_2)$ in the cross section $T_1 \cap \{u_2 = 0\}$ we must have

$$\theta(r_1, 2\pi, z_1) - \theta(r_1, 0, z_1) = \theta(r_2, 2\pi, z_2) - \theta(r_2, 0, z_2) = c$$

for some constant c . Then, calling $\eta = \theta - cu_2$ and integrating by parts the energy integral

$$\int_{T_1} (\nabla\eta)^2$$

yields (24) –since $\Delta\eta = 0$. A unique inner solution is thus specified if value of the total flux of $\mathbf{B}$ over the cross section of the torus is imposed.

As for the outer fields, the ansatz

$$\mathbf{B}_i = \nabla\psi_i \times \nabla u_2 + (\alpha_i\psi_i + F_i)\nabla u_2 \quad (25)$$

($\psi = \psi(r, z)$ and F_i a constant), transforms our equations (15) into the Grad-Shafranov equations

$$-\Delta^*\psi_i = \alpha_i(\alpha_i\psi_i + F_i). \quad (26)$$

Here the operator Δ^* is defined, in cylindrical coordinates (r, u_2, z) by

$$\Delta^* = \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r} \frac{\partial \psi}{\partial r}. \quad (27)$$

Since equation (26) admits no characteristic curves, the field $\mathbf{B}_i$ can be determined, by the Cauchy-Kovalevsky theorem, from analytic Cauchy data for ψ_i and $\frac{\partial \psi_i}{\partial n}$ on the surface S_i (provided the surface S_i is itself analytic).

To determine this Cauchy data we first point out (with Jörgens [24]) that ψ_i must be constant on our surface S_i –as follows from equation (6). Without loss of generality we take

$$\psi_i|_{S_i} = 0 \quad (28)$$

for all $1 \leq i \leq k$. Equations (14) and (16) then give us

$$\left| \frac{\partial \psi_i}{\partial n} \right| = \sqrt{(2\pi r)^2 a_i - |F_i|^2}, \quad (29)$$

and thus the normal derivative of ψ_i –up to a sign; for definiteness, we choose this sign so that

$$\frac{\partial \psi_i}{\partial n} = \sqrt{(2\pi r)^2 a_i - |F_i|^2}; \quad (30)$$

the treatment in the case $\frac{\partial \psi_i}{\partial n} < 0$ is entirely parallel. (This choice of a sign is equivalent to that arising in equation (18). Indeed, note that in the present axisymmetric case we have $g_{12} = 0$.)

We shall always assume F_i is such that the radicand on the right hand side of (30) is positive. By the Cauchy-Kovalevsky theorem, then, equation (26) together with the Cauchy data (28), (30) determine ψ_i and therefore $\mathbf{B}$ in a certain region outside $S_i = \{\psi_i = 0\}$. Since $\frac{\partial \psi_i}{\partial n} > 0$ on $S_i = \{\psi_i = 0\}$, the surfaces $\psi_i = c$ for small positive c form a nested set of toroidal analytic surfaces surrounding S_i . These are all magnetic surfaces, that is, $\mathbf{B}$ is tangent to $\psi_i = c$ for all sufficiently small c . We may thus take any of these to be S_{i+1} , and our construction of sharp boundary solutions then follows inductively.

The solution $\mathbf{B}_i$ above is determined by a choice of the constant F_i satisfying $(2\pi r)^2 a_i - |F_i|^2 > 0$ on S_i . It is important for us to note that the constants F_i and the corresponding rotation numbers of the resulting fields $\mathbf{B}_i^+$ are in one-to-one correspondence, and that, therefore, the prescription of the constant F_i may be

substituted by prescription of the corresponding rotation number. To show this we first remark from equation (25) that the field lines can be parametrized in the form $X(t) = G_0(u_1(t), u_2(t))$, so that

$$\frac{\partial G_0}{\partial u_1} \frac{du_1}{dt} + \frac{\partial G_0}{\partial u_2} \frac{du_2}{dt} = \mathbf{B}(G_0(u_1(t), u_2(t))).$$

Assuming –as we may– that the colinear vectors $\frac{\partial G_0}{\partial u_1}$ and $\nabla \psi_i \times \nabla u_2$ are related by a positive factor, equations (25) and (11) give us

$$\frac{du_2}{du_1} = \frac{l_0 F_i}{2\pi r \sqrt{(2\pi r)^2 a_i - F_i^2}}.$$

It follows that the rotation number $\beta = \beta_i$ of the field lines of the field $\mathbf{B}$ associated with the quantities $F = F_i$ and $a = a_i$ is given by the expression

$$\beta = \int_0^1 \frac{l_0 F}{2\pi r \sqrt{(2\pi r)^2 a - F^2}} du_1, \quad (31)$$

and thus, that $\frac{d\beta_i}{dF_i} > 0$. Defining

$$\Omega_i = \{\beta_i \text{ which correspond to some admissible } F_i\}, \quad (32)$$

we have shown that two different values of F_i necessarily give rise to two different rotation numbers in Ω_i . Together with our remarks on harmonic fields at the beginning of this section, the present arguments establish the existence of axisymmetric solutions containing an arbitrary number of constant-pressure layers, as stated in Theorem 2 below. This theorem contains a statement –which we will need in connection with one of our applications of the KAM theory– on the ratio of curvatures of the axisymmetric torus, see Definition 2. The proof of this additional statement results from a simple continuity argument.

Definition 2 *Let an axisymmetric torus S^0 be given by the parametrization (8), (9), and denote by*

$$R(x) = \frac{1}{\kappa(x)}$$

the radius of curvature of the cross-section $\mathcal{C} = \{(\xi, z)\}$. The curvature ratio of T is defined by

$$A = \min_{x \in \mathcal{C}} \frac{r(x)}{R(x)}.$$

The inductive construction above of axisymmetric solutions with a finite numbers of outer layers gives us the following Theorem

Theorem 2 *Let there be given two sequences of numbers $\alpha_2, \dots, \alpha_j$ and $p_1, \dots, p_j$, $p_{i+1} < p_i$. Let $T_1, \dots, T_j$ be a family of axisymmetric nested tori with analytic boundaries, and let $(\mathbf{B}^j, p)$ be a weak solution of equations (1) in such a way that conditions 1) through 4) of §1.3 are satisfied. Then, we have*

1. *For every pair of real numbers p_{j+1} , α_{j+1} ($p_{j+1} < p_j$) and for every $\beta_j \in \Omega_j$ there exists a neighborhood V of S_j and a function ψ_{j+1} defined in V which satisfies equations (26), (28) and (30). Also, there is a neighborhood J_{j+1} of 0 in the real line such that, for every constant $c \in J_{j+1}$, $c > 0$, the solution set of the equation $\psi_{j+1} = c$ is an analytic toroidal surface nested with and containing T_j .*
2. *Take $c_{j+1} \in J_{j+1}$ and call S_{j+1} the toroidal surface $\psi_{j+1} = c_{j+1}$ and T_{j+1} the corresponding toroidal volume. Define the field $\mathbf{B}^{j+1}$ as equal to $\mathbf{B}^j$ on T_j and as given by equation (25) (with $i = j + 1$) on $T_{j+1} \setminus T_j$. Then $(\mathbf{B}^{j+1}, p, \alpha)$ is a weak solution of equations (1) on T_{j+1} satisfying points 1) through 4) of §1.3.*
3. *If S_j has a convex cross-section and has an curvature ratio larger than one, then for c_{j+1} small enough S_{j+1} also has a convex cross section and an curvature ratio larger than one.*

Our construction of axisymmetric equilibria may be compared with that given by Jörgens [24] in the harmonic case ($\alpha = 0$). This author used an integral representation for the solution outside the discontinuity surface. While the present approach appears more appropriate to our purposes, Jörgens' produces estimates of the domain of existence and analyticity for the solution which may be sharper than those given here by means of the Cauchy-Kowalevsky theorem. For instance in the case of a circular cross section, Jörgens showed that the solution exists everywhere in the exterior of the torus with the exception of the symmetry axis. Like ours, the approach of this author assumes analyticity of the discontinuity surface and initial data.

2 Three-dimensional solutions with non-constant pressure

Given an axisymmetric toroidal surface S , we call S^- the region facing the axis of symmetry, in which the radial derivative $\frac{\partial \psi}{\partial r} < 0$, and we call S^+ the complement of S^- . (Recall we consider the case in which $\nabla \psi$ points in the direction of the outer normal). We begin with a definition which relates our equations to the curvature κ of the cross-section of an axisymmetric toroidal surface S , see §5.2.

Definition 3 *We will say that a pair of real numbers F and α are admissible with respect to the axisymmetric toroidal surface S provided one of the following conditions is satisfied:*

- (i) α and F are both positive or both negative, $\kappa r > -|\alpha|r$ in S^- and $\kappa r > 1 - |\alpha|r$ in S^+ .
- (ii) α and F have different signs, $\kappa r < -1 + |\alpha|r$ in S^- and $\kappa r < |\alpha|r$ in S^+ .

The next Theorem is the most general result we present of existence of solutions containing a single discontinuity surface. Then we give another statement which relates to solutions with an arbitrary number of discontinuity surfaces, and which, for the sake of simplicity, is not given in its most general form. The proof of the second statement, which is based on the results of §3, 4 and 5, is given below. The proof of the first statement is analogous.

Theorem 3 *Let there be given an axisymmetric torus T_1 with analytic boundary S_1 together with a real number $\alpha = \alpha_1$ and a prescription for the sign of the parameter $F = F_1$ in such a way that the admissibility condition described in Definition 3 is satisfied. Let a family of perturbations T_1^ϵ also be given, whose boundaries S_1^ϵ admit parametrizations of the form (12) for a fixed analytic function Q . Further, let c , p_1 , p_2 ($p_1 < p_2$) and α be real numbers. Then, there exists a number $\epsilon_0 > 0$ such that for every ϵ with $|\epsilon| < \epsilon_0$ there is a torus T_2^ϵ with boundary S_2^ϵ , S_2^ϵ enclosing and nested with S_1^ϵ , and a vector field $\mathbf{B}^\epsilon$ defined in T_2^ϵ in such a way that $(\mathbf{B}^\epsilon, p)$ is a weak solution of equations (1), points 1) through 4) of §1.3 are verified, the rotation number $\beta = \beta_1$ of $\mathbf{B}$ on S_1 corresponds to $F = F_1$ through (31), and the potential θ giving $\mathbf{B} = \nabla \theta$ in T_1^ϵ satisfies*

$$(\theta(r, u_1, 1; \epsilon) - \theta(r, u_1, 0; \epsilon)) = c \quad (33)$$

inside T_1^ϵ for $|\epsilon| < \epsilon_0$.

Theorem 4 *Let there be given an axisymmetric torus T_1 with analytic boundary S_1 with a convex cross-section and with curvature ratio larger than one, see Definition 2. Let a family of perturbations T_1^ϵ also be given, whose boundaries S_1^ϵ admit parametrizations of the form (12) for a fixed analytic function Q . Further, let $k \in \mathbb{Z}$, $c \in \mathbb{R}$ and let $p_1, \dots, p_k$ ($p_{i+1} < p_i$) and $\alpha_2, \dots, \alpha_k$ be sequences of real numbers. Then, there exists a number $\epsilon_0 > 0$ such that for every ϵ with $|\epsilon| < \epsilon_0$ there are tori T_i^ϵ with boundaries S_i^ϵ ($2 \leq i \leq k$), S_i^ϵ enclosing and nested with S_{i-1}^ϵ ($2 \leq i \leq k$), and a vector field $\mathbf{B}^\epsilon$ defined in T_k^ϵ such that $(\mathbf{B}^\epsilon, p)$ is a weak solution of equations (1), such that points 1) through 4) of §1.3 are verified, and such that the potential θ giving $\mathbf{B} = \nabla\theta$ in T_1^ϵ satisfies equation (33) inside T_1^ϵ for $|\epsilon| < \epsilon_0$.*

Proof of Theorem 4:

To obtain the inner solution we appeal to Theorem 5 in §3. That theorem tells us that there is a number ϵ_1 such that, for $|\epsilon| < \epsilon_1$, there exists a *single valued* solution $\Psi(r, u_1, u_2; \epsilon)$ of the Neumann problem

$$\Delta\Psi = 0,$$

$$\frac{\partial\Psi}{\partial n}|_{\partial T_1^\epsilon} = -c \frac{\partial u_2}{\partial n}$$

inside T_1^ϵ which is jointly analytic in the spatial variables and ϵ . For points inside T_1^ϵ we define the harmonic function

$$\theta = (\Psi(r, u_1, u_2; \epsilon) + cu_2)$$

so that equation (33) is satisfied. We then set

$$\mathbf{B}(r, u_1, u_2; \epsilon) = \nabla\Phi(r, u_1, u_2; \epsilon) \text{ for } x \in T_1^\epsilon;$$

it follows directly that equation (6) holds. This definition now leads to a values for the quantity a_1 , see (16), which depend analytically on u_1, u_2 and ϵ for $|\epsilon| < \epsilon_1$.

We will now proceed to show inductively that the following statement is true for all $j \leq k$: “there is an increasing sequence $T_1^\epsilon, \dots, T_j^\epsilon$ of nested tori together with an associated weak solution for our problem which satisfies points 1) through 4) of §1.3. Further we have a) the boundaries $S_i^\epsilon = \partial T_i^\epsilon$ and the inside boundary values B_i^- vary analytically with ϵ, u_1 and u_2 for $i \leq j$, and b) for $\epsilon = 0$ the boundaries have convex cross-sections and curvature ratios larger than one”.

Let us assume this statement is true for a certain $j < k$ and let $\beta_j \in \Omega_j$ be a Diophantine irrational (of type $\frac{9}{4}$, see Definition 1 and equation (32)). By Theorem 7 in §4.2 we know that there is a function $\Phi_j(u_1, u_2; \epsilon)$, which satisfies the equation

$$|\nabla_s \Phi_j|^2 = a_j, \tag{34}$$

on S_j^ϵ , which is jointly analytic in its three variables for sufficiently small ϵ , and such that the rotation number of $\nabla_s \Phi_j$ equals β_j .

By Theorem 8 in § 5 there is a jointly analytic solution $\mathbf{B}(x; \epsilon)$ of the system

$$\nabla \times \mathbf{B} = \alpha_{j+1} \mathbf{B}$$

satisfying the initial conditions

$$\mathbf{B}|_{\partial T_j^\epsilon} = \nabla_s \Phi_j.$$

We assume, as we may, that this solution is defined in a neighborhood V of S_j^0 which contains S_j^ϵ for sufficiently small ϵ .

To conclude our inductive step –and the proof of the theorem– we appeal to Appendix § A.2 and to Theorem 9 and its Corollary in §5.2. The results in § A.2 show that the field lines of $\mathbf{B}$ are solutions of a Hamiltonian system with a jointly analytic Hamiltonian. Under our hypothesis, the conditions of monotonicity implied by Theorem 9 permit us to obtain (via the results of §1.4) a jointly analytic invariant curve for the Poincaré map associated with this Hamiltonian system, which is exterior to T_j , and which is defined for sufficiently small ϵ . In particular, this establishes the existence of a jointly analytic toroidal surface $S_{j+1}^\epsilon \subseteq V$ which is a magnetic surface (that is equation (6) is satisfied), which for small ϵ is nested with and contains T_j^ϵ in its interior, and such that S_{j+1}^0 is convex and has curvature ratio larger than one. The proof of the theorem is now complete.

3 The inner volume solution: joint analyticity

In this section we establish that solutions to a Neumann problem for Laplace's equation in a variable domain $T^\epsilon \subseteq \mathbb{R}^3$ depend analytically on the variations of the domain's boundary S^ϵ . A related result was given in [9] in the two dimensional case and for a problem for the Helmholtz equation which admits a unique solution. Our extension of the methods of that paper is not direct, due to the nature of the singular behavior of fundamental solutions in three dimensional space and to the lack of uniqueness of solution for our Neumann problem. Thus, while we use the Fredholm theory in an analytic framework –as proposed in [9]– our actual constructions differ substantially from those given before. For notational simplicity we assume S^ϵ is a toroidal surface in $\mathbb{R}^3$, but our proofs extend easily to general analytic surfaces in $\mathbb{R}^n$ ($n \geq 2$), to either Dirichlet or Neumann problems, and to general analytic perturbations.

Let our ϵ dependent toroidal surface $S^\epsilon = \partial T^\epsilon$ be parametrized by a function f

$$S^\epsilon = \{x \in \mathbb{R}^3 : x = f(u_1, u_2; \epsilon)\}, \quad (35)$$

see e.g. (12), which is periodic of period one in the variables $u = (u_1, u_2)$ and which satisfies

$$P = P(u; \epsilon) = \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \neq 0 \quad (36)$$

for all

$$(u; \epsilon) \in \mathbb{R}^2 \times (-c, c).$$

For definiteness we assume the parametrization f has been chosen so that P is an outer normal to S^ϵ . The unit normal to S^ϵ at a point $x = f(u; \epsilon) \in S^\epsilon$ is then given by

$$n_x = n(x; \epsilon) = m(u; \epsilon), \quad (37)$$

where

$$m(u; \epsilon) = \frac{P(u; \epsilon)}{|P(u; \epsilon)|}. \quad (38)$$

Throughout this section we further assume that the function $f = f(u_1, u_2; \epsilon)$ is analytic for all $(u_1, u_2; \epsilon) \in \mathbb{R}^2 \times (-c, c)$ for some $c > 0$, and we denote $C_{per}^\omega(\mathbb{R}^2 \times (-c, c))$ the space of jointly analytic functions which are periodic of period 1 in their first two variables. Clearly, a function $f \in C_{per}^\omega(\mathbb{R}^2 \times (-c, c))$ can be extended analytically to complex arguments $(u_1 + is_1, u_2 + is_2; \epsilon)$ for sufficiently small real values of s_1 and s_2 , and for complex ϵ with $|\epsilon|$ sufficiently small.

Now, given a jointly analytic family of Neumann data $g(x; \epsilon)$ we seek a function $\Psi(x; \epsilon)$ which, for every fixed $\epsilon \in (-\delta, \delta)$, satisfies the Neumann problem

$$\Delta \Psi = 0,$$

$$\frac{\partial \Psi}{\partial n}|_{S^\epsilon} = g. \quad (39)$$

and which is jointly analytic in x and ϵ inside T^ϵ , and even across $S^\epsilon = \partial T^\epsilon$. The main theorem of this section thus reads:

Theorem 5 *Let $c > 0$, let $f \in C_{per}^\omega(\mathbb{R}^2 \times (-c, c))$ and let $g = g(x; \epsilon)$, defined for*

$|\epsilon| < c$ and for $x \in S^\epsilon$, be such that $g(f(u; \epsilon); \epsilon) \in C_{per}^\omega(\mathbb{R}^2 \times (-c, c))$. Assume

equation (36) is satisfied for $|\epsilon| < c$ and for all $u \in \mathbb{R}^2$. Further, assume g has

zero average on S^ϵ for each fixed real value of ϵ . Then there exist a number ϵ_0 , a neighborhood V of $\cup_{\{|\epsilon| < \epsilon_0\}} T^\epsilon$ and a function $\Psi = \Psi(x; \epsilon) \in C^\omega(V \times (-\epsilon_0; \epsilon_0))$ such that, for each fixed $\epsilon \in (-\epsilon_0; \epsilon_0)$, the restriction of $\Psi(x; \epsilon)$ to the region bounded by S^ϵ satisfies the Neumann problem (39).

The proof of this theorem is given at the end of this section, following a sequence of six preliminary Lemmas. Let us denote by

$$N(x, y) = -\frac{1}{4\pi} \frac{1}{|x - y|}$$

the fundamental solution of Laplace's equation, by $dS(y)$ the surface element on a given surface, and let h be a function defined on the toroidal surface S^ϵ . Then, the function

$$\Psi(x; \epsilon) = L_N[h] = \int_{S^\epsilon} N(x, y) h(y; \epsilon) dS(y) \quad (40)$$

is a solution to the problem (39) if and only if the following integral equation is satisfied:

$$-\frac{1}{2}h + L_K[h] = g, \quad (41)$$

see [33, 18, 17]. Here we have put, for $x \in S^\epsilon$

$$L_K[h](x) = \int_{S^\epsilon} K(x, y; \epsilon) h(y; \epsilon) dS(y) \quad (42)$$

where dS denotes the surface element on S^ϵ , and where

$$K(x, y; \epsilon) = -\frac{1}{4\pi} \frac{\partial}{\partial n_x} \frac{1}{|x - y|} = \frac{1}{4\pi} \frac{n(x; \epsilon) \cdot (x - y)}{|x - y|^3}. \quad (43)$$

Using the variables u and setting $\rho(u; \epsilon) = h(f(u; \epsilon))$, equation (42) can be re-written as

$$L_K[\rho] = \int_{[0,1] \times [0,1]} K(x, f(u; \epsilon); \epsilon) \rho(u; \epsilon) |P(u; \epsilon)| du_1 du_2.$$

Our result of analyticity will result from complexification of the variables u , and ϵ . To begin, let $Q_{\delta,\nu}$ denote the region

$$Q_{\delta,\nu} = \{(u + is; \epsilon) \in \mathbb{C}^3 : s = (s_1, s_2) \in \mathbb{R}^2, |s_1| < \nu, |s_2| < \nu \text{ and } |\epsilon| < \delta\},$$

and, given sets $U, F \subseteq \mathbb{C}^n$, U open, call

$$\mathcal{H}(U) = \{\text{holomorphic functions in } U\}$$

and

$$C(F) = \{\text{continuous functions in } F\}$$

Also, define the Banach spaces

$$\mathcal{H}_{\delta,\nu} = \{\rho : \rho \in \mathcal{H}(Q_{\delta,\nu}) \cap C(\bar{Q}_{\delta,\nu}) \text{ and } \rho \text{ is periodic of period 1 in } u_1 \text{ and } u_2\}$$

with the norms

$$\|\rho\|_{\delta,\nu} = \max_{\bar{Q}_{\delta,\nu}} \{|\rho(u + is; \epsilon)|\}.$$

(Here we have put $\bar{Q}_{\delta,\nu} = \text{closure}(Q_{\delta,\nu})$.)

The operators L_K and L_N can be viewed as acting on the space of bi-periodic functions of period one. More precisely, we can define operators T_K and T_N on this space by

$$T_K[\rho](v; \epsilon) = \int_{[0,1] \times [0,1]} K(f(v; \epsilon), f(u; \epsilon)) \rho(u; \epsilon) |P(u; \epsilon)| du_1 du_2 \quad (44)$$

and

$$T_N[\rho](v; \epsilon) = \int_{[0,1] \times [0,1]} N(f(v; \epsilon), f(u; \epsilon)) \rho(u; \epsilon) |P(u; \epsilon)| du_1 du_2$$

for $v = (v_1, v_2) \in \mathbb{R}^2$ and small $\epsilon \in \mathbb{R}$. Note that, setting

$$A(x) = \frac{x}{4\pi \sqrt{(x \cdot x)}^3}, \quad (45)$$

and using (38), equation (44) can be made to read

$$T_K[\rho](v; \epsilon) = \int_{[0,1] \times [0,1]} m(v; \epsilon) \cdot A(f(v; \epsilon) - f(u; \epsilon)) \rho(u; \epsilon) |P(u; \epsilon)| du_1 du_2. \quad (46)$$

In Lemma 5 below we will show that for δ and ν small enough T_K can be extended to an operator which maps $\mathcal{H}_{\delta,\nu}$ into itself –a similar result, which we do not need, follows analogously for the operator T_N . It can be checked, however, that the simplest device of substituting complex values of v in equation (44) does not lead to an analytic function, due to the branch points which result from the radicals in the integrand. Our construction in Lemma 5 gives the necessary holomorphic extension by means of integration on a complex square which is “parallel” to $[0, 1] \times [0, 1]$, see equation (67). Before Lemma 5, however, we need to establish a number of facts. We begin by showing that for a given function $\rho(u_1, u_2; \epsilon) \in \mathcal{H}_{\delta,\nu}$, $T_K[\rho]$ is in some $\mathcal{H}$ space.

Lemma 1 *Let $\rho \in \mathcal{H}_{\delta,\nu}$. Then, there exist positive numbers δ_1 and ν_1 such that the functions $T_K[\rho](v; \epsilon)$ and $T_N[\rho](v; \epsilon)$ can be extended analytically to elements of $\mathcal{H}_{\delta_1,\nu_1}$.*

Proof: For $y \in S^\epsilon$ define $h(y; \epsilon)$ by

$$h(y; \epsilon) = \rho(u(y; \epsilon); \epsilon) \quad (47)$$

where the (multivalued) coordinates (u_1, u_2) of $u(y; \epsilon)$ are defined by

$$f(u(y; \epsilon); \epsilon) = y.$$

Since ρ is periodic of period 1, the analytic inverse function theorem tells us that h is jointly analytic with respect to $(y; \epsilon)$ for all points on the three-dimensional hypersurface

$$\tilde{S} = \{(y; \epsilon) : y \in S^\epsilon \text{ and } |\epsilon| < \delta\} \subseteq \mathbb{R}^4.$$

(Note that $\tilde{S}$ admits the non-singular parametrization $(f(u; \epsilon); \epsilon)$).

Let us now consider the Cauchy problem *in four dimensional space*:

$$\Delta w = 0$$

$$\frac{\partial w}{\partial n}|_{\tilde{S}} = h \quad (48)$$

and

$$w|_{\tilde{S}} = 0. \quad (49)$$

(Here we have denoted by $\Delta w = w_{y_1 y_1} + w_{y_2 y_2} + w_{y_3 y_3}$ the *three dimensional* Laplacian, which we regard here as an operator which acts on functions w of the four variables $(y; \epsilon)$.) We note that the four dimensional normal vector $\tilde{n}$ to $\tilde{S}$ necessarily contains a non-zero component in the three dimensional y space (since $\tilde{n}$ is orthogonal to $(f_\epsilon; 1)$), and, therefore, that $\tilde{S}$ is non-characteristic for our operator. The Cauchy-Kovalevsky theorem then implies the existence of a solution $w(y; \epsilon)$ which is defined and jointly analytic in a neighborhood W of $\tilde{S}$.

Consider now the surface $S^0 = S^\epsilon|_{\epsilon=0} \subseteq \mathbb{R}^3$ and take δ_0 small enough so that $(y + \delta n(y), 0) \in W$ for all $y \in S^0$ and for all δ with $-\delta_0 \leq \delta \leq \delta_0$. Also, call S^{ext} the toroidal surface $S^{ext} = \{y + \delta_0 n(y) | y \in S^0\}$ and call $F = \{y + \delta n(y) | y \in S^0 \text{ and } 0 \leq \delta \leq \delta_0\}$. Note that, by compactness of F and S^{ext} , there exist positive numbers $\epsilon_1, C > 0$ such that $F \times (-\epsilon_1; \epsilon_1) \subseteq W$ and such that for all $|\epsilon| < \epsilon_1$ the distance $d(S^\epsilon, S^{ext})$ between the surfaces S^ϵ and S^{ext} is larger than C : $d(S^\epsilon, S^{ext}) > C$.

Green's theorem applied to our function w now gives, for $x \in T^\epsilon \setminus S^\epsilon$,

$$\int_{S^\epsilon} \frac{-1}{4\pi|x-y|} h(y; \epsilon) dS(y) = \int_{S^{ext}} \left\{ \frac{-1}{4\pi|x-y|} \frac{\partial w}{\partial n} + w \frac{\partial}{\partial n} \frac{1}{4\pi|x-y|} \right\} dS(y). \quad (50)$$

As is known [33, 18, 17], differentiation of the function on the left hand side of this equation with respect to the outer normal (from inside) at $x = f(v; \epsilon)$ gives $(-\frac{1}{2}I + L_K)[h](f(v; \epsilon))$. The derivative of the right hand function at $x = f(v; \epsilon)$, then, results in an expression which presents $L_K[h](f(u; \epsilon)) = T_K[\rho](u; \epsilon)$ as an analytic function of u and ϵ for all complex values of u_1 and u_2 with sufficiently small imaginary part, and for $|\epsilon| < \epsilon_1$. The proof that $T_N[\rho]$ is in some $\mathcal{H}_{\delta, \nu}$ follows, analogously, from equation (50). This completes the proof of the Lemma.

The following result follows directly from the proof of the previous Lemma.

Lemma 2 *Let h be defined by equation (47) for some $\rho \in \mathcal{H}_{\delta, \nu}$. Then, there exists a toroidal volume T^{ext} , which contains T^ϵ in its interior for real ϵ with $|\epsilon|$ sufficiently small, and such that $L_N[h](x; \epsilon)$ is a jointly analytic function of $(x; \epsilon)$ for all x in the interior of T^{ext} and for sufficiently small $|\epsilon|$.*

In what follows we will use the notation

$$\begin{aligned} a \cdot b &= a_1 b_1 + a_2 b_2 + a_3 b_3 = a_j b_j \in \mathbb{C} \\ \text{and} \quad a^2 &= a_1^2 + a_2^2 + a_3^2 = a_j a_j \in \mathbb{C}. \end{aligned} \quad (51)$$

for $a, b \in \mathbb{C}^3$. Further if $Re(a^2) > 0$ we will write

$$M(a) = \sqrt{(a^2)}, \quad (Re(M(a)) > 0). \quad (52)$$

Note that $a \cdot b$ is *not* the scalar product in $\mathbb{C}^3$. For $a \in \mathbb{R}^3$, $M(a)$ equals the modulus $|a|$ of a .

Since the functions f , m , ρ and P jointly analytic, they can be extended analytically to arguments $(u + is; \epsilon)$ for all real (s_1, s_2) with $|s_i|$ small enough, and for all

sufficiently small *complex* values of ϵ . In order to discuss the analytic properties of the double layer potential we will make use of these extensions. When doing this, however, we will need to guarantee that the argument of the function A in (46) remains inside the regions of analyticity of the corresponding radicand, and we will need an estimate for the quantity A itself. The following two Lemmas will be useful in these connections. In what follows we denote by $d_t(u, v)$ (u and v in $\mathbb{R}^2$) the distance between the points u and v in the torus, which we define by

$$d_t(u, v) = \min\{|(u_1 + k, u_2 + l) - (v_1, v_2)| : k \in \mathbb{Z} \text{ and } l \in \mathbb{Z}\}. \quad (53)$$

Lemma 3 *There exist positive constants ν_2, δ_2 and C such that, for all $u, v, s \in \mathbb{R}^2$ with $|s_1| < \nu_2$ and $|s_2| < \nu_2$, and for all $\epsilon \in \mathbb{C}$ with $|\epsilon| < \delta_2$ we have*

$$\operatorname{Re}\{(f(u + is; \epsilon) - f(v + is; \epsilon)) \cdot (f(u + is; \epsilon) - f(v + is; \epsilon))\} > C\{d_t(u, v)\}^2. \quad (54)$$

Proof: Consider the quantities

$$R = (\operatorname{Re}\{f(u + is; \epsilon) - f(v + is; \epsilon)\})^2, \quad (55)$$

and

$$S = (\operatorname{Im}\{f(u + is; \epsilon) - f(v + is; \epsilon)\})^2 \quad (56)$$

and set

$$t = \frac{u - v}{|u - v|}.$$

By adding to v a vector with integer coordinates, if necessary, we may assume, without loss of generality, that

$$|u - v| = d_t(u, v).$$

By the mean value theorem we thus have

$$R = \left(\sum_{j=1}^3 \{t \cdot \nabla \operatorname{Re}(f_j(v + is + c_j t; \epsilon))\}^2 \right) (d_t(u, v))^2, \quad (57)$$

and

$$S = \left(\sum_{j=1}^3 \{t \cdot \nabla \operatorname{Im}(f_j(v + is + d_j t; \epsilon))\}^2 \right) (d_t(u, v))^2 \quad (58)$$

respectively, for some real c_j and d_j , $0 < c_j$, $d_j < |u - v|$. Now, it follows from equation (36) that none of the directional derivatives of $f(v; \epsilon)$ ever vanish for $v \in \mathbb{R}^2$ and $\epsilon \in \mathbb{R}$, and therefore, that there exists a constant $L' > 0$ such that for all s and ϵ small enough and for all $v \in \mathbb{R}^2$ and $t \in \mathbb{R}^2$ with $|t| = 1$ we have

$$\max_{1 \leq j \leq 3} \{t \cdot \nabla \operatorname{Re}(f_j(v + is; \epsilon))\}^2 > L'. \quad (59)$$

Thus, the quantity in parenthesis on the right hand side of equation (57) is larger than L' for $|u - v|$ small enough. In view of (55) and since the restriction of f to $[0, 1) \times [0, 1) \times \{\epsilon = 0\}$ is real valued and injective, it is easy now to check that there exists a constant L such that for $|s|$ and $|\epsilon|$ small enough we have

$$R > L d_t(u, v)^2 \quad (60)$$

for all $(u, v) \in \mathbb{R}^4$.

Since $\operatorname{Im}(f)$ vanishes when $s = 0$ and $\operatorname{Im}(\epsilon) = 0$, we also have

$$\sum_{1 \leq j \leq 3} \{t \cdot \nabla \operatorname{Im}(f_j(v + is + d_j t; \epsilon))\}^2 < L/2 \quad (61)$$

provided s and $\operatorname{Im}(\epsilon)$ are sufficiently small. Thus, the quotient $\frac{S}{R}$ is less than $1/2$ for ϵ close to the real axis and s small, and we obtain

$$\operatorname{Re}\{(f(u + is; \epsilon) - f(v + is; \epsilon)) \cdot (f(u + is; \epsilon) - f(v + is; \epsilon))\} = R - S > \frac{1}{2}R. \quad (62)$$

The Lemma now follows from equation (60).

Lemma 4 *There is a constant $C > 0$ such that the inequality*

$$|m(v + is; \epsilon) \cdot (f(v + is; \epsilon) - f(u + is; \epsilon))| < C \{d_t(u, v)\}^2 \quad (63)$$

holds for all $u, v \in \mathbb{R}^2$ and for all $s \in \mathbb{R}^2$ and $\epsilon \in \mathbb{C}$ with $|s_i|$ and $|\epsilon|$ small enough.

Proof: Consider Taylor the expansion

$$f(b; \epsilon) - f(a; \epsilon) = \frac{\partial f}{\partial u_j}(a; \epsilon)(b_j - a_j) + R_{j,k}(a, b; \epsilon)(b_j - a_j)(b_k - a_k), \quad (64)$$

and call

$$D_j(a; \epsilon) = m(a; \epsilon) \cdot \frac{\partial f}{\partial u_j}(a; \epsilon). \quad (65)$$

We clearly have $D_j(a; \epsilon) = 0$ for $j = 1, 2$, for all $a \in \mathbb{R}^2$ and all small $\epsilon \in \mathbb{R}$. Since D_j is analytic, we must also have

$$D_j(v + is; \epsilon) = 0 \quad (66)$$

for all $v \in \mathbb{R}^2$ and for sufficiently small $s \in \mathbb{R}^2$ and $\epsilon \in \mathbb{C}$. The Lemma now follows from the integral formula for the remainder $R_{j,k}$ and the analytic properties of f , which insure that expansion (64) must hold not only for real a , b and ϵ , but for all $a = u + is$, $b = v + is$ and $\epsilon \in \mathbb{C}$ provided $|s_i|$ and $|\epsilon|$ are sufficiently small.

We can now establish the analytic properties of the operator T_K .

Lemma 5 *There exist δ_3 and ν_3 such that, for all $\delta < \delta_3$ and $\nu < \nu_3$ the operator T_K maps $\mathcal{H}_{\delta,\nu}$ continuously into itself.*

Proof: $T_K[\rho](v; \epsilon)$ is defined by equation (46) for real v and ϵ . We need to find values of δ and ν such that, for every $\rho \in \mathcal{H}_{\delta,\nu}$, there is an extension of $T_K[\rho]$ to an element of $\mathcal{H}_{\delta,\nu}$. As we shall see, an extension to complex arguments $v + is$, $v \in \mathbb{R}^2$, can be obtained for $|\epsilon|$ and $|s_j|$ small enough by means of the formula

$$\eta(v + is; \epsilon) = \int_{[0,1] \times [0,1]} H(u + is, v + is; \epsilon) \rho(u + is; \epsilon) du, \quad (67)$$

where, using our conventions (51) and (52), we have put

$$H(u + is, v + is; \epsilon) = \frac{m(v + is; \epsilon) \cdot (f(v + is; \epsilon) - f(u + is; \epsilon))}{4\pi \{M((f(v + is; \epsilon) - f(u + is; \epsilon)))\}^3} M(P(u + is; \epsilon)). \quad (68)$$

In what follows u and v will continue to denote elements of $\mathbb{R}^2$.

To check that η is well defined we first notice that, by our hypothesis of analyticity, the quantities m , f , ρ and P are themselves well defined (and indeed, analytic) at $(u + is; \epsilon)$ for all sufficiently small real values of s and for sufficiently small complex values of ϵ . Further, we note from equations (52) and (54) that the expression

$$M(f(v + is; \epsilon) - f(u + is; \epsilon)),$$

can be interpreted as a singly-valued analytic function for all $u \neq v$ in $\mathbb{R}^2$ and for $|s|$ and $|\epsilon|$ small enough. Analogously, and in view of (36), $M(P(u + is; \epsilon))$ is singly valued and analytic for sufficiently small $\epsilon \in \mathbb{C}$ and $s \in \mathbb{R}^2$. Finally, the integrand is an integrable function, as follows immediately from Lemmas 3 and 4.

Instead of showing directly that expression (67) provides the required *analytic* extension, we will establish a corresponding analyticity result for the function

$$\begin{aligned} \xi(v + is; \epsilon) &= \eta(v + is; \epsilon) - \rho(v + is; \epsilon)T_K[1](v + is; \epsilon) = \\ &\int_{[0,1] \times [0,1]} H(u + is, v + is; \epsilon)(\rho(u + is; \epsilon)) - \rho(v + is; \epsilon) du, \end{aligned} \quad (69)$$

which is defined by means of a more regular integrand. It is indeed sufficient to discuss the function ξ since, in view of Lemma 1, $T_K[1]$ lies in some $\mathcal{H}_{\delta, \nu}$.

The property of analyticity of ξ with respect to ϵ follows by differentiation under the integral sign in (69) with respect to the real and imaginary parts of ϵ . As is easily checked, such an operation leads to integrable quantities only (use Lemma 3 and 4), and the Cauchy-Riemann equations corresponding to the variable ϵ are thus verified directly since the integrand is itself analytic with respect to ϵ . To establish the analyticity with respect to the j -th component of $v + is$ we shall show that ξ satisfies the Cauchy-Riemann equations

$$P(D)\xi = \frac{\partial \xi}{\partial v_j} - \frac{1}{i} \frac{\partial \xi}{\partial s_j} = 0.$$

As before, it is readily checked that differentiation under the integral sign with respect to r_j and s_j is permissible. Since H satisfies the Cauchy-Riemann equations with respect to the complex variable $v + is$, we obtain

$$P(D)\xi = - \int_{[0,1] \times [0,1]} \frac{\partial}{\partial u_j} \{H(u + is, v + is; \epsilon)(\rho(u + is; \epsilon)) - \rho(v + is; \epsilon)\} du,$$

and this expression reduces to a boundary integral which vanishes by periodicity. We have thus verified the analyticity of $\xi(v)$, and therefore that of $\eta(v)$.

We also see from the definition that $\|\eta\|_{\delta, \nu} < C \|\rho\|_{\delta, \nu}$ for some $C > 0$ if δ and ν are sufficiently small. The proof of this Lemma is now complete.

The following Lemma provides approximations of the kernel H by appropriate degenerate kernels.

Lemma 6 *Let H be defined by equation (68), and let a constant $C > 0$ be given. Then, there exist real numbers $\delta > 0$, $\nu > 0$ and linearly independent sets of entire functions $\Sigma_\alpha = \{\alpha_1, \dots, \alpha_\ell\}$ and $\Sigma_\beta = \{\beta_1, \dots, \beta_\ell\}$ such that the kernel*

$$\hat{H}(u + is, v + is) = \sum_{j=1}^{\ell} \alpha_j(u + is) \beta_j(v + is) \quad (70)$$

satisfies

$$\int_{[0,1] \times [0,1]} |\{H(u + is, v + is; \epsilon) - \hat{H}(u + is, v + is)\}| du < C \quad (71)$$

for all $v, s \in \mathbb{R}^2$ and $\epsilon \in \mathbb{C}$ with $|s_j| < \nu$ and $|\epsilon| < \delta$.

Proof:

Write $\tau(u, v) = H(u, v; 0) \{d_t(u, v)\}^{\frac{3}{2}}$ and call

$$k_{\delta_1}(u, v) = \begin{cases} \{d_t(u, v)\}^{-\frac{3}{2}} & \text{if } d_t(u, v) > \delta_1 \\ \delta_1^{-\frac{3}{2}} & \text{if } d_t(u, v) \leq \delta_1, \end{cases}$$

see (53). Notice now that τ is a continuous function for $(u, v) \in \mathbb{R}^4$ as follows from (real-variable particular cases of) Lemmas 3 and 4. Further, τ is periodic of period one in each one of its four variables and, in particular, it is bounded: $|\tau(u, v)| \leq C_1$ for all (u, v) . It follows that, for any $v \in \mathbb{R}^2$ the quantity

$$\left| \int_{[0,1] \times [0,1]} \{\tau(u, v) k_{\delta_1}(u, v) - H(u, v; 0)\} du \right| \quad (72)$$

is less than or equal to

$$C_2 \delta_1^{\frac{1}{2}} + \int_{\{[0,1] \times [0,1]\} \cap \{d_t(u, v) < \delta_1\}} |H(u, v; 0)| du.$$

Thus, taking δ_1 small enough we can then make the integral in (72) as small as we please for all v , say

$$\left| \int_{[0,1] \times [0,1]} \{ \tau(u, v) k_{\delta_1}(u, v) - H(u, v; 0) \} du \right| < \frac{C}{2}, \quad (73)$$

since, by Lemmas 3 and 4, $|H(u, v; 0)|$ is dominated by a constant multiple of the integrable function $\{d_t(u, v)\}^{-1}$.

Now, as we pointed out, τ and k_{δ_1} are continuous and periodic functions, of period 1 in each of their variables. Thus, the product $\tau(u, v) k_{\delta_1}(u, v)$ can be approximated uniformly in $\mathbb{R}^4$ by linear combinations with real coefficients of trigonometric monomials, which we may write in the form

$$p_{\tilde{\ell}}(u, v) = \sum_{j=1}^{\tilde{\ell}} \tilde{\alpha}_j(u) \tilde{\beta}_j(v). \quad (74)$$

(Here each one of the functions $\tilde{\alpha}_j(u)$ or $\tilde{\beta}_j(v)$ is itself a multiple of a sine or cosine; e.g. $\tilde{\alpha}_j(u) = a_j \cos(2\pi(n_1^j u_1 + n_2^j u_2))$ or $\tilde{\alpha}_j(u) = a_j \sin(2\pi(n_1^j u_1 + n_2^j u_2))$ where n_1^j and n_2^j are integers and $a_j \in \mathbb{R}$). Taking appropriate trigonometric polynomials $p_{\tilde{\ell}}$, with

$\tilde{\ell}$ large enough, and writing $\hat{H}(u, v) = p_{\tilde{\ell}}(u, v)$ we thus obtain an approximation by less than $\frac{C}{2}$

$$\sup_{(u,v) \in ([0,1] \times [0,1])^2} |\tau(u, v) k_{\delta_1}(u, v) - \hat{H}(u, v)| < \frac{C}{2}. \quad (75)$$

Simple algebraic manipulations now allow us to obtain a representation of the function $\hat{H}$ in terms of products $\alpha_j(u) \beta_j(v)$ of functions of separated variables in which the sets $\{\alpha_j\}$ and $\{\beta_j\}$ are linearly independent. Indeed, unless the family of functions $\tilde{\alpha}_j$ is linearly independent, for example, the number of terms in (74) can be reduced. Extending to complex variables we may thus write

$$\hat{H}(u + is, v + is) = \sum_{j=1}^{\ell} \alpha_j(u + is) \beta_j(v + is), \quad (76)$$

where the sets $\Sigma_\alpha = \{\alpha_1(v + is), \dots, \alpha_\ell(v + is)\}$ and $\Sigma_\beta = \{\beta_1(v + is), \dots, \beta_\ell(v + is)\}$ are linearly independent. It is clear from our construction that the inequality (71) holds for $\epsilon = 0$ and $s = 0$ –see equations (73) and (75). By continuity, there exist positive real numbers δ and ν such that (71) is satisfied for $|s_j| < \nu$ and $|\epsilon| < \delta$, as claimed. This completes the proof of the Lemma.

Proof of Theorem 5:

In what follows the constants δ and ν are assumed to be as small as is necessary to insure the validity of our considerations, and no explicit references to their smallness will be made. Call $\eta(u; \epsilon) = g(f(u; \epsilon); \epsilon) \in \mathcal{H}_{\delta, \nu}$. In view of equations (40)-(41) together with Lemma 2 it is sufficient to find a jointly analytic solution of the integral equation

$$-\frac{1}{2}h + L_K h = g.$$

Equivalently, it will suffice to find a solution $\rho \in \mathcal{H}_{\delta, \nu}$ of the equation

$$-\frac{1}{2}\rho + T_K \rho = \eta. \tag{77}$$

for some $\delta > 0$, $\nu > 0$.

Let now H be defined as in (68), let

$$C < \frac{1}{2},$$

let $\hat{H}$ be the function given by Lemma 6, and define the continuous operator $\hat{T} : \mathcal{H}_{\delta, \nu} \rightarrow \mathcal{H}_{\delta, \nu}$ by

$$\hat{T}[\rho](v + is; \epsilon) = \int_{[0,1] \times [0,1]} \hat{H}(u + is, v + is) \rho(u + is; \epsilon) du. \tag{78}$$

From Lemma 6 it follows that for $\rho \in \mathcal{H}_{\delta, \nu}$

$$\| (T_K - \hat{T})[\rho] \|_{\delta, \nu} < C \| \rho \|_{\delta, \nu},$$

that is, calling $T_1 = (T_K - \hat{T})$ we have

$$\|T_1\|_{\delta, \nu} \leq C < \frac{1}{2}.$$

It follows that $(-\frac{1}{2}I + T_1)$ admits a continuous inverse –which is given by the sum of its Neumann series. This inverse is indeed defined from $\mathcal{H}_{\delta, \nu}$ to itself, since a uniform limit of holomorphic (resp. continuous) functions is holomorphic (resp. continuous). Simple algebraic manipulations following the classical theory of integral equations [33, 18] now show that finding a solution to equation (77) is equivalent to finding a solution to the problem

$$(I + T_2)\rho = \kappa \tag{79}$$

where the operator T_2 is defined by

$$T_2 = (-\frac{1}{2}I + T_1)^{-1}\hat{T}$$

and

$$\kappa = (-\frac{1}{2}I + T_1)^{-1}\eta.$$

Note that, in view of equations (78) and (70) we have

$$T_2[\rho](v + is; \epsilon) = \sum_{j=1}^{\ell} \gamma_j(v + is; \epsilon) \int_{[0,1] \times [0,1]} \alpha_j(u + is) \rho(u + is; \epsilon) du, \tag{80}$$

where

$$\gamma_j = (-\frac{1}{2}I + T_0)^{-1}[\beta_j].$$

The functions γ_j are linearly independent, since so are the functions β_j

Thus, we need to solve the problem

$$\rho(v + is; \epsilon) + \sum_{j'=1}^{\ell} \gamma_{j'}(v + is; \epsilon) \int_{[0,1] \times [0,1]} \alpha_{j'}(u + is) \rho(u + is; \epsilon) du = \kappa(v + is; \epsilon). \tag{81}$$

Multiplying by $\alpha_j(v + is)$ and integrating with respect to v we obtain

$$z_j + \sum_{j'=1}^{\ell} A_{jj'} z_{j'} = a_j$$

where

$$z_j = z_j(\epsilon) = \int_{[0,1] \times [0,1]} \rho(v + is; \epsilon) \alpha_j(v + is) dv,$$

$$A_{jj'} = A_{jj'}(\epsilon) = \int \alpha_j(v + is) \gamma_{j'}(v + is; \epsilon) dv$$

and

$$a_j(\epsilon) = \int \alpha_j(v + is) \kappa(v + is; \epsilon) dv$$

are holomorphic functions of ϵ . Differentiation with respect to s , integration by parts and periodicity show that $A_{jj'}$, z_j and κ do not depend on s .

Denoting by A , a and z the $\ell \times \ell$ matrix $(A_{jj'})$ and the ℓ -dimensional vectors (a_j) and (z_j) respectively, it is easy to see [33, 18] that equation (81) admits a solution if and only if the system

$$(I + A(\epsilon))z(\epsilon) = a(\epsilon) \tag{82}$$

admits a solution; and that if $z(\epsilon)$ is a solution of (82) then a solution ρ is given by

$$\rho(v + is) = \kappa(v + is) - \sum_{j=1}^{\ell} z_j(\epsilon) \gamma_j(v + is; \epsilon).$$

To conclude the proof it is therefore sufficient to find an ϵ dependent family of holomorphic solutions $z(\epsilon)$.

Now, the following two facts are known from the classical theory of integral equations for all real (and small) values of ϵ [33, 18]: 1) The matrix

$$(I + A(\epsilon)),$$

associated with our Neumann problem (39) admits a kernel of dimension one and 2) equation (82) has solutions if and only if the Neumann data g has zero average on S^ϵ —a condition which holds here by hypothesis.

It follows from point 1) that $(I + A(0))$ has a submatrix of rank $(\ell - 1)$. Without loss of generality, we may assume the lower right $(\ell - 1) \times (\ell - 1)$ submatrix is nonsingular; we then call $B = B(\epsilon)$ the $\ell \times \ell$ matrix whose first row equals

$$(1, 0, \dots, 0)$$

and whose remaining $(\ell - 1)$ rows equal the corresponding rows of $I + A(\epsilon)$. Also, denote $b = (0, a_2, a_3, \dots, a_\ell)$.

By continuity of the determinant, the lower right submatrix of $(I + A(\epsilon))$ continues to have rank $(\ell - 1)$ for sufficiently small complex ϵ , and thus, $B(\epsilon)$ is invertible and has a holomorphic inverse for such ϵ . Let now $z(\epsilon) = (B(\epsilon))^{-1}b(\epsilon)$; clearly $z(\epsilon)$ is holomorphic. In order to check that $z(\epsilon)$ satisfies the system (82) it suffices to verify the equality for small real ϵ since both members of this equation are holomorphic functions. Since the last $\ell - 1$ equations are known to be satisfied, it suffices to show that so is the first one –which is linearly dependent with the remaining $\ell - 1$. But the vector a must possess the property that, if a linear combination with coefficients λ_j of the left hand sides of the last $\ell - 1$ equations gives the left hand side of the first equation, then we must have

$$a_1 = \sum_{j=2}^{\ell} \lambda_j a_j.$$

Indeed, this condition is necessary for the system (82) to admit solutions at all. Since we know (82) admits solutions under our hypothesis, it follows that z satisfies this system of equations, and the proof of Theorem 5 is complete.

4 The solution on the current sheet

Our goal in this section is to solve the Hamilton-Jacobi equation

$$|\nabla_s \Phi|^2 = a, \tag{83}$$

see (34), under the assumption that the data $a(x; \epsilon)$ and the torus S^ϵ differ by little from those corresponding to an axisymmetric configuration. In §4.1 we treat this problem in a more general setting corresponding to equation (84) below. No additional complications arise because of this gain in generality. The specifics of our equation (83) are then discussed in §4.2.

4.1 Hamilton-Jacobi Equations on $\mathbb{R}^2/\mathbb{Z}^2$

Given a Hamiltonian

$$h(u_1, u_2, F; \epsilon)$$

we consider the ϵ -parametrized family of Hamilton-Jacobi equations

$$\frac{\partial \Phi}{\partial u_1} = h(u_1, u_2, \frac{\partial \Phi}{\partial u_2}; \epsilon), \quad (84)$$

and we assume conditions 1) to 4) below are satisfied. Our first assumption is

- 1) The function h is defined in a domain $D = \mathbb{R} \times \mathbb{R} \times [\phi_0, \phi_1] \times (-\epsilon_0; \epsilon_0) \subseteq \mathbb{R}^4$ for some fixed $\phi_0 < \phi_1$ and $\epsilon_0 > 0$.

We shall discuss this equation in the torus $(u_1, u_2) \in \mathbb{R}^2/\mathbb{Z}^2$, and we therefore require

- 2) Periodicity: h is periodic of period one with respect to u_1 and u_2 .

We need to study small perturbations from an integrable case (corresponding to $\epsilon = 0$), and we thus assume that for $\epsilon = 0$ h is independent of u_2

- 3) $h(u_1, u_2, F, 0) =: h_0(u_1, F)$.

Further, we shall require h to satisfy certain smoothness assumptions. Using our convention (20) we assume

- 4) $h = h(u_1, u_2, F; \epsilon) \in C^{\eta+1}(D)$.

Under these conditions together with a certain monotonicity assumption (see equation (90) below) we will establish that equation (84) admits solutions with periodic gradients provided ϵ is sufficiently small.

To solve (84) for a certain value of ϵ we consider its characteristic equations, see [14]. Using u_1 as the independent variable the characteristic equations can be reduced to the Hamiltonian system

$$\begin{aligned} \frac{du_2}{du_1} &= -\frac{\partial h}{\partial F} \\ \frac{dF}{du_1} &= \frac{\partial h}{\partial u_2}. \end{aligned} \quad (85)$$

The solution of this system satisfying

$$u_2(0) = b \text{ and } F(0) = c$$

for a given value of ϵ will be denoted by

$$(u_2(u_1, b, c; \epsilon), F(u_1, b, c; \epsilon)).$$

For example, in the unperturbed case $\epsilon = 0$, F is constant, say $F = c$, and u_2 is then given by

$$u_2(u_1, b, c; 0) = b - \int_0^{u_1} \frac{\partial h_0}{\partial F}(t, c) dt. \quad (86)$$

For the case $\epsilon \neq 0$ we have the following Lemma.

Lemma 7 *Let $c_0 \in (\phi_0, \phi_1)$, let $\eta = n \in \mathbb{N}$ or $\eta = \omega$, and assume the Hamiltonian h satisfies the conditions 1) through 4) above. Further, let $g(b; \epsilon) \in C^\eta(\mathbb{R} \times (-\epsilon_0; \epsilon_0))$ be a periodic function of period 1 in the variable b , which, for $\epsilon = 0$ equals the constant c_0*

$$g(b, 0) = c_0.$$

Then there exist $\delta > 0$ and ϵ_1 , $0 < \epsilon_1 < \epsilon_0$, such that

i) For fixed ϵ with $|\epsilon| < \epsilon_1$ the solution

$$S(u_1, b; \epsilon) = (u_2(u_1, b, g(b; \epsilon); \epsilon), F(u_1, b, g(b; \epsilon); \epsilon)) \quad (87)$$

of the system (85) is defined for all $(u_1, b) \in (-\delta, 1 + \delta) \times \mathbb{R}$, and $S \in C^\eta((-\delta, 1 + \delta) \times \mathbb{R} \times (-\epsilon_1; \epsilon_1))$.

ii) The mapping A^ϵ defined in $\mathbb{R}^2$ for each fixed $|\epsilon| < \epsilon_1$ by

$$A_\epsilon(u_1, b) = (A_\epsilon^1, A_\epsilon^2) = (u_1, u_2(u_1, b, g(b; \epsilon); \epsilon)) \quad (88)$$

is bijective from $(-\delta, 1 + \delta) \times \mathbb{R}$ to $(-\delta, 1 + \delta) \times \mathbb{R}$ and its inverse

$$A_\epsilon^{-1} : (-\delta, 1 + \delta) \times \mathbb{R} \rightarrow (-\delta, 1 + \delta) \times \mathbb{R}$$

is of class $C^\eta(\mathbb{R}^2)$ for each fixed $\epsilon \in (-\epsilon_1; \epsilon_1)$. Furthermore, if $A(u_1, b; \epsilon) = A_\epsilon(u_1, b)$ and $C(u_1, b; \epsilon) = A_\epsilon^{-1}(u_1, b)$, then A and C belong to $C^\eta(\mathbb{R}^2 \times (-\epsilon_1; \epsilon_1))$.

Proof:

To establish point i) notice first that for $\epsilon = 0$ we have $F(u_1, b, g(b; \epsilon); \epsilon) = c_0$. The function u_2 is, in turn, given by equation (86), and, in particular, this solution is defined for all u_1 . Since g is periodic in b we conclude [13, pp. 22, 38] that there exist

$\epsilon_1 > 0$ and $\delta > 0$ such that, for all b and for $|\epsilon| < \epsilon_1$, the solution S in equation (87) exists for $-\delta < u_1 < 1 + \delta$. The fact that $S \in C^n$ follows from the corresponding smoothness of the Hamiltonian in (85).

Let us now discuss point *ii*). First note that by periodicity we have $A_\epsilon^2(u_1, b+1) = A_\epsilon^2(u_1, b) + 1$ for any fixed u_1 . By continuity it then follows that, for $|\epsilon| < \epsilon_1$, any point $(u_1, u_2) \in (-\delta, 1 + \delta) \times \mathbb{R}$ is the image under A_ϵ of the point (u_1, b) for some $b \in \mathbb{R}$. Differentiation of the first equation in (85) with respect to b shows that,

$$\frac{\partial u_2}{\partial b}(u_1, b, c; 0) = 1$$

since h does not depend on u_2 for $\epsilon = 0$, and since $u_2(0, b, c; \epsilon) = b$. We conclude that for ϵ small enough the derivative of A_ϵ^2 with respect to b , and therefore the Jacobian of A_ϵ with respect to (u_1, b) , are larger than a constant $K > 0$ for all $(u_1, b) \in (-\delta, 1 + \delta) \times \mathbb{R}$. It follows that A_ϵ^2 increases strictly with b , and A_ϵ is thus bijective. It is clear that the function A is smooth as stated [13]. Finally, the smoothness of C (and therefore that of A_ϵ^{-1}) follow from an application of the inverse function theorem to the mapping

$$(A_\epsilon; \epsilon)$$

whose Jacobian is readily checked to equal that of A_ϵ . The proof of the Lemma is now complete.

We now present the main result of this section. In this theorem we establish the existence of solutions of equation (84) for small ϵ . We do this under the assumption that points 1) through 4) above hold, together with a certain hypothesis of monotonicity of the rotation number in the unperturbed case. Since the solution of the characteristic system in the unperturbed case is given by equation (86), we see the rotation number is given by

$$\beta(c) = - \int_0^1 \frac{\partial h_0}{\partial F}(u_1, c) du_1. \quad (89)$$

Theorem 6 *Assume that $h(u_1, u_2, F; \epsilon)$ verifies the assumptions 1) through 4) above, that the monotonicity condition*

$$k^{-1} \leq \beta'(c) \leq k \quad (90)$$

holds for all $c \in (\phi_0, \phi_1)$ and for some $k > 0$, and let $\eta = 5$ or $\eta = \omega$. Set $\tau = \frac{9}{4}$ and take γ small enough so that there are Diophantine irrationals satisfying the

relation (19) inside the interval $(\beta(\phi_0) + \gamma, \beta(\phi_1) - \gamma)$; call β_0 one such irrational. Then there is a positive number $\epsilon_1 < \epsilon_0$ such that, for $|\epsilon| < \epsilon_1$ there exists a solution $\Phi^\epsilon(u_1, u_2)$ of equation (84) of class $C^{\eta-3}$, with derivatives $\frac{\partial \Phi^\epsilon}{\partial u_1}$ and $\frac{\partial \Phi^\epsilon}{\partial u_2}$ which are periodic functions of period one in u_1 and u_2 , and such that the rotation number of the integral lines $(u_1, u_2(u_1))$ is β_0 . Further, calling $\Phi(u_1, u_2; \epsilon) = \Phi^\epsilon(u_1, u_2)$ we have, in the case $\eta = \omega$, $\Phi(u_1, u_2; \epsilon) \in C^\omega(\mathbb{R}^2 \times (-\epsilon_1; \epsilon_1))$.

Proof:

From the existence theory for first order nonlinear PDEs we learn that, given initial data $F = g(b; \epsilon)$ on the axis $u_1 = 0$, a solution Φ to equation (84) verifying

$$\frac{\partial \Phi}{\partial u_2}(0, b; \epsilon) = g(b; \epsilon)$$

can be found in any domain containing a portion of the axis on which the mapping (88) is a diffeomorphism. Further, in such a domain we have

$$\frac{\partial \Phi}{\partial u_2}(u_1, u_2(u_1, b, g(b; \epsilon); \epsilon); \epsilon) = F(u_1, b, g(b; \epsilon); \epsilon).$$

From Lemma 7 it follows that if g is periodic in b and ϵ is small enough, $|\epsilon| < \epsilon_1$, then a multivalued solution $\Phi(u_1, u_2; \epsilon)$ exists and is defined for all u_2 and for $-\delta < u_1 < 1 + \delta$ for some $\delta > 0$.

From the characteristic equations (85) we see that $\frac{\partial \Phi}{\partial u_2}(u_1, u_2; \epsilon)$ is a periodic function of u_2 provided that this is the case for the initial condition $F(0, b, g(b; \epsilon); \epsilon) = g(b; \epsilon)$. In order to insure periodicity of $\frac{\partial \Phi}{\partial u_2}(u_1, u_2; \epsilon)$ in the variable u_1 , however, we must have

$$F(1, b, g(b; \epsilon); \epsilon) = g(u_2^1; \epsilon) = F(0, u_2^1, g(u_2^1; \epsilon); \epsilon), \quad (91)$$

where we have put $u_2^1 = u_2(1, b, g(b; \epsilon); \epsilon)$. We therefore must find initial data g such that condition (91) is satisfied. In other words, we must find an invariant curve

$$(g(b; \epsilon), b), \quad 0 \leq b \leq 1, \quad g \text{ periodic of period 1 in } b$$

for the period one map

$$(c, b) \rightarrow (F(1, b, c; \epsilon), u_2^1).$$

This is clearly possible for $\epsilon = 0$. Indeed, in this case the period one map maps the circle $\{F = c = \text{constant}, 0 \leq u_2 \leq 1\}$ into itself. The existence of an invariant curve

in the case of small $\epsilon \neq 0$ now follows from the twist map theorem, see §1.4. The self intersection property follows from Liouville's theorem, since (84) is a Hamiltonian system. The monotone twist property is one of our assumptions. Indeed, for $\epsilon = 0$ the period one map reduces to the rotation

$$(c, b) \rightarrow (F(1, b, c; \epsilon) = c, b + \beta(c))$$

where $\beta(c)$ is given by (89). The necessary regularity to apply the results described in §1.4 is guaranteed by Lemma 7.

4.2 Application: The geometrical optics equation on the surface of a nearly axisymmetric torus

An application of the results of the previous section will now allow us to solve equation (83) for configurations which depart slightly from axisymmetry. In what follows we discuss the equation which results from the choice of a plus sign in (18); the case corresponding to the alternative choice can be treated analogously.

Our differential equation (18) coincides with (84) if we set

$$h(u_1, u_2, F; \epsilon) = \frac{g_{12}F + \sqrt{D(g_{22}a - F^2)}}{g_{22}}. \quad (92)$$

We need to insure conditions 1) through 4) of §4.1 are satisfied. To do this we assume the function

$$a(u_1, u_2; \epsilon)$$

satisfies the following conditions for some $\eta = 6$ or for $\eta = \omega$ —recall our convention (20)

- a) $a \in C^\eta(\mathbb{R}^2 \times (-\epsilon_0; \epsilon_0))$,
- b) a is periodic of period 1 in the variables u_1 and u_2 for every fixed ϵ ,
- c)

$$a(u_1, u_2; 0) = a_0(u_1) \in C^\eta(\mathbb{R}),$$

i.e., a does not depend on u_2 for $\epsilon = 0$,

- d) a is strictly positive in its domain of definition and, further, $(g_{22}a - F^2) > 0$ for all (u_1, u_2) , all $\epsilon \in (-\epsilon_0; \epsilon_0)$ and all $F \in (-\phi, \phi)$.

Clearly, assumptions a) through d) above guarantee that our Hamiltonian (92) satisfies conditions 1) through 4) of §4.1. We now have the following theorem.

Theorem 7 *Let the toroidal surfaces S^ϵ be given by the parametrization (12), and assume that the functions G_0 and Q are in $C^{\eta+2}(\mathbb{R}^2)$ for $\eta = 5$ or $\eta = \omega$. Further, assume that $a^\epsilon(u_1, u_2) = a(u_1, u_2; \epsilon)$ verifies the conditions a) through d) above for $\eta = 5$ or $\eta = \omega$. Then there exist positive real numbers γ and ϵ_1 , $0 \leq \epsilon_1 \leq \epsilon_0$ such that, for any β_0 in the interval $(-\beta(\phi) + \gamma, \beta(\phi) - \gamma)$ verifying the diophantine condition (19) with $\tau = \frac{9}{4}$ and for any ϵ with $|\epsilon| < \epsilon_1$, the geometrical optics equation (83) has a solution $\Phi(u_1, u_2; \epsilon)$ whose surface gradient $\nabla_s \Phi$ is periodic of period 1 in u_1 and u_2 and such that the rotation number of the field lines of $\nabla_s \Phi$ is β_0 . Finally, for every fixed ϵ the solution Φ is in $C^{\eta-3}(\mathbb{R}^2)$ for $|\epsilon| < \epsilon_1$ if $\eta = 5$ and $\Phi \in C^\omega(\mathbb{R}^2 \times (-\epsilon_1; \epsilon_1))$ if $\eta = \omega$.*

Proof: According to equations (13) and (9), the diagonal metric coefficients of the axisymmetric torus are given by

$$g_{11}^0 = l_0^2, \quad g_{22}^0 = 4\pi^2[R_0 + \xi(u_1)]^2 = 4\pi^2 r^2,$$

the off-diagonal coefficients vanish, and the determinant is given by

$$D = 4\pi^2 l_0^2 [R_0 + \xi(u_1)]^2 = 4\pi^2 l_0^2 r^2$$

Thus equation (18) in the axisymmetric case becomes

$$\frac{\partial \Phi}{\partial u_1} = \frac{l_0}{2\pi r} \sqrt{4\pi^2 r^2 - \left(\frac{\partial \Phi}{\partial u_2}\right)^2}, \quad (93)$$

and our unperturbed Hamiltonian is given by

$$h_0(u_1, c) = \frac{l_0}{2\pi r(u_1)} \sqrt{4\pi^2 r^2 a_0(u_1) - c^2}.$$

The rotation number (89) is then given by

$$\beta(c) = - \int_0^1 \frac{\partial h_0}{\partial c} du_1 = \int_0^1 \frac{l_0 c}{2\pi r \sqrt{4\pi^2 r^2 a_0 - c^2}} du, \quad (94)$$

in agreement with (31). We therefore have

$$\beta'(c) = 2\pi l_0 \int_0^1 \frac{ra_0(u_1)}{[4\pi r^2 a_0 - c^2]^{\frac{3}{2}}} du_1,$$

and we can thus satisfy condition (90) for some k provided we restrict c , say, to the range

$$|c| < \frac{1}{2} \min_{u_1 \in \mathbb{R}} \{4\pi r^2 \sqrt{a_0(u_1)}\}.$$

The proof now follows by Theorem 5.

5 The solution on the outer shells

The results of this section are used in the proof of Theorems 3 and 4; they show that the solution in the inner volume can be continued to a truly nonlinear weak solution of our problem. We begin in §5.1 by showing that the initial value problem associated with the Beltrami equations admits solutions $\mathbf{B}(x; \epsilon)$ which are jointly analytic in the spatial variables x and the perturbation parameter ϵ . We treat a somewhat more general version of this problem than that described in §1.3, since the additional generality will be needed for our construction of Clebsch potentials in the Appendix. In § 5.2 we show that, in the case $\epsilon = 0$, the rotation numbers associated with the field $\mathbf{B}$ satisfy a certain condition of monotonicity. This condition, in turn, is used in the proof of Theorems 3 and 4 to establish the existence of toroidal magnetic surfaces for the field $\mathbf{B}$, and thus to meet the boundary condition required in our boundary value problem (1).

5.1 Initial value problem for the Beltrami equations: joint analyticity

Consider the initial value problem given in a neighborhood of S^ϵ by the equations

$$\begin{aligned} \nabla \times \mathbf{B}(x; \epsilon) &= \alpha \mathbf{B}(x; \epsilon), \\ \nabla \cdot \mathbf{B}(x; \epsilon) &= 0 \end{aligned} \tag{95}$$

together with the initial conditions

$$\mathbf{B}|_{S^\epsilon} = E(x; \epsilon) = \nabla_s \Phi(x; \epsilon); \tag{96}$$

or, more generally, the problem

$$\begin{aligned}\nabla \times W(x; \epsilon) &= C(x, W(x; \epsilon); \epsilon), \\ \nabla \cdot W(x; \epsilon) &= D(x, W(x; \epsilon); \epsilon)\end{aligned}\tag{97}$$

together with the initial conditions

$$W|_{S^\epsilon} = E(x; \epsilon) = \nabla_s \Phi(x; \epsilon)\tag{98}$$

for some fixed value of the parameter ϵ , see §1.2. (Here Φ is a possibly a multivalued function, but its surface gradient is assumed to be single-valued). Clearly, equations (97)–(98) can only admit solutions if the divergence of $C(x, W(x; \epsilon); \epsilon)$ vanishes. In Theorem 8 we will show these equations do admit solutions, and that these solutions are jointly analytic in x and ϵ provided the initial data and the family S^ϵ are themselves jointly analytic –and provided either 1) C is independent of W , divergence free and satisfies $n \cdot C(x; \epsilon) = 0$ for $x \in S^\epsilon$, or 2) $D = 0$ and $C(x, W; \epsilon) = \alpha W$ for some real constant α . Note that the initial values are given by a surface gradient field, which guarantees, in particular, that $W(x; \epsilon) \cdot n(x; \epsilon) = 0$ for $x \in S^\epsilon$. Like our proof of Lemma 1, the present construction here proceeds by considering a related initial value problem in $(x; \epsilon)$ space.

It is convenient to introduce a coordinate transformation which maps a neighborhood of the ϵ -varying surface onto a fixed domain. To do this consider the map $y = (y_1, y_2, y_3)$ given by

$$y(u_1, u_2, u_3; \epsilon) = G(u_1, u_2; \epsilon) + u_3 \cdot m(u_1, u_2; \epsilon),\tag{99}$$

see §1.2 and equation (37). This map defines the (ϵ -dependent) coordinates $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ we will use on and around the surfaces S^ϵ ; the Jacobian of y will be denoted by

$$J = \frac{\partial(y_1, y_2, y_3)}{\partial(u_1, u_2, u_3)}.\tag{100}$$

In what follows we denote by $W_i = W_i(u_1, u_2, u_3; \epsilon)$ the coordinates of W in the contravariant basis associated with u , that is

$$W(y(u_1, u_2, u_3; \epsilon); \epsilon) = \sum_{i=1}^3 W_i \nabla u_i.\tag{101}$$

The dependence of the quantities C_i and D on the new coordinates is then described by functions $\tilde{C}_i$ and $\tilde{D}$ given by

$$\tilde{C}_i(W_1, W_2, W_3, u; \epsilon) = J \nabla u_i \cdot C(y, W(y; \epsilon); \epsilon)$$

and

$$\tilde{D}(W_1, W_2, W_3, u; \epsilon) = D(y, W(x; \epsilon); \epsilon).$$

As for the boundary values (96), they translate through the Jacobian matrix into data of the form

$$W_i(u_1, u_2, 0; \epsilon) = \tilde{E}_i(u_1, u_2; \epsilon). \quad (102)$$

From relations such as

$$(\nabla u_1) \cdot (\nabla \times W) = J^{-1} \left(\frac{\partial W_3}{\partial u_2} - \frac{\partial W_2}{\partial u_3} \right)$$

it is easily checked that the coordinates B^i satisfy the overdetermined system of four partial differential equations

$$\frac{\partial W_3}{\partial u_2} - \frac{\partial W_2}{\partial u_3} = \tilde{C}_1 \quad (103)$$

$$\frac{\partial W_1}{\partial u_3} - \frac{\partial W_3}{\partial u_1} = \tilde{C}_2 \quad (104)$$

$$\frac{\partial W_2}{\partial u_1} - \frac{\partial W_1}{\partial u_2} = \tilde{C}_3 \quad (105)$$

and

$$\sum_{i=1}^3 \nabla W_i \cdot \nabla u_i + \sum_{i=1}^3 W_i \Delta u_i = \tilde{D} \quad (106)$$

for the three unknowns (W_1, W_2, W_3) . Now, on $S^\epsilon = \{u_3 = 0\}$, ∇u_1 and ∇u_2 are tangential to S^ϵ , while ∇u_3 equals the unit normal $m(u; \epsilon)$. It follows that in a neighborhood of S^ϵ , the first order derivatives with respect to u_3 occurring in equation (106) can be written in the form

$$\frac{\partial W_3}{\partial u_3} + a(w) \frac{\partial W_1}{\partial u_3} + b(w) \frac{\partial W_2}{\partial u_3}$$

with small coefficients $a(w)$ and $b(w)$. Collecting all the terms in equations (103), (104) and (106) which involve derivatives with respect to u_3 and inverting a three by

three matrix (which is a small perturbation of the identity) we thus see the reduced system formed by these three equations can be transformed into a system of the form

$$\partial/\partial u_3 \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = A_1 \partial/\partial u_1 \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} + A_2 \partial/\partial u_2 \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} + F \quad (107)$$

for which $\{u_3 = 0\}$ is a non characteristic surface. Here A_1 , A_2 and C are 3×3 matrices whose entries depend only on u_i , ∇u_i , Δu_i ($i = 1, 2, 3$) and ϵ , whereas $F = F(u, W^i; \epsilon)$ is a vector. Clearly, all these quantities are periodic of period one with respect to u_1 and u_2 —since the right hand sides of the original equation are single-valued functions defined in a neighborhood of S^ϵ , and since the parametrization of S^ϵ itself is given by periodic functions. Thus, equations (107) together with the initial values (102) constitute a well defined initial value problem in the variables $(u; \epsilon)$ around the initial surface $u_3 = 0$. The proof of joint analyticity for solutions of our Beltrami equations, given in Theorem 8 below, is based on consideration of this reduced problem.

After establishing that equations (107)–(102) and therefore the system formed by (102), (103), (104) and (106) admit jointly analytic solutions, the proof of this Theorem proceeds by showing that a solution of the reduced system with boundary values (102) yields also a solution of the full Beltrami system (95)–(96). Theorem 8 includes, of course, the vacuum field case ($C = 0$, $D = 0$) for which our Cauchy problem simplifies to a corresponding Cauchy problem for Laplace's equation. The latter problem was discussed by Kippenhahn [26] who did not consider, however, the analytic dependence of the solution with respect to the parameter ϵ .

Theorem 8 *Assume the family of toroidal surfaces S^ϵ is given by the parametrization (12), with G_0 , $Q \in C_{per}^\omega(\mathbb{R}^2)$, and let the functions C , D and $\nabla \Phi$ belong to $C^\omega(U \times (-\epsilon_0; \epsilon_0))$ for some neighborhood U of S^0 and some $\epsilon_0 > 0$. Further assume that either 1) C does not depend on W , $\nabla \cdot \{C(x; \epsilon)\} = 0$ and $n \cdot C(x; \epsilon) = 0$ for $x \in S^\epsilon$ or that 2) $D = 0$ and $C(x, W; \epsilon) = \alpha W$ for some real constant α . Then there exist a neighborhood V of S^0 and a number $\epsilon_1 > 0$ together with a field $W \in C^\omega(V \times (-\epsilon_1; \epsilon_1))$ such that $W(x; \epsilon)$ solves the Cauchy problem (97)–(98) in V for every fixed $\epsilon \in (-\epsilon_1; \epsilon_1)$.*

Proof:

Consider equations (107)–(102) as a four dimensional initial value problem for our unknown vector $W_i(u; \epsilon)$. Clearly, the initial surface $\Sigma = \{(u; \epsilon) : u_3 = 0, |\epsilon| < \epsilon_0\}$ is non-characteristic for (107). Further, under our assumptions all the coefficients and

right hand sides in these equations are jointly analytic functions, since so is the change of variables $y(u; \epsilon)$. The Cauchy-Kovalevsky theorem then implies the existence of a jointly analytic solution W_i in a neighborhood of the surface Σ . In particular, there exists a neighborhood O of $\{u \in \mathbb{R}^3 : u_3 = 0\}$ and a number $\epsilon_1 > 0$ such that $W_i \in C^\omega(O \times (-\epsilon_1; \epsilon_1))$ solves (107) for every fixed $\epsilon \in (-\epsilon_1; \epsilon_1)$.

Let now

$$\nu = \frac{(\nabla u_1) \times (\nabla u_2)}{\|(\nabla u_1) \times (\nabla u_2)\|};$$

note that ν coincides with the unit normal n on S^ϵ , and that it provides an extension of n which is jointly analytic in x and ϵ in some neighborhood of $S^0 \times \{0\}$. Call now W the vector field given by our analytic solution W_i through equation (101). It is easy to see from our construction that the first two equations (103) and (104) of the reduced system are equivalent to the single vector equation

$$\nu \times (\nabla \times W - C(x, W; \epsilon)) = 0. \quad (108)$$

Thus, to verify that W satisfies the full system (95) in a neighborhood of the surfaces and for sufficiently small ϵ it suffices to check that, defining

$$f(x) = \nu \cdot (\nabla \times W - C(x, W; \epsilon)),$$

we have

$$f = 0. \quad (109)$$

It is easy to show that W satisfies (109) on the initial surface for all sufficiently small ϵ . Indeed, we know by hypothesis that $C \cdot n = 0$ on S^ϵ (note W is tangent to S^ϵ), and, in view of equation (98), Stokes' theorem tells us we also have $n \cdot \nabla \times W = 0$. Our proof now follows from the identity

$$\nu \cdot \nabla f = -(\nabla \cdot \nu)f, \quad (110)$$

which can be viewed as an ordinary differential equation for the quantity f . It implies f vanishes in a neighborhood of S^ϵ since, as we showed above, its initial values on S^ϵ vanish. Thus, to complete our proof we only need establish equation (110).

To do this we note from (108) that calling

$$h = \nabla \times W - C(x, W; \epsilon), \quad (111)$$

we have

$$\nabla \times (\nu \times h) = 0.$$

Using the identity

$$\nabla \times (a \times b) = b \cdot \nabla a - a \cdot \nabla b - b(\nabla \cdot a) + a(\nabla \cdot b)$$

valid for vectors a, b , and since $\nabla \cdot h = 0$ from our hypothesis, (equation (106), which is equivalent to the second of (97), is satisfied) we get

$$(h \cdot \nabla) \nu - (\nu \cdot \nabla) h - h(\nabla \cdot \nu) = 0. \quad (112)$$

Taking dot product of (112) with ν we obtain

$$\nu \cdot (h \cdot \nabla \nu) - \nu \cdot (\nu \cdot \nabla) h - (\nu \cdot h)(\nabla \cdot \nu) = 0. \quad (113)$$

Now, from the identity

$$b \cdot \nabla(a \cdot c) = a \cdot b \cdot \nabla c + c \cdot b \cdot \nabla a$$

and since $\nabla \cdot h = 0$ by hypothesis and $|\nu|^2 = 1$, we obtain the relations

$$\nu \cdot (h \cdot \nabla \nu) = 0 \quad (114)$$

and

$$\nu \cdot (\nu \cdot \nabla) h = (\nu \cdot \nabla)(\nu \cdot h) - h \cdot \{(\nu \cdot \nabla) \nu\} \quad (115)$$

These relations show that (113) can be rewritten as

$$(\nu \cdot \nabla)(\nu \cdot h) = -(\nu \cdot h)(\nabla \cdot \nu) + h \cdot \{(\nu \cdot \nabla) \nu\}.$$

Being $f = \nu \cdot h$, equation (110) now follows from the relation

$$\nu \times (\nabla \times \nu) = -\nu \cdot \nabla \nu. \quad (116)$$

which, in view of (108) and (111), shows that $h \cdot (\nu \cdot \nabla) \nu = 0$. This concludes the proof of the theorem.

The previous theorem provides, of course, a proof of existence of solution of the initial value problem (95)–(96) in the particular case in which the configurations is axisymmetric. Note the initial values are given by a surface gradient. For such axisymmetric cases we also showed that a unique solution for the equations (95) can be found via the ansatz

$$\mathbf{B} = \nabla \psi \times \nabla u_2 + (\alpha \psi + F) \nabla u_2, \quad (117)$$

by providing (axisymmetric) initial values for ψ and $\frac{\partial \psi}{\partial n}$ —the problem then reduces to an initial value problem for the Grad-Shafranov equation

$$-\Delta^* \psi = \alpha(\alpha \psi + F), \quad (118)$$

see (27). We thus have two sets of solutions for our equations (95) in the axisymmetric case. It is important to note that these two sets are in fact identical. Indeed, for a field $\mathbf{B}$ of the form (117) we have

$$\nabla \times \mathbf{B} = -(\Delta^* \psi) \nabla u_2 + \alpha (\nabla \psi \times \nabla u_2) \quad (119)$$

and therefore $(\nabla \times \mathbf{B}) \cdot \mathbf{n} = 0$ on the initial surface, which implies that $\mathbf{B}$ is a surface gradient there (by Stokes' theorem). Conversely, a solution to (95)–(96) is, by definition, a surface gradient on the initial surface, and thus the quantity $F = (2\pi r)^2 \mathbf{B} \cdot \nabla u_2$ is a constant there. (By axisymmetry, F does not depend on u_2 . To show it does not depend on u_1 write F as a path integral on circles $u_1 = \text{const.}$ and use the relation $2\pi r \nabla u_2 d\ell = d\ell$ together with Stokes theorem). Further, multiplying the initial condition (96) by ∇u_1 yields a condition for the normal derivative of ψ which, together with the relation $\psi = 0$, provides a well defined Cauchy problem for (118). The coincidence of the two sets of solutions discussed here now follows by uniqueness of analytic solutions to (95)–(96).

5.2 Monotone twist condition for convex and bean shaped cross sections

Let $\mathbf{B}$ be a solution of (95)–(96) in a case in which both the initial surface and data in (96) are axisymmetric. In what follows we will show that the twist map associated with the field lines of $\mathbf{B}$ (through the Hamiltonian system (138)) satisfies the conditions of monotonicity required by the KAM theory, see §1.4, provided the initial surface satisfies certain geometrical constraints.

In the previous section we showed $\mathbf{B}$ can be represented in the form (117) for a certain function ψ satisfying the Grad-Shafranov equation (118). Further, ψ vanishes on the initial surface, and the magnetic field is tangent to the toroidal surfaces $\psi = c$ for all sufficiently small constants c . As shown in §A.1 in the Appendix, the winding number of the field lines on the surface $\psi = c$ is given by

$$\beta = \mu(c)$$

where

$$2\pi\mu(c) = (F + \alpha c) \int_{\{\psi=c, u_2=0\}} \frac{1}{r|\nabla\psi|} d\ell. \quad (120)$$

Thus, the Poincaré map associated with the Hamiltonian system (138) on the cross-section $\{u_1 = 0\}$ for $\epsilon = 0$ can be written as

$$\begin{aligned} u_2 &\mapsto u_2 - \mu(-E) \\ E &\mapsto E, \end{aligned} \quad (121)$$

and the corresponding KAM monotonicity condition on the surface $\psi = 0$ reads

$$\frac{d\mu}{dc}(0) \neq 0. \quad (122)$$

In what follows we show that under certain hypothesis on the initial surface $\{\psi = 0\}$ we can insure that this condition is satisfied. As in §1.5, the gradient $\nabla\psi$, which does not vanish on $\{\psi = 0\}$, is assumed to point in the direction of the outer normal –of course, the opposite case can be treated analogously.

Using coordinates r and x_3 as defined in §1.2, and given any function $g = g(r, x_3)$ defined in a neighborhood of a given *plane* curve $\{R(r, x_3) = c\}$ (such that the two dimensional gradient $\nabla_{r,x_3} R$ points in the direction of the outer normal to the curve), we have

$$\frac{d}{dc} \int_{\{R(x_1, x_2)=c\}} g d\ell = \int_{\{R(x_1, x_2)=c\}} \nabla_{r,x_3} \cdot \left(g \frac{\nabla_{r,x_3} R}{|\nabla_{r,x_3} R|} \right) \frac{d\ell}{|\nabla_{r,x_3} R|}. \quad (123)$$

This formula can be obtained by means of the divergence theorem (which permits one to write the line integral on the left hand side as a two dimensional integral), and the identity $dr dx_3 = d\ell/|\nabla R|$ for the area of a surface element in a neighborhood of the curve in (r, x_3) -plane; see also [28, 29].

In view of (123) we obtain

$$\begin{aligned} 2\pi\mu'(c) &= \alpha \int_{\{\psi=c, u_2=0\}} \left(\frac{1}{r|\nabla\psi|} \right) d\ell + \\ & (F + \alpha c) \int_{\{\psi=c, u_2=0\}} \nabla_{r,x_3} \cdot \left(\frac{\nabla\psi}{r|\nabla\psi|^2} \right) \frac{d\ell}{|\nabla\psi|} \end{aligned} \quad (124)$$

This expression can be simplified. To do this note first that since the relation $\nabla_{r,x_3}(\psi) = \nabla\psi$ between the two-dimensional and three-dimensional gradient of ψ is satisfied –given that ψ does not depend on u_2 –, the integrand in the second integral can be rewritten as

$$\frac{\Delta_{r,x_3}\psi}{r|\nabla_{r,x_3}\psi|^3} - 2 \frac{(\nabla_{r,x_3}\psi) \cdot \nabla_{r,x_3}(|\nabla_{r,x_3}\psi|)}{r|\nabla_{r,x_3}\psi|^4} - \frac{1}{r^2|\nabla_{r,x_3}\psi|^3} \frac{\partial\psi}{\partial r},$$

where Δ_{r,x_3} is the Laplacian in the variables r, x_3 . Further, simple manipulations on the relation

$$\nabla_{r,x_3} \cdot \left(\frac{\nabla_{r,x_3} \psi}{|\nabla_{r,x_3} \psi|} \right) = \kappa$$

for the curvature κ of a curve $\{\psi = c, u_2 = 0\}$ give

$$\nabla \psi \cdot \nabla_{r,x_3} (|\nabla \psi|) = |\nabla \psi| \Delta_{r,x_3} \psi - \kappa |\nabla \psi|^2.$$

We thus find that (124) may be re-written as

$$2\pi\mu'(c) = \alpha \int_{\{\psi=c, u_2=0\}} \left(\frac{1}{r|\nabla \psi|} \right) d\ell + \\ (F + \alpha c) \int_{\{\psi=c, u_2=0\}} \left[-\Delta_{r,x_3} \psi - \frac{1}{r} \frac{\partial \psi}{\partial r} + 2\kappa |\nabla \psi| \right] \frac{1}{r|\nabla \psi|^3} d\ell.$$

Using the Grad-Shafranov equation (118) to eliminate the Laplacian term we obtain

$$2\pi\mu'(c) = \alpha \int_{\{\psi=c\}} \frac{1}{r|\nabla \psi|} + (\alpha\psi + F) \left[\alpha(\alpha\psi + F) - 2\frac{1}{r} \frac{\partial \psi}{\partial r} + 2\kappa |\nabla \psi| \right] \frac{1}{r|\nabla \psi|^3} d\ell,$$

and thus, at $S = \{\psi = 0\}$

$$2\pi\mu'(0) = \int_{\{\psi=0, u_2=0\}} \alpha \frac{|\nabla \psi|^2 + F^2}{r|\nabla \psi|^3} - \frac{2F}{r|\nabla \psi|^3} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} - \kappa |\nabla \psi| \right) d\ell. \quad (125)$$

Writing

$$h(r, \alpha, F, x, y, \kappa) = \alpha x^2 + 2F(\kappa x - \frac{1}{r}y) + \alpha F^2 \quad (126)$$

we have

$$2\pi\mu'(0) = \int_{\{\psi=0, u_2=0\}} \frac{1}{r|\nabla \psi|^3} h(r, \alpha, F, |\nabla \psi|, \frac{\partial \psi}{\partial r}, \kappa) d\ell.$$

We now study the monotonicity of the rotation numbers μ , and provide conditions insuring that h and therefore $\mu'(0)$ do not vanish. To do this we first establish the following lemma.

Lemma 8 *Let real values of r , α , F and κ be given. Then there exists a positive number ϵ such that for all $x \geq 0$ and all y with $cx \leq y \leq dx$ the inequality $|h| > \epsilon$ holds provided one of the following conditions is satisfied:*

- (i) α and F are both positive or both negative and $\kappa r > d - |\alpha|r$.
- (ii) α and F have different signs and $\kappa r < c + |\alpha|r$.

Proof: We only discuss point (i), since point (ii) follows similarly. Provided $x \geq 0$ we have $h(r, \alpha, F, x, y, \kappa) > h(r, \alpha, F, x, dx, \kappa)$ if the signs of α and F are positive and $h(r, \alpha, F, x, y, \kappa) < h(r, \alpha, F, x, dx, \kappa)$ if the signs of α and F are negative. In either case, it suffices to show that $h(r, \alpha, F, x, dx, \kappa)$ never vanishes for $x > 0$. But

$$h(r, \alpha, F, x, dx, \kappa) = \alpha x^2 + 2F\left(\kappa - \frac{d}{r}\right)x + \alpha F^2,$$

and this is the equation of a parabola whose vertex lies at $x_0 = -\frac{F(\kappa - \frac{d}{r})}{\alpha}$. If $x_0 < 0$ then this curve stays away from the x axis for $x \geq 0$ –since α and F have the same sign. Finally, if $x_0 \geq 0$ the value of the parabola at its vertex is

$$F^2 \left(\alpha - \frac{(\kappa - \frac{d}{r})^2}{\alpha} \right),$$

and this quantity has the same sign as α if and only if $-|\alpha| < (\kappa - \frac{d}{r}) < |\alpha|$. In sum, since α and F have the same sign, the positivity of h follows provided either $(\kappa - \frac{d}{r}) > 0$ or $-|\alpha| < (\kappa - \frac{d}{r}) < |\alpha|$, or, what is equivalent, provided that $(\kappa - \frac{d}{r}) > -|\alpha|$, and the Lemma follows.

The following Theorem provides sufficient conditions for equation (122) to be satisfied. This Theorem differentiates between two regions in our axisymmetric torus S , namely, the region S^- facing the axis of symmetry, in which the radial derivative is negative $\frac{\partial \psi}{\partial r} < 0$, and its complement S^+ , in which $\frac{\partial \psi}{\partial r} > 0$. Further, for $0 \leq a \leq 1$ we will denote by S_a^- (resp. S_a^+) the set of all points in S^- (resp. S^+) for which

$$\left| \frac{\partial \psi}{\partial r} \right| \geq a |\nabla \psi|. \quad (127)$$

Note that $S_0^\pm = S^\pm$. It is important here to recall we consider the case in which $\nabla\psi$ points in the direction of the outer normal; similar results hold in the complementary case.

Theorem 9 *Assume one of the following conditions is satisfied:*

- (i) α and F are both positive or both negative, $\kappa r > 1 - |\alpha|r$ in S^+ and for all a with $0 \leq a \leq 1$ we have $\kappa r > -a - |\alpha|r$ in S_a^- .
- (ii) α and F have different signs, $\kappa r < -1 + |\alpha|r$ in S^- and for all a with $0 \leq a \leq 1$ we have $\kappa r < |\alpha|r + a$ in S_a^+ .

Then, the monotonicity condition (122) holds.

Proof:

The proof follows from the previous Lemma by setting $x = |\nabla\psi|$ and $y = \frac{\partial\psi}{\partial r}$.

In particular, we have the following Corollary.

Corollary 1 *Assume one of the following conditions is satisfied:*

- (i) α and F are both positive or both negative, $\kappa r > -|\alpha|r$ in S^- and $\kappa r > 1 - |\alpha|r$ in S^+ .
- (ii) α and F have different signs, $\kappa r < -1 + |\alpha|r$ in S^- and $\kappa r < |\alpha|r$ in S^+ .

Then, the monotonicity condition (122) holds.

Remark: As an example, note the Corollary tells us that in the harmonic case $\alpha = 0$ the monotonicity condition (122) holds provided the cross section of the torus is convex ($\kappa > 0$) and has curvature ratio $\kappa r > 1$ on S^+ . Notice from Theorem 9 that the monotonicity condition holds for certain non-convex cross-sections as well. Indeed, equation (127) imposes a purely geometrical restriction on the cross-section –namely, that the cosine of a certain angle be larger than a . It is easy to construct non-convex cross-sections (with non-convexity facing the symmetry axis) for which Theorem 9 guarantees that (122) holds. Axisymmetric plasmas whose cross-sections contain non-convexities facing the axis of symmetry have been widely studied. They have come to be known as *Bean Shaped Plasmas*; they form the basis for the PBX-M experiment at Princeton, see [12].

A Appendix: Clebsch potentials and Hamiltonian formulation

In order to represent the Poincaré map induced by the field line flow in the axisymmetric case as a monotone twist map taking circles into circles we use a set of coordinates introduced by Kruskal and Kulrud [28], see §A.1. The corresponding rotation number is then be calculated in terms of one of the associated potentials. In the non-axisymmetric case, on the other hand, we will take advantage of a construction due to Cary-Littlejohn [10] and Boozer [6], which will allow us to show that the trajectories of a general jointly analytic divergence free vector field in three dimensions are solutions of a jointly analytic Hamiltonian system, see §A.2.

A.1 The axisymmetric case: Clebsch representation and winding numbers

Let an axisymmetric divergence free vector field $\mathbf{B}$ of the form (117) be given in a neighborhood of a toroidal surface $S = \{\psi = 0\}$. As in §1.5, $\nabla\psi|_{\{\psi=0\}} \neq 0$ is assumed to point in the direction of the outer normal; as noted there, the set of toroidal magnetic surfaces $\{\psi = c\}$ with small constant values of c cover a neighborhood of S . It is easy to check [4] that the poloidal magnetic flux $q(c)$ between the surfaces S and S_c , which is defined by

$$q(c) = \int_{\{u_1=0\} \cap \{0 \leq \psi \leq c\}} \mathbf{B} \cdot \frac{\nabla u_1}{|\nabla u_1|} dS, \quad (128)$$

is in fact given by $q(c) = c$, that is, ψ at x equals the poloidal flux inside the toroidal surface containing x . (To see this recall (10), and compute the derivative of q with respect to c taking into account that the surface element around the curve $\{u_1 = 0\} \cap \{\psi = c\}$ is given by $drdx_3 = d\ell/|\nabla\psi|$.) In what follows we use the coordinates u_1, u_2 and ψ in a neighborhood of our axisymmetric torus, where u_1, u_2 are the first two of the coordinates associated with the transformation given by (99) *in the case* $\epsilon = 0$. That is, the coordinates u_1 and u_2 for a given point $y_0 \in \mathbb{R}^3$ are defined as the first two entries of the solution of the equation

$$y(u_1, u_2, u_3; 0) = y_0,$$

even when, as in the following section, they are used to discuss a problem with $\epsilon \neq 0$.

Following [28] we can now show that the field (25) can be represented in the form

$$\mathbf{B} = \nabla\psi(x) \times \nabla\nu(x) \quad (129)$$

for some multivalued analytic function $\nu = \nu(x)$. To do this we fix a point $x(c)$ on each surface $\{\psi = c\}$, $(x(c))$ varying analytically with c . Then, choosing for each x on $\{\psi = c\}$ a smooth path within this surface joining $x(c)$ and x , we define

$$\nu(x) = \int_{x(c)}^x w \cdot d\vec{x} \quad (130)$$

where

$$w = \frac{(\mathbf{B} \times \nabla\psi)}{(\nabla\psi)^2}.$$

Note that, by Stokes' theorem, this line integral takes on the same value for any two paths of integration which can be deformed into each other within $\{\psi = c\}$ (since $\nabla \cdot (\nabla\psi \times w) = \nabla \cdot \mathbf{B} = 0$ so that $n \cdot (\nabla \times w) = 0$). It is easy to check that equation (129) holds.

As in [28], one can show the integral representation (130) yields the expression

$$\nu(x) = \lambda(u_1, u_2, \psi(x)) - \mu(\psi)u_1 + u_2 \quad (131)$$

where λ is a single valued function, and μ is given by the line integral

$$\mu(c) = \int_{\{\psi=c, u_2=0\}} \frac{B_{tor}}{|\nabla\psi|} d\ell$$

taken in the direction of increasing u_1 , where

$$B_{tor} = \mathbf{B} \cdot \frac{\nabla u_2}{|\nabla u_2|}.$$

From equation (129) we see that $\mathbf{B} \cdot \nabla\nu = 0$, and thus ν is constant along the field lines. It then follows from equation (131) that the winding number of the field (117) on the surface $\psi = c$ is given by

$$\beta = \mu(c). \quad (132)$$

(See §1.3 for our definition of the winding number β).

Clearly, equation (117) implies that rB_{tor} is constant on each magnetic surface $\psi = c$, more precisely, we have

$$2\pi rB_{tor} = F + \alpha\psi.$$

It follows that

$$2\pi\mu(c) = (F + \alpha c) \int_{\{\psi=c, u_2=0\}} \frac{1}{r|\nabla\psi|} d\ell.$$

A.2 Non-axisymmetric geometries, double Clebsch representations and Hamiltonian formulation

Although a global Clebsch representation (129) can only be given if the magnetic field admits a full set of magnetic surfaces $\{\psi = cst.\}$, it is possible to represent an arbitrary divergence free field in a toroidal geometry by a *double* Clebsch representation. This representation leads to a Hamiltonian description of the magnetic field [6, 10, 42]. Here we show that the Hamiltonian associated with the field-line problem may be chosen in such a way that it varies analytically with the spatial variables and ϵ , and we relate this Hamiltonian to the potential ν discussed in the previous section. In what follows we use the coordinates (u_1, u_2, ψ) of §A.1 in a neighborhood of the unperturbed torus, *even when treating a (nearby) perturbed geometry*, and we follow [42].

We begin by constructing a jointly analytic vector potential $A = A(x; \epsilon)$ for the magnetic field $\mathbf{B}(x; \epsilon)$ which, in the unperturbed case $\epsilon = 0$, coincides with a single-valued potential $A_0(x)$ implied by (129):

$$A_0(x) = \bar{\mu}(\psi(x))\nabla V + \psi(x)\nabla u_2. \quad (133)$$

Here we assumed $\mu(0) \neq 0$ and we used the definitions

$$\bar{\mu}(c) = \int_0^c \mu(s) ds$$

and

$$V(x) = \frac{\lambda(x)}{\mu(\psi(x))} - u_1,$$

see §A.1. (The simpler choice $A_0 = \psi\nabla\nu$ is not adequate since $\nabla\nu$ is not single-valued). Calling

$$d(x) = \nabla \cdot A_0,$$

our vector potential A will be obtained as a solution of the Cauchy problem

$$\begin{aligned}\nabla \times A(x; \epsilon) &= \mathbf{B}(x; \epsilon) \\ \nabla \cdot A(x; \epsilon) &= d(x)\end{aligned}$$

with initial data

$$A(x; \epsilon) = 0 \text{ on } \{(x; \epsilon) : x \in S^\epsilon\}.$$

We know by Theorem 8 that this initial value problem admits a jointly analytic solution in a neighborhood of the fixed surface $S = S^0$ for sufficiently small ϵ . Note that $A_0(x)$ vanishes for $x \in S$, since so does ψ , and that by uniqueness of analytic solution to our Cauchy problem we have

$$A(x; 0) = A_0(x) \tag{134}$$

as desired.

We now proceed to obtain a jointly analytic Hamiltonian formulation for the field lines of $\mathbf{B}$. To do this we first note the identity

$$(\nabla \psi \times \nabla V) \cdot \nabla u_2 = \frac{\mathbf{B} \cdot \nabla u_2}{\mu(\psi)},$$

which holds in the axisymmetric case $\epsilon = 0$. It follows from the inverse function Theorem, see e.g. [15, pp. 151, 152], that provided the toroidal component of our axisymmetric fields never vanish –as it happens in our applications– the triple (V, u_2, ψ) defines a global analytic change of variables in a neighborhood of the surface $\psi = 0$. We may thus express A in the form

$$A = A_\psi \nabla \psi + A_V \nabla V + A_{u_2} \nabla u_2. \tag{135}$$

Defining G by

$$G(V, u_2, \psi; \epsilon) = \int_0^\psi A_\psi(V, u_2, s; \epsilon) ds \tag{136}$$

(for sufficiently small ψ) we have

$$\nabla G = \frac{\partial G}{\partial \psi} \nabla \psi + \frac{\partial G}{\partial V} \nabla V + \frac{\partial G}{\partial u_2} \nabla u_2,$$

which together with (135) gives

$$A = \nabla G + (A_V - \frac{\partial G}{\partial V})\nabla V + (A_{u_2} - \frac{\partial G}{\partial u_2})\nabla u_2.$$

Thus, putting

$$H = A_V - \frac{\partial G}{\partial V}$$

and

$$E = \frac{\partial G}{\partial u_2} - A_{u_2}$$

we obtain

$$A = \nabla G + H\nabla u_1 - E\nabla u_2,$$

and the double Clebsch representation

$$\mathbf{B} = \nabla H \times \nabla u_1 - \nabla E \times \nabla u_2$$

results.

It is easy to check from (133), (134) and (136) that in the axisymmetric case $\epsilon = 0$ the quantities E and H are given by

$$H(V, u_2, \psi; 0) = \bar{\mu}(\psi) \tag{137}$$

and

$$E = -\psi.$$

In particular, the triple (V, u_2, E) defines a jointly analytic coordinate system in a neighborhood of S .

The magnetic field lines in these coordinates are given by the parametrization $E = E(V)$, $u_2 = u_2(V)$, where E and u_2 satisfy the equations

$$\frac{dE}{dV} = \frac{\mathbf{B} \cdot \nabla E}{\mathbf{B} \cdot \nabla V},$$

$$\frac{du_2}{dV} = \frac{\mathbf{B} \cdot \nabla u_2}{\mathbf{B} \cdot \nabla V}$$

or

$$\frac{dE}{dV} = -\frac{\nabla E \cdot (\nabla H \times \nabla V)}{(\nabla E \times \nabla u_2) \cdot \nabla V}$$

$$\frac{du_2}{dV} = -\frac{\nabla u_2 \cdot (\nabla H \times \nabla V)}{(\nabla E \times \nabla u_2) \cdot \nabla V}$$

But the gradient of H is given by

$$\nabla H = \frac{\partial H}{\partial E} \nabla E + \frac{\partial H}{\partial V} \nabla V + \frac{\partial H}{\partial u_2} \nabla u_2.$$

Thus we obtain

$$\begin{aligned} \frac{dE}{dV} &= -\frac{\partial H}{\partial u_2} \\ \frac{du_2}{dV} &= \frac{\partial H}{\partial E} \end{aligned} \tag{138}$$

This is a Hamiltonian system in 1 and 1/2 degrees of freedom which describes the trajectories of the magnetic field. The Hamiltonian $H = H(V, u_2, E; \epsilon)$ is, by construction, jointly analytic in the spatial variables and ϵ . Note that in the axisymmetric case H depends only on E (see (137)), that is, for $\epsilon = 0$ the system (138) is integrable.

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