TRACKING NONPERIODIC TRAJECTORIES WITH THE OVERTAKING CRITERION

By

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1. Introduction

In this paper we study the optimal control problem of tracking a prescribed nonperiodic trajectory on an infinite time interval. The plant is lienar time invariant, and the cost is taken quadratic. It is typical in this situation that all the cost expressions diverge as time increases to infinity. Therefore we adopt the overtaking criterion, which enables to compare plans on $[0,\infty)$ even though the costs grow indefinitely.

In [1] Artstein and Leizarowitz studied the infinite horizon tracking of a periodic trajectory, with the overtaking criterion. It is proved there that there exists a unique overtaking solution for every initial condition, and it is given in a linear closed loop form. The following considerations motivate the generalization of these results. Consider the periodic trajectory which coincides with the tracked trajectory on a large interval $[0,T]$. By the results of [1] there exists a unique solution for tracking this trajectory. On the other hand one does not expect the performance of tracking the original trajectory to be much influenced by its values in the remote future, as long as small time intervals are concerned. Moreover, if we keep enlarging the interval $[0,T]$ then the expressions describing the feedback law and the differential equation for the optimal solutions indeed approach certain meaningful quantities. These are in fact those obtained by formally using the optimal expressions for the periodic case to the nonperiodic one.

In this paper we generalize some of the results of [1] to the nonperiodic case. We prove the existence and uniqueness of an overtaking optimal solution for every initial condition. We show that it is given by the same feedback law as in the periodic case. However, our results are weaker than those of [1] in two respects. First, we consider tracked trajectories which are uniformly continuous on $[0,\infty)$ as well as bounded there. Second, the overtaking criterion that we employ is somewhat weaker than the one used in [1].

One aspect which distinguishes the nonperiodic case from the periodic one is the following. It would be desirable to know that two tracked trajectories which differ only for very large times have respective solutions which are nearly the same for comparatively small times. We show that indeed the overtaking optimal solutions have this property. This is a valuable fact from the application point of view. Suppose that the tracked trajectory is precisely known only for a large time interval $[0,T]$. Then by the above mentioned property we have an approximate solution based on the knowledge of $[0,T]$, which converges to the optimal solution as T grows to infinity. Moreover, if for every t_0 the exact tracked trajectory is known to the controller in the interval $[t_0,t_0 + T]$, then our results enable him to produce a trajectory which deviates from the optimal one by an arbitrarily small quantity, provided that T is sufficiently large.

The paper is organized as follows. In the next section we display some results concerning finite intervals tracking problems. In section 3 we establish some boundedness properties of the optimal trajectories in finite intervals. In section 4 we prove our main result namely, the existence and uniqueness of the overtaking optimal solutions. We show that they are given by a linear feedback law. In the last section we show that if at the time t_0 the tracked trajectory is known only on the interval $[t_0,t_0 + T]$, then a solution can be produced, which is arbitrarily close to the optimal one, uniformly in $[0,\infty)$. This, provided that T is sufficiently large.

2. Notations and preliminary results

The Euclidean n-dimensional space is R^n with column vectors. A prime over a vector or a matrix denotes transposition. Hence $y'x$ is the scalar product of y and x, and $|x| = (x'x)^{1/2}$ is the norm. If Q is an $n \times n$ matrix then $\|x\|_Q^2 = x'Qx$. A dot over a variable denotes differentiation with respect to the time variable t.

We shall denote by $L_p(R^k)$, $1 \leq p < \infty$, the space of all measurable functions $\phi:[0,T] \to R^k$ such that $\int_0^T |\phi(t)|^p dt < \infty$. This, without explicitly mentioning the interval $[0,T]$, and where it will be clear from the context what this interval is. We shall use the notation $L_\infty(R^k)$ in a similar manner.

The system under consideration is

$$\dot{x} = Ax + Bu \tag{2.1}$$

defined for $x \in R^n$, $u \in R^m$ and constant matrices A and B with the appropriate dimensions. Along with (2.1) a trajectory

$$\Gamma: [0,\infty) \to R^n \tag{2.2}$$

is given. It is assumed throughout that $\Gamma(\cdot)$ is uniformly continuous and bounded on $[0,\infty)$.

The admissible controls $u(\cdot)$ are measurable and integrable over finite intervals. Given $u(\cdot)$ and an initial condition $x(0) = x_0$, the system (2.1) has a unique solution, called <u>the response to $u(\cdot)$</u>. We denote it by

$$x(\cdot) = x(\cdot;u,x_0).$$

However, the dependence on $u(\cdot)$ and x_0 will be suppressed when no confusion may arise. A response to an admissible control will sometimes be called <u>a solution of (2.1)</u> or <u>a solution</u>.

The cost of using the control $u(\cdot)$ with the response $x(\cdot)$ over the interval $[t_1,t_2]$ is given by

$$c_{t_1,t_2}(u) = \int_{t_1}^{t_2} [\|x(t) - \Gamma(t)\|_Q^2 + \|u(t)\|_R^2]dt \tag{2.3}$$

where R is a given positive definite symmetric matrix, and Q is a positive semidefinite symmetric matrix. The following is assumed throughout.

A Standing Hypothesis. The pair (A,B) is controllable and the pair (A,Q) is observable.

Typically, unless $\Gamma(\cdot)$ has a very special form, $c_{0,T}(u)$ in (2.3) will diverge to infinity as T grows indefinitely, for every choice of $u(\cdot)$. Therefore we wish to apply an optimality criterion which is capable of comparing costs which tend to infinity. Such is the overtaking criterion, originated in the economic literature, see Gale [3], von-Weizsacker [5].

Definition. Let x_0 be a fixed initial condition. A control $u_1(\cdot)$ overtakes the control $u_2(\cdot)$ if for every $\varepsilon > 0$ there is a time t_0, depending on ε, such that

$$c_{0,t}(u_1) < c_{0,t}(u_2) + \varepsilon$$

for all $t > t_0$. The control $u^*(\cdot)$ is overtaking optimal if it overtakes any other admissible control.

A Remark. Note that the definition is different from the one in [1], where the ε does not appear in the definition and the inequality.

We shall display now some auxiliary results, needed in the sequel, about tracking a trajectory $\gamma(\cdot)$ on the finite interval $[0,T]$. For y and z fixed in R^k consider the problem

$$\text{(2.4)} \qquad \text{minimize} \int_0^T [\|x(t) - \gamma(t)\|_Q^2 + \|u(t)\|_R^2]dt$$

subject to $x(0) = y$, $x(T) = z$. The minimization is performed over all admissible controls and responses on $[0,T]$ which satisfy (2.1). We denote the infimal value of problem (2.4) by $v(y,z,\gamma)$. In particular, if the $\gamma(\cdot)$ is given by $\gamma(t) = 0$ for all $0 \leq t \leq T$ then we denote the infimal value by $v(y,z,0)$.

Let B denote the closed unit ball in $L_\infty(R^k)$ and consider the minimization problem (2.4) for trajectories $\gamma(\cdot)$ in B.

Lemma 2.1. There is a constant $\lambda_0 > 0$ and a constant $r_0 > 0$ such that

$$v(y,z,\gamma) \geq \lambda_0(|y|^2 + |z|^2) \tag{2.5}$$

for all $\gamma(\cdot)$ in B and for all $|y|^2 + |z|^2 \geq r_0^2$.

Proof: It is well known (see e.g. Lee and Markus [4] page 217) that the minimum in problem (2.4) for $\gamma \equiv 0$ is attained, and clearly the minimal value $v(y,z,0)$ is convex and continuous on $R^n \times R^n$. Therefore

$$\mu_0 = \min\{v(y,z,0): |y|^2 + |z|^2 = 1\} \tag{2.6}$$

is a positive number (recall that (A,Q) is observable). Let $|y|^2 + |z|^2 = r^2$ be such that $r > 0$, and let $\gamma(\cdot)$ belong to B. It is easy to see that

$$v(y,z,\gamma) \geq r^2 \cdot \inf\{v(y_0,z_0,\tfrac{\gamma}{r}): |y_0|^2 + |z_0|^2 = 1\}. \tag{2.7}$$

We claim that if r is sufficiently large then

$$v(y_0,z_0,\tfrac{\gamma}{r}) \geq \tfrac{1}{2}\mu_0, \quad \text{for all } \gamma \in B,\ |y_0|^2 + |z_0|^2 = 1. \tag{2.8}$$

Otherwise there is a sequence $r_k \to \infty$ with $|y_k|^2 + |z_k|^2 = 1$, controls $u_k(\cdot)$ with responses $x_k(\cdot)$ in $[0,T]$ and $\gamma_k(\cdot) \in B$ satisfying

$$\int_0^T [\|x_k - \frac{\gamma_k}{r_k}\|_Q^2 + \|u_k\|_R^2]dt < \tfrac{1}{2}\mu_0 . \tag{2.9}$$

It can be assumed that $\{u_k\}_{k=1}^\infty$ converges weakly in $L_2(R^m)$, say to $u_0(\cdot)$, and then $\{x_k(\cdot)\}_{k=1}^{\infty}$ converges uniformly on $[0,T]$ to $x_0(\cdot)$, the response to $u_0(\cdot)$. Also $\{\frac{\gamma_k}{r_k}(\cdot)\}_{k=1}^{\infty}$ converges weakly in $L_2(R^n)$ to zero and all this implies, in view of (2.9), that

$$\int_0^T [\|x_0(t)\|_Q^2 + \|u_0(t)\|_R^2]dt \leq \frac{1}{2}\mu_0, \text{ hence}$$

$$v(y_0,z_0,0) \leq \frac{1}{2}\mu_0$$

where $y_0 = x_0(0)$ and $z_0 = x_0(T)$, which contradicts (2.6). Then (2.8) must hold if $r > r_0$ for some constant r_0. Choosing $\lambda_0 = \frac{1}{2}\mu_0$, the assertion (2.5) follows from (2.7). □

Let us denote by B_M the closed ball of radius $M > 0$ about the origin in $L_\infty(R^n)$.

Proposition 2.2. For every fixed $\gamma \in B_M$ the problem (2.4) has a unique solution. The function $v(y,z,\gamma)$ is strictly convex on $R^n \times R^n$. There are constants $\lambda > 0$ and $r > 0$ such that

$$v(y,z,\gamma) \geq \lambda[|y|^2 + |z|^2] \tag{2.10}$$

for all $\gamma \in B_M$ and all $|y|^2 + |z|^2 \geq r^2$.

Proof. The first two claims are standard (see e.g. Lee and Markus [4] and Artstein and Leizarowitz [1]). The last assertion follows from Lemma 2.1.

□

Proposition 2.3.

(i) A control $u_0(\cdot)$ is the optimal control of (2.4) and $x_0(\cdot)$ is its response if and only if

$$u_0(t) = R^{-1}B'y_0(t)$$

where $x_0(\cdot)$, $y_0(\cdot)$ solve the system

$$\dot{x} = Ax + BR^{-1}B'y \qquad (2.11)$$
$$\dot{y} = Qx - A'y - Q\gamma$$

(ii) For every $\bar{y},\bar{z} \in R^n$ there is a unique $y(\cdot)$ such that $x(\cdot)$ and $y(\cdot)$ solve (2.11) with the boundary constraints $x(0) = \bar{y}$, $x(T) = \bar{z}$.

Proof: This result is standard too (see e.g. Lee and Markus [4] and Artstein and Leizarowitz [1]). □

Let K be the positive definite solution of the Riccati equation

$$-KA - A'K + KBR^{-1}B'K - Q = 0. \qquad (2.12)$$

We denote

$$F = A - BR^{-1}B'K \qquad (2.13)$$

and it is well known that F is stable whenever (A,B) is controllable and (A,Q) is observable (see Athens and Falb [2] page 773).

Proposition 2.4.

(i) If $x(\cdot)$ and $y(\cdot)$ satisfy (2.11) then the function $g(\cdot)$ which is defined by

$$g(t) = y(t) + Kx(t) \qquad (2.14)$$

verifies the equation

$$\dot{g}(t) = -F'g(t) - Q\gamma(t). \qquad (2.15)$$

(ii) If $g(\cdot)$ satisfies (2.15) and $x(\cdot)$ is a solution of

$$\dot{x}(t) = Fx(t) + BR^{-1}B'g(t) \tag{2.16}$$

then $x(\cdot)$ and $y(\cdot)$ verify equations (2.11), where $y(\cdot)$ is defined by

$$y(t) = g(t) - Kx(t).$$

Proof: The proof is by a straightforward computation, using (2.12).

□

Given a bounded uniformly continuous tracked trajectory $r\colon [0,\infty) \to R^n$ we define

$$g(t) = \int_t^\infty e^{(s-t)F'}Qr(s)ds, \quad t \geq 0. \tag{2.17}$$

Let $x^*(\cdot)$ be a solution of (2.16) with this $g(\cdot)$. (Notice that both $g(\cdot)$ and $x^*(\cdot)$ are bounded on $[0,\infty)$.) The response $x^*(\cdot)$ is the candidate to be the overtaking optimal solution, and (2.16) with (2.17) define the linear feedback rule. We conclude this section with a rather mild optimality property of $x^*(\cdot)$:

Theorem 2.5: For every $0 \leq t_1 < t_2 < \infty$, $x^*(\cdot)$ restricted to the interval $[t_1,t_2]$ is the solution of the problem

$$\text{minimize } c_{t_1,t_2}(u), \text{ subject to} \tag{2.18}$$
$$x(t_1) = x^*(t_1), \quad x(t_2) = x^*(t_2)$$

where $x(\cdot)$ is the response to $u(\cdot)$ in $[t_1,t_2]$.

Proof: The function $g(\cdot)$ in (2.17) is a solution of equation (2.15), while $x^*(\cdot)$ solves (2.16). By Proposition 2.4 (ii), $x^*(\cdot)$ and $y^*(\cdot)$ verify (2.11) where

$$y^*(t) = g(t) - Kx^*(t).$$

Then by Proposition 2.3 $x^*(\cdot)$ is the optimal solution to the problem (2.18).

□

3. Boundedness properties of responses

In this section we shall prove that we can restrict our attention only to responses $x(\cdot)$ which are contained in a fixed ball in R^n (once the initial value x_0 is fixed). We consider the optimization problem (2.4) with the end points y,z in a fixed bounded set. We shall show that the corresponding function $g(\cdot)$ defined in (2.14) have a uniform bound, for all the possible choices of y,z and $\gamma(\cdot) \in B_M$.

For a fixed $T > 0$ we divide the $[0,\infty)$ interval into segments $I_k = [kT,(k + 1)T]$, $k > 0$, and given a tracked trajectory $\Gamma: [0,\infty) \to R^n$ we define $\Gamma_k: [0,T] \to R^n$ by

(3.1) $$\Gamma_k(t) = \Gamma(t + kT).$$

For every k we consider the minimization problem (2.4) with $\gamma(\cdot) = \Gamma_k(\cdot)$. We denote

$$v_k(y,z) = v(y,z,\Gamma_k).$$

By the uniform boundedness of $\{\Gamma_k(\cdot)\}_{k=0}^{\infty}$ and by Proposition 2.2 there are $\lambda > 0$ and $r > 0$ such that

(3.2) $$v_k(y,z) > \lambda[|y|^2 + |z|^2] \text{ for all } |y|^2 + |z|^2 > r^2.$$

We consider now a discrete version of the tracking problem. For any sequence $\sigma = (y_0, y_1, \ldots)$ of states in R^n we associate the sequence of costs

$$c_m(\sigma) = \sum_{k=0}^{m-1} v_k(y_k, y_{k+1}), \quad m = 1,2,\ldots \tag{3.3}$$

with the interpretation: $c_m(\sigma)$ is the optimal cost of tracking $\Gamma(\cdot)$ on $[0,mT]$ with the constraints $x(kT) = y_k$ for $k = 0,1,\ldots,m$.

We denote by $B(0,\rho)$ the closed ball of radius ρ about the origin in R^n.

Lemma 3.1. Let the initial value be x_0. Given an $L > 0$ there is a number $b > 0$ with the property: If $\sigma = \{y_0, y_1, y_2, \ldots\}$ is a sequence in R^n, $y_0 = x_0$, and σ is not contained in $B(0,b)$, then there is a sequence $\sigma' = \{z_0, z_1, z_2, \ldots\}$, $z_0 = x_0$, such that σ' is contained in $B(0,b)$ and

$$c_m(\sigma') < c_m(\sigma) - L \tag{3.4}$$

for every $m > m_0$, for some m_0.

Proof: The sequence $\{v_k(.,.)\}_{k=1}^{\infty}$ is uniformly bounded on compact sets in $R^n \times R^n$. In particular the number β which is defined by

$$\beta = \sup_k v_k(0,0)$$

is finite. Let a be such that $a > |x_0|$ and if $(y,z) \notin B(0,a) \times B(0,a)$ then $v_k(y,z) > \beta + 1$, for all $k \geq 1$. Such an a exists by (3.2). If $\sigma = \{y_0, y_1, y_2, \ldots\}$ is such that for some k_0 $y_k \notin B(0,a)$ for all $k > k_0$, then $\sigma' = \{x_0, 0, 0, \ldots\}$ will satisfy (3.4), for every L. Assume then that σ keeps returning to $B(0,a)$. Let α be defined by

$$\alpha = \sup_k \{ \max_{y,z \in B(0,a)} u_k(y,z) \}.$$

Given an L then if $N > 2\alpha + L$ and $\{y_k, y_{k+1}, \ldots, y_{k+N}\}$ is such that

y_k, $y_{k+N} \in B(0,a)$ but

(3.5) $$y_{k+i} \notin B(0,a) \quad \text{for} \quad 1 \leq i \leq N-1$$

then

$$\sum_{j=k}^{k+M-1} v_j(y_j, y_{j+1}) > M(\beta + 1) > [2\alpha + M\beta] + L$$

for every $2\alpha + L < M \leq N$. It follows from this inequality that every such finite sequence $\{y_k, y_{k+1}, \ldots, y_{k+N}\}$ with $N > 2\alpha + L$ can be replaced by $\{y_k, 0, \ldots, 0, y_{k+N}\}$ and thus lowering the costs for all times $k + M$, $M > 2\alpha + L$.

Let $b > a$ be such that if $(y,z) \notin B(0,b) \times B(0,b)$ then

$$v_k(y,z) > 2\alpha + (2\alpha + L) \cdot \beta + L.$$

Then if $\{y_k, y_{k+1}, \ldots, y_{k+N}\}$ is such that $y_k, y_{k+N} \in B(0,a)$ and $y_{k+i} \notin B(0,b)$ for some $1 \leq i \leq N-1$, and $N \leq 2\alpha + L$, then

$$\sum_{j=k}^{k+N-1} v_j(y_j, y_{j+1}) > 2\alpha + (2\alpha + L) \cdot \beta + L \geq (2\alpha + N\beta) + L.$$

Thus the finite sequence $\{y_k, \ldots, y_{k+N}\}$ can be replaced by $\{y_k, 0, \ldots, 0, y_{k+N}\}$ and thus lowering the cost of all times $M \geq k + N$ by at least L. Now, given $\sigma = \{y_0, y_1, y_2, \ldots\}$ we replace every finite sequence $\{y_k, \ldots, y_{k+N}\}$ of length $N \geq 2$ which satisfies $y_k, y_{k+N} \in B(0,a)$ and (3.5) and $y_{k+i} \notin B(0,b)$ for some $1 \leq i \leq N-1$ by the finite sequence $\{y_k, 0, \ldots, 0, y_{k+N}\}$. Call the sequence thus obtained σ'. The above discussion implies that σ' satisfies (3.4).

$\square$

There is a correspondence between solutions of (2.1) and sequences in R^n. To a solution $x(\cdot)$ we associate the sequence $\{x(kT)\}_{k=0}^{\infty}$, while to a sequence $\{x_k\}_{k=0}^{\infty}$ we assocsiate the solution $x(\cdot)$ which coincides on the interval $I_k = [kT, (k+1)T]$ with the solution of the problem

$$(3.6)\qquad \text{minimize} \int_{kT}^{(k+1)T} [\|x(t) - \Gamma(t)\|_Q^2 + \|u(t)\|_R^2]dt$$

subject to $x(kT) = x_k$, $x((k + 1)T) = x_{k+1}$.

Let a be as in lemma 3.1 and as before let α be given by

$$\alpha = \sup_k [\max_{y,z \in B(0,a)} u_k(y,z)].$$

Let in Lemma 3.1 $L = \alpha$ and let b be as asserted in this Lemma. There is a number c' such that if $x(\cdot)$ solves problem (3.6) with $y,z \in B(0,a)$ then $x(t) \in B(0,c')$ for $kT \leq t \leq (k + 1)T$. Also, there is a number d' such that if $x(\cdot)$ solves problem (3.6) with $y,z \in B(0,b)$ then $x(t) \in B(0,d')$ for $kT \leq t \leq (k + 1)T$. Let $c > c'$ and $d > d'$. We claim the following:

<u>Theorem 3.2</u>. Let c and d be as above. If $x(\cdot)$ is a response to $u(\cdot)$ on $[0,\infty)$ such that $x(\tau) \notin B(0,d)$ for some $\tau > 0$ then there is a response $x_0(\cdot)$ to $u_0(\cdot)$ such that $x_0(t) \in B(0,d)$ for all $t \geq 0$ and $u_0(\cdot)$ overtakes $u(\cdot)$.

<u>Proof</u>: Let $x(\cdot)$ be a solution of (2.1) and consider the sequence $\sigma = \{x(kT)\}_{k=0}^{\infty}$ corresponding to it. If it is not contained in $B(0,b)$, then by Lemma 3.1 there is a sequence σ' contained in $B(0,b)$ such that $c_m(\sigma') < c_m(\sigma) - \alpha$ for large m. If $x'(\cdot)$ corresponds to σ' and is a response to $u'(\cdot)$, then $u'(\cdot)$ overtakes $u(\cdot)$. The reason for this is: If $kT \leq t < (k + 1)T$ and $x(\cdot)$ is not equal to $x'(\cdot)$ at both times kT and $(k + 1)T$, then $x'(kT)$ and $x'((k + 1)T)$ are in $B(0,a)$, therefore

$$c_{0,t}(u') \leq c_{0,kT}(u') + \alpha = c_k(\sigma') + \alpha < c_k(\sigma) \leq c_{0,t}(u)$$

whenever k is sufficiently large.

Now suppose that σ is contained in $B(0,b)$. There are two cases:

(i) First possibility: There is a t_0 such that $x(t) \in B(0,d)$ for all $t \geq t_0$, but $x(\tau) \notin B(0,d)$ for some $0 < \tau < t_0$. Then for all intervals $I_k = [kT,(k+1)T]$ such that $kT \leq t_0$ we replace $x(\cdot)$ in I_k by the optimal solution to problem (3.6) with the boundary values equal to those of $x(\cdot)$. We thus obtain a solution $x_0(\cdot)$ corresponding to $u_0(\cdot)$, and $u_0(\cdot)$ overtakes $u(\cdot)$. Clearly $x_0(t) \in B(0,d)$ for all $t \geq 0$.

(ii) Second possiblity: There are arbitrarily large times τ such that $x(\tau) \notin B(0,d)$ holds. Let $x_0(\cdot)$ correspond to $\{x(kT)\}_{k=0}^{\infty}$ and be a response to $u_0(\cdot)$. We claim that there is an $\varepsilon > 0$, independent of k, such that if $x(\tau) \notin B(0,d)$ with $kT < \tau < (k+1)T$ then

$$(3.7) \qquad \int_{kT}^{(k+1)T} [\|x(t) - r(t)\|_Q^2 + \|u(t)\|_R^2]dt > v_k(x(kT),x(k+1)T)) + \varepsilon.$$

This is implied by the following consideration. The number d' has the property that a solution $x(\cdot)$ of (3.6) with $y,z \in B(0,b)$ satisfies $x(t) \in B(0,d')$ for $kT \leq t \leq (k+1)T$. Let $\gamma(\cdot)$ be in the $L_\infty(R^n)$ closure of the set $\{r_k(\cdot): k \geq 0\}$ (recall (3.1)). Then if $\xi(\cdot)$ solves problem (2.4) with $y,z \in B(0,b)$ it follows that $\xi(t) \in B(0,d')$ for $0 \leq t \leq T$. This, combined with the uniqueness of solutions to problem (2.4) and the fact that $d > d'$, implies (3.7). It follows from (3.7) that $u_0(\cdot)$ overtakes $u(\cdot)$, thus the proof of the Theorem is complete. □

As a result of Theorem 3.2 we shall consider only responses which are contained in $B(0,d)$, which we shall denote henceforth by Θ. Once we have proved the existence of an overtaking optimal solution among these responses, it will be an overtaking optimal solution among all the admissible solutions, as implied by Theorem 3.2.

<u>Proposition 3.3</u>. Let $x(\cdot)$ be a solution to problem (3.6) such that $x(t) \in \Theta$ for all $kT < t < (k+1)T$. Let $y(\cdot)$ in $[kT,(k+1)T]$ be the unique function guaranteed in Proposition 2.3. Then there is a constant H, independent of k, such that

$$|y(t)| < H \quad \text{for all} \quad kT \leq t \leq (k+1)T.$$

<u>Proof</u>: If the assertion of the Proposition is false then there is a sequence of pairs $(x_k(\cdot),y_k(\cdot))$ such that

(3.8)
$$\dot{x}_k(t) = Ax_k + BR^{-1}B'y_k$$
$$\dot{y}_k(t) = Qx_k - A'y_k - Q\Gamma_k$$

for $0 < t < T$, where $\Gamma_k(\cdot)$ is as in (3.1), and

$$\lambda_k = \max_{0 \leq t \leq T} |y_k(t)| \to \infty \quad \text{as} \quad k \to \infty.$$

We define

$$\psi_k(t) = \frac{1}{\lambda_k} x_k(t), \quad \phi_k(t) = \frac{1}{\lambda_k} y_k(t)$$

then we get by (3.8)

(3.9)
$$\dot{\psi}_k = A\psi_k + BR^{-1}B'\phi_k$$

(3.10)
$$\dot{\phi}_k = -A'\phi_k + [Q\psi_k - \frac{1}{\lambda k} Q\,\Gamma_k].$$

By (3.10) $\{\phi_k(t)\}$ converges uniformly in $[0,T]$ to $\phi(t)$ where $\phi(\cdot)$ is a function which verifies

(3.11)
$$\dot{\phi}(t) = -A'\phi(t),$$

and
$$\sup_{0 \leq t \leq T} |\phi(t)| = 1 .$$

Since $\psi_k(t) \to 0$ uniformly on $[0,T]$, it follows from (3.9) that also $\dot{\psi}_k(t)$ converges uniformly on $[0,T]$ to zero. Thus we obtain that $B'\phi(t) = 0$ for $0 \leq t \leq T$ which, combined with (3.11), contradicts the controllability of (A,B). This concludes the proof. □

We shall need the following result:

Theorem 3.4. Let $x(\cdot)$ be the solution of the problem (2.4) with

(3.12) $\quad T \leq t_2 - t_1 \leq 2T$ and $x(t) \in \Theta$ for all $t_1 \leq t \leq t_2$.

Let $y(\cdot)$ be related to $x(\cdot)$ as in (2.11), and $g(\cdot)$ defined by (2.14). Then there is a constant P which does not depend on $x(\cdot)$, t_1 and t_2 such that

$$|y(t)|, \quad |g(t)| \leq P$$

for all $t_1 \leq t \leq t_2$.

Proof: The bound on $|y(t)|$ follows from the assumed bound on $|x(t)|$ and from Proposition 3.3. The assertion concerning the boundedness of $|g(t)|$ follows then from (2.14) and the boundedness of $|x(t)|$ and $|y(t)|$.

4. The main result: overtaking optimality of $x^*(\cdot)$

Proposition 4.1. Let x_0 be fixed. Let $g(\cdot)$ be given by (2.17) and $x^*(\cdot)$ be the solution in $[0,\infty)$ of (2.16) with $x(0) = x_0$. Let $x(\cdot)$ be a response to the control $u(\cdot)$. Then there is a constant μ, which does not depend on $u(\cdot)$, such that

(4.1) $$c_{0,T}(u^*) \leq c_{0,T}(u) + \mu$$

for all $T \geq T_0$, for some $T_0 > 0$, and where $x^*(\cdot)$ is the response to $u^*(\cdot)$.

Proof: By Theorem 3.2 we can assume that $x^*(t), x(t) \in \Theta$ for all $t \geq 0$, where the ball Θ depends only on the initial value x_0. We denote by $v_{t_0}(y,z)$ the minimal value for the minimization problem

$$\text{minimize} \int_{t_0}^{t_0+1} [\|x(t) - \Gamma(t)\|_Q^2 + \|u(t)\|_R^2]dt$$

subject to $x(t_0) = y$, $x(t_0 + 1) = z$.

Define

$$\mu = \sup\{v_{t_0}(y,z): t_0 \geq 0, y \in \Theta, z \in \Theta\} \tag{4.2}$$

and it follows from the boundedness of $\Gamma(\cdot)$ that $\mu < \infty$. We claim that μ in (4.2) satisfies (4.1) for all $T > 1$. If not then

$$c_{0,T}(u) < c_{0,T}(u^*) - \mu, \text{ which implies}$$

$$c_{0,T-1}(u) < c_{0,T}(u^*) - \mu \tag{4.3}$$

for some $T > 1$. By (4.2) and the assumption $x(T - 1), x^*(T) \in \Theta$ we have

$$v_{T-1}(x(T - 1),x^*(T)) \leq \mu.$$

This, combined with (4.3) yields

$$c_{0,T-1}(u) + v_{T-1}(x(T - 1),x^*(T)) < c_{0,T}(u^*). \tag{4.4}$$

The left hand side of (4.4) is $c_{0,T}(\tilde{u})$ where $\tilde{u}$ is a control with a response $\tilde{x}(\cdot)$ which satisfies $\tilde{x}(0) = x_0$ and $\tilde{x}(T) = x^*(T)$. The inequality (4.4) thus contradicts Theorem 2.5, concluding the proof of the Proposition. □

The following Theorem asserts that not only can we consider only responses $x(\cdot)$ which remain in Θ for all times, but we can consider only such responses $x(\cdot)$ which satisfy

$$(4.5) \qquad |x(t) - x^*(t)| \to 0 \quad \text{as} \quad t \to \infty.$$

Theorem 4.2. Let $x^*(\cdot)$ satisfy (2.16) with $g(\cdot)$ as in (2.17). Let $x(\cdot)$ be a response to $u(\cdot)$ and suppose it does not verify (4.5). Then

$$\lim_{T\to\infty} [c_{0,T}(u) - c_{0,T}(u^*)] = \infty.$$

To the proof of Theorem 4.2 we shall need the following two Lemmas.

Lemma 4.3. Given an $\varepsilon > 0$ there exist constants $\delta > 0$ and $\nu > 0$ such that if $x_i(\cdot)$, $i = 1,2$, are solutions of the minimizing problems

$$\text{minimize} \int_{t_0-\delta}^{t_0+\nu} [\|x(t) - r(t)\|_Q^2 + \|u(t)\|_R^2]dt$$

subject to $x_i(t_0 - \delta) = y_i$, $x_i(t_0 + \nu) = z_i$, $i = 1,2$ where $y_i, z_i \in \Theta$ and $t_0 > \delta$, then

$$|x_1(t_0) - x_2(t_0)| < \varepsilon.$$

In particular, if $x^*(\cdot)$ is as in Theorem 4.2, then $|x_1(t_0) - x^*(t_0)| < \varepsilon$.

Proof: By Proposition 2.4 the solutions $x_i(\cdot)$ are given by

$$(4.6) \qquad x_i(t_0) = e^{\delta F}x_i(t_0 - \delta) + \int_{t_0-\delta}^{t_0} e^{(t_0 - t)F} BR^{-1}B'g_i(t)dt$$

where $g_i(\cdot)$ is some solution of (2.15). If δ is large enough then the first term in (4.6) can be estimated as follows

$$|e^{\delta F}y_i| < \frac{1}{4}\varepsilon \quad \text{for all} \quad y_i \in \Theta,$$

using the stability of F. We fix such a δ, and consider the restriction of $g_i(\cdot)$ to $[t_0 - \delta, t_0]$ when we take ν to be very large. We claim the following:

(4.7) $$\max_{t_0-\delta<t<\delta} |g_1(t) - g_2(t)| \to 0 \quad \text{as} \quad \nu \to \infty$$

and this, uniformly in t_0. This is a consequence of the fact that both $g_1(\cdot)$ and $g_2(\cdot)$ solve equation (2.15), hence their difference $g = g_1 - g_2$ solves the equation

$$\dot{g} = -F'g$$

in $[t_0 - \delta, t_0 + \nu]$. By Theorem 3.4 there is a bound on $|g(t_0 + \nu)|$, which does not depend on t_0, and thus (4.7) follows from the stability of F'. This implies that

$$\left|\int_{t_0-\delta}^{t_0} e^{(t_0 - t)F} BR^{-1}B'(g_1(t) - g_2(t))dt\right| < \frac{1}{2}\varepsilon$$

if ν is sufficiently large. In view of the bound on $|e^{\delta F}y_i|$, this concludes the proof of the Lemma. □

<u>Lemma 4.4.</u> Let ε, δ and ν be as in Lemma 4.3 and let $\alpha > \varepsilon$. Then there is a $\beta > 0$ such that if $\tilde{x}(\cdot)$ is a response to $\tilde{u}(\cdot)$ in $[t_0 - \delta, t_0 + \nu]$ with $\tilde{x}(t_0 - \delta), \tilde{x}(t_0 + \nu) \in \Theta$ and if $x^*(\cdot)$ is the optimal solution of

$$\text{minimize} \quad c_{t_0-\delta,t_0+\nu}(u)$$

subject to $x(t_0 - \delta) = \tilde{x}(t_0 - \delta)$, $x(t_0 + \nu) = \tilde{x}(t_0 + \nu)$ and if

$$|\tilde{x}(t_0) - x^*(t_0)| > \alpha,$$

then

$$c_{t_0-\delta,t_0+\nu}(\tilde{u}) > c_{t_0-\delta,t_0+\nu}(u^*) + \beta$$

where $x^*(\cdot)$ is the response of $u^*(\cdot)$.

Proof: Assume that the assertion is false. Then there is a sequence $\{x_k(\cdot)\}_{k=1}^{\infty}$ of responses to controls $\{u_k(\cdot)\}_{k=1}^{\infty}$ in $[0,\delta+\nu]$, such that

(4.8) $$|x_k(\delta) - x_k^*(\delta)| > \alpha \quad \text{and}$$

(4.9) $$c_{0,\delta+\nu}(u_k) - c_{0,\delta+\nu}(u_k^*) \to 0 \quad \text{as} \quad k \to \infty,$$

where the costs in the last expression are of tracking a trajectory $\gamma_k(\cdot)$ in $[0,\delta+\nu]$, $x_k(\cdot)$ and $x_k^*(\cdot)$ have the same boundary values, and $x_k^*(\cdot)$ is the optimal solution.

Since $\{u_k(\cdot)\}_{k=1}^{\infty}$ is bounded in $L_2(R^m)$ we can assume that

(4.10) $$u_k(\cdot) \to u_0(\cdot) \quad \text{as} \quad k \to \infty \ \text{weakly in} \ L_2(R^m)$$

and

(4.11) $$x_k(\cdot) \to x_0(\cdot) \quad \text{as} \quad k \to \infty, \ \text{uniformly in} \ [0,\delta+\nu],$$

where $x_0(\cdot)$ is the response to $u_0(\cdot)$. Since the $\{\gamma_k(\cdot)\}_{k=1}^{\infty}$ are restricstions to intevals of length $\nu+\delta$ of the function $\Gamma(\cdot)$, which is uniformly continuous and bounded on $[0,\infty)$, we can assume that also

(4.12) $$\gamma_k(t) \to \gamma_0(t) \quad \text{as} \quad k \to \infty, \ \text{uniformly in} \ [0,\delta+\nu].$$

It follows from (4.10), (4.11) and (4.12) that

(4.13) $$c_{0,\delta+\nu}(u_0) \leq \liminf_{k\to\infty} c_{0,\delta+\nu}(u_k).$$

Now let $v(y,z,\gamma)$ be the minimal value of problem (2.4) with $T = \alpha + \nu$. Let y and z vary in Θ and γ take values in a ball B_M about the origin in $C[0,\delta+\nu]$, the space of the continuous functions from $[0,\delta+\nu]$ to R^n endowed with the $L_\infty(R^n)$ norm. Then the function

(4.14) $$(y,z,\gamma) \to v(y,z,\gamma)$$

is continuous in $\gamma(\cdot)$, uniformly in $y,z \in \Theta$, and it is continuous in (y,z) for every fixed γ in B_M. Therefore it is jointly continuous in (y,z,γ). Similar to (4.10) and (4.11) we have

$$(4.15) \qquad u_k^*(\cdot) \to u_0^*(\cdot) \quad \text{as } k \to \infty, \text{ weakly in } L_2(R^m)$$

$$(4.16) \qquad x_k^*(t) \to x_0^*(t) \quad \text{as } k \to \infty, \text{ uniformly in } [0,\delta + \nu],$$

where $x_0^*(\cdot)$ is the response to $u_0^*(\cdot)$ in $[0,\delta + \nu]$. From the continuity of the function in (4.14), together with (4.12), (4.15) and (4.16) we get that

$$c_{0,\delta+\nu}(u_0^*) = \lim_{k\to\infty} c_{0,\delta+\nu}(u_k^*)$$

This equality, combined with (4.9) and (4.13) yields

$$c_{0,\delta+\nu}(u_0) = c_{0,\delta+\nu}(u_0^*)$$

while from (4.8), (4.11) and (4.16) we have

$$\max_{0<t\leq\delta+\nu} |x_0(\delta) - x_0^*(\delta)| \geq \alpha$$

which contradicts the uniqueness of the optimal solutions. □

<u>Proof of Theorem 4.2</u>: Under the assumptions of the Theorem there is an $\varepsilon > 0$ and a sequence of times $t_k \to \infty$ such that

$$|x^*(t_k) - x(t_k)| \geq 2\varepsilon$$

and the intervals

$$I_k = [t_k - \delta, t_k + \nu]$$

are mutually disjoint, where δ and ν are as in Lemma 4.3. In each interval I_k we replace $x(\cdot)$ by the optimal tracking solution in I_k which satisfies the same boundary conditions as $x(\cdot)$ in I_k. Call the solution thus obtained $y(\cdot)$, and let it be the response to the control $w(\cdot)$. Then it follows from Lemma 4.4 that

$$\lim_{T\to\infty} [c_{0,T}(u) - c_{0,T}(w)] = \infty. \tag{4.17}$$

In view of Proposition 4.1 this implies that

$$\lim_{T\to\infty} [c_{0,T}(u) - c_{0,T}(u^*)] = \infty$$

which is the assertion of the Theorem. □

The following Theorem is our main result:

Theorem 4.5. Let the initial condition x_0 be fixed, and $\Gamma(\cdot)$ is uniformly continuous and bounded on $[0,\infty)$. Let $x^*(\cdot)$ satisfy (2.16) with $g(\cdot)$ as in (2.17). Then $x^*(\cdot)$ is the unique overtaking optimal solution.

Proof: We first prove the uniqueness. If $x^*(\cdot)$ is an overtaking optimal solution then it must verify (2.16) on $[0,\infty)$ with a bounded function $g(\cdot)$, which satisfies (2.15) on $[0,\infty)$. But then $g(\cdot)$ is given by

$$g(t) = e^{-tF'}[g(0) - \int_0^t e^{sF'}Q\Gamma(s)ds]$$

and since F' is stable, there is only one choice of $g(0)$ for which $g(\cdot)$ becomes bounded, namely

$$g(0) = \int_0^\infty e^{sF'}Q\Gamma(s)ds.$$

This proves the uniqueness.

Let Z_0 be the collection of continuous functions

$$\gamma: [0,1] \to R^n$$

such that there is some $t_0 > 0$ so that

$$\gamma(t) = \Gamma(t_0 + t) \quad \text{for} \quad 0 \leq t \leq 1.$$

Consider Z_0 as a subset of $C([0,1])$, the space of continuous functions from $[0,1]$ to R^m, with the $L_\infty(R^n)$ norm. Let Z be the closure of Z_0 in $C([0,1])$, then Z is compact since $\Gamma(\cdot)$ is uniformly continuous and bounded on $[0,\infty)$. We define the function

$$m: \Theta \times \Theta \times Z \to R^1$$

such that $m(y,z,\gamma)$ is the minimal cost for problem (2.4) with $T = 1$. Arguing as in the proof of the continuity of $v(.,.,.)$ in (4.14) it follows that $m(.,.,.)$ is continuous, and by the compactness of its domain it is uniformly continuous. Therefore, given an $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|m(y_1,z_1,\gamma) - m(y_2,z_2,\gamma)| < \varepsilon \tag{4.18}$$

for all $|y_1 - y_2|$, $|z_1 - z_2| < \delta$, $y_1,y_2,z_1,z_2 \in \Theta$, $\gamma \in Z$.

Let $x(\cdot)$ be a response to $u(\cdot)$ in $[0,\infty)$. If $|x(t) - x^*(t)|$ does not tend to zero as t grows to infinity, then $u^*(\cdot)$ overtakes $u(\cdot)$ by Theorem 4.2. If $|x(t) - x^*(t)| \to 0$ as $t \to \infty$, then there is a t_0 such that

$$|x(t) - x^*(t)| < \delta \quad \text{for} \quad t > t_0. \tag{4.19}$$

Let $t > t_0$ and $\gamma(\cdot)$ be given by

$\gamma(s) = \Gamma(t + s)$ for $0 \leq s \leq 1$. Using (4.18) and (4.19) we estimate as follows:

$$\begin{aligned} c_{0,t+1}(u) &\geq c_{0,t}(u) + m(x(t),x(t+1),\gamma) > \\ &> c_{0,t}(u) + m(x(t),x^*(t+1),\gamma) - \varepsilon \geq \\ &\geq c_{0,t+1}(u^*) - \varepsilon \end{aligned}$$

with the last inequality following from the optimality of $x^*(\cdot)$ on finite intervals. This concludes the proof of the Theorem. □

5. A case of limited knowledge about the tracked trajectory

In this section we consider the following situation: At every time t the controller knows what are the values of the tracked trajectory in the interval $[t,t + T]$, but not for larger times. Another information available is that $r(\cdot)$ is bounded with a known bound

$$(5.1) \qquad |r(t)| \leq g \quad \text{for all} \quad t \geq 0,$$

and is uniformly continuous on $[0,\infty)$.

Let $r_1(\cdot)$ and $r_2(\cdot)$ both satisfy (5.1) and suppose that they coincide on the $[0,T]$ interval. Inspecting the expressions for the overtaking optimal solutions we see that for a fixed initial value x_0 and for a fixed interval $[0,\tau]$ we obtain

$$\max_{0 \leq t \leq \tau} |x_1^*(t) - x_2^*(t)| \to 0 \quad \text{as} \quad T \to \infty$$

and this uniformly for pairs $(r_1(\cdot),r_2(\cdot))$ which satisfy (5.1). This suggests the following claim:

<u>Theorem 5.1</u>: Let the initial condition x_0 be fixed and the tracked trajectory be uniformly continuous and bounded on $[0,\infty)$. Assume that at every $t \geq 0$ it is known in the interval $[0,t + T]$, $T > 0$ fixed. Let $x^*(\cdot)$ be the overtaking optimal solution. Given an $\varepsilon > 0$ there is a $T > 0$ and a response $x_T(\cdot)$ to a control $u_T(\cdot)$ such that

$$\sup_{0 \leq t < \infty} |x^*(t) - x_T(t)| < \varepsilon$$

and the value $u_T(t)$ is a function of the values of $\Gamma(\cdot)$ in the interval $[0,t + T]$.

<u>Proof</u>: We define

$$(5.2) \qquad g_T(t) = \int_t^{t+T} e^{(s-t)F'} Q\Gamma(s)ds$$

and let $x_T(\cdot)$ be the solution of (2.1) with the feedback control

$$u_T(t) = R^{-1}B'[g_T(t) - Kx(t)].$$

Then $x_T(\cdot)$ satisfies the equation

$$(5.3) \qquad \dot{x}_T = FX_T + BR^{-1}B'g_T, \quad x_T(0) = x_0.$$

It follows from the boundedness of $g_T(\cdot)$ and $g(\cdot)$ and the stability of F that there is a bound b such that

$$(5.4) \qquad |x^*(t)|, \ |x_T(t)| \leq b$$

for all $t \geq 0$ and $T > 0$.

Let $\tau > 0$ be such that

$$(5.5) \qquad |e^{Ft}y| < \frac{1}{4}\varepsilon$$

whenever $|y| \leq b$ and $t \geq \tau$. Denote

$$\Delta x_T(t) = x^*(t) - x_T(t)$$

then by (2.16) and (5.3)

$$\frac{d}{dt}\Delta x_T = F(\Delta x_T) + BR^{-1}B'(g - g_T), \ \Delta x_T(0) = 0$$

and $\Delta x_T(t)$ is given as follows:

$$(5.6)\qquad \Delta x_T(t) = \begin{cases} \int_0^t e^{(t-s)F}BR^{-1}B'[g(s) - g_T(s)]ds & \text{if } 0 \leq t \leq \tau \\ e^{\tau F}[\Delta x_T(t-\tau)] + \int_{t-\tau}^t e^{(t-s)F}BR^{-1}B'[g(s) - g_T(s)]ds & \text{if } t>\tau. \end{cases}$$

By (2.17) and (5.2) we have

$$g(t) - g_T(t) = e^{TF'} \int_0^\infty e^{sF'} Q r(t + T + s)ds$$

and therefore

$$|g(t) - g_T(t)| \to 0 \quad \text{as} \quad T \to \infty$$

uniformly in t. Thus we can find a T so large that

$$\left| \int_{t-\tau}^t e^{(t-s)F}BR^{-1}B'[g(s) - g_T(s)]ds \right| < \frac{\varepsilon}{2}$$

whenever $t > \tau$, and also

$$\left| \int_0^t e^{(t-s)F}BR^{-1}B'[g(s) - g_T(s)]ds \right| < \frac{\varepsilon}{2}$$

whenever $0 \leq t < \tau$.

Then in (5.6) the integral terms in both cases of definition are less than $\frac{1}{2}\varepsilon$ in absolute value. The term $|e^{\tau F}[\Delta x_F(t - \tau)]|$ is less than $\frac{1}{2}\varepsilon$ by (5.4) and (5.5). This concludes the proof of the Theorem. □

References

[1] Z. Artstein and A. Leizarowitz (1985). Tracking periodic signals with the overtaking criterion. Submitted.

[2] M. Athens and P.L. Falb (1966). Optimal Control. McGraw-Hill, New York.

[3] D. Gale (1967). On optimal development in multi-sector economy. Rec. Econ. Stud. 37, pp. 1-19.

[4] E.B. Lee and L. Markus (1967). Foundation of Optimal Control Theory. Wiley, New York.

[5] C.C. van Weizsacker (1965). Existence of optimal programs of accumulation for an infinite time horizon. Rev. Econ. Stud. 32, pp. 85-104.

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