Pointwise Fourier Inversion:

a Wave Equation Approach

Mark A. Pinsky[1]

Northwestern University
Evanston IL 60208

Michael E. Taylor[2]

University of North Carolina
Chapel Hill NC 27599

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Introduction

Recently one of us [P2,P3] found a necessary and sufficient condition for the inversion of the Fourier transform of a function f at a pre-assigned point, within the class of piecewise smooth functions on Euclidean space. This allows one to construct elementary (spherically symmetric) examples of divergent Fourier integrals in dimensions three and higher. The same criterion extends without change to Fourier expansions on the sphere and on hyperbolic space, while partial extensions are available for multiple Fourier series on the torus [PST] and for certain eigenfunction expansions for the Laplacian on bounded domains, with the Dirichlet boundary condition.

Various examples considered in these papers gave a striking impression that there should be a connection between the issue of convergence or divergence of Fourier inversion on the one hand and focusing of waves on the other. However, the analytical techniques employed did not provide a direct connection.

The purpose of this paper is to give a direct analysis of pointwise Fourier inversion in a fashion that makes explicit use of the wave equation. In §1 we give a rather general result, relating the convergence of $S_R f = \chi_R(A) f$ at a point x to the behavior of the solution $u(t,x)$ to the wave equation

$$(\partial_t^2 - L)u = 0, \quad u(0,x) = f(x), \ u_t(0,x) = 0. \tag{0.1}$$

where L is a second order negative elliptic partial differential operator on a manifold M (such as the Laplace operator Δ), $A = \sqrt{-L}$, and χ_R is the characteristic function of $[0, R]$. It is noted that, when $M = \mathbb{R}$ and $L = d^2/dx^2$, the criterion reduces to the classical Dini criterion for convergence.

In §§2-4, we work out the implications for Fourier inversion of functions on $\mathbb{R}^n$, first for $n = 3$, then for $n = 2$, then for general n, making use of the classical Poisson-Kirchhoff formula for the solution of the wave equation on $\mathbb{R} \times \mathbb{R}^n$. This technique allows one to retrieve various results of [P2,P3]. Furthermore, the hypotheses that f is piecewise smooth and has compact support are here generalized to square integrability at infinity and a Dini condition at x. We also briefly discuss alternatives to the Dini condition. The formula for the solution to the wave equation naturally leads to expressions involving spherical means, parallel to those derived by other means in [P2,P3]. Appreciating this fact leads to results in Appendix A, which we will mention a little further on.

Section 5 deals with Fourier inversion for functions defined on spheres S^n. The general set-up of §1 does not quite apply to this case, since the decay hypothesis (1.6) is violated. However, we use the special behavior of the spectrum of the Laplace operator on S^n to 'compactify' harmonic analysis on spheres. We then implement the modified version of the approach of §1, making use of the exact fundamental solution to the wave equation (0.1) on $\mathbb{R} \times S^n$, with $L = \Delta - \left(\frac{n-1}{2}\right)^2$. It is worth pointing out that, with this approach, one avoids appeal to special function theory. In §6, we consider other compact rank one symmetric spaces, such as complex projective space and quaternionic projective space.

Section 7 considers Fourier inversion for functions defined on hyperbolic space $\mathcal{H}^n$. We use the exact fundamental solution of the shifted wave equation (0.1) on $\mathbb{R} \times \mathcal{H}^n$, with $L = \Delta + \left(\frac{n-1}{2}\right)^2$, obtaining results quite parallel to those for Euclidean space $\mathbb{R}^n$. We pass on to more general considerations, first mentioning other rank one symmetric spaces of noncompact type. A special property of this class of spaces is that, given a function f, and a point x, convergence $S_R f(x) \to f(x)$ is equivalent to $S_R \Phi(x) \to \Phi(x)$, where Φ is the radial symmetrization of f about x. Here, we do not bring in explicit formulas for the solution to wave equations. Instead, we make use of the Hadamard parametrix to establish a result on convergence $S_R f(x) \to f(x)$ for radial functions on a more general class of manifolds.

Section 8 provides the denouement. The use of the Hadamard parametrix is extended to study Fourier inversion for non-radial functions, on various classes of manifolds. Also, other results from microlocal analysis are brought to bear, in situations where the Hadamard parametrix fails. It is in this section that the connection between convergence $S_R f(x) \to f(x)$ and focusing of waves appears most clearly. In this section we also discuss sharp results on the rate of convergence of $S_R f(x)$ to $f(x)$, for a significant class of functions f, consisting of classical conormal distributions. In addition, we consider a class of manifolds with boundary, on which a Dirichlet or Neumann boundary condition is imposed for the Laplace operator.

Section 9 contains some results on a different but related question, pointwise convergence of the expansion of a function on $\mathbb{R}^n$ in terms of Hermite functions. As an analytical tool, we use the unitary group generated by the Hermite operator, and analyze it via Mehler's formula. It is interesting that, in this case, convergence of $S'_M f(x)$ to $f(x)$ is influenced by phenomena involving focusing, both at x and at $-x$.

Another related topic is considered in §10, namely nonradial Fourier inversion. We treat this as a study of spectral projections for an operator $\lambda(D)$, given $\lambda(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$, positive and homogeneous of degree one. We make use of the pseudodifferential equation $u_t - i\lambda(D)u = 0$, and obtain some results analogous to those in §§2-4, under the hypothesis that the set $\{\xi : \lambda(\xi) < 1\}$ is symmetric and strongly convex.

In §§2-10 there are a number of results on pointwise convergence of $S_R f(x)$ at a point $x \in \Sigma$, a smooth surface across which f has a simple jump. Section 11 takes up the natural question of analyzing the Gibbs phenomenon in such a context. We also analyze a version of the Gibbs phenomenon when there is a boundary, and the Dirichlet boundary condition is imposed, but f does not vanish on the boundary. Progressing wave expansions for solutions to the wave equation provide an essential tool in these analyses.

At the end of the paper are three appendices. The first two provide derivations of the classical Poisson-Kirchhoff formula for the solution to the wave equation on $\mathbb{R} \times \mathbb{R}^n$ (for n odd; the case n even is, as ever, amenable to the method of descent). In Appendix A, we establish a recursion formula for the Dirichlet kernel, representing the operator S_R, and then derive the formula for the wave kernel, making use of the direct link between these two, provided by the operator identity (1.3). The resulting argument seems to us more direct than, for example, the derivation of the wave kernel on pp.683-686 of [CH], which also makes use of the Dirichlet kernel. In Appendix B, a variant of this argument is used to derive the wave kernel from the heat kernel, an object whose calculation involves a well-known manipulation of Gaussian integrals. Both of these methods strike us as pedagogically useful. In addition, there is Appendix C, in which we study the Fourier transform of some distributions which are oscillatory at the origin, which arise in the analysis of §9. In some model cases, we relate the computations to some done in Appendix A.

We end this introduction with a few general comments about this paper. First, we deal with convergence of spectral projections applied to f, not with various summability methods, such as Riesz means. There is a considerable literature on the behavior of Riesz means, some of which touches on wave equation methods; see, e.g., [Sog1] and references given there. Second, we deal with convergence $S_R f(x) \to f(x)$ at specified points, not with such matters as pointwise convergence on a large but unspecified set. Third, while we state a number of results in terms of functions f belonging to various Banach spaces of functions, our emphasis is on results which are illustrated in interesting and non-trivial fashions by quite 'nice' functions. In particular, examples illustrating non-convergence, or relatively slow convergence, use functions which have 'simple' singularities, and are not at all pathological.

To be sure, one person's pathological function might seem to someone else to be quite natural. To give an example, functions of the form $t^\mu \sin(1/t)$ arise in elementary analysis, as counterexamples to various assertions, and have perhaps acquired a reputation for being pathological. However, these functions are seen to arise quite naturally in the analysis of Schrödinger equations, and in this capacity they play a role in §9 in the study of pointwise convergence of Hermite series. On the other hand, these functions are not brought to bear as examples to illustrate convergence and divergence phenomena in this paper, though the industrious reader will find it possible to bend them to that purpose.

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1. A general criterion for pointwise Fourier inversion

Let L be a second order elliptic differential operator on a Riemannian manifold M. Assume $-L$ is positive and self adjoint, such as perhaps $-L = -\Delta$, and let $A = (-L)^{\frac{1}{2}}$ be its positive square root. If φ is an even function of a real variable, we have

$$\varphi(A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(t) \ \cos tA \ dt, \tag{1.1}$$

where $\hat{\varphi}(t) = \int \varphi(\lambda) e^{-i\lambda t} \, d\lambda$ is the Fourier transform of φ. If we take $\varphi(\lambda) = \chi_{\{|\lambda| \le R\}} = \chi_R(\lambda)$, then

$$\hat{\chi}_R(t) = \int_{-R}^{R} \cos \lambda t \ d\lambda = 2 \ \frac{\sin Rt}{t}, \tag{1.2}$$

so $S_R = \chi_R(A)$ is given by

$$S_R f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rt}{t}\ u(t,x)\ dt, \tag{1.3}$$

where $u(t,x) = \cos tA\ f(x)$. Note that $u(t,x)$ solves the hyperbolic equation

$$(\partial_t^2 - L)u = 0, \quad u(0,x) = f(x),\ u_t(0,x) = 0. \tag{1.4}$$

For now we will make the following hypothesis, which will be verified in many special cases:

$$\begin{gathered} \text{Given } x \in M, \text{ there exists } T_0 > 0, \text{ such that} \\ \int_{T_0}^{\infty} t^{-1}\,|u(t,x)|\ dt < \infty. \end{gathered} \tag{1.5}$$

In particular, the conclusion of (1.5) holds if, for some $a > 0,\ C < \infty$,

$$|t| > T_0 \Longrightarrow |u(t,x)| \le C|t|^{-a}. \tag{1.6}$$

Basic results on hyperbolic equations imply that, for any fixed $x \in M$, $u(\cdot,x) \in \mathcal{D}'(\mathbb{R})$, and together with (1.5) this implies that (1.3) is well defined pointwise.

Certainly $S_R f \to f$ in $L^2(M)$ if $f \in L^2(M)$. We want to study when $S_R f(x) \to f(x)$ for some given x. To study this, we will pick $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(0) = 1$, and add $0 = \varphi(t)f(x) - \varphi(t)f(x)$ to $u(t,x)$ in (1.3). Note that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rt}{t}\ \varphi(t)f(x)\ dt = S_R\varphi(0) \cdot f(x), \tag{1.7}$$

where S_R here acts on φ, so we know that $\lim\limits_{R\to\infty} S_R\varphi(0) = \varphi(0) = 1$. Now (1.3) yields

$$S_R f(x) = S_R\varphi(0) \cdot f(x) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t,x) - \varphi(t)f(x)}{t}\ \sin Rt\ dt. \tag{1.8}$$

We have then the following criterion for pointwise convergence of $S_R f(x)$.

Proposition 1.1. *Let $f \in L^2(M)$ and fix $x \in M$. Then $S_R f(x) \to f(x)$ as $R \to \infty$, if and only if*

$$\int_{-\infty}^{\infty} \frac{u(t,x) - \varphi(t)f(x)}{t}\ \sin Rt\ dt \to 0 \quad \text{as} \quad R \to \infty. \tag{1.9}$$

Applying the Riemann-Lebesgue Lemma, we have the following *sufficient* condition for convergence:

Corollary 1.2. *Given $x \in M$, we have $S_R f(x) \to f(x)$, provided*

$$\frac{u(t,x)-\varphi(t)f(x)}{t} \in L^1(\mathbb{R}). \tag{1.10}$$

Note that, when $M = \mathbb{R}$ (with its standard metric) then the quantity in (1.10) is equal to

$$\frac{1}{2}\,\frac{f(x+t)+f(x-t)-2\varphi(t)f(x)}{t}. \tag{1.11}$$

In such a case, the condition (1.10) is a classical condition (of Dini) guaranteeing convergence $S_R f(x) \to f(x)$.

Note that, if (1.10) holds, then the hypothesis (1.5) must hold. Conversely, if (1.5) holds, then we can localize the hypothesis (1.10).

The analysis above also has implications for uniform convergence of $S_R f$, on a set $S \subset M$. Clearly, $S_R f(x) \to f(x)$ uniformly for $x \in S$ if and only if (1.9) holds, uniformly for $x \in S$. Thus, in the spirit of Corollary 1.2, we have:

Corollary 1.3. *Given $S \subset M$, we have $S_R f(x) \to f(x)$ uniformly for $x \in S$, provided*

$$\Big\{\frac{u(t,x)-\varphi(t)f(x)}{t} : x \in S\Big\} \ \text{is compact in} \ \ L^1(\mathbb{R}). \tag{1.12}$$

Proof. One need only note that, if $\mathcal{K} \subset L^1(\mathbb{R})$ is compact, then $\hat{g}(t) \to 0$ as $|t| \to \infty$, uniformly for $g \in \mathcal{K}$.

Let us also remark that, since $S_R\varphi(0)$ tends to $\varphi(0)$ rapidly as $R \to \infty$, the rate of convergence of $S_R f(x)$ is controlled by the rate of convergence in (1.9). We will draw some explicit conclusions from this observation in §8.

It is useful to define $S_R f$ for some elements f that do not belong to $L^2(M)$. We can do the following. For $s > 0$, let $\mathcal{D}_s$ be the domain of the self adjoint operator A^s, with the graph topology. Then, let $\mathcal{D}_{-s}$ be the dual of $\mathcal{D}_s$. Set $\mathcal{D}_0 = L^2(M)$. We have natural containments

$$\mathcal{D}_s \subset \mathcal{D}_t, \quad -\infty < t < s < \infty. \tag{1.13}$$

Also, by elliptic regularity, for any $s \in \mathbb{R}$,

$$f \in H^s(M), \ \ \text{supp}\, f \text{ compact } \Longrightarrow f \in \mathcal{D}_s. \tag{1.14}$$

Furthermore, for any bounded φ, we have

$$\varphi(A) : \mathcal{D}_s \longrightarrow \mathcal{D}_s, \quad \forall\, s \in \mathbb{R}. \tag{1.15}$$

For $s \geq 0$, this holds by the Spectral Theorem, and for $s < 0$ it holds by duality. Now it follows that $S_R f$ is defined for any $f \in \mathcal{D}_s$, for any $s \in \mathbb{R}$, and $S_R f \to f$ in the topology of $\mathcal{D}_s$. By (1.14), we have

$$L^1_{\text{comp}}(M) \subset \mathcal{D}_s, \quad \forall\, s < -\tfrac{n}{2}, \tag{1.16}$$

so $S_R f$ is well defined for any $f \in L^1(M)$ with compact support. As is known from the work [F], (even for $M = \mathbb{R}^2$), we do not generally have $S_R f \to f$ in L^p norm, for all $f \in L^p_{\text{comp}}(M)$, for any $p \neq 2$. (The case $M = \mathbb{R}$ is an exception; then any $p \in (1, \infty)$ is okay.) However, it is still meaningful to inquire about various types of convergence, including pointwise convergence, particularly for integrable functions with 'simple' singularities. We will say more about this in §8.

2. Pointwise Fourier inversion on $\mathbb{R}^n$ $(n = 3)$

If $f \in L^2(\mathbb{R}^n)$, then

$$S_R f(x) = (2\pi)^{-n} \int\limits_{|\xi| \le R} \hat{f}(\xi) e^{ix\cdot\xi} \, d\xi, \tag{2.1}$$

where $\hat{f}(\xi) = \int f(x) e^{-ix\cdot\xi} \, dx$, is also given by

$$S_R f = \chi_R(\sqrt{-\Delta}) f, \quad \Delta = \partial_1^2 + \cdots + \partial_n^2, \tag{2.2}$$

so we have the formulas (1.3) and (1.7), with $u(t, x)$ solving the standard wave equation

$$(\partial_t^2 - \Delta) u = 0, \quad u(0, x) = f(x), \; u_t(0, x) = 0. \tag{2.3}$$

The solution to (2.3) is given by

$$u(t) = \partial_t R(t) f, \tag{2.4}$$

where $R(t)$ is the fundamental solution operator, for which there are explicit formulas. For example, when $n = 3$,

$$R(t) f(x) = \frac{t}{4\pi} \int\limits_{S^2} f(x - t\omega) \, dS(\omega) = t \overline{f}_x(|t|), \tag{2.5}$$

where $\overline{f}_x$, is defined by the last identity. Thus, the quantity in (1.9) is equal to

$$\int_{-\infty}^{\infty} \frac{\overline{f}_x(|t|) - \varphi(t) f(x)}{t} \sin Rt \; dt + \int_{-\infty}^{\infty} \big(\partial_t \overline{f}_x(|t|)\big) \sin Rt \; dt. \tag{2.6}$$

In order to localize the analysis of the first term in (2.6), and also for use in subsequent sections, we establish some general results on $\overline{f}_x(t)$, defined for $f \in L^2(\mathbb{R}^n)$ by

$$\overline{f}_x(t) = \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} f(x + t\omega) \, dS(\omega), \tag{2.7}$$

where A_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

Lemma 2.1. *Let $f \in L^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$. Then $\overline{f}_x(t)$ is well defined for a.e. $t \in \mathbb{R}^+$, and*

$$\int_0^\infty t^{n-1}|\overline{f}_x(t)|^2 \, dt \le A_{n-1}^{-1}\|f\|_{L^2}^2. \tag{2.8}$$

Also, for any $T > 0$,

$$\int_T^\infty t^{-1}|\overline{f}_x(t)| \, dt \le \left(nT^n A_{n-1}\right)^{-\frac{1}{2}}\|f\|_{L^2}. \tag{2.9}$$

Proof. That $\overline{f}_x(t)$ is well defined for a.e. $t \in \mathbb{R}^+$ follows from Fubini's theorem. Cauchy's inequality implies

$$|\overline{f}_x(t)|^2 \le \frac{1}{A_{n-1}} \int_{S^{n-1}} |f(x+t\omega)|^2 \, dS(\omega). \tag{2.10}$$

Then, by Fubini's theorem,

$$\int_0^\infty t^{n-1}|\overline{f}_x(t)|^2 \, dt \le \frac{1}{A_{n-1}} \int_0^\infty \int_{S^{n-1}} t^{n-1}|f(x+t\omega)|^2 \, dS(\omega) \, dt = A_{n-1}^{-1}\|f\|_{L^2}^2.$$

This gives (2.8). In particular, $|\overline{f}_x(t)| < \infty$ for a.e. t.

Again by Cauchy's inequality,

$$\begin{aligned}\left(\int_T^\infty t^{-1}|\overline{f}_x(t)| \, dt\right)^2 &= \left(\int_T^\infty t^{\frac{n-1}{2}} \, |\overline{f}_x(t)| \, t^{-\frac{n+1}{2}} \, dt\right)^2 \\ &\le \left(\int_T^\infty t^{n-1}|\overline{f}_x(t)|^2 \, dt\right)\left(\int_T^\infty t^{-n-1} \, dt\right) \\ &\le \frac{1}{nT^n}\int_0^\infty t^{n-1}|\overline{f}_x(t)|^2 \, dt.\end{aligned}$$

Applying (2.8) to the last expression, we have (2.9).

It follows from Lemma 2.1 and the Riemann-Lebesgue Lemma that the first term in (2.6) converges to 0 as $R \to \infty$ as long as, for some $T > 0$, (so supp $\varphi \subset [-T, T]$),

$$\int_0^T t^{-1}\left|\overline{f}_x(|t|) - f(x)\right| \, dt < \infty. \tag{2.11}$$

When (2.11) holds, we say $\overline{f}_x$ satisfies the Dini condition at $t = 0$. A sufficient condition for this to hold is that $\overline{f}_x(|t|)$ is Hölder continuous at $t = 0$, where by convention we take $\overline{f}_x(0) = f(x)$.

We thus have the following convergence result:

Proposition 2.2. *Let $f \in L^2(\mathbb{R}^3)$ and fix $x \in \mathbb{R}^3$. Assume $\overline{f}_x$ satisfies the Dini condition (2.11) at 0. Then, as $R \to \infty$,*

$$S_R f(x) \to f(x) \Longleftrightarrow \int_{-\infty}^{\infty} \left(\partial_t \overline{f}_x(|t|)\right) \ \sin Rt \ dt \to 0. \tag{2.12}$$

To take an explicit example, let $x = 0$ and let $f(x) = \chi_{B_1}(x)$ be the characteristic function of the unit ball in $\mathbb{R}^3$, centered at 0. Then

$$\overline{f}_0(|t|) = 1 \ \text{ if } \ |t| < 1, \quad 0 \ \text{ if } \ |t| > 1, \tag{2.13}$$

so

$$\partial_t \overline{f}_0(|t|) = \delta(t+1) - \delta(t-1), \tag{2.14}$$

and

$$\int_{-\infty}^{\infty} \left(\partial_t \overline{f}_0(|t|)\right) \ \sin Rt \ dt = -2 \ \sin R, \tag{2.15}$$

which oscillates as $R \to \infty$, so $S_R \chi_{B_1}(0)$ does *not* converge to 1 as $R \to \infty$. On the other hand, with $f(x) = \chi_{B_1}(x)$ on $\mathbb{R}^3$, one sees that, whenever $x \neq 0$, $f_x(|t|)$ is Lipschitz. This is true even for $|x| = 1$, provided we make the convention that

$$|x| = 1 \Longrightarrow \chi_{B_1}(x) = \frac{1}{2}. \tag{2.16}$$

Then we have

$$x \neq 0 \Longrightarrow S_R \chi_{B_1}(x) \to \chi_{B_1}(x). \tag{2.17}$$

Let us pause to record a simple sufficient condition for $S_R f(x) \to f(x)$, as a corollary of Proposition 2.2.

Corollary 2.3. *Let $f \in L^2(\mathbb{R}^3)$ and fix $x \in \mathbb{R}^3$. Then $S_R f(x) \to f(x)$ as $R \to \infty$, provided*

$$\begin{aligned} &\overline{f}_x \ \ \textit{satisfies the Dini condition at } 0, \\ &\qquad \partial_t \overline{f}_x(|t|) \in L^1(\mathbb{R}). \end{aligned} \tag{2.18}$$

Proof. You can apply the Riemann-Lebesgue Lemma to the integral in (2.12).

We now discuss a family of examples more general than χ_{B_1}, which was mentioned in [P1] (and which attracted the interest of the second author). Suppose $g \in C^\infty(\overline{B}_1)$, where $B_1 = \{x : |x| \leq 1\}$. Set

$$\begin{aligned} f(x) = \ & g(x), && |x| < 1, \\ & \tfrac{1}{2} g(x), && |x| = 1, \\ & 0, && |x| > 1. \end{aligned} \tag{2.19}$$

Then $\overline{f}_x$ satisfies the conditions of Corollary 2.3 whenever $x \neq 0$. If $x = 0$, we have $\overline{f}_0(t)$ smooth on $[0,1)$, zero on $(1,\infty)$, and $\overline{f}_0(1)$ is half the mean value of $g\big|_{S^2}$. Thus $\overline{f}_0$ is Lipschitz near $t = 1$ if this mean value a is 0, and $\overline{f}_0$ jumps at $t = 1$ if $a \neq 0$, so

$$\partial_t \overline{f}_0(|t|) = \partial_t \overline{g}_0(|t|) + a\delta(t+1) - a\delta(t-1). \tag{2.20}$$

Consequently, for such a function f,

$$S_R f(0) \to f(0) \Longleftrightarrow \int_{S^2} g(x)\ dS(x) = 0. \tag{2.21}$$

We now give another sufficient condition for $S_R f(x) \to f(x)$. First, we assume f is Hölder continuous and has compact support. Then Corollary 2.3 will apply provided $\partial_t \overline{f}_x(|t|) \in L^1(\mathbb{R})$. We use the estimate

$$4\pi \int_0^\infty |\partial_t \overline{f}_x(t)|\ dt \le \int_{\mathbb{R}^3} |\nabla f(x-y)| \cdot |y|^{-2}\ dy. \tag{2.22}$$

Hence we have:

Proposition 2.4. *Let $f \in L^2(\mathbb{R}^3)$ and assume $f \in C^r(\mathbb{R}^3)$ for some $r > 0$. Then $S_R f(x) \to f(x)$ as $R \to \infty$, provided*

$$\int_{\mathbb{R}^3} |\nabla f(y)| \cdot |x-y|^{-2}\ dy \le B(x) < \infty. \tag{2.23}$$

We then have a result on a.e. convergence.

Corollary 2.5. *Let $f \in L^2(\mathbb{R}^3)$ and assume $f \in C^r(\mathbb{R}^3)$ for some $r > 0$. If also*

$$\nabla f \in L^1(\mathbb{R}^3), \tag{2.24}$$

then $S_R f(x) \to f(x)$ for almost all $x \in \mathbb{R}^3$.

Proof. Note that

$$|x|^{-2} \in L^{q_1}(\mathbb{R}^3) + L^{q_2}(\mathbb{R}^3), \tag{2.25}$$

whenever $q_1 < \frac{3}{2} < q_2$. Hence the hypothesis (2.24) implies that the convolution integral in (2.23) belongs to $L^{q_1}(\mathbb{R}^3) + L^{q_2}(\mathbb{R}^3)$, and hence is finite a.e.

In fact, we can generalize the hypothesis (2.24) to

$$\nabla f \in L^1(\mathbb{R}^3) + L^p(\mathbb{R}^3), \quad p \in [1,3), \tag{2.26}$$

since we then have $L^p * L^{q_j} \subset L^{r_j}$, $q_j \le r_j < \infty$, provided q_2 is close enough to $3/2$.

It is also convenient to phrase results in terms of Φ_x, defined on $\mathbb{R}^3$ by

$$\Phi_x(y) = \int_{SO(3)} f(x+\rho y)\ d\rho = \overline{f}_x(|y|). \tag{2.27}$$

For example, we can replace Proposition 2.4 by:

Proposition 2.6. *Let $f \in L^2(\mathbb{R}^3)$, and fix $x \in \mathbb{R}^3$. Assume Φ_x is Hölder continuous at the origin. Then $S_R f(x) \to f(x)$ as $R \to \infty$, provided*

$$\int_{\mathbb{R}^3} |\nabla \Phi_x(y)| \cdot |y|^{-2}\, dy \le B(x) < \infty. \tag{2.28}$$

We make some remarks about the role of the Dini condition (2.11) in the work above. Its role was simply to guarantee that

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \frac{\sin Rt}{t}\, \overline{f}_x(|t|)\, dt = \overline{f}_x(0) = f(x). \tag{2.29}$$

As an alternative to requiring the Dini condition, one could more generally hypothesize (2.29) (which might reasonably be called the inversion condition on $\overline{f}_x$ at 0). Part of the classical lore are some other conditions on a function on $\mathbb{R}$ that are sufficient to assure such a result. In particular, the classical Dirichlet-Jordan test (cf. [Z]), put together with Riemann's localization result, implies that, provided $\overline{f}_x$ is in $L^1(\mathbb{R}, \langle t\rangle^{-1} dt)$ and has bounded variation on a neighborhood of the origin, then the limit on the left side of (2.29) exists, and is equal to $\lim_{t\to 0}\ \overline{f}_x(|t|)$. (We use a common notation: $\langle t\rangle = \sqrt{t^2+1}$.)

As indicated in the Introduction, our primary goal in this paper is to demonstrate the flexibility of the wave equation approach to questions of pointwise Fourier inversion, rather than to strive for maximally sharp convergence results in specific cases. However, we must succumb to the temptation to use the observation of the previous paragraph, as it yields a desirable sharpening of Corollary 2.3, with little additional effort.

In fact, the second part of the hypothesis (2.18), i.e., $\partial_t \overline{f}_x(|t|) \in L^1(\mathbb{R})$, implies that $\overline{f}_x$ has bounded variation. Since we know that, for any $f \in L^2(\mathbb{R}^3)$, (2.9) holds, we see that (2.29) holds, provided we replace the right side by $\lim_{t\to 0}\ \overline{f}_x(|t|)$. Hence, we have the following improvement of Corollary 2.3:

Proposition 2.7. *Let $f \in L^2(\mathbb{R}^3)$ and fix $x \in \mathbb{R}^3$. Provided*

$$\partial_t \overline{f}_x(|t|) \in L^1(\mathbb{R}), \tag{2.30}$$

we have

$$\lim_{R\to\infty}\ S_R f(x) = \lim_{t\searrow 0}\ \frac{1}{V_3 t^3} \int_{|y|\le t} f(x+y)\, dy; \tag{2.31}$$

in particular, the limits in (2.31) exist. If also x is a Lebesgue point of f, then $S_R f(x) \to f(x)$ as $R \to \infty$.

Thus, in Proposition 2.4, we can omit the hypothesis that $f \in C^r(\mathbb{R}^3)$, and deduce that (2.31) holds wherever (2.23) holds. This allows us to improve Corollary 2.5:

Proposition 2.8. *The result of Corollary 2.5 still holds when one omits the hypothesis* $f \in C^r(\mathbb{R}^3)$.

Proof. The hypothesis $f \in C^r(\mathbb{R}^3)$ played no role in showing that (2.23) holds for a.e. $x \in \mathbb{R}^3$. Thus, for a.e. $x \in \mathbb{R}^3$, (2.31) holds. Lebesgue's theorem implies that the right side of (2.31) is equal a.e. to $f(x)$.

3. Fourier inversion on $\mathbb{R}^2$

Here we consider the $n = 2$ case of (2.1)-(2.4). Thus, we have, in place of (2.5),

$$R(t)f(x) = \frac{t}{2\pi} \int\limits_{D_1} f(x - ty)\bigl(1 - |y|^2\bigr)^{-\frac{1}{2}}\, dy, \tag{3.1}$$

where $D_1 = \{y \in \mathbb{R}^2 : |y| < 1\}$. Thus

$$R(t)f(x) = t \int_0^1 \overline{f}_x(|t|r)(1 - r^2)^{-\frac{1}{2}} r\, dr, \tag{3.2}$$

where, as in (2.5), $\overline{f}_x(|t|)$ is the radial symmetrization of f about $x \in \mathbb{R}^2$, here given by

$$\overline{f}_x(|t|) = \frac{1}{2\pi} \int\limits_{S^1} f(x - t\omega)\, ds(\omega). \tag{3.3}$$

Thus, the solution to the wave equation (2.3) is

$$u(t, x) = \int_0^1 \overline{f}_x(|t|r)(1 - r^2)^{-\frac{1}{2}} r\, dr + t\partial_t\Bigl(\int_0^1 \overline{f}_x(|t|r)(1 - r^2)^{-\frac{1}{2}} r\, dr\Bigr), \tag{3.4}$$

and this time, into (1.8) goes the quantity

$$\begin{aligned}\frac{u(t, x) - \varphi(t)f(x)}{t} &= \frac{\int_0^1 \overline{f}_x(|t|r)(1 - r^2)^{-\frac{1}{2}} r\, dr - \varphi(t)f(x)}{t} \\ &\quad + \partial_t\Bigl(\int_0^1 \overline{f}_x(|t|r)(1 - r^2)^{-\frac{1}{2}} r\, dr\Bigr).\end{aligned} \tag{3.5}$$

Note that

$$h_x(|t|) = \int_0^1 \overline{f}_x(|t|r)(1 - r^2)^{-\frac{1}{2}} r\, dr \tag{3.6}$$

is a *weighted average* of dilates of $\overline{f}_x(|t|)$. Hence h_x satisfies the Dini condition (2.11) if $\overline{f}_x$ does. Parallel to Proposition 2.1, we have:

Proposition 3.1. *Let $f \in L^2(\mathbb{R}^2)$ and fix $x \in \mathbb{R}^2$. Assume $\overline{f}_x$ satisfies the Dini condition (2.11). Then, as $R \to \infty$,*

$$S_R f(x) \to f(x) \Longleftrightarrow \int_{-\infty}^{\infty} \partial_t \Bigl(\int_0^1 \overline{f}_x(|t|r)(1-r^2)^{-\frac{1}{2}} r \, dr \Bigr) \ \sin Rt \ dt \to 0. \tag{3.7}$$

In particular, the condition (3.7) is fulfilled, provided

$$\partial_t \Bigl(\int_0^1 \overline{f}_x(|t|r)(1-r^2)^{-\frac{1}{2}} r \, dr \Bigr) \in L^1(\mathbb{R}). \tag{3.8}$$

Proof. Given the discussion above, the only point it remains to check is that, for any $T > 0$,

$$\int_T^{\infty} t^{-1} |h_x(t)| \ dt < \infty. \tag{3.9}$$

To see this, write

$$t^{-1}|h_x(t)| \le \int_0^1 (rt)^{-1} \, |\overline{f}_x(tr)| \, (1-r^2)^{-\frac{1}{2}} r^2 \ dr.$$

Hence

$$\|t^{-1}h_x\|_{L^1([T,\infty))} \le \int_0^1 r^{-1} \|t^{-1}\overline{f}_x\|_{L^1([rT,\infty))} (1-r^2)^{-\frac{1}{2}} r^2 \ dr.$$

If we apply (2.9), with T replaced by rT, we dominate this by

$$CT^{-1} \Bigl(\int_0^1 (1-r^2)^{-\frac{1}{2}} \ dr \Bigr) \|f\|_{L^2},$$

since $n = 2$ here, and this yields (3.9).

In this case, if we set $f(x) = \chi_{D_1}(x)$, the characteristic function of the unit disk in $\mathbb{R}^2$, then the condition (3.8) is fulfilled for each $x \in \mathbb{R}^2$ (even $x = 0$), and so $S_R \chi_{D_1}(x) \to \chi_{D_1}(x)$ for each $x \in \mathbb{R}^2$ (even for $|x| = 1$, if, as in (2.11), we set $\chi_{D_1}(x) = \frac{1}{2}$ for $x \in \partial D_1$). Rather than make an explicit analysis of (3.8) for $f = \chi_{D_1}$, we will establish a general result.

To do this, we study the operator T, defined by

$$Tg(t) = \int_0^1 g(tr)(1-r^2)^{-\frac{1}{2}} r \ dr. \tag{3.10}$$

In our application of course $g(t) = \overline{f}_x(t), \ t \ge 0$. Thus, we want to estimate $\partial_t Tg$. We begin with the following simple estimate.

Lemma 3.2. *Given $p \in (1,\infty)$, we have (with $C = C_p$)*

$$\|\partial_t T g\|_{L^p([\frac{1}{2},1])} \le C\|g\|_{L^1([0,1])} + C\|Kg\|_{L^p([\frac{1}{4},2])}, \tag{3.11}$$

where

$$Kg(t) = \partial_t \kappa * g(t), \tag{3.12}$$

with

$$\kappa(t) = t^{-\frac{1}{2}} \ \text{ for } \ t > 0, \quad 0 \ \text{ for } \ t < 0.$$

Proof. This follows from the fact that the integral kernel of T has the behavior of a pseudodifferential operator of order $-\frac{1}{2}$, away from $t = 0$.

We want to estimate $\partial_t T g(t)$ for $t \in [\frac{1}{2}\rho, \rho]$. We use the dilation group

$$D_\rho g(r) = g(\rho r), \tag{3.13}$$

and the relations

$$\begin{gathered} TD_\rho = D_\rho T, \quad \partial_t T D_\rho = \rho D_\rho \partial_t T, \\ KD_\rho = \rho^{\frac{1}{2}} D_\rho K, \quad \|D_\rho g\|_{L^p} = \rho^{-1/p}\|g\|_{L^p}. \end{gathered} \tag{3.14}$$

Thus, we have

$$\begin{aligned} \|\partial_t T g\|_{L^1([\frac{1}{2}\rho,\rho])} &= \rho\|D_\rho \partial_t T g\|_{L^1([\frac{1}{2},1])} = \|\partial_t T(D_\rho g)\|_{L^1([\frac{1}{2},1])} \\ &\le C\|D_\rho g\|_{L^1([0,1])} + C\|KD_\rho g\|_{L^p([\frac{1}{4},2])} \\ &= C\rho^{-1}\|g\|_{L^1([0,\rho])} + C\rho^{\frac{1}{2}-\frac{1}{p}}\|Kg\|_{L^p([\frac{1}{4}\rho,2\rho])}. \end{aligned} \tag{3.15}$$

Let us take $p \in (1,\infty)$, $\rho = 2^{-k}$, and then sum. Thus

$$\begin{aligned} \|\partial_t T g\|_{L^1([0,1])} \le C &\sum_{k\ge 0} 2^k \|g\|_{L^1([0,2^{-k}])} \\ &+ C\sum_{k\ge 0} 2^{-k(\frac{1}{2}-\frac{1}{p})}\|Kg\|_{L^p([2^{-k-2},2^{-k+1}])}. \end{aligned} \tag{3.16}$$

The first term on the right side of (3.16) is not well behaved, but we can improve matters by considering

$$\gamma(t) = g(0)\varphi(t), \tag{3.17}$$

with $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(t) = 1$ for $|t| \le 1$. Then

$$T\gamma(t) = g(0) \ \text{ for } \ 0 \le t \le 1, \quad \partial_t T\gamma(t) = 0 \ \text{ for } \ 0 \le t \le 1, \tag{3.18}$$

and hence, if we replace g by $g-\gamma$ in (3.16),

$$\begin{aligned}\|\partial_t Tg\|_{L^1([0,1])} \le C|g(0)| &+ C\sum_{k\ge 0} 2^k \|g-g(0)\|_{L^1([0,2^{-k}])} \\ &+ C\sum_{k\ge 0} 2^{-k(\frac{1}{2}-\frac{1}{p})}\|Kg\|_{L^p([2^{-k-2},2^{-k+1}])}.\end{aligned} \tag{3.19}$$

Note that

$$\sum_{k\ge 0} 2^k \|g-g(0)\|_{L^1([0,2^{-k}])} \approx \left\|t^{-1}\big(g-g(0)\big)\right\|_{L^1([0,1])}. \tag{3.20}$$

We also need to estimate $\partial_t Tg$ on $[1,\infty)$. The following works when g has support in $[0,\frac{1}{2}]$.

Lemma 3.3. *If supp $g\subset[0,\frac{1}{2}]$, then*

$$\|\partial_t Tg\|_{L^1([1,\infty))} \le C\|g\|_{L^1}. \tag{3.21}$$

Proof. If supp $g\subset[0,\frac{1}{2}]$ and $t\ge 1$, we have

$$\partial_t Tg(t) = -\frac{1}{t}\int_0^{\frac{1}{2}} g(tr)r\varphi(r)\ dr,$$

where

$$r\varphi(r) = \partial_r\big[(1-r^2)^{-\frac{1}{2}}r^2\big],\quad \varphi\in C^\infty\big([0,\tfrac{1}{2}]\big).$$

Hence, with D_r as in (3.13)

$$\begin{aligned}\|\partial_t Tg\|_{L^1([1,\infty))} &\le \int_0^{\frac{1}{2}} \|D_r g\|_{L^1(\mathbb{R}^+)} r|\varphi(r)|\ dr \\ &= \int_0^{\frac{1}{2}} |\varphi(r)|\ dr\cdot\|g\|_{L^1(\mathbb{R}^+)},\end{aligned}$$

which implies the lemma.

Combining this with (3.19)-(3.20), we have an estimate on $\|\partial_t Tg\|_{L^1(\mathbb{R}^+)}$ when g has support in $[0,\frac{1}{2}]$. Using (3.14) again, we can estimate this quantity for g with support in $[0,\rho]$, for any $\rho\in(0,\infty)$. This yields the following result.

Proposition 3.4. *Let $f\in L^2(\mathbb{R}^2)$ have compact support. Fix $x\in\mathbb{R}^2$. Then $S_R f(x)\to f(x)$ as $R\to\infty$, provided the following two properties hold:*

$$t^{-1}\Big(\overline{f}_x(|t|)-\varphi(t)f(x)\Big)\in L^1(\mathbb{R}), \tag{3.22}$$

and, for some $p > 1$,

$$\sum_{k\geq 0} 2^{-k(\frac{1}{2}-\frac{1}{p})} \|K\overline{f}_x\|_{L^p([2^{-k-1}A,2^{-k}A])} < \infty, \tag{3.23}$$

where A *is large enough that supp* $f \subset B_{A/2}(x)$.

Note that the exponent of 2 in (3.23) is positive for $p < 2$ and negative for $p > 2$. In particular, (3.23) holds provided

$$K\overline{f}_x \in L^p([0, A]), \quad p > 2. \tag{3.24}$$

Hence we have:

Corollary 3.5. *Let* $f \in L^2(\mathbb{R}^2)$ *have support in* $B_{A/2}(x)$ *and assume* $\overline{f}_x$ *satisfies the Dini condition (2.11). Then* $S_R f(x) \to f(x)$ *as* $R \to \infty$, *provided that, for some* $p > 1$, $q > 2$, $C \in (0, A)$, *we have*

$$\overline{f}_x \in H^{\frac{1}{2},p}\big([-A, A]\big), \quad \overline{f}_x\big|_{[-C,C]} \in H^{\frac{1}{2},q}\big([-C, C]\big). \tag{3.25}$$

Proof. Write f as a sum of two terms, one vanishing near x and the other supported in $B_{C/2}(x)$, in such a fashion that Proposition 3.4 applies to each term.

To apply this to $f = \chi_{D_1}$, note that $\overline{f}_0 = \chi_{[-1,1]}$ in this case, and

$$\chi_{[-1,1]} \in H^{\frac{1}{2},p}(\mathbb{R}), \quad \forall\, p \in [1, 2). \tag{3.26}$$

Let us also make note of a connection between the $n = 2$ and $n = 3$ cases of pointwise Fourier inversion, following a suggestion made to one of us by C.D.Hill. Pick $\psi \in \mathcal{S}(\mathbb{R})$ such that $\psi(0) = 1$ and supp $\hat{\psi} \subset [-1, 1]$, and set

$$F(x) = f(x')\psi(x_3), \quad x = (x', x_3) = (x_1, x_2, x_3). \tag{3.27}$$

Proposition 3.6. *Assume* $f \in L^1(\mathbb{R}^2)$, *and fix* $x' \in \mathbb{R}^2$. *Then, as* $R \to \infty$,

$$S_R f(x') \to f(x') \Longleftrightarrow S_R F(x', 0) \to F(x', 0). \tag{3.28}$$

Proof. Without loss of generality, we consider the case $x' = 0$. Then

$$S_R f(0) = S_R F(0) + \int_{\mathcal{O}(R)} \widehat{F}(\xi)\, d\xi, \tag{3.29}$$

where

$$\mathcal{O}(R) = \{\xi \in \mathbb{R}^3 : |\xi| \geq R,\ |\xi'| \leq R,\ |\xi_3| \leq 1\}. \tag{3.30}$$

Note that

$$\text{Vol } \mathcal{O}(R) \le C_0 < \infty, \tag{3.31}$$

with C_0 independent of R, so the integral in (3.29) is bounded in absolute value by

$$C_0 \sup_{\mathcal{O}(R)} |\widehat{F}(\xi)| \le C_0 \sup_{|\xi|\ge R} |\widehat{F}(\xi)|. \tag{3.32}$$

By the Riemann-Lebesgue lemma, this goes to 0 as $R \to \infty$, since $F \in L^1(\mathbb{R}^3)$. This completes the proof.

One could re-derive Proposition 3.4 as a consequence of Proposition 3.6, though we will not do this. We will use Proposition 3.6 and the results of §2 to establish the following result of [Bo]:

Proposition 3.7. *Let $f \in L^2(\mathbb{R}^2)$, and fix $x' \in \mathbb{R}^2$. If x' is a Lebesgue point of f and $\overline{f}_{x'}$ has bounded variation on $\mathbb{R}$, then $S_R f(x') \to f(x')$ as $R \to \infty$.*

Proof. If you set $F(x', x_3) = f(x')\psi(x_3)$ as in (3.27), then the hypotheses imply that $\overline{F}_{(x',0)}$ has bounded variation. Also, one readily verifies that $\partial_t \overline{F}_{(x',0)}(|t|)$ is better than L^1 on $\mathbb{R} \setminus 0$, and it cannot have an atom at 0, so in fact $\partial_t \overline{F}_{(x',0)} \in L^1(\mathbb{R})$. By Proposition 2.7, it follows that $S_R F(x', 0) \to F(x', 0)$, so Proposition 3.6 implies $S_R f(x') \to f(x')$.

4. Fourier inversion on $\mathbb{R}^n$ (general n)

We extend results of §§2-3 by using the formula for $\cos t\sqrt{-\Delta}$ on functions on $\mathbb{R}^n$. First, we consider the case $n = 2k+1$, odd. Then

$$\cos t\sqrt{-\Delta}\ f(x) = C_k t \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^k \big(t^{2k-1} \overline{f}_x(|t|)\big), \tag{4.1}$$

with $C_k = 1/(2k-1)!!$, where $(2k-1)!! = 3 \cdot 5 \cdots (2k-1)$. We can use the Leibniz formula to apply $\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^k$ to the product of t^{2k-1} and $\overline{f}_x(|t|)$, obtaining

$$\cos t\sqrt{-\Delta}\ f(x) = \overline{f}_x(|t|) + \sum_{j=1}^k \gamma_{kj} t^{2j} \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^j \overline{f}_x(|t|). \tag{4.2}$$

Thus, the quantity (1.8) is equal to

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{\overline{f}_x(|t|) - \varphi(t) f(x)}{t}\ \sin Rt\ dt \\ &+ \sum_{j=1}^k \gamma_{kj} \int_{-\infty}^{\infty} t^{2j-1} \Big\{ \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^j \overline{f}_x(|t|)\Big\}\ \sin Rt\ dt.\end{aligned} \tag{4.3}$$

Compare (2.6), which treats the case $k = 1$ ($n = 3$). Extending Proposition 2.1, we have

Proposition 4.1. *Let $f \in L^2(\mathbb{R}^{2k+1})$ and fix $x \in \mathbb{R}^{2k+1}$. Assume $\overline{f}_x$ satisfies the Dini condition (2.11). Then, as $R \to \infty$,*

$$S_R f(x) \to f(x) \iff \sum_{j=1}^{k} \gamma_{kj} \int_{-\infty}^{\infty} t^{2j-1} \Big\{ \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^j \overline{f}_x(|t|) \Big\} \ \sin Rt \ dt \to 0. \tag{4.4}$$

We can rewrite the quantity in (4.4), using

$$t^{2j-1}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^j = \sum_{\ell=1}^{j} \alpha_{j\ell} t^{\ell-1} \partial_t^\ell \tag{4.5}$$

for certain constants $\alpha_{j\ell}$. Hence

$$\begin{aligned} \sum_{j=1}^{k} \gamma_{kj} t^{2j-1}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^j &= \sum_{j=1}^{k}\sum_{\ell=1}^{j} \gamma_{kj}\alpha_{j\ell} t^{\ell-1}\partial_t^\ell \\ &= \sum_{\ell=1}^{k} \beta_{k\ell} t^{\ell-1} \partial_t^\ell . \end{aligned} \tag{4.6}$$

Thus, under the hypotheses of Proposition 4.1,

$$S_R f(x) \to f(x) \iff \sum_{\ell=1}^{k} \beta_{k\ell} \int_{-\infty}^{\infty} t^{\ell-1} \big(\partial_t^\ell \overline{f}_x(|t|)\big) \ \sin Rt \ dt \to 0. \tag{4.7}$$

We have the following extension of Corollary 2.2

Corollary 4.2. *Let $f \in L^2(\mathbb{R}^{2k+1})$ and fix $x \in \mathbb{R}^{2k+1}$. Then $S_R f(x) \to f(x)$ as $R \to \infty$, provided*

$$\overline{f}_x(|t|) \ \textit{ satisfies the Dini condition } (2.11) \tag{4.8}$$

and

$$t^{\ell-1}\partial_t^\ell \overline{f}_x(|t|) \in L^1(\mathbb{R}), \quad 1 \le \ell \le k. \tag{4.9}$$

We can also produce an extension of Proposition 2.4, using the following estimate, extending (2.22):

$$A_{2k} \int_0^1 |\partial_t^\ell \overline{f}_x(t)| t^{\ell-1} \ dt \le \int\limits_{R^{2k+1}} |\nabla^\ell f(x-y)| \cdot |y|^{\ell-2k-1} \ dy. \tag{4.10}$$

Hence we have:

Proposition 4.3. *Let $f \in L^2(\mathbb{R}^{2k+1})$ and assume $f \in C^r(\mathbb{R}^{2k+1})$ for some $r > 0$. Then $S_R f(x) \to f(x)$ as $R \to \infty$, provided*

$$\sum_{|\alpha|=\ell} \int_{\mathbb{R}^{2k+1}} |D^\alpha f(x-y)| \cdot |y|^{\ell-2k-1} \, dy \le B(x) < \infty, \quad 1 \le \ell \le k. \tag{4.11}$$

Of course, we can also symmetrize about x, as in (2.27), and produce an extension of Proposition 2.6. Thus, if we set

$$\Phi_x(y) = \int_{SO(n)} f(x + \rho y) \, d\rho = \overline{f}_x(|y|), \tag{4.12}$$

for f defined on $\mathbb{R}^n$ (here, $n = 2k+1$), we replace (4.11) by

$$\sum_{|\alpha|=\ell} \int_{\mathbb{R}^{2k+1}} |D^\alpha \Phi_x(y)| \cdot |y|^{\ell-2k-1} \, dy \le B(x) < \infty, \quad 1 \le \ell \le k. \tag{4.13}$$

Next, we consider $n = 2k$, even. Then

$$\cos t\sqrt{-\Delta} \, f(x) = \tilde{C}_k t \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^k \Big(t^{2k-1} \int_0^1 \overline{f}_x(r|t|)(1-r^2)^{-\frac{1}{2}} r^{2k-1} \, dr\Big). \tag{4.14}$$

When $k = 1$, this becomes equivalent to (3.4), with $\tilde{C}_1 = 1$. Let us set

$$T_k g(t) = \frac{1}{\tau_k} \int_0^1 g(rt)(1-r^2)^{-\frac{1}{2}} r^{2k-1} \, dr, \tag{4.15}$$

where $\tau_k = \int_0^1 (1-r^2)^{-\frac{1}{2}} r^{2k-1} \, dr$. Then $T_1 = T$, given by (3.9). Parallel to (4.2), we have

$$\begin{aligned} \cos t\sqrt{-\Delta} \, f(x) &= T_k \overline{f}_x(|t|) + \sum_{j=1}^k \tilde{\gamma}_{kj} t^{2j} \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^j T_k \overline{f}_x(|t|) \\ &= T_k \overline{f}_x(|t|) + \sum_{\ell=1}^k \tilde{\beta}_{k\ell} t^\ell \Big(\partial_t^\ell T_k \overline{f}_x(|t|)\Big), \end{aligned} \tag{4.16}$$

the latter identity parallel to (4.6). In this case, the quantity (1.8) is equal to

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{T_k \overline{f}_x(|t|) - \varphi(t) f(x)}{t} \sin Rt \, dt \\ &\quad + \sum_{\ell=1}^k \tilde{\beta}_{k\ell} \int_{-\infty}^{\infty} t^{\ell-1} \Big(\partial_t^\ell T_k \overline{f}_x(|t|)\Big) \sin Rt \, dt. \end{aligned} \tag{4.17}$$

Note that $T_k \overline{f}_x(|t|)$ is a weighted average of dilates of $\overline{f}_x(|t|)$. Parallel to Proposition 4.1, we have:

Proposition 4.4. *Let $f \in L^2(\mathbb{R}^{2k})$ have compact support. Fix $x \in \mathbb{R}^{2k}$ and assume $\overline{f}_x$ satisfies the Dini condition (2.11). Then, as $R \to \infty$,*

$$S_R f(x) \to f(x) \iff \sum_{\ell=1}^{k} \tilde{\beta}_{k\ell} \int_{-\infty}^{\infty} t^{\ell-1}\big(\partial_t^\ell T_k \overline{f}_x(|t|)\big) \ \sin Rt \ dt \to 0. \tag{4.18}$$

Proof. As in the proof of Proposition 3.1, it remains only to show that, for $T > 0$,

$$\int_T^\infty t^{-1} |T_k \overline{f}_x(t)| \ dt < \infty, \tag{4.19}$$

and, as in that argument, this can be established using the estimate (2.9).

Note that the condition (4.18) is fulfilled, provided

$$t^{\ell-1} \partial_t^\ell T_k \overline{f}_x \in L^1(\mathbb{R}), \quad 1 \le \ell \le k. \tag{4.20}$$

An equivalent condition is

$$\partial_t (t\partial_t)^j T_k \overline{f}_x \in L^1(\mathbb{R}), \quad 0 \le j \le k-1. \tag{4.21}$$

Since $t\partial_t$ commutes with T_k, another equivalent condition is:

$$\partial_t T_k (t\partial_t)^j \overline{f}_x \in L^1(\mathbb{R}), \quad 0 \le j \le k-1. \tag{4.22}$$

Now the analysis (3.10)-(3.20) of $T = T_1$ extends with essentially no change to each T_k. Thus we have the following extension of Corollary 3.5.

Proposition 4.5. *Let $f \in L^2(\mathbb{R}^{2k})$ have support in $B_{A/2}(x)$, and assume $\overline{f}_x$ satisfies the Dini condition (2.11). Then $S_R f(x) \to f(x)$ as $R \to \infty$, provided that, for some $a > 1,\ b > 2,\ C \in (0, A)$, we have*

$$(t\partial_t)^j \overline{f}_x \in H^{\frac{1}{2},a}\big([-A, A]\big) \cap H^{\frac{1}{2},b}\big([-C, C]\big), \quad 0 \le j \le k-1. \tag{4.23}$$

5. Fourier inversion on spheres

Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, with its standard metric and Laplace operator Δ. It is convenient to replace the use of $\sqrt{-\Delta}$ by that of

$$A = \sqrt{-\Delta + \Big(\frac{n-1}{2}\Big)^2}. \tag{5.1}$$

Then

$$\text{Spec } A = \Big\{\frac{n-1}{2} + \nu : \nu = 0, 1, 2, 3, \dots\Big\}, \tag{5.2}$$

so, if $n = 2k+1$ is odd,

$$u(t) = (\cos tA)f \Longrightarrow u(t+2\pi, x) = u(t,x). \tag{5.3}$$

We will consider the case $n = 2k$, even, below.

Now, it is convenient to replace (1.1) by

$$\varphi(A) = \frac{1}{2\pi} \int_{S^1} \hat{\varphi}(t) \ \cos tA \ dt, \tag{5.4}$$

for an even function $\varphi : \mathbb{Z} \to \mathbb{C}$, where, now

$$\hat{\varphi}(t) = \sum_{k=-\infty}^{\infty} \varphi(k) e^{-ikt}, \tag{5.5}$$

and we identify S^1 with $\mathbb{R}/(2\pi\mathbb{Z})$. In particular, with $\varphi(k) = \chi_N(k) = \chi_{\{|k|\le N\}}$, we have

$$\hat{\chi}_N(t) = D_N(t) = \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t}, \tag{5.6}$$

and hence, for $S_N f = \chi_N(A)f$,

$$S_N f(x) = \frac{1}{2\pi} \int_{S^1} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t} \ u(t,x) \ dt. \tag{5.7}$$

Parallel to our passage from (1.3) to (1.7), we add $0 = f(x) - f(x)$ to $u(t,x)$ in (3.7), this time obtaining

$$S_N f(x) = f(x) + \frac{1}{2\pi} \int_{S^1} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t} \ \big[u(t,x) - f(x)\big] \ dt. \tag{5.8}$$

Thus we have the following criterion for convergence $S_N f(x) \to f(x)$.

Proposition 5.1. *Let $f \in L^2(S^n)$, $n = 2k+1$, and fix $x \in S^n$. Then, as $N \to \infty$,*

$$S_N f(x) \to f(x) \Longleftrightarrow \int_{S^1} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t} \ \big[u(t,x) - f(x)\big] \ dt \to 0. \tag{5.9}$$

Again using the Riemann-Lebesgue Lemma, we have the following sufficient condition for convergence:

Corollary 5.2. *Given $x \in S^n$, $n = 2k+1$, we have $S_N f(x) \to f(x)$ provided*

$$u(\cdot, x) \in L^1(S^1) \tag{5.10}$$

and

$$\frac{u(t,x) - f(x)}{t} \in L^1(-\pi, \pi). \tag{5.11}$$

Now, there are formulas for $u(t,x) = (\cos tA) f(x)$ similar to those for Euclidean space. We have

$$(\cos tA) f(x) = C_k (\sin t) \Bigl(\frac{1}{\sin t} \frac{\partial}{\partial t} \Bigr)^k \bigl(\sin^{2k-1} t \ \overline{f}_x(|t|) \bigr), \tag{5.12}$$

with $C_k = 1/(2k-1)!!$, where $(2k-1)!! = 3 \cdot 5 \cdots (2k-1)$. A derivation can be found in Chapter 4 of [T2].

In particular, when $n = 3$ ($k = 1$), we have

$$\begin{aligned} n = 3 \Longrightarrow (\cos tA) f(x) &= \partial_t \bigl(\sin t \ \overline{f}_x(|t|) \bigr) \\ &= (\cos t) \overline{f}_x(|t|) + (\sin t) \partial_t \overline{f}_x(|t|). \end{aligned} \tag{5.13}$$

Thus, for $f \in L^2(S^3)$, the integral in (5.9) is equal to

$$\begin{aligned} &\int_{S^1} \frac{\sin(N + \frac{1}{2})t}{\sin \frac{1}{2} t} \bigl[(\cos t) \overline{f}_x(|t|) - f(x) \bigr] \ dt \\ &+ \int_{S^1} \sin(N + \tfrac{1}{2})t \ \frac{\sin t}{\sin \frac{1}{2} t} \ \partial_t \overline{f}_x(|t|) \ dt. \end{aligned} \tag{5.14}$$

In order to obtain a result parallel to Proposition 2.2, we want to dispose of the first integral in (5.14). The Dini hypothesis controls $(\sin \frac{1}{2} t)^{-1} \bigl[(\cos t) \overline{f}_x(|t|) - f(x) \bigr]$ for t close to zero. We also need to control this quantity for t close to π. Note that, for f defined on S^n, if $x \in S^n$, with antipodal point $\overline{x}$, we have

$$\int_{\pi - \delta}^{\pi} |\overline{f}_x(t)| \ dt \sim \int_{B_\delta(\overline{x})} |f(y)| \ \mathrm{dist}(\overline{x}, y)^{1-n} \ dS(y), \tag{5.15}$$

where $B_\delta(\overline{x}) = \{ y \in S^n : \mathrm{dist}(\overline{x}, y) \le \delta \}$. We have the following:

Proposition 5.3. *Let $f \in L^2(S^3)$, and fix $x \in S^3$. Assume $\overline{f}_x$ has the Dini property (2.11). Also assume*

$$\int_{S^3} |f(y)| \ dist(\overline{x}, y)^{-2} \ dS(y) < \infty, \tag{5.16}$$

where $\overline{x}$ is the point antipodal to x. Then, as $N \to \infty$,

$$S_N f(x) \to f(x) \Longleftrightarrow \int_{S^1} \sin(N+\tfrac{1}{2})t \; \frac{\sin t}{\sin \frac{1}{2}t} \; \partial_t \overline{f}_x(|t|) \; dt \to 0. \tag{5.17}$$

In particular, (5.17) holds, provided

$$\begin{aligned} \partial_t \overline{f}_x(|t|) &\in L^1\big([-\tfrac{3}{4}\pi, \tfrac{3}{4}\pi]\big), \\ |t-\pi| \cdot \partial_t \overline{f}_x(|t|) &\in L^1\big([\tfrac{1}{2}\pi, \tfrac{3}{2}\pi]\big). \end{aligned} \tag{5.18}$$

Let us consider some examples. Take $S^3 = \{x \in \mathbb{R}^4 : |x| = 1\}$. Given $a \in (-1, 1)$, set

$$\begin{aligned} f_a(x) = 1 &\text{ if } x_4 > a, \\ \tfrac{1}{2} &\text{ if } x_4 = a, \\ 0 &\text{ if } x_4 < a. \end{aligned} \tag{5.19}$$

Then $S_N f(x) \to f(x)$ for each $x \in S^3$ *except* the two poles, where $x_4 = \pm 1$. The analysis of this family of examples is very similar to the analysis of $f = \chi_{B_1}$ in §2.

To illustrate that the set of points of convergence need not always be invariant under the antipodal map, consider the following. Take $p = (0,0,0,1) \in S^3$, and let f be the restriction to S^3 of a function of x_4, having the property that

$$\overline{f}_p(t) = t^{-\frac{1}{2}}, \quad 0 \le t \le \pi. \tag{5.20}$$

Then, as $N \to \infty$,

$$S_N f(x) \to f(x) \Longleftrightarrow x \neq p. \tag{5.21}$$

Note that, for general n,

$$\int_{B_\delta(\overline{x})} \text{dist}(\overline{x}, y)^{q(1-n)} \; dS(y) \sim \int_0^\delta s^{q(1-n)+n-1} \; ds, \tag{5.22}$$

which is finite if and only if $q(1-n) + n > 0$, i.e., $q < n/(n-1)$. Thus (5.15) is finite provided

$$f \in L^p\big(B_\delta(\overline{x})\big), \quad p > n. \tag{5.23}$$

We can obtain a formula analogous analogous to (5.14) for more general $n = 2k+1$, as follows. As in (4.2), we can use the Leibniz formula to apply $\big(\frac{1}{\sin t}\frac{\partial}{\partial t}\big)^k$ to the product of $\sin^{2k-1} t$ and $\overline{f}_x(|t|)$, obtaining

$$\cos t\sqrt{-\Delta} = \gamma_{k0}(t)\overline{f}_x(|t|) + \sum_{j=1}^{k} \gamma_{kj}(t)(\sin^{2j} t)\Big(\frac{1}{\sin t}\frac{\partial}{\partial t}\Big)^j \overline{f}_x(|t|), \tag{5.24}$$

where this time, $\gamma_{kj}(t)$ are polynomials in $\cos t$, and $\gamma_{k0}(0) = 1$. Here, for $f \in L^2(S^{2k+1})$, the integral in (5.9) is equal to

$$\begin{aligned}&\int\limits_{S^1} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t}\left[\gamma_{k0}(t)\overline{f}_x(|t|) - f(x)\right]\, dt \\ &+ \sum_{j=1}^{k} \int\limits_{S^1} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t}\gamma_{kj}(t)(\sin^{2j} t)\Big(\frac{1}{\sin t}\frac{\partial}{\partial t}\Big)^j \overline{f}_x(|t|)\, dt.\end{aligned} \tag{5.25}$$

Making use of this, one can obtain results analogous to those for functions on $\mathbb{R}^{2k+1}$ in §4. We will forego the details.

When $n = 2k$ is even, we must replace (5.3) by

$$u(t) = (\cos tA)f \Longrightarrow u(t+4\pi, x) = u(t, x), \tag{5.26}$$

so, instead of (5.4), we use

$$\varphi(A) = \frac{1}{4\pi}\int\limits_{\widetilde{S}} \hat{\varphi}(t)\ \cos tA\ dt, \tag{5.27}$$

for an even function $\varphi : \frac{1}{2}\mathbb{Z} \to \mathbb{C}$, where $\widetilde{S} = \mathbb{R}/(4\pi\mathbb{Z})$ and

$$\hat{\varphi}(t) = \sum_{k=-\infty}^{\infty} \varphi(k/2)e^{-ikt/2}. \tag{5.28}$$

In particular, with $\varphi(k/2) = \chi_N(k/2) = \chi_{\{|k|\le N\}}$, we have

$$\hat{\chi}_N(t) = \sum_{k=-N}^{N} e^{-ikt/2} = D_N(t/2) = \frac{\sin\frac{1}{2}(N+\frac{1}{2})t}{\sin\frac{1}{4}t}, \tag{5.29}$$

and hence, for $S_N f = \chi_N(A)f,\ f \in L^2(S^{2k})$,

$$S_N f(x) = \frac{1}{4\pi}\int\limits_{\widetilde{S}} \frac{\sin\frac{1}{2}(N+\frac{1}{2})t}{\sin\frac{1}{4}t}\ u(t,x)\ dt. \tag{5.30}$$

Using this, one can obtain results analogous to those for functions on $\mathbb{R}^{2k}$ in §§3-4. Again, we omit the details.

Further results on Fourier inversion on spheres can be found in sections 8 and 11.

6. Fourier inversion on complex projective space, and variants

The complex projective space $P_n(\mathbb{C})$ can be defined as the quotient of S^{2n+1} (viewed as a submanifold of $\mathbb{C}^n$) by the group of complex numbers of modulus 1, acting multiplicatively. As such it acquires a Riemannian structure from S^{2n+1} and a canonical Laplacian. The eigenvalues of the negative Laplacian of this compact Riemannian manifold have been computed as

$$\lambda_k = 4k(n+k) = (2k+n)^2 - n^2. \tag{6.1}$$

See [BGM], pp.172-173; see also the material below.

We consider the operator

$$A = \sqrt{-\Delta + n^2} \tag{6.2}$$

Then

$$\text{Spec}\,(A) = \{2k+n,\ k = 0,1,2,\dots\}. \tag{6.3}$$

Thus, as in (5.4), we write

$$\varphi(A) = \frac{1}{2\pi} \int\limits_{S^1} \hat{\varphi}(t)\ \cos tA\ dt, \tag{6.4}$$

for an even function $\varphi : \mathbb{Z} \to \mathbb{C}$, with $\hat{\varphi}(t)$ as in (5.5). In particular, with $\varphi(k) = \chi_N(k) = \chi_{\{|k| \le N\}}$, we have $S_N f = \chi_N(A) f$ given by

$$S_N f(x) = f(x) + \frac{1}{2\pi} \int\limits_{S^1} \frac{\sin(N+\frac{1}{2})t}{\sin \frac{1}{2}t}\ \big[u(t,x) - f(x)\big]\ dt, \tag{6.5}$$

with $u(t,x) = (\cos tA) f(x)$, solving

$$u_{tt} - (\Delta - n^2)u = 0, \quad u(0,x) = f(x),\ u_t(0,x) = 0. \tag{6.6}$$

Again, (5.9) gives a criterion for convergence $S_N f(x) \to f(x)$, and we have the following result, parallel to Corollary 5.2:

Proposition 6.1. *Let $f \in L^2(P_n(\mathbb{C}))$ and fix $x \in P_n(\mathbb{C})$. Then as $N \to \infty$ we have $S_N f(x) \to f(x)$ provided that*

$$t \to u(t,x) \in L^1(S^1), \tag{6.7}$$

and

$$t \to \frac{u(t,x) - f(x)}{t} \in L^1(-\pi,\pi). \tag{6.8}$$

There are a number of ways to analyze the solution to the wave equation (6.6) in this case. One way is by 'descent' from the wave equation on $\mathbb{R} \times S^{2n+1}$. Before discussing this approach, we mention a direct link between Fourier inversion on $P_n(\mathbb{C})$ and on S^{2n+1}.

Proposition 6.2. *Let $\pi : S^{2n+1} \to P_n(\mathbb{C})$ be the standard projection. Let $f \in L^2(P_n(\mathbb{C}))$ and fix $x \in P_n(\mathbb{C})$. Suppose $x = \pi(y),\ y \in S^{2n+1}$. Set*

$$g = f \circ \pi \in L^2(S^{2n+1}). \tag{6.9}$$

Then, as $N \to \infty$,

$$S_N f(x) \to f(x) \iff S_N g(y) \to g(y). \tag{6.10}$$

We can put Proposition 6.2 in the following more general context. Consider $\pi : P \to M$, a principal G-bundle, where M is a compact Riemannian manifold and G is a compact Lie group. Assume P has a G-invariant Riemannian metric such that, if $y \in P$, then H_y, the subspace of T_yP orthogonal to the fiber through y, is mapped isometrically onto T_xM by $D\pi$, where $x = \pi(y)$. Thus H_y is the horizontal space of a connection on P. In the context of Proposition 6.2, $S^{2n+1} \to P_n(\mathbb{C})$ is a principal S^1-bundle, with this structure.

Now both P and M have Laplace-Beltrami operators, Δ_P and Δ_M, determined by their Riemannian metrics. Also, $\pi^* f(y) = f\big(\pi(y)\big)$ gives an isomorphism of $L^2(M)$ onto the linear subspace of $L^2(P)$ consisting of functions invariant under the action of G. Furthermore, we have the relation

$$\varphi(\Delta_P)\pi^* f = \pi^* \varphi(\Delta_M) f. \tag{6.11}$$

In particular,

$$\chi_R(\sqrt{-\Delta_P})\pi^* f(y) = \chi_R(\sqrt{-\Delta_M}) f(x), \quad y = \pi(x), \tag{6.12}$$

and hence, as $R \to \infty$,

$$\chi_R(\sqrt{-\Delta_P})\pi^* f(y) \to \pi^* f(y) \iff \chi_R(\sqrt{-\Delta_M}) f(x) \to f(x), \tag{6.13}$$

a result that contains (6.10).

Generally, P is a more complicated space than M, and hence (6.13) might not provide a useful reduction of the problem of pointwise Fourier inversion on M. Of course, Proposition 6.2 treats a case where P is simpler than M.

Note that, as a consequence of (6.11), we have

$$\text{Spec}\ \Delta_M \subset \text{Spec}\ \Delta_P. \tag{6.14}$$

Hence, when $M = P_n(\mathbb{C})$, the result that every eigenvalue of $-\Delta_M$ has the form (6.1) is a consequence of the results on the spectrum of Δ on spheres given in §5. Also, we can use (6.11) with $\varphi(\Delta_M) = \cos t\sqrt{-\Delta_M + n^2}$ to analyze the wave equation (6.6) on $\mathbb{R} \times P_n(\mathbb{C})$.

A similar analysis works on quaternionic projective space $P_n(\mathbb{H})$. We can take the unit sphere S^{4n+3} in $\mathbb{H}^{n+1} \approx \mathbb{R}^{4n+4}$, and divide by the action of the group of unit quaternions, a group isomorphic to $SU(2)$, to obtain $S^{4n+3} \to P_n(\mathbb{H})$, a principal $SU(2)$-bundle. Thus we have analogues of Propositions 6.1-6.2 for $P_n(\mathbb{H})$.

7. Fourier inversion on hyperbolic space, and variants

Let $\mathcal{H}^n$ be n-dimensional hyperbolic space, a simply connected, complete Riemannian manifold with sectional curvature -1. If Δ is the Laplace operator on $\mathcal{H}^n$, then, parallel to (5.1), it is convenient to use

$$A = \sqrt{-\Delta - \Big(\frac{n-1}{2}\Big)^2}. \tag{7.1}$$

It is known [P4] that Spec $(-\Delta) = [\frac{1}{4}(n-1)^2, \infty)$; hence A in (5.1) is the positive square root of a positive self adjoint operator. The considerations of §1 apply here, and we also have explicit formulas for $\cos tA\ f(x) = u(t,x)$.

If $n = 2k+1$ is odd, we have, parallel to (5.12),

$$\cos tA\ f(x) = C_k (\sinh t) \Big(\frac{1}{\sinh t}\frac{\partial}{\partial t}\Big)^k \big(\sinh^{2k-1} t\ \overline{f}_x(|t|)\big). \tag{7.2}$$

One can also find a derivation of this formula in Chapter 4 of [T2].

In particular, when $n = 3$ ($k = 1$), we have

$$\begin{aligned} \cos tA\ f(x) &= \partial_t \big(\sinh t\ \overline{f}_x(|t|)\big) \\ &= (\cosh t)\overline{f}_x(|t|) + (\sinh t)\partial_t \overline{f}_x(|t|). \end{aligned} \tag{7.3}$$

Parallel to Lemma 2.1, we have

Lemma 7.1. *Let $f \in L^2(\mathcal{H}^n)$. Then*

$$\int_1^\infty \sinh^{n-1} t\ |\overline{f}_x(t)|^2\ \ dt < \infty, \qquad \int_1^\infty \frac{\cosh^{\frac{n-1}{2}} t}{t}\ |\overline{f}_x(t)|\ \ dt < \infty.$$

Proof. In terms of geodesic polar coordinates (t,ω) with respect to $x \in \mathcal{H}^n$, the condition $f \in L^2(\mathcal{H}^n)$ is written $\int_0^\infty \int_{S^{n-1}} |f(\exp_x(t\omega))|^2 \sinh^{n-1} t\, dt\, d\omega < \infty$. Applying Cauchy-Schwarz as before we have

$$|\overline{f}_x(t)|^2 \le \frac{1}{A_{n-1}} \int_{S^{n-1}} |f(\exp_x(t\omega))|^2\ d\omega,$$

$$\int_1^\infty |\overline{f}_x(t)|^2 \sinh^{n-1} t\, dt \le \frac{1}{A_{n-1}} \int_1^\infty \int_{S^{n-1}} |f(\exp_x(t\omega))|^2 \sinh^{n-1} t\, dt\, d\omega < \infty.$$

which gives the first estimate. To get the second, again use Cauchy-Schwarz:

$$\begin{aligned} \Big(\int_1^\infty \frac{\cosh^{\frac{n-1}{2}} t}{t} |\overline{f}_x(t)|\, dt\Big)^2 &= \Big(\int_1^\infty \frac{\coth^{\frac{n-1}{2}} t}{t}\ \sinh^{\frac{n-1}{2}} t |\overline{f}_x(t)|\, dt\Big)^2 \\ &\le \Big(\int_1^\infty \frac{\coth^{n-1} t}{t^2}\, dt\Big)\Big(\int_1^\infty \sinh^{n-1} t |\overline{f}_x(t)|^2\, dt\Big) < \infty. \end{aligned}$$

Thus, parallel to Proposition 2.2, we have:

Proposition 7.2. *Let $f \in L^2(\mathcal{H}^3)$. Fix $x \in \mathcal{H}^3$ and assume $\overline{f}_x$ satisfies the Dini condition (2.11). Then, as $R \to \infty$,*

$$S_R f(x) \to f(x) \iff \int_{-\infty}^{\infty} \frac{\sinh t}{t} \left(\partial_t \overline{f}_x(|t|)\right) \sin Rt \; dt \to 0. \tag{7.4}$$

In particular, if $\overline{f}_x(t)$ has compact support in t for given x, we have an obvious analogue of Corollary 2.2:

Corollary 7.3. *Let $f \in L^2(\mathcal{H}^3)$ have compact support. Fix $x \in \mathcal{H}^3$. Then $S_R f(x) \to f(x)$ as $R \to \infty$, provided $\overline{f}_x(|t|)$ satisfies the Dini condition (2.11) and is absolutely continuous on $\mathbb{R}$.*

Note the similarity of the conditions (7.4) and (2.12), the latter arising in the analysis of functions on $\mathbb{R}^3$. In fact (granted the hypothesis of compact support), the conditions (2.12) and (7.4) are *equivalent.* To see this, note that the factor $t^{-1} \sinh t$ is a smooth nonvanishing function. Furthermore, if $\partial_t \overline{f}_x(|t|) \in \mathcal{E}'(\mathbb{R})$, this factor (or its inverse) can be cut off. Thus, the equivalence of these conditions follows from the $k = 1$ case of:

Lemma 7.4. *Given $\varphi \in C_0^\infty(\mathbb{R}^k)$, $\psi \in \mathcal{E}'(\mathbb{R}^k)$, we have,*

$$\hat{\psi}(\xi) \to 0 \text{ as } |\xi| \to \infty \Longrightarrow (\varphi\psi)\hat{}(\xi) \to 0 \text{ as } |\xi| \to \infty. \tag{7.5}$$

Proof. This is immediate from the identity

$$(\varphi\psi)\hat{}(\xi) = \int \hat{\varphi}(\eta)\hat{\psi}(\xi - \eta) \, d\eta, \tag{7.6}$$

plus the facts that, under our hypotheses, $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^k)$ and $\hat{\psi} \in L^\infty(\mathbb{R}^k)$.

Rather than explicitly recording the parallels with results in previous sections for Fourier inversion of functions on $\mathcal{H}^n$ for other values of n, we will move on to more general considerations.

Hyperbolic space $\mathcal{H}^n$ is a rank one symmetric space. Generally, a Riemannian manifold M is a rank one symmetric space precisely when, for any $x \in M$, the group G_x of isometries of M fixing x acts transitively on the unit sphere in $T_x M$. If M is such a space, $f \in L^2(M)$, and $x \in M$, we can form

$$\Phi(y) = \int_{G_x} f(g \cdot y) \, dg. \tag{7.7}$$

Choosing a constant $K \geq \sup \operatorname{spec} \Delta$, set $S_R = \chi_R(A)$, $A = \sqrt{-\Delta + K}$. Since the action of G_x on $L^2(M)$ commutes with functions of the Laplace operator, we have, as $R \to \infty$,

$$S_R f(x) \to f(x) \iff S_R \Phi(x) \to \Phi(x). \tag{7.8}$$

Other examples of rank one symmetric spaces of noncompact type include complex hyperbolic spaces and quaternionic hyperbolic spaces:

$$\mathcal{H}_n(\mathbb{C}) = U(n,1)/U(n)\times U(1), \quad \mathcal{H}_n(\mathbb{H}) = Sp(n,1)/Sp(n)\times Sp(1),$$

which are 'duals' of the spaces $P_n(\mathbb{C})$ and $P_n(\mathbb{H})$ discussed in §6. There is also a 16-dimensional hyperbolic Cayley space. See [Hel].

In fact, we will take the following more general setting for our next proposition. Let M be a complete, connected, n-dimensional Riemannian manifold, and let x be a point in M. Assume that the exponential map $\mathrm{Exp}_x : T_xM \to M$ is a diffeomorphism of T_xM onto M. This holds for the symmetric spaces mentioned above, which all have negative sectional curvature. Let $A = \sqrt{-\Delta + K}$, for some constant $K \geq \sup \operatorname{spec} \Delta$, and $S_R = \chi_R(A)$.

Proposition 7.5. *Let M satisfy the hypotheses of the previous paragraph. Let $f \in L^2(M)$ be a radial function, i.e., of the form*

$$f(y) = \varphi(r), \quad r = \operatorname{dist}(x,y). \tag{7.9}$$

Assume there exists $T_0 < \infty$ such that the decay property (1.6) holds, with $u(t,x) = \cos tA\ f(x)$. Then $S_Rf(x) \to f(x)$, provided φ has the following two properties. First,

$$\varphi \in H^{\frac{1}{2}(n-1),p}_{\mathrm{loc}}(\mathbb{R}\setminus 0), \tag{7.10}$$

with $p = 1$ if n is odd and some $p > 1$ if n is even. Second, dilates $\varphi_\nu(r) = \varphi(2^\nu r)$ satisfy, for $\nu \geq 0$,

$$\sum_{\nu\geq 0}\Big\{\big\|\varphi_\nu - \varphi(0)\big\|_{H^{\frac{1}{2}(n-1),p}([\frac{1}{4},1])} + \big\|\varphi_\nu - \varphi(0)\big\|_{L^1([0,1])}\Big\} < \infty. \tag{7.11}$$

Proof. We study the map

$$\begin{gathered}\varphi \mapsto \psi, \\ \psi(t) = \cos tA\ f(x).\end{gathered} \tag{7.12}$$

We want to show that, under the hypotheses of the proposition,

$$\int_{-T_0}^{T_0} t^{-1}\big[\psi(t) - \psi(0)\big]\ \sin Rt\ dt \to 0 \tag{7.13}$$

as $R \to \infty$. Note that, by finite propagation speed, $\psi\big|_{[-\tau,\tau]}$ depends only on $\varphi\big|_{[0,\tau]}$. Also, the general parametrix construction for the hyperbolic operator $\partial_t^2 - (\Delta - K)$ implies that, for any $\varepsilon > 0$,

$$\varphi \mapsto \psi\big|_{[-T_0,T_0]\setminus[-\varepsilon,\varepsilon]} \tag{7.14}$$

is a pseudodifferential operator of order $\frac{1}{2}(n-1)$. If n is odd, it is a sum of a differential operator of order $\frac{1}{2}(n-1)$ and a pseudodifferential operator of order -1. See §8 for a description of the parametrix and other implications of this nature.

Now the hypotheses on f imply that $\psi \in L^p\big([-T_0, T_0] \setminus [-\varepsilon, \varepsilon]\big)$. Furthermore, if we dilate a small neighborhood of x, we obtain an estimate

$$\begin{aligned} \big\|t^{-1}\big(\psi - \psi(0)\big)\big\|_{L^1([2^{-\nu-1}, 2^{-\nu}])} &\leq C\|\psi_\nu - \psi(0)\|_{L^1([\frac{1}{2},1])} \\ &\leq C\|\varphi_\nu - \varphi(0)\|_{H^{\frac{1}{2}(n-1),p}([\frac{1}{4},1])} + \|\varphi_\nu - \varphi(0)\|_{L^1([0,1])}. \end{aligned} \tag{7.15}$$

Thus, under the hypotheses of the proposition, we actually have

$$\int_0^{T_0} t^{-1}\,|\psi(t) - \psi(0)|\; dt < \infty, \tag{7.16}$$

and the proposition is proved.

When $n = 2k+1$ is odd, we have, in (7.11),

$$\begin{aligned} \|\varphi_\nu - \varphi(0)\|_{H^{k,1}([\frac{1}{4},1])} &= \|\varphi_\nu - \varphi(0)\|_{L^1([\frac{1}{4},1])} + \sum_{j=1}^{k} \|\partial_r^j \varphi_\nu\|_{L^1([\frac{1}{4},1])} \\ &= 2^\nu \|\varphi - \varphi(0)\|_{L^1([2^{-\nu-2}, 2^{-\nu}])} + \sum_{j=1}^{k} 2^{\nu - j\nu} \|\partial_r^j \varphi\|_{L^1([2^{-\nu-2}, 2^{-\nu}])}. \end{aligned} \tag{7.17}$$

Thus:

Proposition 7.6. *When $n = 2k+1$ is odd, the condition (7.11) is equivalent to*

$$\big\|r^{-1}\big(\varphi - \varphi(0)\big)\big\|_{L^1([0,1])} + \sum_{j=1}^{k} \|r^{j-1}\partial_r^j \varphi\|_{L^1([0,1])} < \infty. \tag{7.18}$$

Note the agreement of this condition with (4.8)-(4.9), in the analysis of functions on $\mathbb{R}^{2k+1}$.

8. Fourier inversion on strongly scattering manifolds

Let us assume, as in §7, that M is a complete, connected, n-dimensional Riemannian manifold, x is a point in M, and $\mathrm{Exp}_x : T_xM \to M$ is a diffeomorphism of T_xM onto M. (Later in this section, we will drop this last assumption.) Let $A = \sqrt{-\Delta + K}$, for some constant $K \geq \sup \operatorname{spec} \Delta$, and set $S_R = \chi_R(A)$.

Proposition 8.1. *Let M satisfy the hypotheses of the previous paragraph. Let $f \in L^2(M)$. Assume there exists $T_0 < \infty$ such that the decay property (1.6) holds, with $u(t,x) = \cos tA\ f(x)$. Take C^∞ functions $V_j(x,y)$ (to be described below), for $j = 0,\dots,[n/2]$, and set*

$$\varphi_j(r) = \ Avg_{S_r(x)}\ V_j(x,y)f(y), \quad S_r(x) = \{y \in M : dist(x,y) = r\}, \tag{8.1}$$

the average with respect to spherical measure, in exponential coordinates. Assume that

$$\varphi_j \in H^{\frac{1}{2}(n-1)-j,p}_{\mathrm{loc}}(\mathbb{R} \setminus 0), \tag{8.2}$$

with $p = 1$ if n is odd, and some $p > 1$ if n is even. Furthermore, assume dilates $\varphi_{j\nu}(r) = \varphi_j(2^\nu r)$ satisfy

$$\sum_{\nu \geq 0} \Big\{ \big\|\varphi_{j\nu} - \varphi_j(0)\big\|_{H^{\frac{1}{2}(n-1)-j,p}([\frac{1}{4},1])} + \big\|\varphi_{j\nu} - \varphi_j(0)\big\|_{L^1([0,1])} \Big\} < \infty. \tag{8.3}$$

Then $S_R f(x) \to f(x)$, as $R \to \infty$.

Note that this result extends Proposition 7.5, in that f is not required to be radial. The functions φ_j are to some degree replacements for the process of taking radial symmetrization.

Proof. We make use of the Hadamard parametrix. As described in Proposition 17.4.3 of [Ho], we have

$$A^{-1} \sin tA\ f(x) = \sum_{j=1}^{N} \int f(y) U_j(x,y) E_j\big(t, \mathrm{dist}(x,y)\big)\ dy + R_N, \tag{8.4}$$

the 'integration' being a pairing of distributions, in the exponential coordinate system. Here, U_j are C^∞ functions, and

$$E_j(t,x) = c_{jn} \chi_+^{j-\frac{1}{2}(n-1)}\big(t^2 - |x|^2\big), \tag{8.5}$$

where χ_+^a is a family of distributions on $\mathbb{R}$, given by

$$\chi_+^a = \Gamma(1+a)^{-1} x_+^a, \tag{8.6}$$

for Re $a > -1$, and by analytic continuation for all $a \in \mathbb{C}$. We have

$$\partial_x \chi_+^a = \chi_+^{a-1}, \quad \chi_+^{-k} = \delta^{(k-1)}. \tag{8.7}$$

In (8.4), R_N has arbitrarily high regularity, if N is taken sufficiently large.

Now $\cos tA\ f(x)$ is obtained by taking the t-derivative of (8.4). Hence, given $f \in L^2(M)$, we can write, mod $L^2_{\text{loc}}(\mathbb{R})$,

$$\cos tA\ f(x) = \sum_{j=0}^{[n/2]} B_j \varphi_j(t), \tag{8.8}$$

where φ_j are as in (8.1) and B_j are linear operators with the following properties. First, for any $\tau > 0$, $B_j\varphi_j\big|_{[-\tau,\tau]}$ depends only on $\varphi_j\big|_{[-\tau,\tau]}$. Second, for any $\varepsilon > 0$, $B_j\varphi_j\big|_{[-T_0,T_0]\setminus[-\varepsilon,\varepsilon]}$ is given by the action of a pseudodifferential operator of order $\frac{1}{2}(n-1)-j$. If n is odd, it is the sum of a differential operator of order $\frac{1}{2}(n-1)-j$ and a pseudodifferential operator of order -1.

From here, the argument that hypotheses (8.2)-(8.3) imply $S_R f(x) \to f(x)$ is exactly as in the proof of Proposition 7.5.

Results affirming cases in which the decay hypothesis (1.6) is satisfied belong to scattering theory. Typically they require of M that there be no trapped rays, i.e., no trapped geodesics. For example, results and methods of [Mor] imply that, if one has a compactly supported perturbation of the flat metric on $\mathbb{R}^n$, having no trapped rays, then there is such decay, provided f has compact support. A variant of such arguments can be used to show that, if M is obtained from hyperbolic space $\mathcal{H}_n$ by perturbing the metric over a compact region, in such a fashion that there are no trapped rays, then such a property holds. Related variants of this argument are given in [MjT]. This is not an exhaustive account of cases for which such decay holds (see in particular the comments below regarding domains with boundary), and indeed the question of when one has such decay deserves further study.

We now consider a more general class of Riemannian manifolds. We continue to assume M is complete, connected, and n-dimensional. We also assume that, if $f \in L^2(M)$ has compact support, then the decay hypothesis (1.6) holds. As before, $A = \sqrt{-\Delta + K}$, with $K \geq \sup \operatorname{spec} \Delta$. However, we do not assume that $\operatorname{Exp}_x : T_xM \to M$ is a diffeomorphism. As a consequence, we allow situations where the formation of caustics can lead to the breakdown of the Hadamard parametrix (8.4).

Rather than extend Proposition 8.1, we will analyze convergence of $S_R f(x)$ for a special class of functions f. We will assume $f \in L^1(M)$ has compact support, and that f is a classical conormal distribution, with singularity along a smooth hypersurface $\Sigma \subset M$. We will say $f \in I^0_0(M,\Sigma)$ if f is piecewise smooth with a simple jump across Σ. We say $f \in I^\mu(M,\Sigma)$ if f has compact support and $f = Ag$, for some $g \in I^0_0(M,\Sigma)$, $A \in OPS^\mu$. Then $I^\mu(M,\Sigma) \subset L^1(M)$ for $\mu < 1$, and more generally, if $1 \leq p < \infty$, $I^\mu(M,\Sigma) \subset L^p(M)$ for $\mu < 1/p$. This notation is a variant of that used in [Ho]; in particular, we have a different definition of the 'order' μ.

For small $|t|$, $u(t,\cdot) = \cos tA\ f$ is a sum of elements of $I^\mu(M,\Sigma_t)$ and $I^\mu(M,\Sigma_{-t})$, where $\Sigma_{\pm t}$ are the hypersurfaces obtained from Σ via flow along geodesics normal to Σ. For larger $|t|$, the surfaces $\Sigma_{\pm t}$ might develop singularities. The union of the singularities forms the *caustic set* $\mathcal{C} \subset M$.

Assume $x \notin \Sigma$, so the quantity (1.10) is well behaved for $|t|$ close to zero. If also $x \notin \mathcal{C}$, then $u(\cdot,x)$ belongs to $I^\mu(\mathbb{R},\mathcal{T}_x)$, where $\mathcal{T}_x \subset \mathbb{R}$ consists of a finite set of points. Hence, as long as $\mu < 1$, $S_R f(x) \to f(x)$ as $R \to \infty$. A little further on, we will discuss the rate of convergence.

If x belongs to the caustic set $\mathcal{C}$, then $u(\cdot, x)$ can be a more singular distribution in $\mathcal{D}'(\mathbb{R})$. If $\mathcal{C}$ is a 'simple caustic,' in the sense of [Dui1], p.186, then, associated to any $x \in \mathcal{C}$ is an order $\kappa(x)$, generally of the form

$$\kappa(x) = \frac{1}{2} - \frac{1}{\nu(x)}, \tag{8.9}$$

with $\nu(x) \in \mathbb{Z}^+$, and we have

$$u(\cdot, x) \in I^{\mu+\kappa(x)}(\mathbb{R}, \mathcal{T}_x), \tag{8.10}$$

again for some finite subset $\mathcal{T}_x \subset \mathbb{R}$. To give some examples, if x is a fold point on $\mathcal{C}$, then $\kappa(x) = 1/6$, while if x is a cusp point on $\mathcal{C}$, then $\kappa(x) = 1/4$. A further discussion of the simple caustics can be found in [Dui1,Dui2]; see also [AVG] and [GS]. In addition, there are unstable caustics which have fairly simple behavior, in particular, the perfect focus caustic, which is produced by the function $f = \chi_{B_1}$ studied in §2, for example. If $n = \dim M$, and $\mathcal{C} = \{x\}$ is a perfect focus caustic, then $\kappa(x) = \frac{1}{2}(n-1)$, which is larger than the numbers in (8.9). There exist smooth perturbations of unstable caustics with more exotic behavior, and examples for which $u(\cdot, x)$ does not belong to any space $I^{\mu+\kappa}(\mathbb{R}, \mathcal{T}_x)$; in fact the singular support could have accumulation points. We will not dwell on these cases.

Let us record a convergence result, which follows from (8.10).

Proposition 8.2. *Under the hypotheses above, assume $x \notin \Sigma$ and that either $x \notin \mathcal{C}$ (then take $\kappa(x) = 0$) or that $x \in \mathcal{C}$ is a caustic point of order $\kappa(x)$. Then, as $R \to \infty$, $S_R f(x) \to f(x)$ for all $f \in I^\mu(M, \Sigma)$, if and only if*

$$\mu + \kappa(x) < 1. \tag{8.11}$$

We remark that, for most $f \in I^\mu(M, \Sigma)$, the principal symbol (in the sense of [Ho]) of $u(\cdot, x)$ will be nonvanishing, so that $u(\cdot, x)$ will not belong to $I^{\mu+\kappa(x)-\varepsilon}(\mathbb{R}, \mathcal{T}_x)$ for any $\varepsilon > 0$. However, for some f, the principal symbol might vanish, and we will have $u(\cdot, x) \in I^{\mu+\kappa(x)-1}(M, \Sigma)$. This phenomenon arises in the study of the function (2.19); in that case, the principal symbol is proportional to the mean value of g on the unit sphere $|x| = 1$.

In Proposition 8.2, we can also allow $x \in \Sigma$, in some cases. All we need is that $t^{-1}\big[u(t, x) - f(x)\big]$ be integrable for t in some neighborhood of the origin. For example, if $f \in I^\mu(M, \Sigma)$ and $\mu < 0$, then the analysis works. Also, if $f \in I_0^0(M, \Sigma)$, i.e., f has a simple jump across Σ, it follows that $u(t, x)$ is a sum of two terms with simple jumps across the characteristic hypersurfaces through Σ described above, for small $|t|$, and one has the desired integrability near $t = 0$, as long as $f(x)$ is equal to the mean of the limits of $f(y)$ as $y \to x$ from each side of Σ. See §11 for a more detailed discussion of convergence on and around Σ, in this case.

We now discuss the rate of convergence of $S_R f(x)$ to $f(x)$. For this, we strengthen the decay hypothesis (1.6). There are results on exponential decay of $u(t, x)$ and all its derivatives, as $|t| \to \infty$, for various classes of manifolds M with no trapped rays, when

$n = \dim M$ is odd. When n is even, there are results on algebraic decay. Both sorts of results imply the following: for some $B \in C_0^\infty(\mathbb{R})$,

$$(8.12) \qquad \big(1 - B(t)\big)u(t,x) \text{ has rapidly decreasing } t\text{-Fourier transform.}$$

Under such an hypothesis, the study of the rate of convergence of $S_R f(x)$ to $f(x)$, which has been reduced to the study of the rate of decay of (1.9), is localized. We have the following.

Proposition 8.3. *Under the hypotheses of Proposition 8.2, and the hypothesis (8.12), given $f \in I^\mu(M, \Sigma)$, $x \in M \setminus \Sigma$, of order $\kappa(x)$, if $\mu + \kappa(x) < 1$, we have*

$$(8.13) \qquad |S_R f(x) - f(x)| \le C R^{-a}, \quad a = 1 - \mu - \kappa(x).$$

Proof. It is elementary that, if $v \in \mathcal{E}'(\mathbb{R})$, then

$$(8.14) \qquad v \in I^{1-b}(\mathbb{R}, \mathcal{T}_x) \Longrightarrow |\hat{v}(R)| \le C R^{-b},$$

and this gives (8.13).

We illustrate Proposition 8.3 with some examples, exhibiting varieties of caustic points that arise in simple situations.

First, consider $\Sigma \subset \mathbb{R}^2$, an ellipse, given by

$$(8.15) \qquad \left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 = 1,$$

The caustic set $\mathcal{C}$ is constructed by travelling from a point $y \in \partial\Omega$ inward along the normal to $\partial\Omega$ at y, a distance equal to the radius of curvature of $\partial\Omega$ at y. As y runs over $\partial\Omega$, this locus sweeps out $\mathcal{C}$. If $0 < a_1 < a_2$, this is a curve which is smooth except at 4 cusps. Each point in the smooth part $\mathcal{C}_0$ of $\mathcal{C}$ is a caustic point of order $1/6$, and each cusp is a caustic point of order $1/4$.

Now, suppose $\overline{\Omega}$ is the compact region bounded by Σ, $g \in C^\infty(\overline{\Omega})$, and let $f \in L^2(\mathbb{R}^2)$ be equal to g in Ω and to 0 on $\mathbb{R}^2 \setminus \overline{\Omega}$. If $x \in \mathbb{R}^2 \setminus (\partial\Omega \cup \mathcal{C})$, then

$$(8.16) \qquad f(x) = S_R f(x) + O(R^{-1}).$$

Here, $a = 1 - 0 - 0$ in (8.13). If $x \in \mathcal{C}_0$ (and $x \notin \partial\Omega$), then

$$(8.17) \qquad f(x) = S_R f(x) + O(R^{-5/6}).$$

Here, $a = 1 - 0 - \frac{1}{6}$ in (8.13). If x is a cusp of $\mathcal{C}$, then

$$(8.18) \qquad f(x) = S_R f(x) + O(R^{-3/4}).$$

Here, $a = 1 - 0 - \frac{1}{4}$ in (8.13). For general g, these remainder estimates are sharp. In fact, the hypothesis of (8.14) implies that $\hat{v}(R) \sim C_0 R^{-b}$, and C_0 is nonzero unless actually $v \in I^{-b}(\mathbb{R}, \mathcal{T}_x)$, in which case $\hat{v}(R) \sim C_1 R^{-b-1}$.

If $a_1 = a_2$ in (8.15), then Ω is a disk, and $\mathcal{C}$ consists of one point, the center of the disk, a caustic point of order 1/2. Then, if x is the center, we have

$$f(x) = S_R f(x) + O(R^{-1/2}). \tag{8.19}$$

Here, $a = 1 - 0 - \frac{1}{2}$ in (8.13). For other $x \in \Omega$, of course we have the behavior (8.16).

Next, consider $\Sigma \subset \mathbb{R}^3$, an ellipsoid, given by

$$\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 = 1. \tag{8.20}$$

If $0 < a_1 < a_2 < a_3$, then the caustic set $\mathcal{C}$ consists entirely of fold points, where $\kappa(x) = 1/6$, and cusp points, where $\kappa(x) = 1/4$. Suppose $0 < a_1 = a_2 < a_3$, in which case Σ is an ellipsoid of revolution (but not a sphere). Then $\mathcal{C}$ also contains three points of unstable type, at which $\kappa(x) = 1/2$. All three points are on the axis of revolution, and one of them is the origin.

Here is another family of examples. Let $\Sigma \subset \mathbb{R}^n$ be given by

$$|x|^2 + a|x - e|^{2k} = 1, \tag{8.21}$$

where we have chosen $a > 0$, $k \in \mathbb{Z}^+$, and e is a unit vector in $\mathbb{R}^n$. We will not analyze the entire caustic set $\mathcal{C}$ for such a surface, but take a look at the origin, which belongs to $\mathcal{C}$, as long as $k \geq 2$, since Σ is tangent to the unit sphere at e, with order of contact $2k$. The origin is a caustic point of unstable type, but one directly sees that

$$\kappa(0) = \tfrac{1}{2}(n-1) - \tfrac{1}{2k}(n-1) = \tfrac{1}{2}(n-1)(1 - \tfrac{1}{k}). \tag{8.22}$$

If f is the characteristic function of the set bounded by Σ (and $0 \notin \Sigma$), then $S_R f(0) \to f(0)$ if $n = 3$, for any k. If $n = 4$, we have $S_R f(0) \to f(0)$ if $k = 2$ (in which case $\kappa(0) = 3/4$), but not if $k \geq 3$. If $n \geq 5$, then $S_R f(0)$ does not converge if $k \geq 2$.

We now give a brief discussion of a class of domains with boundary. Let K be a compact subset of $\mathbb{R}^n$ with smooth boundary. We consider the Laplace operator Δ on $\Omega = \mathbb{R}^n \setminus K$, with either Dirichlet or Neumann boundary conditions on ∂K. We will draw some conclusions about the pointwise convergence of $S_R f(x) = \chi_R(\sqrt{-\Delta}) f(x)$, as $R \to \infty$.

Assume that $f \in L^2(\Omega)$ has compact support, and that the region $\Omega = \mathbb{R}^n \setminus K$ has no trapped rays. Here a ray which hits ∂K reflects by the rule that angle of incidence equals angle of reflection. Then, fundamental results in scattering theory, including particularly [MRS], [Ral], and [Mel], imply that the condition (1.6) holds for $u(t,x) = \cos t\sqrt{-\Delta}\, f(x)$. In fact, it holds locally uniformly in x, and, if n is odd, one actually has local exponential decay. Thus, results of §1 imply the following.

Proposition 8.4. *Assume $f \in L^2(\Omega)$ has support in a compact set $S \subset \overline{\Omega}$. Fix $x \in \overline{\Omega}$. Assume that, for any ray $\gamma(t)$ lying over a point in S at $t = 0$, $\gamma(t)$ is disjoint from x whenever $|t| \geq T_0$ (where $T_0 < \infty$). Then $S_R f(x) \to f(x)$ as $R \to \infty$, if and only if*

$$\int_{-T_0}^{T_0} t^{-1}\big[u(t,x) - f(x)\big]\ \sin Rt\ dt \to 0, \quad \text{as } R \to \infty. \tag{8.23}$$

In particular, such convergence holds provided

$$t^{-1}\big[u(t,x)-f(x)\big]\in L^1\big([-T_0,T_0]\big). \tag{8.24}$$

Note that, under the hypotheses made above, the 'integrand' in (8.23) is a sum of a distribution in $\mathcal{E}'\big((-T_0,T_0)\big)$ and a function in $C^\infty\big([-T_0,T_0]\big)$.

We next examine convergence of $S_R f(x)$ when $f\in C^\infty(\overline{\Omega})$, with compact support. If f vanishes on a neighborhood of $\partial\Omega$, then $f\in\mathcal{D}(\Delta^k)$ for all k, and then uniform convergence $S_R f\to f$ follows from the Sobolev imbedding theorem. However, in general, an element $f\in C^\infty(\overline{\Omega})$ does not belong to $\mathcal{D}(\Delta)$, unless the appropriate boundary condition is satisfied. To examine $\cos t\sqrt{-\Delta}\ f(x)=u(t,x)$, we do the following. Extend f to an element of $C_0^\infty(\mathbb{R}^n)$, and let $u^0(t,x)$ solve the free space wave equation, with Cauchy data $u^0(0,x)=f(x),\ \partial_t u^0(0,x)=0$. Then

$$u(t,x)=u^0(t,x)-v(t,x), \tag{8.25}$$

where $v(t,x)$ is constructed as follows. If we use the Dirichlet boundary condition on $\partial\Omega$, then solve the outgoing problem

$$\begin{gathered}(\partial_t^2-\Delta)w=0,\quad w(t,x)=u^0(t,x)\ \text{for}\ t\ge 0,\ x\in\partial\Omega,\\ w(t,x)=0\ \ \text{for}\ \ t<0,\end{gathered} \tag{8.26}$$

and set

$$v(t,x)=w(t,x)+w(-t,x). \tag{8.27}$$

If we use the Neumann boundary condition, then solve the outgoing problem

$$\begin{gathered}(\partial_t^2-\Delta)w=0,\quad \partial_\nu w(t,x)=\partial_\nu u^0(t,x)\ \text{for}\ t\ge 0,\ x\in\partial\Omega,\\ w(t,x)=0\ \ \text{for}\ \ t<0,\end{gathered} \tag{8.28}$$

and again use (8.27).

Note that the boundary data in (8.26) and (8.28) have simple jumps across $\{t=0\}$. No glancing rays are involved in the local parametrix constructions for (8.26) and (8.28), which can be done by methods given in Chapter 9 of [T1]. Let us make the hypothesis that rays going into $\overline{\Omega}$, normal to $\partial\Omega$, never again hit $\partial\Omega$. It follows that, for a neighborhood $\overline{\mathcal{O}}$ of $\partial\Omega$ in $\overline{\Omega}$, we have

$$u\big|_{\mathbb{R}\times\mathcal{O}}\subset I^0(\mathbb{R}\times\mathcal{O},\Sigma), \tag{8.29}$$

for the Dirichlet problem, and

$$u\big|_{\mathbb{R}\times\mathcal{O}}\in I^{-1}(\mathbb{R}\times\mathcal{O},\Sigma), \tag{8.30}$$

for the Neumann problem. Here, Σ is the union of of the two characteristic surfaces for $\partial_t^2-\Delta$, passing through $\{(0,x):x\in\partial\Omega\}$. Under the ray hypothesis made above, it follows that the nature of the singularities of u in all of $\mathbb{R}\times\Omega$ is determined from (8.29) (or from (8.30)), by the same rules as in the free space case. Thus, the argument yielding Proposition 8.3 also works to give:

Proposition 8.5. *Let $\Omega = \mathbb{R}^n \setminus K$, where K is compact and ∂K is smooth. Assume Ω has no trapped rays, and assume rays into $\overline{\Omega}$ normal to $\partial\Omega$ never again hit $\partial\Omega$. Take $f \in C^\infty(\overline{\Omega})$, with compact support. Let $\mathcal{C} \subset \Omega$ denote the caustic set derived from $\partial\Omega$.*

Given the Dirichlet boundary condition on Δ, then $S_R f(x) \to f(x)$ whenever $x \in \Omega \setminus \mathcal{C}$, and whenever $x \in \mathcal{C}$ has order $\kappa(x) < 1$. In that case,

$$|S_R f(x) - f(x)| \le C R^{-(1-\kappa(x))}. \tag{8.31}$$

Given the Neumann boundary condition, then $S_R f(x) \to f(x)$ whenever $x \in \Omega \setminus \mathcal{C}$, and whenever $x \in \mathcal{C}$ has order $\kappa(x) < 2$. In that case,

$$|S_R f(x) - f(x)| \le C R^{-(2-\kappa(x))}. \tag{8.32}$$

If we allowed rays into $\overline{\Omega}$, normal to $\partial\Omega$, to return to $\partial\Omega$, but insisted that those that did so, hit $\partial\Omega$ either transversally or with simple tangency, staying in $\overline{\Omega}$, then the results and methods of [MeT] should allow a fairly precise extension of Proposition 8.5 to this more general situation. We will not pursue this here.

We consider another phenomenon regarding the behavior of $S_R f$ for f defined on $\Omega = \mathbb{R}^n \setminus K$, that has no analogue in the case of complete Riemannian manifolds without boundary. Assume $f \in L^2(\Omega)$ has support in a compact subset S of Ω. Then the wave front set $WF(f)$ is a subset of $T^*\Omega$ lying over S. We say a compact set $T \subset \overline{\Omega}$ belongs to the *shadow region* of f if *no rays* join any point of $WF(f)$ to a point lying over T. A sequence of results on propagation of singularities, culminating in [MeS], implies that $\cos t\sqrt{-\Delta}\, f$ is smooth on $\mathbb{R} \times T$, whenever T belongs to the shadow region. Using Corollary 1.3, we have:

Proposition 8.6. *Under the hypotheses of Proposition 8.4, assume $f \in L^2(\Omega)$ is supported in a compact set $S \subset \Omega$, and assume $T \subset \overline{\Omega}$ is a compact set, in the shadow region of f. Then, as $R \to \infty$,*

$$S_R f \to f \quad \text{uniformly on} \quad T. \tag{8.33}$$

Furthermore, the convergence is rapid in R, i.e., $S_R f - f = O(R^{-N})$ on T, for each N.

If K does not have smooth boundary, the phenomenon of diffraction by edges, corners, etc., could affect the behavior of $S_R f(x)$ for x in the shadow region. Rather than enter into a general study, we confine attention to a special case.

Let $L \subset \mathbb{R}^2$ be a line segment; say

$$L = \{(x, 0) : -1 \le x \le 1\},$$

and set $\Omega = \mathbb{R}^2 \setminus L$. Let p be a point on the negative y-axis, say $p = (0, -q)$. Let Σ be the circle of radius a centered at p; assume $a < q$, so Σ lies in the lower half plane. If $f \in I^\mu(\Omega, \Sigma)$, the shadow region $\mathcal{S}$ for f is the open region bounded to the left by the ray from p through $(-1, 0)$, to the right by the ray from p through $(1, 0)$, and from below by L. Now, the theory of diffraction by a slit, due to [Som], and extended in a number of works, including [Fr] and [CT], shows that, in $\mathbb{R}\times\mathcal{S}$, cos $t\sqrt{-\Delta}\, f$ belongs to $I^{\mu-\frac{1}{2}}(\mathbb{R}\times\mathcal{S},\mathcal{F})$, where $\mathcal{F}$ is $\mathbb{R}\times\mathcal{S}$ intersected with the union of four light cones, with vertices at $(\pm r_0,(\pm 1,0))$, where $r_0=\sqrt{1+q^2}-a$. We thus see that, for such $f\in I^{\mu}(\Omega,\Sigma)$,

$$\mu<\frac{3}{2},\ x\in\mathcal{S}\Longrightarrow S_Rf(x)\to f(x), \tag{8.34}$$

and, under these hypotheses,

$$|S_Rf(x)-f(x)|\le CR^{-(\frac{3}{2}-\mu)}. \tag{8.35}$$

We end this section with a comment on some results similar to those above, for a class of manifolds far from being strongly scattering, namely the spheres, with their standard metrics. Combining the techniques of §5 with those of this section, we have the following analogue of Proposition 8.3, for Fourier analysis on spheres.

Proposition 8.7. *Let Σ be a smooth hypersurface in S^n. Given $f\in I^{\mu}(S^n,\Sigma)$, $x\in S^n\setminus\Sigma$, of order $\kappa(x)$, if $\mu+\kappa(x)<1$, then*

$$|S_Nf(x)-f(x)|\le CR^{-a},\quad a=1-\mu-\kappa(x). \tag{8.36}$$

In connection with this result, we mention the paper [Dar]. There one finds an examination of rotationally invariant $f\in I^{\mu}(S^2,\Sigma)$, when Σ is a circle of constant latitude and $0<\mu<1$. Convergence is examined when x is a pole (so $\kappa(x)=\frac{1}{2}$), and the results for this case are in agreement with Proposition 8.7.

9. Hermite expansions and the Schrödinger equation

In this section we extend the previous discussions to study the partial sums of the multidimensional Hermite expansion. The treatment is somewhat parallel to that of the spherical harmonics. A Hermite expansion is written

$$f(x)\sim\sum_{k_1,\dots,k_n}a_{k_1\dots k_n}H_{k_1}(x_1)\cdots H_{k_n}(x_n) \tag{9.1}$$

where $H_k(x), k=0,1,2,\dots$ is the Hermite polynomial, solution of $\frac{1}{2}y''-xy'+ky=0$, normalized so that $\int_{-\infty}^{\infty}H_k(x)^2e^{-x^2}dx=\sqrt{\pi}2^kk!$. We adhere to the notations of Szegö [Sz].

The associated Schrödinger equation is written

$$i\frac{\partial u}{\partial t}=Lu=\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2u}{\partial x_j^2}-\sum_{j=1}^{n}x_j\frac{\partial u}{\partial x_j} \tag{9.2}$$

If $f(x_1, \dots, x_n)$ has an L^2-convergent Hermite expansion as above, then

$$u(t,x) \sim \sum_{k_1,\dots,k_n} a_{k_1\dots k_n} H_{k_1}(x_1)\cdots H_{k_n}(x_n) e^{i(k_1+\cdots+k_n)t}. \tag{9.3}$$

This is clearly 2π-periodic in time, so that we can recover the coefficients by Fourier-series:

$$\frac{1}{2\pi}\int_0^{2\pi} u(t,x)e^{-ipt}dx = \sum_{k_1+\cdots+k_n=p} a_{k_1\dots k_n} H_{k_1}(x_1)\cdots H_{k_n}(x_n) \tag{9.4}$$

and the partial sum of the Hermite series

$$\begin{aligned} S_M f(x) &= \sum_{0\le k_1+\cdots+k_n\le M} a_{k_1\dots k_n} H_{k_1}(x_1)\cdots H_{k_n}(x_n) \\ &= \frac{1}{2\pi}\sum_{p=0}^{M}\int_0^{2\pi} u(t,x)e^{-ipt}\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(t,x)e^{-iMt/2}\;\frac{\sin(M+1)t/2}{\sin(t/2)}\,dt \end{aligned} \tag{9.5}$$

Since $-L$ is positive semidefinite, we can instead sum over $-M \le p \le M$ in (9.5), obtaining

$$S_M f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(M+\frac{1}{2})t}{\sin\frac{1}{2}t}\,u(t,x)\;dt. \tag{9.6}$$

As before, we can use the Rieman-Lebesgue Lemma to deduce a sufficient condition for pointwise convergence of (9.1).

Proposition 9.1. *Suppose that $f \in L^2(\mathbb{R}^n, e^{-|x|^2}dx)$, and $u(t,x)$ is the associated solution of the Schrödinger equation $i\,u_t = Lu$. Then the partial sum of the Hermite series is represented by the integral on the last line of (9.5). If furthermore $t \to u(t,x)$ is integrable and*

$$\int_0^{2\pi}\left|\frac{u(t,x)-f(x)}{t}\right|dt < \infty, \tag{9.7}$$

then $\lim_{M\to\infty} S_M f(x) = f(x)$.

As we will see below, (9.7) actually fails in many simple cases where pointwise convergence holds, and it is not as pertinent as was its analogue in previous sections.

We note that a study of eigenfunction expansions for L is equivalent to a study of eigenfunction expansions for the quantum harmonic oscillator:

$$H = -\Delta + |x|^2, \tag{9.8}$$

which is self adjoint on $L^2(\mathbb{R}^n)$. In fact, we have a unitary operator:

$$M : L^2(\mathbb{R}^n, e^{-|x|^2}dx) \to L^2(\mathbb{R}^n), \quad Mf(x) = e^{-|x|^2/2}f(x), \tag{9.9}$$

and a calculation gives

$$(H-n)Mf = -2MLf. \tag{9.10}$$

Thus, a study of (9.6) is equivalent to

$$S'_M f(x) = \chi_{2M}(H-n)f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(M+\frac{1}{2})t}{\sin\frac{1}{2}t}\, e^{i(H-n)t/2} f(x)\ dt. \tag{9.11}$$

Now Mehler's formula for e^{-tH}, plus analytic continuation, implies

$$e^{itH/2}f(x) = \int \kappa(t,x,y)f(y)\ dy, \tag{9.12}$$

where

$$\begin{aligned} \kappa(t,x,y) &= (-2\pi i\ \sin t)^{-\frac{n}{2}}\, e^{-i[\frac{1}{2}(\cos t)(|x|^2+|y|^2)-x\cdot y]/(\sin t)} \\ &= (-2\pi i\ \sin t)^{-\frac{n}{2}}\, e^{-\frac{1}{2}i[(\cot t)|x-y|^2-\beta(t)x\cdot y]}, \end{aligned} \tag{9.13}$$

where

$$\beta(t) = 2\,\frac{1-\cos t}{\sin t} = 2\ \tan\frac{t}{2}. \tag{9.14}$$

Proofs can be found in [Leb], §4.11, or Chapter 1 of [T2].

The behavior of (9.13) is simplest at $x = 0$:

$$e^{itH/2}f(0) = (-2\pi i\ \sin t)^{-\frac{n}{2}} \int f(y)e^{-i(\cot t)|y|^2/2}\ dy. \tag{9.15}$$

If we set $s = \tan t$, we can write

$$\begin{aligned} e^{itH/2}f(0) &= (\cos t)^{-\frac{n}{2}}(-2\pi i s)^{-\frac{n}{2}} \int f(y)e^{-i|y|^2/2s}\ dy \\ &= (\cos t)^{-\frac{n}{2}}\, e^{-is\Delta/2}f(0). \end{aligned} \tag{9.16}$$

It is convenient to use (9.16) for t close to 0 or to $\pm\pi$, while (9.15) is more convenient for t close to $\pm\frac{1}{2}\pi$.

In terms of the radial symmetrization $\overline{f}_0$, we have

$$\begin{aligned} e^{-is\Delta/2}f(0) &= (-2\pi i\, s)^{-\frac{n}{2}} \int_0^\infty \overline{f}_0(r)\, e^{-ir^2/2s}\, r^{n-1}\ dr \\ &= \tfrac{1}{2}(-2\pi i\, s)^{-\frac{n}{2}} \int_0^\infty \overline{f}_0(\sqrt{\tau})\, e^{-i\tau/2s}\, \tau^{\frac{n}{2}-1}\ d\tau. \end{aligned} \tag{9.17}$$

We look at the asymptotic behavior of this quantity as $s \to 0$.

Lemma 9.2. *Assume that f has compact support and vanishes near 0. Assume that $\overline{f}_0 \in I^{\mu}(\mathbb{R}^+, \{a\})$, with $a > 0$. Then*

$$e^{-is\Delta/2} f(0) = \Psi(s) e^{-ib/s}, \tag{9.18}$$

where $b = a^2/2$, and

$$\Psi \in I^{\frac{n}{2}+\mu-1}(\mathbb{R}, \{0\}), \tag{9.19}$$

having the behavior

$$\Psi(s) \sim a_0^{\pm} s^{-\frac{n}{2}-\mu+1} + a_1^{\pm} s^{-\frac{n}{2}-\mu+2} + \cdots, \quad \pm s \searrow 0. \tag{9.20}$$

Proof. The last integral in (9.17) is a multiple of $\hat{w}(\xi)$, with $\xi = 1/2s$. Here, $w(\tau) = \overline{f}_0(\sqrt{\tau})\tau^{\frac{n}{2}-1}$ has compact support in $(0, \infty)$, and a simple singularity, at $\tau = a^2$, and (9.20) follows from the asymptotic behavior of $\hat{w}(\xi)$, as $\xi \to \pm\infty$.

The behavior (9.20) clearly implies (9.19) when $\mu + \frac{n}{2} - 1 < 1$. For the general case, write $f = \Delta^k g + h$, with k large and g, h compactly supported, vanishing near 0, and satisfying $\overline{g}_0, \overline{h}_0 \in I^{\mu-2k}(\mathbb{R}^+, \{a\})$, and use the identity

$$e^{-is\Delta/2} \Delta^k g(0) = \Big(-\frac{2}{i}\frac{\partial}{\partial s}\Big)^k e^{-is\Delta/2} g(0). \tag{9.21}$$

If in Lemma 9.2 the hypothesis that f vanishes near 0 is changed to f smooth near 0, then we can write

$$e^{-is\Delta/2} f(0) = \Psi(s) e^{-ib/s} + \Phi(s), \quad \Phi(0) = f(0), \tag{9.22}$$

where Ψ is as above and Φ is smooth. This follows by writing $f = g + h$, with $h \in C_0^{\infty}(\mathbb{R}^n)$ and g vanishing near the origin, and noting that $e^{-is\Delta/2} h$ is smooth.

The following complements Lemma 9.2:

Lemma 9.3. *Assume that f has compact support, that*

$$F(x) = \int_{SO(n)} f(gx)\, dg \tag{9.23}$$

is C^{∞} on $\mathbb{R}^n \setminus 0$, and that

$$\overline{F}_0(r) \sim b_0 r^{-\mu} + b_1 r^{-\mu+1} + \cdots, \quad r \searrow 0, \tag{9.24}$$

for some $\mu < n$. Then

$$e^{-is\Delta/2} f(0) = \Psi(s), \tag{9.25}$$

where

$$\Psi \in I^{\frac{\mu}{2}}(\mathbb{R}, \{0\}) \tag{9.26}$$

has the behavior

$$\Psi(s) \sim a_0 s^{-\frac{\mu}{2}} + a_1 s^{-\frac{\mu}{2}+1} + \cdots, \quad s \to 0. \tag{9.27}$$

Proof. We have (9.17), and $\overline{f}_0 = \overline{F}_0$. Hence, as $\tau \searrow 0$,

$$\overline{f}_0(\sqrt{\tau}) \sim b_0 \tau^{-\frac{\mu}{2}} + \cdots, \tag{9.28}$$

and if we set $w(\tau) = \overline{f}_0(\sqrt{\tau})\tau^{\frac{n}{2}-1}$, for $\tau > 0$, zero for $\tau < 0$, then $w \in I^{\frac{\mu}{2}-\frac{n}{2}+1}(\mathbb{R}, \{0\})$, provided $\mu < n$. It follows that

$$\hat{w}(\xi) \sim c_0 \xi^{\frac{\mu}{2}-\frac{n}{2}} + \cdots, \quad \xi \to \infty, \tag{9.29}$$

which implies (9.27), and hence (9.26).

Of course, by translation invariance of the Laplace operator Δ, we have similar results on $e^{-is\Delta/2} f(x)$, for any $x \in \mathbb{R}^n$. We can apply this to an analysis of the following type of function. Pick $a > 0$, $p \in \mathbb{R}^n$. Let $g \in C^\infty\big(\overline{B_a(p)}\big)$, where $B_a(p) = \{x \in \mathbb{R}^n : |x-p| < a\}$. As in (2.19), set

$$\begin{aligned} f(x) = g(x), &\quad x \in B_a(p) \\ \tfrac{1}{2} g(x), &\quad x \in \partial B_a(p) \\ 0, &\quad x \notin \overline{B_a(p)}. \end{aligned} \tag{9.30}$$

We have from Lemmas 9.2-9.3 that

$$e^{-is\Delta/2} f(p) = \Psi(s,p) e^{-ib(p)/s} + \Phi(s,p), \tag{9.31}$$

where $\Phi(s,p)$ is smooth in s and $\Psi(s,p)$ has the form (9.20), with

$$\mu = \mu(p) = 0. \tag{9.32}$$

In addition,

$$e^{-is\Delta/2} f(x) = \sum_{j=1}^{2} \big[\Psi_j(s,x) e^{ib_j(x)/s} + \Phi_j(s,x)\big], \quad x \notin \partial B_a(p) \cup \{p\}. \tag{9.33}$$

where, for each such x, $\Phi_j(s,x)$ is smooth in s, $b_j(x) = a_j(x)^2/2$ where $a_j(x)$ are the critical values of $|x-y|$, $y \in \partial B_a(p)$, and $\Psi_j(s,x)$ have the form (9.20), with

$$\mu = \mu_j(x) = -\tfrac{1}{2}(n-1), \quad x \notin \partial B_a(p) \cup \{p\}. \tag{9.34}$$

Finally,

$$e^{-is\Delta/2}f(x)=\Psi_1(s,x)+\Psi_2(s,x)e^{ib_2(x)/s}+\Phi_2(s,x),\quad x\in\partial B_a(p), \tag{9.35}$$

where $\Phi_2(s,x)$ is smooth in s, $b_2(x)=2a^2$, and $\Psi_2(s,x)$ has the form (9.20), with

$$\mu=\mu_2(x)=-\tfrac{1}{2}(n-1),\quad x\in\partial B_a(p), \tag{9.36}$$

and $\Psi_1(s,x)$ has the form (9.27), with

$$\mu=\mu_1(x)=0. \tag{9.37}$$

In other words, for $x\in\partial B_a(p)$,

$$\begin{aligned}\Psi_1(s,x)&\sim a_0(x)+a_1(x)s+\cdots,\\ \Psi_2(s,x)&\sim a_0'(x)s^{\frac{1}{2}}+a_1'(x)s^{\frac{3}{2}}+\cdots.\end{aligned} \tag{9.38}$$

Incorporating this analysis into (9.15)-(9.16), we have the following.

Proposition 9.4. *Let f be given by (9.30), with $g\in C^\infty\big(\overline{B_a(p)}\big)$. Then, for any n,*

$$S_M'f(0)\longrightarrow f(0)\quad \text{as}\quad M\to\infty, \tag{9.39}$$

provided $p\neq 0$.

Proof. Making use of the estimates above, we see that, for $|t|\le\pi/4$,

$$\Big|e^{it(H-n)/2}f(0)-f(0)\Big|\le C|t|^\lambda, \tag{9.40}$$

where

$$\begin{aligned}|p|\neq a,\ p\neq 0&\Longrightarrow\lambda=\frac{n}{2},\\ |p|=a&\Longrightarrow\lambda=\min\big(1,\frac{n}{2}\big).\end{aligned} \tag{9.41}$$

Similarly, we obtain estimates of $\big|e^{it(H-n)/2}f(0)\big|$ on the rest of $\mathbb{R}/(2\pi\mathbb{Z})$. Together, these estimates show that Proposition 9.1 (or rather its analogue with L replaced by $-(H-n)/2$) is applicable, and (9.39) follows.

Note that, complementing (9.41), we have

$$p=0\Longrightarrow\lambda=-\frac{n}{2}+1. \tag{9.42}$$

Thus, in this case, we can also apply Proposition 9.1 to get (9.39), when $n=1$. However, the integrability of $t^{-1}\big|e^{it(H-n)/2}f(0)-f(0)\big|$ fails in the case $n=2$. On the other hand, it follows from the results of Appendix C that the condition for (9.6) to converge to $f(x)$, when $x=p=0$, holds here, if $n=2$, but not if $n\ge 3$. Thus, complementary to Proposition 9.4, we have:

Proposition 9.5. *Let f be given by (9.30), with $g \in C^\infty\big(\overline{B_a(p)}\big)$, and $p = 0$. Then, as $M \to \infty$,*

$$S'_M f(0) \to f(0) \tag{9.43}$$

for all such functions, if and only if $n \le 2$.

We now extend our study to that of $e^{itH/2} f(x)$ for $x \neq 0$. For t close to 0, it is convenient to use (9.12)-(9.13) to write

$$e^{itH} f(x) = (\cos t)^{-\frac{n}{2}} e^{-is\Delta/2} (E_{\beta(t)x} f)(x), \tag{9.44}$$

where

$$E_z f(y) = e^{\frac{1}{2} iz\cdot y} f(y). \tag{9.45}$$

Hence

$$e^{itH/2} f(x) = (\cos t)^{-\frac{n}{2}} e^{-is\Delta/2} f(x) + (\cos t)^{-\frac{n}{2}} \sum_{k=1}^{\infty} \frac{i^k}{k! 2^k} \big[e^{-is\Delta/2} \varphi^{k,x}(x)\big] \beta(t)^k, \tag{9.46}$$

where

$$\varphi^{k,x}(y) = (x \cdot y)^k f(y). \tag{9.47}$$

As before, $s = \tan t$ and $\beta(t) = 2 \tan \frac{1}{2} t$. As long as f has compact support, the series (9.46) is convergent and has good asymptotic properties as $t \to 0$, in view of the elementary estimate

$$\|e^{-is\Delta/2} \varphi\|_{L^\infty} \le C |s|^{-\frac{n}{2}} \|\varphi\|_{L^1}. \tag{9.48}$$

In particular, if $f \in L^1(\mathbb{R}^n)$ has compact support, the sum over $k > [n/2]$ in (9.46) belongs to $L^1\big([-\frac{\pi}{4}, \frac{\pi}{4}]\big)$, in t, for each $x \in \mathbb{R}^n$.

Having the analysis above for $e^{it(H-n)/2} f(x)$ for t near zero, we need to consider t in the rest of $\mathbb{R}/(2\pi\mathbb{Z})$. Now, if f has compact support, this is smooth for t away from $\{k\pi : k \in \mathbb{Z}\}$, so it remains to analyze it for t near π. This can be done using

$$e^{i\pi(H-n)/2} f(x) = f(-x), \tag{9.49}$$

hence

$$e^{it(H-n)/2} f(x) = e^{i(t-\pi)(H-n)/2} f(-x). \tag{9.50}$$

Thus we are reduced to the previous analysis, with $f(y)$ replaced by $\check{f}(y) = f(-y)$. Thus, we have the following result.

Theorem 9.6. *Let $f \in L^1(\mathbb{R}^n)$ have compact support, and fix $x \in \mathbb{R}^n$. Then $S'_M f(x) \to f(x)$ as $M \to \infty$ if and only if*

$$C(M,x) = A_0(M,x) + \sum_{k=1}^{[n/2]} A_k(M,x) + B_0(M,x) + \sum_{k=1}^{[n/2]} B_k(M,x) \tag{9.51}$$

satisfies $C(M,x) \to 0$ as $M \to \infty$, where, with $s(t) = \tan t$, $\beta(t) = 2\tan \frac{1}{2}t$,

$$A_0(M,x) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin(M+\frac{1}{2})t}{\sin \frac{1}{2}t} \left[e^{-int/2}(\cos t)^{-\frac{n}{2}} e^{-is\Delta/2} f(x) - f(x)\right] dt, \tag{9.52}$$

and, for $k \geq 1$,

$$A_k(M,x) = \frac{i^k}{k!2^k} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin(M+\frac{1}{2})t}{\sin \frac{1}{2}t} e^{-int/2}(\cos t)^{-\frac{n}{2}} \beta(t)^k e^{-is(t)\Delta/2} \varphi^{k,x}(x)\, dt, \tag{9.53}$$

with

$$\varphi^{k,x}(y) = (x \cdot y)^k f(y), \tag{9.54}$$

and

$$B_0(M,x) = \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\sin(M+\frac{1}{2})t}{\sin \frac{1}{2}t} e^{-in(t-\pi)/2}(\cos t)^{-\frac{n}{2}} e^{-is(t-\pi)\Delta/2} f(-x)\, dt, \tag{9.55}$$

and, for $k \geq 1$,

$$\begin{aligned} B_k(M,x) = \frac{i^k}{k!2^k} \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} & \frac{\sin(M+\frac{1}{2})t}{\sin \frac{1}{2}t} e^{-in(t-\pi)/2}(\cos t)^{-\frac{n}{2}} \\ & \times \beta(t-\pi)^k e^{-is(t-\pi)\Delta/2} \psi^{k,x}(x)\, dt, \end{aligned} \tag{9.56}$$

where

$$\psi^{k,x}(y) = (x \cdot y)^k f(-y). \tag{9.57}$$

Note how the behavior of various functions at $-x$ plays a role in Theorem 9.6, as well as behavior at x. Compare Theorem 5.4.3 in [Th], discussing Riesz means of Hermite expansions.

Using Theorem 9.6 and the estimates in Lemmas 9.2-9.3, and in Appendix C, we have the following, complementing Propositions 9.4-9.5.

Proposition 9.7. *Let f be given by (9.30), with $g \in C^\infty\big(\overline{B_a(p)}\big)$. Then, as $M \to \infty$,*

$$n \le 2 \Longrightarrow S'_M f(x) \to f(x), \quad \forall x \in \mathbb{R}^n, \tag{9.58}$$

$$n \le 4 \Longrightarrow S'_M f(x) \to f(x), \quad \forall x \ne p, \tag{9.59}$$

$$n \ge 5 \Longrightarrow S'_M f(x) \to f(x), \quad \forall x \notin \{p, -p\}. \tag{9.60}$$

Proof. We do the analysis for $x \notin \partial B_a(p)$; similar arguments, making use of Lemma 9.3, work for $x \in \partial B_a(p)$.

For such f, as described in Proposition 9.7, the quantity $A_0(M,x)$ in (9.52) has the form

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin(M+\frac{1}{2})t}{\sin\frac{1}{2}t}\, e^{-int/2}(\cos t)^{-\frac{n}{2}}\Psi\big(s(t)\big)e^{-ib/s(t)}\, dt, \tag{9.61}$$

plus smaller terms. Here, by Lemma 9.2, supplemented by (9.22), $\Psi(s)$ has the form

$$\Psi(s) \sim a_0^{\pm} s^{-\frac{n}{2}-\mu+1} + \cdots, \quad \pm s \searrow 0, \tag{9.62}$$

More precisely, if $x \ne p$, one has a sum of two terms of this form, with different values of b, the two critical values of $|x-y|^2/2$, as y runs over $\partial B_a(p)$, and $\mu = -(n-1)/2$. If $x = p$, one has one term, and $b = a^2/2$, and $\mu = 0$.

Taking account of the factor $\sin\frac{1}{2}t$ in the denominator in (9.62), and applying the results of Appendix C, we have for (9.61) the asymptotic behavior

$$\begin{aligned} &\sim A\,\hat{\omega}_{-\frac{n}{2}-\mu}(bM) \\ &= Ac_n\, M^{\frac{n}{4}+\frac{\mu}{2}-\frac{1}{2}}\, J_{\frac{n}{2}+\mu-1}(2\sqrt{bM}) \\ &\sim Ac'_n\, M^{\frac{n}{4}+\frac{\mu}{2}-\frac{3}{4}}\, \cos\big(2\sqrt{bM} - \tfrac{1}{2}(\tfrac{n}{2}+\mu-1) - \tfrac{1}{4}\pi\big), \end{aligned} \tag{9.63}$$

plus lower order terms. For general g, the coefficient A will be nonzero.

As mentioned, if $x \ne p$, then $\mu = -(n-1)/2$, so the factor of M to a power in the last line of (9.63) is $M^{-\frac{1}{2}}$. The terms $A_k(M,x)$ in (9.53) are all of lower order.

If $x = p$, then $\mu = 0$ in (9.63), so the factor of M to a power is $M^{(n-3)/4}$. This vanishes as $M \to \infty$ if $n \le 2$. When $A \ne 0$, the term of the form (9.63), dominates all the other terms in (9.51) (at $x = p$), so if $n \ge 3$, then for general g there is not convergence at $x = p$.

A similar analysis applies to the terms (9.55)-(9.56). One difference is that $\sin\frac{1}{2}t$ does not vanish at $t = \pi$, so instead of (9.63), the principal terms have the form

$$\sim \hat{\omega}_{-\frac{n}{2}-\mu+1}(bM) \sim Bc''_n\, M^{\frac{n}{4}+\frac{\mu}{2}-\frac{5}{4}}\, \cos\Big(2\sqrt{bM} - \tfrac{1}{2}(\tfrac{n}{2}+\mu-2) - \tfrac{1}{4}\pi\Big), \tag{9.64}$$

plus lower order terms. Again, for general g, the coefficient B is nonzero. This time, if $x \ne -p$, then $\mu = -(n-1)/2$, so (9.64) tends to zero. If $x = -p$, then $\mu = 0$, and this term tends to zero if and only if $n \le 4$. Other terms in (9.55)-(9.56) are of lower order, so the analysis of (9.58)-(9.60) is done.

We remark that, as a consequence of the proof of Lemma 9.2, the following results hold. In the asymptotic analysis of $A_0(x,M)$ in (9.63), at $x=p$, the factor A is proportional to $\overline{f}_p(a)$. Similarly, in the asymptotic analysis of $B_0(x,M)$ in (9.64), at $x=-p$, the factor B is proportional to $\overline{f}_p(a)$.

We mention a couple of specific examples, to illustrate Proposition 9.7. If f is the characteristic function of a ball centered at p in $\mathbb{R}^3$, re-defined on the boundary as in (9.30), then $S'_M f(x) \to f(x)$ for all $x \neq p$. If we consider $S'_M f(p)$, then (9.63) applies, with $n=3,\ \mu=0$. Hence, $S'_M f(p)$ has bounded oscillatory divergence, of a nature very similar to that considered in (2.13)-(2.15).

Suppose on the other hand that f is the characteristic function of a ball centered at p in $\mathbb{R}^5$, re-defined as usual on the boundary of the ball. Then, by (9.63), with $n=5,\ \mu=0$, we see that $S'_M f(p)$ blows up as $M \to \infty$. If $p \neq 0$, then the dominant part of (9.51) at $x=-p$ is given by (9.64). We have $n=5,\ \mu=0$ there; hence $S'_M f(-p)$ has bounded oscillatory divergence.

10. Nonspherical Fourier inversion on $\mathbb{R}^n$

In this section we study some phenomena regarding Fourier inversion on $\mathbb{R}^n$, that arise when S_R is replaced by the following. Let $\mathcal{C}$ be a smoothly bounded, strongly convex set in $\mathbb{R}^n$, which is symmetric, i.e., $x \in \mathcal{C} \Rightarrow -x \in \mathcal{C}$. We define

$$\widetilde{S}_{R\mathcal{C}} f(x) = (2\pi)^{-n} \int\limits_{\{R^{-1}\xi \in \mathcal{C}\}} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{10.1}$$

We can analyze this in a way parallel to (2.2). In fact, define $\lambda \in C^\infty(\mathbb{R}^n \setminus 0)$ to have the properties

$$\begin{gathered} \lambda \ \text{ is homogeneous of degree one in } \ \xi, \\ \lambda(\xi) = 1 \ \text{ for } \ \xi \in \partial\mathcal{C}. \end{gathered} \tag{10.2}$$

Then we have

$$\widetilde{S}_{R\mathcal{C}} f(x) = \chi_R\big(\lambda(D)\big) f(x). \tag{10.3}$$

Parallel to (1.1)-(1.3), we can write

$$\widetilde{S}_{R\mathcal{C}} f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rt}{t}\, u(t,x)\, dt, \tag{10.4}$$

where

$$u(t,x) = e^{it\lambda(D)} f(x). \tag{10.5}$$

In this case, $\lambda(D)$ is a pseudodifferential operator, of order 1, with real symbol, and $u(t,x)$ solves the pseudodifferential equation

$$\frac{\partial u}{\partial t} = i\lambda(D)u, \quad u(0,x) = f(x). \tag{10.6}$$

As in (1.1), we could replace $e^{it\lambda(D)}$ by $\cos t\lambda(D)$ in (10.5), and we will find it convenient to do so below. In general, solutions to $u_{tt} + \lambda(D)^2 u = 0$ do not enjoy finite propagation speed, unless $\lambda(D)^2$ is actually a differential operator.

Now, results from microlocal analysis are applicable to (10.6). We can write

$$u(t,x) = \rho_t * f(x), \quad \rho_t(x) = e^{it\lambda(D)}\,\delta(x), \tag{10.7}$$

where ρ_t is a smooth function of t with values in $\mathcal{S}'(\mathbb{R}^n)$. Since $t\lambda(D) = \lambda(tD)$ for $t > 0$, we have

$$\rho_t(x) = t^{-n}\rho_1(x/t), \quad \rho_{-t}(x) = t^{-n}\rho_{-1}(x/t), \tag{10.8}$$

for $t > 0$. Furthermore, the theory of Fourier integral operators implies

$$\rho_{\pm 1} \in I^{\frac{1}{2}(n+1)}(\mathbb{R}^n, \partial\mathcal{B}), \tag{10.9}$$

where $\mathcal{B}$ is the (open) convex subset of $\mathbb{R}^n$ *dual* to $\mathcal{C}$. This is due to the fact that, under our hypotheses,

$$\partial\mathcal{B} = \{\nabla\lambda(\xi) : \xi \in \partial\mathcal{C}\} = \{-\nabla\lambda(\xi) : \xi \in \partial\mathcal{C}\}. \tag{10.10}$$

From (10.8) it clearly follows that, whenever f has compact support, the decay property (1.6) holds. Thus the analysis of $\widetilde{S}_{RC}f(x)$ is localizable, and (1.7)-(1.10) are effective in this analysis.

We concentrate on the following case. Take $g \in C^\infty(\mathbb{R}^n)$, and set

$$f(x) = \begin{cases} g(x), & x \in \mathcal{B}, \\ \frac{1}{2}g(x), & x \in \partial\mathcal{B}, \\ 0, & x \notin \overline{\mathcal{B}}. \end{cases} \tag{10.11}$$

Thus $f \in I^0(\mathbb{R}^n, \partial\mathcal{B})$. We see that $u(t,x)$, given by (10.5), has a perfect focus caustic at $x = 0$, $t = \pm 1$. Away from there, it belongs to $I^0(\mathbb{R}^n, \Sigma_1) + I^0(\mathbb{R}^n, \Sigma_2)$, where Σ_j are swept out by $\partial\mathcal{B}$, by the bicharacteristic flow for $\partial_t - i\lambda(D)$, in the usual manner of geometrical optics. We therefore have the following.

Proposition 10.1. *Let f be given by (10.11), with $g \in C^\infty(\mathbb{R}^n)$. Then, as $R \to \infty$,*

$$n \le 2 \Longrightarrow \widetilde{S}_{RC}f(x) \to f(x), \quad \forall\, x \in \mathbb{R}^n, \tag{10.12}$$

$$n \ge 3 \Longrightarrow \widetilde{S}_{RC}f(x) \to f(x), \quad \forall\, x \ne 0. \tag{10.13}$$

Proof. The discussion above yields everything but the stated convergence at $x \in \partial\mathcal{B}$. For this, it suffices to establish the integrability of

$$\frac{\cos t\lambda(D)\, f(x) - f(x)}{t} \tag{10.14}$$

near $t = 0$. When $\cos t\lambda(D) f$ is calculated for small t by the progressing wave expansion of geometrical optics, the symmetry $\lambda(\xi) = \lambda(-\xi)$ implies that the principal terms will have simple jumps across characteristic surfaces, and will lack log terms. Since $\cos t\lambda(D)\, f(x)$ is an even function of t, for fixed x, there cannot be a nonzero jump at $t = 0$; hence, for each $x \in \partial\mathcal{B}$, $\cos t\lambda(D)\, f(x)$ is at least log-Lipschitz in t near $t = 0$. This observation suffices to establish the integrability of (10.14) near $t = 0$, and hence complete the proof.

It is desirable to capture more precisely the principal singularity of the distributions $\rho_{\pm 1}$. One way ths can be done is via a stationary phase analysis of the radial Fourier transform of these distributions, cut off away from their singular support $\partial\mathcal{B}$. We have

$$\rho_{\pm 1}(x) = (2\pi)^{-n} \int e^{\pm i\lambda(\xi) + ix\cdot\xi}\, d\xi, \tag{10.15}$$

so, with $x = r\omega,\ \omega \in S^{n-1}$, consider

$$\begin{aligned} \sigma_{\pm}(\omega, \tau) &= \iint e^{\pm i\lambda(\xi) + ir\omega\cdot\xi - ir\tau}\, a(r)\ dr\, d\xi \\ &= \tau^n \iint e^{i\tau[\pm\lambda(\zeta) + r\omega\cdot\zeta - r]}\, a(r)\ dr\, d\zeta. \end{aligned} \tag{10.16}$$

Choose $a \in C_0^\infty(\mathbb{R}^+)$, equal to 1 on a neighborhood of $\{r : r\omega \in \partial\mathcal{B} \text{ for some } \omega \in S^{n-1}\}$. The phase function $\varphi_{\pm}(r, \zeta) = \pm\lambda(\zeta) + r\omega\cdot\zeta - r$ has (for given ω and given choice of sign) a critical point at $r_0 = r_{\pm}(\omega)$, $\zeta_0 = \zeta_{\pm}(\omega)$, determined by

$$\nabla\lambda(\zeta_0) = \mp r_0\omega, \quad \omega\cdot\zeta_0 = 1. \tag{10.17}$$

Let us note that, if $\mu(\xi)$ is another real function in $C^\infty(\mathbb{R}^n \setminus 0)$, homogeneous of degree 1 in ξ, and if μ and λ agree to third order at ζ_0, then the quantity obtained by substituting μ for λ in (10.16) has the same leading asymptotic behavior as $\tau \to \infty$ as does $\sigma_{\pm}(\omega, \tau)$, for the fixed ω for which (10.17) holds.

This observation enables us to compare $\rho_{\pm 1}$ with $e^{\pm i\sqrt{-\Delta}}\,\delta(x)$, as follows. For each $\zeta_0 \in \partial\mathcal{C}$, we can find an ellipsoid $\mathcal{E}$, centered at the origin, tangent to $\partial\mathcal{C}$ at ζ_0, and having the same second fundamental form as $\partial\mathcal{C}$ at ζ_0. This ellipsoid is then a level set $\{\xi : Q_{\zeta_0}(\xi) = 1\}$ for a quadratic form Q_{ζ_0}, and we see that $\lambda(\xi)$ and $\sqrt{Q_{\zeta_0}(\xi)}$ agree to third order at ζ_0. Now

$$q_{\pm 1} = e^{\pm i\sqrt{Q_{\zeta_0}(D)}}\,\delta(x) \tag{10.18}$$

is singular on the dual ellipsoid $\widetilde{\mathcal{E}}$, which is tangent to $\partial\mathcal{B}$ at $x_0 = \nabla\lambda(\zeta_0) = r_0\omega$ (and at $-x_0$), and the radial Fourier transforms of $q_{\pm 1}(r\omega)$ and of $\rho_{\pm 1}(r\omega)$ have the same leading terms. Since $q_{\pm 1}(x)$ is obtained from $e^{\pm i\sqrt{-\Delta}}\,\delta(x)$ via a linear change of coordinates, we have the following.

Proposition 10.2. *If $n = 2k+1$, then there exists a positive $A \in C^\infty(\partial\mathcal{B})$ such that*

$$\tfrac{1}{2}(\rho_1 + \rho_{-1}) = A(x)(r\partial_r)^k \omega_{\partial\mathcal{B}} \mod I^{\frac{1}{2}(n-1)}(\mathbb{R}^n, \partial\mathcal{B}), \tag{10.19}$$

where $\omega_{\partial\mathcal{B}}$ is surface area on $\partial\mathcal{B}$.

Corollary 10.3. *If f is as in Proposition 10.1, with $g > 0$ on $\overline{\mathcal{B}}$, then $\widetilde{S}_{R\mathcal{C}} f(0)$ is divergent, as $R \to \infty$, whenever $n \geq 3$ is odd.*

Proof. Say $n = 2k+1$, $k \geq 1$. Using (10.8) and (10.19), we see that the leading singularity near $t = 1$ of $\frac{1}{2}\langle \rho_t + \rho_{-t}, f\rangle$ is given by

$$(-1)^k \int_{\partial\mathcal{B}} A(x)\,(r\partial_r)^k f(tx)\ dS(x) = (-\partial_t)^k \int_{\partial\mathcal{B}} A(x) f(tx)\ dS(x). \tag{10.20}$$

This last integral has a jump at $t = 1$, of magnitude

$$\alpha(g) = \int_{\partial\mathcal{B}} A(x) g(x)\ dS(x). \tag{10.21}$$

As long as this is nonzero, we see that, as $R \to \infty$,

$$\widetilde{S}_{R\mathcal{C}} f(0) = c_n \alpha(g)\, R^{k-1}\ \cos R + o(R^{k-1}). \tag{10.22}$$

There is a similar treatment for n even, whose details we omit.

11. Gibbs phenomena on manifolds

Let M be a strongly scattering manifold, as in §8. Let $\overline{\Omega}$ be a compact region in M, with smooth boundary, $g \in C^\infty(\overline{\Omega})$, and define f as follows:

$$\begin{aligned} f(x) = \ & g(x), & x \in \Omega, \\ & \tfrac{1}{2} g(x), & x \in \partial\Omega, \\ & 0, & x \notin \overline{\Omega}. \end{aligned} \tag{11.1}$$

The pointwise behavior of $S_R f(x)$ has been described in Proposition 8.2 and the remarks following it. Here we want to examine the nature of the convergence on a neighborhood of $\partial\Omega$, obtaining a precise description of the analogue in this case of the well known Gibbs phenomenon for 1-dimensional Fourier analysis, i.e., when $M = \mathbb{R}$, $\Omega = (a, b)$.

The Gibbs phenomenon is controlled by the behavior of the integrand in (1.3) for small $|t|$. We will thus analyze

$$S_R^\beta f(x) = \frac{1}{\pi} \int \frac{\sin\ Rt}{t}\, u(t,x)\, \beta(t)\ dt, \tag{11.2}$$

with $\beta \in C_0^\infty(\mathbb{R})$, $\beta(t) = 1$ for $|t| \le a$, 0 for $|t| \ge 2a$. We pick $a > 0$ sufficiently small that the progressing wave expansion of geometrical optics provides a parametrix for $u(t,x) = \cos tA\, f(x)$, for $|t| \le 2a$, so in particular there are no caustics in this region.

Thus, for $|t| \le 2a$ and x in a neighborhood $\mathcal{O}$ of $\partial\Omega$, we can write

$$
\begin{aligned}
u(t,x) = A_0(t,x)\chi_+\bigl(t-\psi(x)\bigr) &+ \sum_{j=1}^{k} A_j(t,x)\chi_+^j\bigl(t-\psi(x)\bigr) \\
+ B_0(t,x)\chi_+\bigl(t+\psi(x)\bigr) &+ \sum_{j=1}^{k} B_j(t,x)\chi_+^j\bigl(t+\psi(x)\bigr) + R_k(t,x),
\end{aligned}
\tag{11.3}
$$

with terms having the following nature. The function χ_+ is the characteristic function of $\mathbb{R}^+$, and χ_+^j are as in (8.6). The function $\psi(x)$ solves the eikonal equation near $\partial\Omega$:

$$
|\nabla\psi(x)|^2 = 1, \quad \psi(x) = 0 \text{ for } x \in \partial\Omega. \tag{11.4}
$$

We take ψ to be positive in Ω; then ψ is uniquely specified, and $-\psi$ also solves (11.4). The functions $A_j(t,x)$ and $B_j(t,x)$ are C^∞ and are obtained by solving certain transport equations. The remainder R_k has the property

$$
R_k \in C^k([-2a,2a] \times \mathcal{O}). \tag{11.5}
$$

We remark that comparing jumps across $\partial\Omega$ at $t = 0$ yields the relation

$$
A_0(t,x) - B_0(t,x) = g(x), \quad x \in \partial\Omega. \tag{11.6}
$$

Let us look at the contribution of the first term on the right side of (11.3) to S_R^β. We have

$$
A_R^0 f(x) = \frac{1}{\pi} \int_{\psi(x)}^{\infty} \frac{\sin Rt}{t}\, A_0(t,x)\beta(t)\; dt. \tag{11.7}
$$

Writing $A_0(t,x) = A_0(x) + tA_{01}(t,x)$, we have

$$
A_R^0 f(x) = A_R^{00} f(x) + A_R^{01} f(x), \tag{11.8}
$$

with

$$
A_R^{00} f(x) = \frac{A_0(x)}{\pi} \int_{\psi(x)}^{\infty} \frac{\sin Rt}{t} \beta(t)\; dt. \tag{11.9}
$$

Suppose the neighborhood $\mathcal{O}$ of $\partial\Omega$ is sufficiently small that $|\psi(x)| \le a$ for $x \in \mathcal{O}$. Then

$$
A_R^{00} f(x) = \frac{A_0(x)}{\pi} \int_{\psi(x)}^{\infty} \frac{\sin Rt}{t}\; dt + O(R^{-\infty}), \tag{11.10}
$$

uniformly for $x \in \mathcal{O}$, since the difference between the integrals in (11.9) and (11.10) is equal to $\int_0^\infty (\sin Rt)t^{-1}[1-\beta(t)]\,dt$ and $t^{-1}\chi_+(t)[1-\beta(t)] \in S^{-1}_{1,0}(\mathbb{R})$.

We can express this in terms of the special function

$$G(\tau) = \frac{2}{\pi}\int_0^\tau \frac{\sin t}{t}\,dt. \tag{11.11}$$

A residue calculation shows that $G(\tau) \to 1$ as $\tau \to \infty$, so (11.10) implies

$$A_R^{00} f(x) = \tfrac{1}{2}\bigl[1 - G\bigl(R\psi(x)\bigr)\bigr] A_0(x) + O(R^{-\infty}). \tag{11.12}$$

As is well known, the graph of $G(\tau)$ gives the prototypical profile of the Gibbs phenomenon. While $G(\tau) \to 1$ as $\tau \to +\infty$, we have

$$G_{\max} = G(\pi) \approx 1.178979744472167\cdots. \tag{11.13}$$

Similarly $G(\tau) \to -1$ as $\tau \to -\infty$, and $G_{\min} = G(-\pi) = -G(\pi)$. This extra 17% on the loci $R\psi(x) = \pm\pi$ is the contribution of $A_R^{00} f(x)$ to the Gibbs phenomenon in our case.

The other term in (11.8) is

$$A_R^{01} f(x) = \frac{1}{\pi}\int_{\psi(x)}^\infty A_{01}(t,x)\beta(t)\sin Rt\,dt. \tag{11.14}$$

Clearly

$$A_R^{01} f(x) = O(R^{-1}), \tag{11.15}$$

uniformly on $\mathcal{O}$.

The next terms in $S_R^\beta f(x)$ arising from (11.3) have the form

$$A_R^j f(x) = \frac{1}{\pi}\int \frac{\sin Rt}{t}\,A_j(t,x)\chi_+^j\bigl(t-\psi(x)\bigr)\beta(t)\,dt, \tag{11.16}$$

for $1 \le j \le k$. This is equal to

$$S_R\Psi_{j,x}(0), \tag{11.17}$$

where

$$\Psi_{j,x}(t) = A_j(t,x)\chi_+^j\bigl(t-\psi(x)\bigr)\beta(t). \tag{11.18}$$

These functions are all uniformly Lipschitz in t, with support in $-2a \le t \le 2a$. Hence, as $R \to \infty$, (11.17) converges to $\Psi_{j,x}(0)$, uniformly for $x \in \mathcal{O}$, i.e.,

$$A_R^j f(x) \to A_j(0,x)\chi_+^j\bigl(-\psi(x)\bigr), \text{ uniformly for } x \in \mathcal{O}. \tag{11.19}$$

We can make a similar analysis of the remaining terms in $S_R^\beta f(x)$ arising from (11.3). In particular, we have

$$\begin{aligned} B_R^{00} f(x) &= \frac{B_0(x)}{\pi}\int_{-\psi(x)}^\infty \frac{\sin Rt}{t}\beta(t)\,dt \\ &= \tfrac{1}{2}\bigl[1 + G\bigl(R\psi(x)\bigr)\bigr] B_0(x) + O(R^{-\infty}). \end{aligned} \tag{11.20}$$

All other terms in $S_R^\beta f(x)$ arising from (11.3) converge uniformly on $\mathcal{O}$, as $R \to \infty$.

Note that

$$A_R^{00} f(x) + B_R^{00} f(x) = \tfrac{1}{2}[A_0(x) + B_0(x)] - \tfrac{1}{2}G\bigl(R\psi(x)\bigr)[A_0(x) - B_0(x)] + O(R^{-\infty}). \tag{11.21}$$

We have the following conclusion.

Proposition 11.1. *Let $\overline{\Omega}$ be a compact region with smooth boundary in a strongly scattering manifold M. Let f be given by (11.1), with $g \in C^\infty(\overline{\Omega})$. Then $\partial\Omega$ has a neighborhood $\mathcal{O}$ such that, as $R \to \infty$,*

$$S_R^\beta f(x) = -\tfrac{1}{2} G\big(R\psi(x)\big)\,[A_0(x) - B_0(x)] + S_R^\# f(x), \tag{11.22}$$

for $x \in \mathcal{O}$, and $S_R^\# f(x)$ converges uniformly as $R \to \infty$.

Recall from (11.6) that $A_0(x) - B_0(x) = g(x)$ for $x \in \partial\Omega$.

Considerations of propagation of singularities imply that $S_R^\beta f(x)$ coverges uniformly on any compact subset of $M \setminus \partial\Omega$. The analysis of $S_R f(x) - S_R^\beta f(x)$ is amenable to the methods of §8. One has

$$S_R f(x) - S_R^\beta f(x) \to 0, \tag{11.23}$$

uniformly, on any compact set disjoint from the set $\mathcal{K}$ of caustic points of order ≥ 1.

There are a number of variants of Proposition 11.1 which can be treated by a similar wave equation analysis. For one, we can suppose Ω is a domain with smooth boundary in a sphere S^n. We take $g \in C^\infty(\overline{\Omega})$ and again define f by (11.1). This time, $S_N f(x)$ is given by (5.7) if n is odd, or more generally by (5.30). More generally, instead of S^n, we can take a compact rank one symmetric space, as in §6, and also have such a formula for $S_N f(x)$. In this case, the Gibbs phenomenon arises from the behavior of

$$S_N^\beta f(x) = \frac{1}{4\pi} \int \frac{\sin \frac{1}{2}(N + \frac{1}{2})t}{\sin \frac{1}{4} t}\, u(t,x)\, \beta(t)\, dt, \tag{11.24}$$

where, as above, $\beta \in C_0^\infty(\mathbb{R})$, $\beta(t) = 1$ for $|t| \leq a$, 0 for $|t| \geq 2a$, and we take $a << \pi$. An analysis parallel to that in (11.3)-(11.21) holds here, so we have:

Proposition 11.2. *Let Ω be a region with smooth boundary in a compact rank one symmetric space X. Let f be given by (11.1), with $g \in C^\infty(\overline{\Omega})$. Then $\partial\Omega$ has a neighborhood $\mathcal{O}$ such that, as $R = \frac{1}{2}N \to \infty$, (11.22) holds, and $S_R^\# f(x)$ converges uniformly.*

As above, $S_N f(x) - S_N^\beta f(x) \to 0$ uniformly on every compact subset of X disjoint from the set $\mathcal{K}$ of caustic points of order ≥ 1.

We mention that the Gibbs phenomenon was studied in [W] in the special case $X = S^2$.

Next, consider Δ on $M = \mathbb{R}^n \setminus K$, with the Dirichlet boundary condition on ∂K. As in §8, we assume $\mathbb{R}^n \setminus K$ has no trapped rays. If $f \in C^\infty(\overline{M})$ has compact support, then $u(t,x)$ is given by (8.26)-(8.30). In this case, a parametrix for $w(t,x)$, for small $|t|$, and for x in a small neighborhood $\mathcal{O}$ of ∂K in $\overline{M}$, has a form which is a variant of (11.3):

$$w(t,x) = A_0(t,x)\chi_+\big(t - \psi(x)\big) + \sum_{j=1}^{k} A_j(t,x)\chi_+^j\big(t - \psi(x)\big) + R_k(x). \tag{11.25}$$

Here, ψ satisfies the eikonal equation (11.4) (with $\partial\Omega$ replaced by ∂K), and $\psi(x) \geq 0$ on $\mathcal{O}$. The functions $A_j(t,x)$ are C^∞, and R_k is of class C^k, as in (11.5). This time, comparing jumps across $t = 0$ at ∂K, we have

$$A_0(x) = f(x), \quad x \in \partial K, \tag{11.26}$$

where, as above, $A_0(x) = A_0(0,x)$. We have the following result.

Proposition 11.3. *Assume $M = \mathbb{R}^n \setminus K$ has no trapped rays, and let Δ have the Dirichlet boundary condition. Let $f \in C^\infty(\overline{M})$ have compact support. There is a neighborhood $\mathcal{O}$ of ∂K in $\overline{M}$ such that*

$$S_R^\beta f(x) = G\big(R\psi(x)\big)\, A_0(x) + S_R^\# f(x), \tag{11.27}$$

for $x \in \mathcal{O}$, and $S_R^\# f(x)$ converges uniformly on $\mathcal{O}$ as $R \to \infty$.

A. The Dirichlet kernel and the wave equation

In this section we give a self-contained treatment of the Dirichlet kernel of the multiple Fourier integral and its application to a new proof of the representation formula (4.1) for the wave equation. One can compare the derivation on pp. 683-686 of [CH].

The n-dimensional Dirichlet kernel is defined by the integral

$$\mathcal{D}_n^M(x) = \frac{1}{(2\pi)^n} \int_{|\xi| \le M} e^{ix\cdot\xi}\, d\xi. \tag{A.1}$$

When $n = 1$ we have the elementary computation

$$\mathcal{D}_1^M(x) = \frac{\sin Mx}{\pi x}, \qquad x \neq 0. \tag{A.2}$$

Proposition A.1. *For $n \ge 1$, $\mathcal{D}_n^M(x) = D_n^M(|x|)$ where $D_n^M : [0,\infty) \to \mathbb{R}$ and we have the reduction formula*

$$D_{n+2}^M(r) = -\frac{1}{2\pi r}\frac{d}{dr} D_n^M(r), \qquad r > 0. \tag{A.3}$$

In particular for $n = 2k+1$,

$$D_n^M(r) = \Big(-\frac{1}{2\pi r}\frac{d}{dr}\Big)^k \Big(\frac{\sin Mr}{\pi r}\Big). \tag{A.4}$$

Proof. Taking $x = re_1$ in (A.1), we have

$$D_n^M(r) = (2\pi)^{-n} V_{n-1} \int_{-M}^{M} e^{ir\xi_1} (M^2 - \xi_1^2)^{\frac{n-1}{2}}\, d\xi_1, \tag{A.5}$$

where V_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$ (with $V_0 = 1$). Thus,

$$\frac{d}{dr} D_n^M(r) = (2\pi)^{-n} V_{n-1} i \int_{-M}^{M} e^{irs} s (M^2 - s^2)^{\frac{n-1}{2}}\, ds. \tag{A.6}$$

On the other hand,

$$rD_{n+2}^M(r) = (2\pi)^{-n-2}V_{n+1}\int_{-M}^{M} re^{irs}(M^2-s^2)^{\frac{n+1}{2}}\,ds. \tag{A.7}$$

If we write $re^{irs} = (1/i)\partial_s(e^{irs})$ and integrate by parts, then, since

$$\partial_s(M^2-s^2)^{\frac{n+1}{2}} = -(n+1)s(M^2-s^2)^{\frac{n-1}{2}},$$

we get, upon comparing the result with (A.6),

$$rD_{n+2}^M(r) = C_n\frac{d}{dr}D_n^M(r), \tag{A.8}$$

with

$$C_n = -\frac{(n+1)(2\pi)^{-n-2}V_{n+1}}{(2\pi)^{-n}V_{n-1}} = -\frac{n+1}{4\pi^2}\frac{V_{n+1}}{V_{n-1}}.$$

Using the well-known formula

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)},$$

we see that $C_n = -1/2\pi$, so (A.3) is proved.

We now use the form of the Dirichlet kernel to identify the fundamental solution of the wave equation, by comparing two formulas for the spherical partial sum operator. On the one hand, we have

$$\begin{aligned} S_Mf(x) &= \int_{\mathbb{R}^n} f(x+y)\mathcal{D}_n^M(y)\,dy \\ &= \int_0^\infty \Big(\int_{S^{n-1}} f(x+r\omega)d\omega\Big)D_n^M(r)r^{n-1}\,dr \\ &= A_{n-1}\int_0^\infty \bar{f}_x(r)D_n^M(r)r^{n-1}\,dr. \end{aligned} \tag{A.9}$$

As in (2.7), A_{n-1} is the area of the unit sphere $S^{n-1}\subset\mathbb{R}^n$. If f is smooth with compact support, the following computations can be made pointwise in the classical sense for $n \geq 3$:

$$\begin{aligned} S_Mf(x) &= A_{n-1}\int_0^\infty r^{n-1}\bar{f}_x(r)\frac{-1}{2\pi r}\frac{d}{dr}D_{n-2}^M(r)\,dr \\ &= \frac{A_{n-1}}{2\pi}\int_0^\infty (d/dr)[r^{n-2}\bar{f}_x(r)]D_{n-2}^M(r)\,dr. \end{aligned} \tag{A.10}$$

Repeating this procedure k times, we have for $n \geq 2k+1$,

$$S_Mf(x) = \frac{A_{n-1}}{(2\pi)^k}\int_0^\infty r\Big(\frac{d}{r\,dr}\Big)^k[r^{n-2}\bar{f}_x(r)]D_{n-2k}^M(r)\,dr.$$

If $n = 2k+1$ is odd, we have

$$S_M f(x) = \frac{A_{n-1}}{(2\pi)^k} \int_0^\infty r\Big(\frac{d}{r\,dr}\Big)^k [r^{n-2}\bar{f}_x(r)] \cdot \frac{\sin Mr}{\pi r}\,dr. \tag{A.11}$$

On the other hand, as seen in §1, for any L^2 function

$$S_M f = \chi_M(\sqrt{-\Delta})f = \frac{2}{\pi}\int_0^\infty u(t,\cdot)\frac{\sin Mt}{t}\,dt \tag{A.12}$$

By the uniqueness of one-dimensional Fourier transforms, we conclude that

$$u(t,x) = \frac{A_{n-1}}{2(2\pi)^k} t\Big(\frac{d}{t\,dt}\Big)^k [t^{n-2}\bar{f}_x(t)], \qquad n = 2k+1. \tag{A.13}$$

B. The heat kernel and the wave kernel

Here we give another derivation of the formula for $u(s,x) = \cos s\sqrt{-\Delta}\; f(x)$, by comparing two formulas for $e^{t\Delta}f(x)$. The first is

$$\begin{aligned} e^{t\Delta}f(x) &= (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|y|^2/4t} f(x-y)\,dy \\ &= (4\pi t)^{-\frac{n}{2}} A_{n-1} \int_0^\infty \overline{f}_x(r) r^{n-1} e^{-r^2/4t}\,dr, \end{aligned} \tag{B.1}$$

where A_{n-1} is the area of the unit sphere S^{n-1} in $\mathbb{R}^n$. This follows, via Fourier analysis, from the evaluation of the Gaussian integral

$$\int_{\mathbb{R}^n} e^{-t|\xi|^2 + ix\cdot\xi}\,d\xi = \Big(\frac{\pi}{t}\Big)^{\frac{n}{2}} e^{-|x|^2/4t}. \tag{B.2}$$

The second formula is

$$e^{t\Delta}f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{h}_t(s)u(s,x)\,ds, \tag{B.3}$$

with $h_t(\sigma) = e^{-t\sigma^2}$, hence, by (B.2), $\hat{h}_t(s) = (2\pi/\sqrt{4\pi t})e^{-s^2/4t}$. Setting $4t = 1/\lambda$ and comparing these formulas, we have

$$\int_0^\infty u(s,x)e^{-\lambda s^2}\,ds = \tfrac{1}{2}\pi^{-\frac{n-1}{2}} A_{n-1}\lambda^{\frac{n-1}{2}} \int_0^\infty \overline{f}_x(r) r^{n-1} e^{-\lambda r^2}\,dr, \tag{B.4}$$

for all $\lambda > 0$. The key to getting a formula for $u(s,x)$ from this is to make the factor $\lambda^{\frac{n-1}{2}}$ on the right side of (B.4) disappear.

Let us assume that $n = 2k+1$, and use

$$\text{(B.5)} \qquad -\frac{1}{2r}\frac{d}{dr}e^{-\lambda r^2} = \lambda e^{-\lambda r^2}$$

to write the right side of (B.4) as

$$\text{(B.6)} \qquad C_n \int_0^\infty r^{2k}\overline{f}_x(r)\Big(-\frac{1}{2r}\frac{d}{dr}\Big)^k e^{-\lambda r^2}\ dr.$$

Repeated integration by parts shows that this is equal to

$$\text{(B.7)} \qquad C_n \int_0^\infty r\Big(\frac{1}{2r}\frac{d}{dr}\Big)^k \big[r^{2k-1}\overline{f}_x(r)\big]\ e^{-\lambda r^2}\ dr.$$

Now it follows from uniqueness of Laplace transforms that

$$\text{(B.8)} \qquad u(t,x) = C_n t\Big(\frac{1}{2t}\frac{d}{dt}\Big)^k \big[t^{2k-1}\overline{f}_x(t)\big],$$

for well behaved functions f on $\mathbb{R}^n$, when $n = 2k+1$. By (B.4), we have

$$C_n = \frac{1}{2}\pi^{-(n-1)/2}A_{n-1}.$$

We can also compute C_n directly in (B.8), by considering $f = 1$. Then $\overline{f}_x = 1$ and $u = 1$, so

$$1 = C_n t\Big(\frac{1}{2t}\frac{d}{dt}\Big)^k t^{2k-1} = C_n(k-\tfrac{1}{2})(k-\tfrac{3}{2})\cdots\tfrac{1}{2},$$

i.e.,

$$\text{(B.9)} \qquad C_n = \frac{1}{(k-\frac{1}{2})(k-\frac{3}{2})\cdots\frac{1}{2}}, \quad n = 2k+1.$$

This simply means

$$A_{2k} = \frac{2\pi^k}{(k-\frac{1}{2})(k-\frac{3}{2})\cdots\frac{1}{2}},$$

a formula which is frequently derived by looking at Gaussian integrals.

C. Distributions oscillatory at the origin

Here we study oscillatory integrals of the form

$$\text{(C.1)} \qquad g(R) = \int_0^\infty \varphi(t)t^\mu e^{-i/\sigma(t)}e^{-iRt}\ dt,$$

given $\varphi \in C_0^\infty(\mathbb{R})$, $\sigma : \mathbb{R} \to \mathbb{R}$ a diffeomorphism (at least on supp φ), such that $\sigma(0) = 0$, and given $\mu \in \mathbb{C}$. If we write

$$\sigma(t) = \frac{t}{\tau(t)}, \tag{C.2}$$

with $\tau(0) = \sigma'(0)^{-1} = b$, then

$$\frac{1}{\sigma(t)} = \frac{b}{t} + \sigma_0(t), \tag{C.3}$$

where $\sigma_0(t)$ is smooth, so

$$g(R) = \int_0^\infty \psi(t) t^\mu e^{-ib/t} e^{-iRt}\, dt, \tag{C.4}$$

with $\psi(t) = \varphi(t) e^{-i\sigma_0(t)}$ in $C_0^\infty(\mathbb{R})$.

We can analyze this in terms of

$$\hat{\omega}_\mu(R) = \int_{-\infty}^\infty t^\mu e^{-i/t} e^{-iRt}\, dt, \tag{C.5}$$

where

$$\omega_\mu(t) = t^\mu e^{-i/t}. \tag{C.6}$$

For Re $\mu > -1$, ω_μ is locally integrable and of tempered growth, hence in $\mathcal{S}'(\mathbb{R})$. There is a unique holomorphic continuation $\mu \mapsto \omega$, satisfying

$$\omega_\mu = i(\mu + 2)\omega_{\mu+1} - i\partial_t \omega_{\mu+2}. \tag{C.7}$$

Thus

$$\hat{\omega}_\mu(R) = i(\mu + 2)\hat{\omega}_{\mu+1}(R) + R\hat{\omega}_{\mu+2}(R). \tag{C.8}$$

Contrast this with the family t^μ, whose continuation is merely meromorphic in μ.

Now (if $b = 1$) (C.4) is given by

$$g(R) = \hat{\psi} * \hat{\omega}_\mu(R), \tag{C.9}$$

and, given $\psi \in C_0^\infty(\mathbb{R})$, we have

$$\hat{\psi} \in \mathcal{S}(\mathbb{R}), \quad \int \hat{\psi}(R)\, dR = c\psi(0). \tag{C.10}$$

The slowly oscillatory behavior of $\hat{\omega}_\mu(R), R \to \infty$ we establish below will also hold for $g(R)$. We now show that $\hat{\omega}_\mu(R)$ can be computed in terms of a standard Bessel function

Proposition C.1. *We have*

$$\hat{\omega}_\mu(R) = c_\mu R^{-\frac{1}{2}(\mu+1)} J_{-\mu-1}(2\sqrt{R}), \quad R > 0. \tag{C.11}$$

Proof. By the Paley-Wiener Theorem, $\hat{\omega}_\mu(R)$ vanishes for $R < 0$. There are several ways to analyze it for $R > 0$. For one, we can use the ODE satisfied by ω_μ :

$$t^2 \partial_t \omega_\mu = (i + \mu t)\omega_\mu, \tag{C.12}$$

to obtain

$$R\partial_R^2 \hat{\omega}_\mu + (2+\mu)\partial_R \hat{\omega}_\mu + \hat{\omega}_\mu = 0. \tag{C.13}$$

Now (C.11) follows from an investigation of (C.13). To see this, set $h_\mu(s) = \hat{\omega}_\mu(s^2)$. Then (C.13) implies for h_μ the differential equation

$$h_\mu''(s) + \frac{3+2\mu}{s} h_\mu'(s) + 4h_\mu(s) = 0.$$

This is readily transformed into a Bessel equation, and one gets

$$\hat{\omega}_\mu(R) = R^{-\frac{1}{2}(\mu+1)} \big[c(\mu) J_{-\mu-1}(2\sqrt{R}) + d(\mu) Y_{-\mu-1}(2\sqrt{R}) \big], \quad R > 0.$$

The coefficients $c(\mu)$, $d(\mu)$ are uniquely determined, hence holomorphic in μ. Now, for $\mu < -1$, since ω_μ is integrable near infinity, $\hat{\omega}_\mu$ is bounded on $R \in (0, 1]$. This forces $d(\mu) = 0$ for $\mu < -1$, hence, by holomorphy, $d(\mu) = 0$ for all μ, and we have (C.11).

It is interesting to consider an alternative derivation of some special cases of this proposition, namely the important cases $\mu = -\frac{n}{2}$, $n = 1, 2, 3, \ldots$. We can directly relate $\hat{\omega}_{-\frac{n}{2}}(R)$ to the Dirichlet kernel, as follows. We have

$$\omega_{-\frac{n}{2}}(t) = t^{-\frac{n}{2}} e^{-i/t} = c e^{-it\Delta} \delta(x), \quad |x| = 2. \tag{C.14}$$

Hence

$$\begin{aligned} \hat{\omega}_{-\frac{n}{2}}(R) &= c \int_{-\infty}^{\infty} e^{-it(\Delta+R)}\, dt\ \delta(x) \\ &= c\delta(-\Delta - R)\delta(x) \\ &= c\partial_R S_{\sqrt{R}} \delta(x), \end{aligned} \tag{C.15}$$

where $S_{\sqrt{R}}$ is the partial sum operator on $\mathbb{R}^n$. Hence

$$\hat{\omega}_{-\frac{n}{2}}(R) = c_n \partial_R D_n^{\sqrt{R}}(2), \tag{C.16}$$

where the Dirichlet kernel, at $r = 2$, appears on the right side. Recalling (A.2), we have

$$\hat{\omega}_{-\frac{1}{2}}(R) = c_1 \partial_R \Big(\frac{\sin 2\sqrt{R}}{2\pi} \Big) = c_1' \frac{\cos 2\sqrt{R}}{\sqrt{R}}, \tag{C.17}$$

and, by (A.4),

$$\begin{aligned}\hat{\omega}_{-k-\frac{1}{2}}(R) &= \Big(-\frac{1}{2\pi r}\frac{\partial}{\partial r}\Big)^k \Big[c_1\partial_R\Big(\frac{\sin r\sqrt{R}}{\pi r}\Big)\Big]\Big|_{r=2} \\ &= c_1'\Big(-\frac{1}{2\pi r}\frac{\partial}{\partial r}\Big)^k \frac{\cos r\sqrt{R}}{\sqrt{R}}\Big|_{r=2}.\end{aligned} \tag{C.18}$$

In particular,

$$\hat{\omega}_{-\frac{3}{2}}(R) = c_3'' \ \sin 2\sqrt{R}. \tag{C.19}$$

A well-known general formula, valid for both even n and odd n, is

$$\partial_\rho D_n^\rho(r) = c_n \rho^{n-1} \int\limits_{S^{n-1}} e^{i\rho r\omega\cdot\omega'} \ dS(\omega), \tag{C.20}$$

valid for any $\omega' \in S^{n-1}$. Hence,

$$\partial_\rho D_n^\rho(r) = c\rho^{n-1}(\rho r)^{1-\frac{1}{2}n} J_{\frac{n}{2}-1}(\rho r), \tag{C.21}$$

where J_ν is the Bessel function. Thus

$$\partial_R D_n^{\sqrt{R}}(2) = \frac{1}{2\sqrt{R}}\partial_\rho D_n^\rho(2)\big|_{\rho=\sqrt{R}} = cR^{\frac{n}{4}-\frac{1}{2}} J_{\frac{n}{2}-1}(2\sqrt{R}). \tag{C.22}$$

In other words, for $R > 0$,

$$\hat{\omega}_{-\frac{n}{2}}(R) = cR^{\frac{n}{4}-\frac{1}{2}} J_{\frac{n}{2}-1}(2\sqrt{R}). \tag{C.23}$$

Thus we have another derivation of (C.11), when $\mu = -\frac{n}{2}$. Using the identities

$$J_{-\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \cos z, \quad J_{\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \sin z, \tag{C.24}$$

we recover (C.17) and (C.19) from (C.23).

Corresponding to the case $n = 2$, we have

$$\hat{\omega}_{-1}(R) = cJ_0(2\sqrt{R}). \tag{C.25}$$

In view of the asymptotic behavior

$$J_\nu(s) \sim \Big(\frac{2}{\pi s}\Big)^{\frac{1}{2}} \cos(s - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O(s^{-3/2}), \quad s \to +\infty, \tag{C.26}$$

valid for any fixed $\nu > -\frac{1}{2}$ (see, e.g., [Leb]), we have

$$\hat{\omega}_{-1}(R) \sim CR^{-\frac{1}{4}} \cos(2\sqrt{R} - \tfrac{1}{4}\pi), \quad R \to \infty. \tag{C.27}$$

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