**LORENZ EQUATIONS PART I: EXISTENCE AND
NONEXISTENCE OF HOMOCLINIC ORBITS**

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Lorenz Equations

Part I: Existence and Nonexistence of Homoclinic Orbits

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Abstract. The Lorenz equations are a system of three ordinary differential equations

$$x' = s(y - x), \quad y' = Rx - y - xz, \quad z' = xy - qz$$

where s , R , and q are positive parameters. We show that this system has homoclinic orbits associated with the origin (i.e., orbits that tend to the origin as $t \rightarrow \pm\infty$) if and only if $s > (2q+1)/3$. The method is based on Liapunov functions and a shooting argument used previously by Hastings and Troy in studying homoclinic orbits of the Lorenz equations.

Keywords. Lorenz equations, homoclinic orbits, shooting methods, Liapunov functions, asymptotic behavior.

AMS subject classifications (1991). 34C37.

1. Introduction.

The Lorenz equations we studied here are a system of ordinary differential equations

$$\begin{cases} x' = s(y - x), \\ y' = Rx - y - xz, \\ z' = xy - qz \end{cases} \quad (1.1)$$

where $' = \frac{d}{dt}$ and s , R , and q are positive parameters. This system was first presented in 1963 by E. N. Lorenz [11] in studying fluid convection in a two dimensional layer heated from below. In the last decades, there has been an immense amount of interest generated by these equations due to the fact that for some parameter values, numerical computed solutions oscillate in the pseudo-random way which people call “chaotic”. For more detailed description of the observed “chaotic” behavior and mathematical theories built upon (1.1) such as the geometric models of the Lorenz equations, Sil’nikov-type or homoclinic bifurcations, and averaging methods, see [3, 5, 13, 14, 15, 16, 18], a review book of Sparrow [17], a mathematical text book of Guckenheimer & Holmes [4], and the references therein.

Generally speaking, to apply certain well-developed theories to the concrete example of (1.1), certain hypotheses have to be verified. One way to achieve this would be by a computer simulation, but there are few rigorous results. Recently, Hastings & Troy [8, 9],

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Hassard, Hastings, Troy, & Zhang [7], and Mischaikow & Mrozek [12] built up mathematical theories for certain characterizations of the chaotic behavior of (1.1) and implemented rigorous arithmetic numerical schemes to verify the validity of their hypotheses, and hence lead to affirmative conclusions on the chaotic behavior they studied for the solutions of (1.1) for certain parameter values (s, R, q) .

Among all the solutions of (1.1), a very special one is a homoclinic orbit associated with the origin, which is a solution having the property that it approaches the origin as $t \rightarrow \pm\infty$. In developing geometric model theories or bifurcation theories, homoclinic orbits play an essential role. An example of its fascination and importance can be read from the phase “homoclinic explosion”, which is used to refer to the appearance of various kinds of chaotic behavior when parameters are perturbed from the values where there is a homoclinic orbit; see, for example, Sparrow [17] and Robinson [14]. Therefore, it is important to find these homoclinic orbits. The existence of homoclinic orbits associated with the origin for (1.1) can be seen from strong numerical evidence. The first pure mathematical proof is given by Hastings and Troy [10]. Using a shooting argument and a pioneer, though tedious, mathematical analysis on the nature of the solution when $(s, R, q) = (10, 1000, 1)$, they were able to show that for each (s, q) in some neighborhood of the point $(10, 1)$ there is an R in the interval $(1, 1000)$ such that the Lorenz equations has a homoclinic orbit associated with the origin. A rigorous numerical implementation of a similar method for the existence of a homoclinic orbit was used by Hassard and Zhang [6] to pin down the value of the parameter R . They showed that when $s = 10$ and $q = 8/3$, R is between 13.9265 and 13.927.

In this paper, we shall study the existence and non-existence of homoclinic orbits of (1.1). In particular we prove the following theorem:

Theorem 1.1. *Assume that s and q are given positive constants. Then the Lorenz equations (1.1) have at least one homoclinic orbit associated with the origin for some $R \in (0, \infty)$ if and only if $s > \frac{2q+1}{3}$.*

For the existence part, the proof is based on a shooting argument developed in [10], where the shooting parameter is R . In completing this shooting argument, one of the essential difficulties lies in the study of the behavior of the solution for large R . Impressed by the magnitude of the value $R = 1000$ taken in [10], here we replace their analysis for $R = 1000$ by studying the asymptotic behavior of the solution as $R \rightarrow \infty$. Though substantial information, compared to that obtained in [10] for $R = 1000$, was lost, the key properties needed to complete the shooting argument were not. The main advantage of our method is that we can establish the existence of homoclinic orbits for a large set of values of (s, q) .

The proof of the non-existence is solely based on Liapunov functions. Fortunately, the result obtained compliments perfectly the existence result, as is seen from Theorem 1.1.

We shall prove the non-existence part of Theorem 1.1 in §2 and the existence part in §3 and §4.

Remark 1.2. As will be seen in the proof of Theorem 1.1, the homoclinic orbit established in this paper is the simplest one among all homoclinic orbits the Lorenz system

could have; namely, its x coordinate does not change sign and has only one local extreme, so that in the x - x' phase plan, the orbit forms a single loop on the right or the left half plane.

In a forthcoming paper [1], we shall prove the existence of homoclinic orbits having the property that the x coordinate can change any prescribed number of times, and in each time interval where x does not change sign x' can change arbitrary prescribed odd number of times (except in the first two intervals). Also we prove the existence of orbits having the property that the x coordinate changes sign infinitely many times and during each time interval where x does not change, x' changes a certain number of signs according to a prescribed double bounded odd integer sequence.

In another forthcoming paper [2], we shall prove the existence of homoclinic explosion for the Lorenz system for certain parameters; namely, we shall prove the existence of a horseshoe like Poincare map.

2. Non-existence of homoclinic orbits.

First we recall a boundedness lemma originated from Lorenz [11].

Lemma 2.1. *The region $D := \{(x, y, z) \mid x^2 + \frac{1}{R}y^2 + \frac{1}{R}(z - (1+s)R)^2 \leq \frac{qR(1+s)^2+1}{\min\{2s, 2, q\}}\}$ is positively invariant; that is, if a trajectory enters D at sometime, it will stay there forever. In addition, every trajectory will enter D in finite time.*

Proof. Set $V = x^2 + \frac{1}{R}y^2 + \frac{1}{R}(z - (1+s)R)^2 - \frac{qR(1+s)^2+1}{\min\{2s, 2, q\}}$. Then along any trajectory of (1.1),

$$V' = -2sx^2 - \frac{2}{R}y^2 - \frac{q}{R}(z - (1+s)R)^2 - \frac{q}{R}z^2 + qR(1+s)^2 \leq -\min\{2s, 2, q\}V - 1 - \frac{q}{R}z^2.$$

The assertion of the lemma thus follows. $\square$

The following lemma shows that the case $R \in (0, 1]$ is trivial.

Lemma 2.2. *Assume that $R \in (0, 1]$. Then for every positive s and q , every solution of (1.1) approaches, as $t \rightarrow \infty$, to $(0, 0, 0)$ when $R \in (0, 1)$ or to $(x^*, x^*, 0)$ for some $x^* \in (-\infty, \infty)$ when $R = 1$. Consequently, there are no homoclinic orbits associated with any stationary points of (1.1).*

Proof. Define $V(x, y, z) = x^2 + sy^2 + sz^2$. Since $R \in (0, 1]$, along any trajectory of (1.1),

$$V' = -2s\left\{ \left[x - \frac{1+R}{2}y\right]^2 + \left[1 - \left(\frac{1+R}{2}\right)^2\right]y^2 + qz^2 \right\} \leq 0$$

where the equal sign is taken only at the stationary points of the Lorenz system. The assertion of the lemma thus follows from a standard theory of Liapouov functions. $\square$

In the sequel, we shall always assume that

$$s > 0, \quad q > 0, \quad R > 1.$$

In this case, (1.1) has three stationary points located at

$$C^0 = (0, 0, 0), \quad C_R^\pm = (\pm\sqrt{q(R-1)}, \sqrt{q(R-1)}, R-1).$$

When $R > 1$, the origin C^0 is non-stable and the linearized flow near the origin has three real eigenvalues,

$$\lambda_i^0 = \frac{1}{2} \left\{ -(s+1) - (-1)^i \sqrt{(s+1)^2 + 4s(R-1)} \right\} \quad (i = 1, 2), \quad \lambda_3^0 = -q. \quad (2.1)$$

The eigenvectors corresponding to λ_1^0 and λ_2^0 lie on the x - y plane whereas the eigenvalue λ_3^0 corresponds to the two trajectories being the positive and negative z -axis. Since λ_1^0 is positive and λ_2^0 and λ_3^0 are negative, at C^0 , there is a two dimensional stable manifold which contains the z -axis, and a one dimension unstable manifold $\gamma = \gamma^+ \cup \gamma^- \cup C^0$ where γ^- is the reflection of γ^+ across the z axis and γ^+ initially points into the positive octant. In fact, the trace of γ^+ is a smooth curve starting from the origin having the asymptotic behavior $\gamma = (x, (1 + \frac{\lambda_1^0}{s})x + O(x^2), \frac{\lambda_1^0 + s}{s(2\lambda_1^0 + q)}x^2 + O(x^3))$ for small positive x .

When $R > 1$, the characteristic function for the eigenvalues of the linearized flow near the stationary point C^+ and C^- is

$$\lambda^3 + (1 + s + q)\lambda^2 + (s + R)q\lambda + 2sq(R - 1) = 0.$$

This equation has a negative root, and, when the real part of the other two roots (which are complex conjugate to each other) are non-negative, then its imaginary part is non-zero. That is, either $C^\pm$ is stable, or $C^\pm$ has a one dimensional stable manifold and a two dimensional unstable manifold on which the flow is of spiral type.

It is convenient to introduce

$$Q := z - \frac{1}{2s}x^2, \quad \mu = 1 - \frac{q}{2s}.$$

Then, one can directly verify that the Lorenz equations (1.1) are equivalent to the following system:

$$\begin{cases} x'' + (1 + s)x' = s \left\{ (R - 1) - Q - \frac{1}{2s}x^2 \right\} x & (= s(R - 1 - z)x), \\ Q' + qQ = \mu x^2. \end{cases} \quad (2.2)$$

To show that (2.2) has no homoclinic orbits associated with the origin when $s \leq \frac{2q+1}{3}$, we considered three cases:

$$(1) \ 0 < s < \frac{q}{2}, \quad (2) \ s = \frac{q}{2}, \quad (3) \ \frac{q}{2} < s \leq \frac{2q+1}{3}.$$

First we consider the case $0 < s < \frac{q}{2}$.

Lemma 2.3. *Assume that $0 < s < \frac{q}{2}$. Then for any $R > 1$, (1.1) has no homoclinic orbits associated with any stationary points. In addition, as $t \rightarrow \infty$, every trajectory has a limit being one of the stationary points of (1.1).*

Proof. Define $V = x'^2 + \frac{s}{2|\mu|q}Q'^2 + \frac{s}{2q}(x^2 - (R-1)q)^2$. Since $\mu = 1 - \frac{q}{2s} \leq 0$, substituting Q' by $-qQ + \mu x^2$ yields

$$V = x'^2 + sx^2 \left[\frac{1}{4s}x^2 + Q - (R - 1) \right] - \frac{sq}{2\mu}Q^2 + \frac{s(R-1)^2q}{2}.$$

It then follows that along any trajectory,

$$\begin{aligned} V' &= 2x' \left\{ x'' + sx \left[\frac{1}{2s} x^2 + Q - (R-1) \right] \right\} + \frac{s}{\mu} Q' \{ \mu x^2 - qQ \} \\ &= -2(1+s)x'^2 + \frac{s}{\mu} Q'^2 \end{aligned}$$

by using the differential equations in (2.2). Since $\mu < 0$, $V' \leq 0$ for all $t \in (-\infty, \infty)$ and in addition, $V' = 0$ if and only if $x' = 0$ and $Q' = 0$. The assertions of the Lemma thus follows by the properties of the Liapouov function V . $\square$

Next we consider the case $s = \frac{q}{2} > 0$.

Lemma 2.4. *Assume that $s = \frac{q}{2} > 0$. Then the assertion of Lemma 2.2 holds.*

Proof. Note that when $s = \frac{q}{2}$, $\mu = 0$, so that $Q' = -qQ$. Consequently, $Q(t) = Q(0)e^{-qt}$. In particular, on γ^+ , $Q(t) \equiv 0$ so that $x'' + (1+s)x' = sx[(R-1) - \frac{1}{2s}x^2]$. Set $V = x'^2 + \frac{1}{4}[x^2 - 2s(R-1)]^2$. Then $V' = -2(1+s)x'^2 \leq 0$ and therefore γ^+ approaches C^+ as $t \rightarrow \infty$. (Note that the set $\{(x, y, z) | V < 0\}$ consists of two disjoint domains with C^+ and C^- in each of them.)

For any other trajectories, $Q(t) \rightarrow 0$ as $t \rightarrow \infty$, and one can show that every trajectory has to tend to one of the stationary points. Details are omitted. $\square$

Finally, we consider the case $\frac{q}{2} < s \leq \frac{2q+1}{3}$.

Lemma 2.5. *Assume that $0 < \frac{q}{2} < s \leq \frac{2q+1}{3}$. Then for any $R > 1$, the Lorenz equations (1.1) have no homoclinic orbits associated with the origin.*

Proof. We need only show that $\gamma^\pm$ does not approach the origin as $t \rightarrow \infty$. By symmetry, we need only consider γ^+ .

First, notice that $\mu = 1 - \frac{q}{2s} > 0$, so that $Q' + qQ = \mu x^2 \geq 0$. It then follows that on γ^+ , $Q(t) > 0$ for all t .

Define

$$V = x'^2 + sx^2 \left\{ \frac{1}{4s} x^2 + Q - (R-1) \right\} + (1+s)xx'.$$

Then, along γ^+ ,

$$\begin{aligned} V' &= 2x' \left\{ x'' + sx \left[\frac{1}{2s} x^2 + Q - (R-1) \right] \right\} + sx^2 Q' + (1+s)x'^2 + (1+s)xx'' \\ &= 2x' \left\{ -(1+s)x' \right\} + sx^2 [-qQ + \mu x^2] + (1+s)x'^2 + \\ &\quad + (1+s)x \left\{ -(1+s)x' + s[(R-1) - Q - \frac{1}{4s}x^2 - \frac{1}{4s}x^2] \right\} \\ &= -(1+s) \left\{ x'^2 + (1+s)xx' + sx^2 \left[\frac{1}{4s} x^2 + Q - (R-1) \right] \right\} + \left(\mu s - \frac{1+s}{4} \right) x^4 - sqQx^2 \\ &= -(1+s)V + \frac{3}{4} \left(s - \frac{2q+1}{3} \right) x^4 - sqQx^2 \end{aligned}$$

where we have used the equations in (2.2) in the second equality, and the definition of V and the relation $\mu s - \frac{1+s}{4} = (1 - \frac{q}{2s})s - \frac{1+s}{4} = \frac{3}{4}(s - \frac{2q+1}{3})$ in the last equality.

Since $s \leq \frac{2q+1}{3}$ and $Q > 0$ on $\gamma^\pm$, the last equation yields

$$V'(t) + (1+s)V(t) \leq 0 \quad \text{on } \gamma^+ \text{ for all } t \in (-\infty, \infty).$$

In particular, since $V(-\infty) = 0$, we have that $V(t) \leq 0$ for all t . Since Gronwall's inequality implies that $V(t) \leq V(0)e^{-(1+s)t}$ for all $t > 0$, we have that $|V(t)| \geq |V(0)|e^{-(1+s)t}$ for all $t > 0$.

We now claim that γ^+ cannot approach $C^0 = (0, 0, 0)$ as $t \rightarrow \infty$. In fact, if a trajectory is on the stable manifold of the origin and is not the z axis, then for t sufficiently large, $x'' = -(1+s)x' + (R-1+o(1))x$, so that $x, x', x'' = O(e^{[\lambda_2^0+o(1)]t})$ as $t \rightarrow \infty$. That is, if γ^+ approaches the origin as $t \rightarrow \infty$, then $V(t) = O(e^{2[\lambda_2+o(1)]t})$. But this is impossible since $|V(t)| \geq |V(0)|e^{-(1+s)t}$ for all $t > 0$ and, from the expression of λ_2^0 in (2.1), $\lambda_2^0 < -(1+s)$. Therefore, γ^+ cannot approach the origin; namely, there is no homoclinic orbit associated with the origin. $\square$

Summarizing Lemmas 2.2–2.5, we now can conclude the following:

Theorem 2.6. *Assume that $0 < s \leq \frac{2q+1}{3}$. Then for any $R \in (0, \infty)$, the Lorenz equations (1.1) have no homoclinic orbits associated with the origin.*

Remark. Lemmas 2.3 and 2.4 tell that when $R > 1$ and $s \leq \frac{q}{2}$, $\gamma^\pm$ approaches $C^\pm$ as $t \rightarrow \infty$ and every other trajectories of (1.1) approaches one of the stationary points. However, we are unable to show the same conclusion for the case $\frac{q}{2} < s \leq \frac{2q+1}{3}$.

3. Existence of homoclinic orbits associated with the origin.

We now begin to show the existence part of Theorem 1.1; namely, we show the following theorem:

Theorem 3.1. *Assume that $q > 0$ and $s > \frac{2q+1}{3}$. Then there exists $R \in (1 + \frac{q}{2s}, \infty)$ such that the Lorenz equations (1.1) have a homoclinic orbit associated with the origin and lying on the half space $\{(x, y, z) | x > 0\}$.*

Clearly, Theorem 1.1 follows from Theorems 2.6 and 3.1.

In the sequel, we shall always assume that s , R and q satisfy

$$q > 0, \quad s > \frac{2q+1}{3}, \quad R > 1.$$

Notice that the first two conditions imply that $\mu = 1 - \frac{q}{2s} > 0$.

The proof is based on a shooting argument first used by Hastings and Troy [10] and later on used by Hassard and Zhang [6]; here R is the shooting parameter. In the sequel, we shall use γ_R^+ to denote the trajectory of the stable manifold associated with the origin which initially enters the positive octant. Also we use x_R, y_R, z_R, Q_R to denote the solution with the trajectory γ_R^+ . To set up the shooting argument, we define a set $\mathfrak{R}$ as follows:

$$\mathfrak{R} = \left\{ R > 1 \left| \begin{array}{l} \text{(i)} \exists T_R^1 \text{ such that } x'_R(T_R^1) = 0, x''_R(T_R^1) < 0, \text{ and } x'_R > 0 \text{ in } (-\infty, T_R^1); \\ \text{(ii)} \exists T_R^2 > T_R^1 \text{ such that } x'_R(T_R^2) = -sx(T_R^2) \text{ and } x''_R < -sx'_R \text{ in } [T_R^1, T_R^2]; \\ \text{(iii)} \exists T_R^3 > T_R^2 \text{ such that } x_R(T_R^3) = 0 \text{ and } x'_R < -sx_R \text{ in } (T_R^2, T_R^3]. \end{array} \right. \right\}$$

Observe that $x'_R + sx_R = sy_R$, so that the second condition in the definition means that $y_R(T_R^2) = 0$ and $y'_R < 0$ in $[T_R^1, T_R^2]$, and the third condition means that $y_R < 0$ in $(T_R^2, T_R^3]$.

Lemma 3.2. *The set $\mathfrak{R}$ is open.*

Proof. Assume that $\bar{R} \in \mathfrak{R}$. Since $x''_{\bar{R}}(T_{\bar{R}}^1) \neq 0$, $(x'_{\bar{R}} + sx_{\bar{R}})'(T_{\bar{R}}^2) \neq 0$ and $x'_{\bar{R}}(T_{\bar{R}}^3) \neq 0$, by the continuity of the trajectory $\gamma_{\bar{R}}^+$ with respect to R and the Implicit Function Theorem, for all R sufficiently close to $\bar{R}$, there exist T_R^1, T_R^2 , and T_R^3 satisfying the three conditions in the definition of $\mathfrak{R}$; that is, $R \in \mathfrak{R}$. Hence, $\mathfrak{R}$ is open. $\square$

Lemma 3.3. *Assume that $R \in (1, 1 + \frac{q}{2s}]$. Then the “box” $B := \{(x, y, z) \mid 0 < x < \sqrt{q}, 0 < y < \sqrt{q}, \frac{1}{2s}x^2 < z < 1\}$ is positively invariant. Since initially γ_R^+ stays in B , γ_R^+ stays in B for all $t \in (-\infty, \infty)$. Consequently, $R \notin \mathfrak{R}$ and there is no homoclinic orbits associated with the origin.*

Proof. The argument given below is adapted from [10].

To show that B is positive invariant, we need only check the direction of the vector field of (1.1) on the boundary of B .

On the bottom face of B , $Q = 0$. Since $Q' + qQ = \mu x^2 \geq 0$, we know that once Q is positive, it will remain positive. It then follows that any trajectory can not exit B from the face $\{z = \frac{1}{2s}x^2\}$.

On the top face of B , $z = 1$. It follows that $z' = xy - qz = xy - q < 0$ except at $(x, y) = (\sqrt{q}, \sqrt{q})$. When $(x, y, z) = (\sqrt{q}, \sqrt{q}, 1)$, $y' = (R - 2)\sqrt{q} < 0$, $z' = 0$, $z'' = \sqrt{q}y' < 0$, $x' = 0$, $x'' = (R - 2)\sqrt{q} < 0$. Hence, any trajectory cannot exit from the top face of B . Similarly, one can show that any trajectory cannot exit B from the left face $\{x = 0\}$, the right face $\{x = \sqrt{q}\}$, and the front face $\{y = 0\}$ of B . It remains to consider the back face (without edges) of B .

On the back face $\{y = \sqrt{q}, 0 < x < \sqrt{q}, \frac{1}{2s}x^2 < z < 1\}$ of B , $y' = (R - z)x - \sqrt{q}$. Since $R \leq 1 + \frac{q}{2s}$ and $z > \frac{1}{2s}x^2$, $y' < x(1 + \frac{q}{2s} - \frac{1}{2s}x^2) - \sqrt{q} = (\sqrt{q} - x)(\frac{x(x + \sqrt{q})}{2s} - 1) \leq (\sqrt{q} - x)(\frac{q}{s} - 1) \leq 0$. We then conclude that any trajectory cannot exit from B on the face $\{y = \sqrt{q}\}$ also.

In conclusion, B is positive invariant and $R \notin \mathfrak{R}$. Since an orbit cannot enter the origin from the inside of B , there are no homoclinic orbits associate with the origin. This completes the proof of the Lemma. $\square$

Lemma 3.3. *There exists $R_0 \gg 1$ such that $[R_0, \infty) \in \mathfrak{R}$.*

This is a technical lemma and we shall leave the proof to the next section. In fact, this is the major difference of our proof from that of Hastings and Troy [10] and Hassard and Zhang [6].

Assume for the moment that Lemma 3.3 has been proven. We can now complete the proof of Theorem 3.1. The argument below is essentially the same as that in [10] and [6]. Here we provide it for completeness, as well as for reader's convenience.

Proof of Theorem 3.1. Define $R^* \equiv \inf\{R \mid R \in \mathfrak{R}\}$. From Lemmas 3.2 and 3.3, $R^* \in [1 + \frac{q}{2s}, \infty)$, so that $\gamma_{R^*}^+$ is well-defined. Since $\mathfrak{R}$ is open, $R^* \notin \mathfrak{R}$. We now show that $\gamma_{R^*}^+$ is a homoclinic orbit, via the following steps:

1. there exists $T_{R^*}^1$ satisfying the first condition in the definition of $\mathfrak{R}$; namely, $x_{R^*} > 0$ in $(-\infty, T_{R^*}^1)$, $x'_{R^*}(T_{R^*}^1) = 0$, and $x''_{R^*}(T_{R^*}^1) < 0$;
2. there exists $T_{R^*}^2$ satisfying the second condition in the definition of $\mathfrak{R}$; namely, $(x'_{R^*} + sx_{R^*})' < 0$ on $[T_{R^*}^1, T_{R^*}^2]$ and $(x_{R^*} + sx_{R^*})(T_{R^*}^2) = 0$;

3. $x_{R^*} > 0$ for all $t \in (-\infty, \infty)$, $x'_{R^*} < -sx_{R^*}$ for all $t \in (T_{R^*}^2, \infty)$, and $\gamma_{R^*}^+$ approaches the origin as $t \rightarrow \infty$.

The first step: Since γ_{R^*} is on the unstable manifold of the origin, $x_{R^*} > 0$ and $x'_{R^*} > 0$ for all t sufficiently negative large. We claim that there is a first time such that $x'_{R^*} = 0$. In fact, if it is not true, then $x'_{R^*} > 0$ for all t , which implies that as $t \rightarrow \infty$, γ_{R^*} approaches the stationary point $C_{R^*}^+ \equiv (\sqrt{q(R^* - 1)}, \sqrt{q(R^* - 1)}, R^* - 1)$ (Recall that all the trajectories are bounded for positive t). There are only two possibilities: (i) C_{R^*} is stable; (ii) C_{R^*} is non-stable with a two dimensional unstable manifold on which the flow is of spiral type. In the first case, there is a small ball centered at C^* such that the vector fields of (1.1) intersect the boundary of the ball non-tangentially and inwards. Consequently, when R is sufficiently closed to R^* , the vector fields with the parameter R also intersect the boundary of the ball non-tangentially and inwards, so that the ball is positively invariant. Since for all R sufficiently close to R^* , γ_R^+ is close to $\gamma_{R^*}^+$ up to the time when γ_R^+ enters the ball, which means that γ_R^+ will be trapped into the ball before the possibility for x_R to reach its first zero. Clearly, this violates the definition of R^* . In the second case, there are two small balls B_1 and B_2 , both centered at C^* and $B_1 \subset B_2$ such that any trajectory starting from any point on the boundary of B_1 has to wind around $\gamma_{R^*}^+$ at least two times before it could possibly exit from the boundary of B_2 . This implies that for R sufficiently close to R^* , any trajectory hits B_1 has to wind around the stable manifold of the stationary point C_R^+ at least one time before it could possibly hit the boundary of B_2 , which means that x'_R has to change sign at least two times before x_R could possibly reach its first zero. Clearly, this is impossible for R in $\mathfrak{R}$. Therefore, there is a first time $T_{R^*}^1$ such that x'_{R^*} takes its first zero.

Since $T_{R^*}^1$ is the first zero of x'_{R^*} , $x'_{R^*} > 0$ in $(-\infty, T_{R^*}^1)$ and $x''_{R^*}(T_{R^*}^1) \leq 0$. Therefore $Q''_{R^*} + qQ'_{R^*} = 2\mu x_{R^*} x'_{R^*} > 0$ in $(-\infty, T_{R^*}^1)$, so that $Q'_{R^*} > 0$ on $(-\infty, T_{R^*}^1]$, and $z'_{R^*} = Q'_{R^*} + \frac{1}{s} x_{R^*} x'_{R^*} > 0$ in $(-\infty, T_{R^*}^1]$ as well. This excludes the possibility from $x''_{R^*}(T_{R^*}^1)$ being zero since otherwise $x''_{R^*}(T_{R^*}^1) \geq 0$ and from the equation for x_{R^*} , one can derive that $z_{R^*} = R^* - 1$ and $z'_{R^*} \leq 0$, which is impossible. Hence $x''_{R^*}(T_{R^*}^1) < 0$. Consequently, $x'_{R^*} + sx_{R^*} > 0$ and $(x'_{R^*} + sx_{R^*})' < 0$ at $T_{R^*}^1$.

The second step: Since the solution of (1.1) is autonomous, we can shift t such that $T_R^1 = 0$ for all $R \in \mathfrak{R} \cup R^*$. Under this normalization, since $x''_R(0) \neq 0$, by the Implicit Function Theorem, the solution of (1.1) is both smooth in time t and in the parameter R . We claim that $T_{R^*}^2 = \liminf_{R \in \mathfrak{R}, R \searrow R^*} T_R^2$ is finite. In fact if $T_{R^*}^2 = \infty$, then by the continuity of the trajectory in R and the definition of T_R^2 , $(x'_{R^*} + sx_{R^*})' \leq 0$ and $(x'_{R^*} + sx_{R^*}) \geq 0$ for all $t > 0$. It then follows that $x'_{R^*} + sx_{R^*}$ is positive and monotonically decreasing in $[0, \infty)$ so that either x'_{R^*} reaches zero and then becomes positive at some time in $(0, \infty)$ before x_{R^*} reaches its first zero in $(-\infty, \infty)$ or $sx_{R^*} > -x'_{R^*} \geq 0$ for all $t > T_{R^*}^1$. The first case cannot happen since R^* is a limit point of $\mathfrak{R}$ in which x_R obtains its first zero before x'_R obtains its first zero in $(0, \infty)$. The second case implies that on the phase plane of x vis x' , (x_{R^*}, x'_{R^*}) approaches $(0, 0)$ from the sector bounded by the positive x -axis and the line $x' + sx = 0$ (the possibility that x_{R^*} monotonically decreases to $\sqrt{q(R^* - 1)}$ can be excluded in a similar way as in the first step). This is impossible since the direction that a trajectory can approach the origin on the x - x' phase plane is $(1, \lambda_2^0)$ but $\lambda_2^0 < -(1 + s)$. Hence, $T_{R^*}^2 < \infty$.

By the definition of $T_{R^*}^2$, the continuity of solutions in R , and the second condition in the definition of $\mathfrak{R}$, $x'_{R^*} + sx_{R^*} = 0$ at $T_{R^*}^2$ and $(x'_{R^*} + sx_{R^*})' \leq 0$ in $[0, T_{R^*}^2]$. If $(x'_{R^*} + sx_{R^*})' = 0$ at some point $T \in (0, T_{R^*}^2)$, then $(x'_{R^*} + sx_{R^*})(T) > 0$ and $(x'_{R^*} + sx_{R^*})''(T) = 0$. From the differential equation $(x' + sx)' + (x' + sx) = s(R - z)x$, we then have that $z_{R^*}(T) < R^*$ and $z'_{R^*}(T) \geq 0$. In addition, from $z'' + qz' = \frac{1}{s}[x(x' + sx)]' = x'(x' + sx) + x(x' + sx)' \leq 0$ in $[T, T_{R^*}^2]$, we have that $z'_{R^*} < 0$ in $(T, T_{R^*}^2]$. Thus $z_{R^*}(T_{R^*}^2) < R^*$. However, at $T_{R^*}^2$, $sx_{R^*}(R^* - z_{R^*}) = (x'_{R^*} + sx_{R^*})' + (x'_{R^*} + sx_{R^*}) \leq 0$ which implies that $z_{R^*}(T_{R^*}^2) \geq R^*$; we get a contradiction. This contradiction shows that $(x'_{R^*} + sx_{R^*}) < 0$ in $(0, T_{R^*}^2)$. Now if $(x'_{R^*} + sx_{R^*})' = 0$ at $T_{R^*}^2$, then since by continuity and the definition of R^* , $x'_{R^*} + sx_{R^*}$ is positive in $[0, T_{R^*}^2)$ and non-positive in $[T_{R^*}^2, T_{R^*}^2 + \delta)$ for some $\delta > 0$, we must have $(x_{R^*} - sx_{R^*})'' = 0$. This is equivalent to, recall that $y = \frac{1}{s}(x' + sx)$, $y_{R^*} = y'_{R^*} = y''_{R^*} = 0$ at $T_{R^*}^2$, which, in turn, implies that $x_{R^*} = y_{R^*} = z_{R^*} = 0$; clearly this is impossible. Therefore, $T_{R^*}^2$ satisfies the second condition in the definition of $\mathfrak{R}$.

The final step: Now let $T_{R^*}^3 = \liminf_{R \in \mathfrak{R}, R \searrow R^*} T_{R^*}^3$. We first claim that $T_{R^*}^3 = \infty$. Assume that this is not true. Then $x(T_{R^*}^3) = 0$ and $x_{R^*} \geq 0$ for all $t \in (-\infty, T_{R^*}^3]$. From the continuity and the definition of R^* , $x'_{R^*} + sx_{R^*} \leq 0$ for all $t \in [T_{R^*}^2, T_{R^*}^3]$. If $(x'_{R^*} + sx_{R^*})(T) = 0$ for some T in $(T_{R^*}^2, T_{R^*}^3)$, then $(x'_{R^*} + sx_{R^*})'(T) = 0$, $(x'_{R^*} + sx_{R^*})''(T) \leq 0$, and $x_{R^*}(T) > 0$. But from the equations $(x'_{R^*} + sx_{R^*})' + (x'_{R^*} + sx_{R^*})' = s(R^* - z_{R^*})x_{R^*}$ and $z'_{R^*} = \frac{1}{s}x_{R^*}(x'_{R^*} + sx_{R^*}) - qz_{R^*}$, this implies that $z_{R^*}(T) = R^*$ and $z'_{R^*}(T) < 0$. Consequently, $(x'_{R^*} + sx_{R^*})''(T) = -z'_{R^*}(T)x_{R^*}(T) > 0$, which is a contradiction. This contradiction shows that $x'_{R^*} + sx_{R^*} < 0$ in $(T_{R^*}^2, T_{R^*}^3)$. Also, since $x'_{R^*}(T_{R^*}^3) = 0$ implies that the trajectory is the z -axis, we must have $x'(T_{R^*}^3) < 0$. Therefore, $T_{R^*}^3$ satisfies the third condition in the definition of $\mathfrak{R}$. Hence, from the first two steps, we have that $R^* \in \mathfrak{R}$, again a contradiction. This contradiction shows that $T_{R^*}^3 = \infty$. Hence, we have that $x_{R^*} > 0$ for all t and, and by a similar argument as above, $x'_{R^*} < -sx_{R^*} < 0$ for all $t > T_{R^*}^2$. Therefore, as $t \rightarrow \infty$, x_{R^*} and x'_{R^*} approach zero. That is, $\gamma_{R^*}^+$ approaches the origin.

This completes the proof of Theorem 3.1. $\square$

4. Solutions for large R .

To prove Lemma 3.3, we now study the behavior of the solutions of (1.1) for large R . To this end, we introduce a variation of a transformation which was first used by Robbins [13] and then by Swinnerton-Dyer [18] to establish, among other things, the existence of periodic solutions of (1.1) for large R . The transformation is as follows (assume that $q \neq 2s$):

$$\begin{aligned} \tau &= \sqrt{s(R-1)}t, \quad \varepsilon = (1+s)/\sqrt{s(R-1)}, \quad \alpha = 4\mu s/(1+s), \quad \beta = q/(1+s), \\ x_R(t) &= 2\sqrt{s(R-1)}x_\varepsilon(\tau), \quad Q_R(t) = 4\mu\sqrt{s(R-1)}Q_\varepsilon(\tau). \end{aligned} \quad (4.1)$$

Under this transformation, the system (1.1) or (2.2) becomes

$$\begin{cases} \ddot{x}_\varepsilon = x_\varepsilon(1 - 2x_\varepsilon^2) - \varepsilon(\dot{x}_\varepsilon + \alpha x_\varepsilon Q_\varepsilon), \\ \dot{Q}_\varepsilon = x_\varepsilon^2 - \varepsilon\beta Q_\varepsilon \end{cases} \quad (4.2)$$

where $\dot{\cdot} = \frac{d}{d\tau}$.

In the sequel, we shall use the phase space $(x, \dot{x}, Q)$, so that a trajectory of (4.2) refers to its trace of points $(x_\varepsilon(\tau), \dot{x}_\varepsilon(\tau), Q_\varepsilon(\tau))$ for all $\tau \in (-\infty, \infty)$.

When $\varepsilon = 0$, the system (4.2) can be solved by quadrature. A solution of interest is given in the following lemma.

Lemma 4.1. *When $\varepsilon = 0$, the system (4.2) has a unique trajectory γ_0^+ which approaches the origin as $\tau \rightarrow -\infty$ and lies in the first octant for sufficient negative large τ . Its corresponding solution is given by*

$$X^0(\tau) = (\cosh \tau)^{-1}, \quad Q^0(\tau) = 1 + \tanh \tau, \quad t \in (-\infty, \infty). \quad (4.3)$$

The assertion of the Lemma follows by a direct calculation and details are omitted.

Note that in (4.3), we have normalized the solution such that X^0 obtains its maximum at $\tau = 0$.

One may notice that the stationary point $(0, 0, 0)$ of (4.2) changes from saddle type to degenerate type when ε is changed from positive to zero, so that the continuous dependence of the unstable manifold of the origin with respect ε has to be rigorous verified. Hence, we provide the following lemma for completeness.

Lemma 4.2. (1) *For every $\varepsilon \geq 0$, (4.2) has a unique trajectory γ_ε^+ which approaches the origin as $\tau \rightarrow -\infty$ and initially x_ε is positive.*

(2) *There exists a positive constant δ_1 such that for every $\varepsilon \in [0, 1]$, γ_ε^+ has the expansion*

$$x_\varepsilon = \xi, \quad \dot{x}_\varepsilon = P(\varepsilon, \xi), \quad Q_\varepsilon = Q(\varepsilon, \xi) \quad \text{for all } \xi \in (0, \delta_1]$$

where $P(\varepsilon, \xi)$ and $Q(\varepsilon, \xi)$ are analytic functions of ξ and ε in an open neighborhood containing $[0, \delta_1] \times [0, 1]$. In addition, there exists a positive constant C_1 such that for all $\varepsilon \in [0, 1]$ and all $\xi \in [0, \delta_1]$, $P(\varepsilon, \xi)$ and $Q(\varepsilon, \xi)$ satisfy the following estimates:

$$\begin{aligned} |P(\varepsilon, \xi) - \lambda_\varepsilon^P \xi| &\leq C_1 \xi^2 & \text{where } \lambda_\varepsilon^P &:= \sqrt{1 + (\varepsilon/2)^2} - \varepsilon/2, \\ |Q(\varepsilon, \xi) - \lambda_\varepsilon^Q \xi| &\leq C_1 \xi^2 & \text{where } \lambda_\varepsilon^Q &:= 1/(2\lambda_\varepsilon^P + \beta\varepsilon). \end{aligned} \quad (4.4)$$

(3) *There exists a small positive constant $\varepsilon_1 \in (0, 1]$ such that for all $\varepsilon \in [0, \varepsilon_1]$, the x_ε component of γ_ε^+ can obtain its first maximum. In addition, if one normalize the solution $(X^\varepsilon, \dot{X}^\varepsilon, Q^\varepsilon)$ of (4.2) corresponding γ_ε^+ such that X^ε obtains its first maximum at $\tau = 0$, then as a function of variables ε and t , $(X^\varepsilon(\tau), \dot{X}^\varepsilon(\tau), Q^\varepsilon(\tau))$ is analytic in $[0, \varepsilon_1] \times (-\infty, \infty)$. Consequently, for all $T > 0$, there exists a constant $C_2(T)$ such that*

$$\|X^\varepsilon - X^0\|_{C^2([-T, T])} + \|Q^\varepsilon - Q^0\|_{C^1([-T, T])} \leq C_2(T)\varepsilon.$$

Proof. (1) When $\varepsilon > 0$, (4.2) is equivalent to (1.1) so that γ_ε^+ is well-defined, whereas when $\varepsilon = 0$, γ_0^+ is given by Lemma 4.1.

(2) In a small neighborhood of the origin, γ_ε^+ can be obtained by solving the system of two ordinary differential equations:

$$\frac{d}{dx}P = x(1 - 2x^2)/P - \varepsilon(1 + \alpha Q/P), \quad \frac{d}{dx}Q = x^2/P - \varepsilon\beta Q/P$$

where x is an independent variable. One can verify that this new system has a convergent power series solution $P = P(\varepsilon, x) := \sum_{i=0}^{\infty} p_i(\varepsilon)x^i$, $Q = Q(\varepsilon, x) := \sum_{i=0}^{\infty} q_i(\varepsilon)x^i$ in some interval $[0, \delta_1]$ which is independent of $\varepsilon \in [0, 1]$, provided that one takes $p_0 = 0, p_1 = \lambda_\varepsilon^P$, $q_0 = q_1 = 0$, and $q_2 = \lambda_\varepsilon^Q$ where λ_ε^P and λ_ε^Q are defined in (4.4), whereas all the other coefficients are uniquely determined by the differential equations. Clearly, this solution corresponds to γ_ε^+ . In addition, by adding a dummy equation $\frac{d}{dx}\varepsilon = 0$ if necessary, one can show by the Implicit Function Theorem that $P(\varepsilon, \xi)$ and $Q(\varepsilon, x)$ are analytic in ε and x . This proves the second assertion of the lemma.

(3) From (2), the curve $(\delta_1, P(\varepsilon, \delta_1), Q(\varepsilon, \delta_1))$ is analytic in $\varepsilon \in [0, 1]$. Taking this as initial conditions for (4.2) at $\tau = 0$, we have a solution $(\hat{x}_\varepsilon(\tau), \hat{x}_\varepsilon'(\tau), \hat{Q}_\varepsilon(\tau))$ which is analytic in both ε and τ . In addition, since at the point where $\hat{x}_0$ obtains its maximum, $\hat{x}_0'' = -1 \neq 0$, for all small ε , we can use the Implicit Function Theorem to solve, for T_ε , the equation $\hat{x}_\varepsilon'(T_\varepsilon) = 0$ near the place where $\hat{x}_0$ obtains its maximum. The function T_ε so obtained is analytic in ε in $[0, \varepsilon_1]$ for some small positive ε_1 . Shift the time phase of the solution $(\hat{x}_\varepsilon, \hat{x}_\varepsilon', \hat{Q}_\varepsilon)$ by T_ε , we then obtain the third assertion of the lemma. $\square$

In the sequel, we shall always use the representation $(X^\varepsilon, \dot{X}^\varepsilon, Q^\varepsilon)$ stated in Lemma 4.2 (3) for γ_ε^+ where $\varepsilon \in [0, \varepsilon_1]$.

Lemma 4.3. *There exists a constant $\varepsilon_2 \in (0, \varepsilon_1]$ such that the following holds:*

- (i) *For every $\varepsilon \in [0, \varepsilon_2]$, $\dot{X}^\varepsilon > 0$ in $(-\infty, 0)$, $\dot{X}^\varepsilon(0) = 0$, and $\dot{X}^\varepsilon(0) < -1/2$.*
- (ii) *For every $\varepsilon \in (0, \varepsilon_2]$, there exists $T_\varepsilon^2 \in (0, \sqrt{\varepsilon}]$ such that*

$$(\dot{X}^\varepsilon + \frac{s\varepsilon}{1+s}X^\varepsilon)(T_\varepsilon^2) = 0, \quad \frac{d}{d\tau}(\dot{X}^\varepsilon + \frac{s\varepsilon}{1+s}X^\varepsilon) < -1/4 \text{ in } [0, T_\varepsilon^2].$$

Proof. Since $\ddot{X}^0(0) = -1$ the first assertion follows from Lemma 4.2, whereas the second assertion follows by the continuity of the solution with respect to ε and τ and the Implicit Function Theorem. $\square$

Observe that under the transformation (4.1), the quantity $x_R' + sx_R$ in the original variables becomes $\frac{2(1+s)^2}{\varepsilon^2}(\dot{x}_\varepsilon + \frac{s\varepsilon}{1+s}x_\varepsilon)$ so that when $R \in [\frac{1}{s}(\frac{1+s}{\varepsilon_2})^2, \infty)$, γ_R^+ defined in section 3 satisfies the first two conditions in the definition of $\mathfrak{R}$. We now proceed to show that the third condition in the definition of $\mathfrak{R}$ is also satisfied by γ_R^+ for large R , or equivalently, by γ_ε^+ for small ε .

Lemma 4.4. (1) *Define $H_\varepsilon = \dot{x}_\varepsilon^2 + x_\varepsilon^4 - x_\varepsilon^2$. Then, along any trajectory of (4.2),*

$$\frac{1}{2\varepsilon}\dot{H}_\varepsilon = -\dot{x}_\varepsilon^2 - \alpha Q_\varepsilon x_\varepsilon \dot{x}_\varepsilon, \quad \text{for all } \varepsilon \geq 0, \tau \in (-\infty, \infty). \quad (4.5)$$

(2) *Define $m := -\int_{-\infty}^{\infty} [(\dot{X}^0)^2 + \alpha Q^0 X^0 \dot{X}^0] d\tau$. Then $m = \frac{2}{1+s}[s - \frac{2q+1}{3}]$.*

(3) *For any $T > 0$, there exists $C_3(T) > 0$ such that for all $\varepsilon \in [0, \varepsilon_2]$*

$$\left| \int_{-\infty}^T [(\dot{X}^\varepsilon)^2 + \alpha Q^\varepsilon X^\varepsilon \dot{X}^\varepsilon - (\dot{X}^0)^2 - \alpha Q^0 X^0 \dot{X}^0] d\tau \right| \leq C_3(T)\varepsilon.$$

Proof. The first two assertions follow by the differential equations in (4.2), the explicit expressions of X^0 and Q^0 in Lemma 4.1, the definition of α in (4.1) and directly computation; details are omitted. We now prove (3).

Let $T_\varepsilon(\delta_1) < 0$ be the time such that $X^\varepsilon(T_\varepsilon(\delta_1)) = \delta_1$. Since at $T_\varepsilon(\delta_1)$, $\dot{X}^\varepsilon > 0$, $T_\varepsilon(\delta_1)$ is analytic in ε . Hence, the integral of $(\dot{X}^\varepsilon)^2 + \alpha Q^\varepsilon X^\varepsilon \dot{X}^\varepsilon$ from $[T_\varepsilon(\delta_1), T]$ differs from the integral of $(\dot{X}^0)^2 + \alpha Q^0 X^0 \dot{X}^0$ from $[T_0(\delta_1), T]$ by a quantity of order ε . Using the change of variables $\xi = X^\varepsilon(\tau)$ we have that

$$\begin{aligned} \int_{-\infty}^{T_\varepsilon(\delta_1)} [(\dot{X}^\varepsilon)^2 + \alpha Q^\varepsilon X^\varepsilon \dot{X}^\varepsilon] d\tau &= \int_0^{\delta_1} [P(\varepsilon, \xi) + \alpha Q(\varepsilon, \xi)\xi] d\xi \\ &= \int_0^{\delta_1} [P(0, \xi) + \alpha Q(0, \xi)\xi] d\xi + O(\varepsilon) = \int_{-\infty}^{T_0(\delta_1)} [(\dot{X}^0)^2 + \alpha Q^0 X^0 \dot{X}^0] d\tau + O(\varepsilon). \end{aligned}$$

The third assertion of the lemma thus follows. $\square$

Now we are ready to prove the following lemma.

Lemma 4.5. *Assume that $s > \frac{2q+1}{3}$. Then there exists $\varepsilon_0 \in (0, \varepsilon_2]$ such that for every $\varepsilon \in (0, \varepsilon_0]$, there exists $T_\varepsilon^3 > T_\varepsilon^2$ such that the following holds:*

$$X^\varepsilon(T_\varepsilon^3) = 0, \quad \dot{X}^\varepsilon < 0 \text{ in } (0, T_\varepsilon^3], \quad \text{and } (\dot{X}^\varepsilon + \frac{s\varepsilon}{1+s}X^\varepsilon) < 0 \text{ in } (T_\varepsilon^2, T_\varepsilon^3].$$

Proof. Since $s > \frac{2q+1}{3}$, m defined in Lemma 4.4 is positive. Let T_m be a large positive constant such that

$$\int_{T_m}^{\infty} [(\dot{X}^0)^2 + \alpha|Q^0 X^0 \dot{X}^0|] d\tau < \frac{1}{8}m, \quad (4.6)$$

$$\delta_m := X^0(T_m) < \min\{\frac{1}{8}, \frac{1}{12}m\}. \quad (4.7)$$

By the continuous dependence of the solution on ε , we can find a positive constant $\varepsilon_0 \in (0, \min\{1/\sqrt{2}, \varepsilon_2, m(8C_3(T_m))^{-1}\}]$ such that for all $\varepsilon \in [0, \varepsilon_0]$,

$$\dot{X}^\varepsilon < -\frac{s\varepsilon}{1+s}X^\varepsilon < 0 \quad \text{in } (T_\varepsilon^2, T_m + 1], \quad (4.8)$$

$$|X^\varepsilon(T_m) - X^0(T_m)| + |Q^\varepsilon(T_m) - Q^0(T_m)| \leq \frac{1}{2}\delta_m. \quad (4.9)$$

Denote the value of H_ε on γ_ε^+ by H^ε . Since $H^\varepsilon(-\infty) = 0$, integrating (4.5) along γ_ε^+ from $-\infty$ to T_m and using Lemma 4.4, we have that

$$\begin{aligned} \frac{1}{2\varepsilon}H^\varepsilon(T_m) &= \int_{-\infty}^{T_m} [- (\dot{X}^\varepsilon)^2 - \alpha Q^\varepsilon X^\varepsilon \dot{X}^\varepsilon] d\tau \\ &= \int_{-\infty}^{\infty} [- (\dot{X}^0)^2 - \alpha Q^\varepsilon X^0 \dot{X}^0] d\tau + \int_{T_m}^{\infty} [(\dot{X}^0)^2 + \alpha Q^\varepsilon X^0 \dot{X}^0] d\tau \\ &\quad + \int_{-\infty}^{T_m} [(\dot{X}^\varepsilon)^2 + \alpha Q^\varepsilon X^\varepsilon \dot{X}^\varepsilon - (\dot{X}^0)^2 - \alpha Q^0 X^0 \dot{X}^0] d\tau \\ &\geq m - \frac{1}{8}m - C_3(T_m)\varepsilon \geq \frac{6}{8}m \quad \forall \varepsilon \in [0, \varepsilon_0]. \end{aligned}$$

Hence, $H^\varepsilon(T_m) \geq \frac{3}{2}m\varepsilon$ for all $\varepsilon \in [0, \varepsilon_0]$.

Now for all $\varepsilon \in (0, \varepsilon_0]$, define

$$T_\varepsilon^3 := \inf \left\{ \tau > T_m \mid X^\varepsilon > 0, \dot{X}^\varepsilon < 0 \text{ and } H^\varepsilon > m\varepsilon \text{ in } [T_m, \tau] \right\}.$$

In view of (4.8), T_ε^3 is well-defined and $T_\varepsilon^3 > T_m + 1$.

Since $\dot{X}^\varepsilon < 0$ in $[T_m, T_\varepsilon^3]$, we have that $0 \leq X^\varepsilon(\tau) \leq X^\varepsilon(T_m) \leq \frac{3}{2}\delta_m \leq \frac{3}{16}$ for all $\tau \in [T_m, T_\varepsilon^3]$. In addition, from the definition of H^ε , in $[T_m, T_\varepsilon^3]$,

$$-\dot{X}^\varepsilon = \sqrt{H^\varepsilon + (X^\varepsilon)^2[1 - (X^\varepsilon)^2]} \geq \sqrt{m\varepsilon + \frac{1}{2}(X^\varepsilon)^2}$$

Therefore, we have that $-\dot{X}^\varepsilon \geq \sqrt{m\varepsilon}$ in $[T_m, T_\varepsilon^3]$ which implies that $T_\varepsilon^3 \leq X^\varepsilon(T_m)/\sqrt{m\varepsilon} < \infty$. Also we have that $-\dot{X}^\varepsilon > X^\varepsilon/\sqrt{2}$ in $[T_m, T_\varepsilon^3]$, which, together with (4.8) and the fact that $\varepsilon_0 \leq 1/\sqrt{2}$, yields

$$\dot{X}^\varepsilon + \frac{s\varepsilon}{1+s}X^\varepsilon < 0 \text{ in } (T_\varepsilon^2, T_\varepsilon^3], \quad \text{for all } \varepsilon \in [0, \varepsilon_0].$$

Hence, to finish the proof, we need only show that $X^\varepsilon(T_\varepsilon^3) = 0$.

At T_ε^3 , there are only two possibilities, either $X^\varepsilon(T_\varepsilon^3) = 0$ or $H^\varepsilon(T_\varepsilon^3) = m\varepsilon$. We shall show that the second alternative will not happen by integrating the identity (4.5) from T_m to T_ε^3 . To do this, we need an estimate for the maximum of $\dot{X}^\varepsilon$ in $[T_m, T_\varepsilon^3]$. (Though this can be directly verified by using Lemma 2.1, we provide another proof here for possible other applications.)

Since $Q^\varepsilon > 0$ for all $\tau \in (-\infty, \infty)$ and all $\varepsilon \geq 0$, $\dot{Q}^\varepsilon = -\varepsilon\beta Q^\varepsilon + (X^\varepsilon)^2 < (X^\varepsilon)^2$. Consequently, for all $\tau \in (T_m, T_\varepsilon^3]$,

$$\begin{aligned} Q^\varepsilon(\tau) &< Q^\varepsilon(T_m) + \int_{T_m}^\tau (X^\varepsilon)^2 = Q^\varepsilon(T_m) + \int_{X^\varepsilon(T_m)}^{X^\varepsilon(\tau)} \frac{X^\varepsilon}{\dot{X}^\varepsilon} dX^\varepsilon \\ &\leq Q^\varepsilon(T_m) + \int_0^{X^\varepsilon(T_m)} \frac{x}{\sqrt{m\varepsilon + x^2/2}} dx \leq Q^\varepsilon(T_m) + \sqrt{2}X^\varepsilon(T_m) \leq 4 \end{aligned}$$

since $Q^\varepsilon(T_m) = Q^0(T_m) + \frac{1}{2}\delta_m \leq \frac{5}{2}$ and $X^\varepsilon(T_m) \leq \frac{3}{2}\delta_m < 1/2$. Integrating (4.5) from T_m to $\tau \in [T_m, T_\varepsilon^3]$ then yields

$$\frac{1}{2\varepsilon}[H^\varepsilon(\tau) - H^\varepsilon(T_m)] \leq -\alpha \int_{T_m}^\tau Q^\varepsilon X^\varepsilon \dot{X}^\varepsilon \leq -4\alpha \int_{T_m}^{T_\varepsilon^3} X^\varepsilon \dot{X}^\varepsilon d\tau \leq 2\alpha(X^\varepsilon(T_m))^2 \leq 1/2$$

for all $\varepsilon \in [0, \varepsilon_0]$, since $\alpha = \frac{4\mu s}{1+s} < 4$ and $X^\varepsilon(T_m) \leq \frac{3}{2}\delta_m \leq 1/4$. It then follows from the definition of H^ε that $|\dot{X}^\varepsilon| \leq \sqrt{H^\varepsilon + X^2} \leq 1$ for all $\tau \in [T_m, T_\varepsilon^3]$.

Having the estimate for $|\dot{X}^\varepsilon|$ in $[T_m, T_\varepsilon^3]$, we can now integrating the identity (4.5) from T_m to T_ε^3 and using the fact that $-\alpha Q^\varepsilon X^\varepsilon \dot{X}^\varepsilon \geq 0$ in $(0, T_\varepsilon^3)$, obtaining that

$$\begin{aligned} \frac{1}{2\varepsilon}[H^\varepsilon(T_\varepsilon^3) - H^\varepsilon(T_m)] &\geq -\int_{T_m}^{T_\varepsilon^3} (\dot{X}^\varepsilon)^2 d\tau = -\int_{X^\varepsilon(T_m)}^{X^\varepsilon(T_\varepsilon^3)} \dot{X}^\varepsilon dX^\varepsilon \\ &\geq -X^\varepsilon(T_m) \geq -\frac{3}{2}\delta_m \quad \forall \varepsilon \in [0, \varepsilon_0] \end{aligned}$$

Therefore, $H^\varepsilon(T_\varepsilon^3) \geq H^\varepsilon(T_m) - 3\varepsilon\delta_m \geq \frac{3}{2}m\varepsilon - 3\varepsilon\delta_m \geq \frac{5}{4}m\varepsilon$ by the smallness of δ_m . Hence, we must have $X^\varepsilon(T_\varepsilon^3) = 0$. This completes the proof of the lemma. $\square$

As mentioned early, since $X'_R + sX_R = \frac{2(1+s)^2}{\varepsilon^2}[\dot{x}_\varepsilon + \frac{s\varepsilon}{1+s}x_\varepsilon]$, Lemmas 4.3 and 4.5 imply that for R large enough, γ_R^+ satisfies the three conditions in the definition $\mathfrak{R}$; that is, $R \in \mathfrak{R}$. The assertion of Lemma 3.3 thus follows, thereby completing the proof of Theorem 3.1.

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