THE STRONG STABILITY OF A SEMIGROUP ARISING FROM A COUPLED HYPERBOLIC/PARABOLIC SYSTEM

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The Strong Stability of a Semigroup Arising from a coupled Hyperbolic/Parabolic System

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Abstract

We consider here a coupled system of hyperbolic and parabolic PDE's which arises in a given fluid/structure interaction. The system is transformed into an abstract differential equation, and from this operator theoretic model, questions of strong stability for the equation are addressed. A distinctive feature of the problem is that the resolvent of the operator is not compact, and hence a treatment with the standard Nagy–Foias theory or Lasalle Invariance Priniciple is not available. Instead, we show that a powerful stability result of Arendt–Batty applies, and which consequently proves strong decay of the energy functional.

1 Introduction

1.1 Statement of Problem and Motivation

Let Ω be an bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with Lipshitz boundary Γ, and Γ_0 a segment of Γ. We consider here the problem of finding functions $z(t,x)$ and $v(t,x)$ which solve the following system comprised of a "coupling" between a wave and elastic plate–like equation:

$$\begin{cases} z_{tt} = \Delta z & \text{on } (0,\infty)\times\Omega \\ z(0,x) = z^0, z_t(0,x) = z^1 & \text{on } \Omega \\ z(t,x) = 0 & \text{on } (0,\infty)\times\Gamma\setminus\Gamma_0 \\ \dfrac{\partial z(t,x)}{\partial\nu} + \alpha z_t(t,x) = v_t & \text{on } (0,\infty)\times\Gamma_0 \text{ with } \alpha \geq 0; \end{cases} \tag{1}$$

$$\begin{cases} v_{tt} = -\Delta^2 v - \Delta^2 v_t - z_t & \text{on } (0,\infty)\times\Gamma_0 \\ v(0,x) = v^0, v_t(0,x) = v^1 & \text{on } \Gamma_0 \\ v(t,x) = \dfrac{\partial v(t,x)}{\partial\nu} = 0 & \text{on } (0,\infty)\times\partial\Gamma_0. \end{cases} \tag{2}$$

We are interested here in the stability of the solutions $[\overrightarrow{z}, \overrightarrow{v}]^T \equiv [z, z_t, v, v_t]^T$ to (1)–(2); viz. we are concerned with the decay of the "energy" $E(\overrightarrow{z}, \overrightarrow{v}, t)$ of the system (1)–(2) as $t \to \infty$, where

$$E(\overrightarrow{z}, \overrightarrow{v}, t) = \int_\Omega \left[|\nabla z(t)|^2 + |z_t(t)|^2\right] d\Omega + \int_{\Gamma_0} \left[|\Delta v(t)|^2 + |v_t(t)|^2\right] d\Gamma_0. \tag{3}$$

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The "structural acoustics" model (1)–(2) mathematically describes the interaction of a vibrating beam/plate in an enclosed acoustic field or chamber. The model has been an object of intensive experimental and numerical studies at the NASA Langley Research Center (see [4],[5] and references therein), particularly in the context of smart material technology.

The mathematical challenge in this problem stems from the fact that we are dealing with a system comprised of coupled hyperbolic and "parabolic-like" equations, where the coupling takes place on the boundary through appropriate trace operators. It will be shown below that these couplings are modelled by *unbounded and unclosable* operators. As a result, many interesting–and mostly open–issues arise, including those of controllability, stabilizability, and optimal boundary control.

Of additional mathematical interest here is that the available theories in the literature which pertain to boundary control (see [6] and [13]) *are not* applicable for the problem as these works rely heavily on the system being of either hyperbolic or parabolic character. Most recently, the authors were able to develop an optimal control theory for this model (with $\alpha = 0$), relying heavily on the "sharp" trace regularity of hyperbolic solutions in combination with the propagation of smoothing effects associated with a structurally damped plate (see [2]).

The aim of this paper is to address issues regarding the boundary stability of (1)–(2). As it will be shown below, the given model corresponds to a contractive semigroup for $\alpha \geq 0$, with the energy of the system consequently being nondecreasing. Moreover, we prove that if $\alpha \geq 0$, the corresponding system exhibits the property of *strong* stability. This result is optimal, since it is known that uniform (exponential) stability cannot be achieved with only a small portion of the boundary subjected to damping (even if $\alpha > 0$); in our case, the active Γ_0 can be an arbitrarily small subportion of the boundary Γ.

Finally, we note that the main technical difficulty associated with our strong stability problem is the lack of compactness of the resolvent. For this reason, the usual approaches via Nagy–Foias theory and the LaSalle Invariance Principle are inappropriate. To cope with this complication, we will resort to a powerful and very interesting semigroup result due to Arendt–Batty (also proved independently in [14]), and due to this result, our task here will boil down to eliminating all three parts of the spectrum (of the generator) from the imaginary axis.

1.2 Preliminaries

In dealing with (1)–(2), we will find it useful to work with an associated abstract evolution equation for which we will need the following facts and definitions:

- Let the operator $A : L^2(\Omega) \supset D(A) \to L^2(\Omega)$ be defined by

$$Az = -\Delta z, D(A) = \left\{ z : \Delta z \in L^2(\Omega);\ z|_{\Gamma\setminus\Gamma_0} = 0,\ \frac{\partial z}{\partial \nu}\Big|_{\Gamma_0} = 0 \right\}. \tag{4}$$

  Note that A is self-adjoint, positive definite, and hence the fractional powers of A are well defined.

- By [10], we have the following characterization:

$$\begin{aligned} &D(A^{\frac{1}{2}}) = H^1_{\Gamma\setminus\Gamma_0}(\Omega) = \left\{ z \in H^1(\Omega) \ni z = 0 \text{ on } \Gamma \setminus \Gamma_0 \right\}, \\ &\text{with } \|z\|^2_{D(A^{\frac{1}{2}})} = \left\| A^{\frac{1}{2}} \right\|^2_{L^2(\Omega)} = \textstyle\int_\Omega |\nabla z|^2 \, d\Omega = \|z\|^2_{H^1_{\Gamma\setminus\Gamma_0}(\Omega)} \ \forall\, z \in D(A^{\frac{1}{2}}), \end{aligned} \tag{5}$$

  where the last equality in (5) follows from Poincaré's inequality.

- We define the map N by

$$z = Ng \Longleftrightarrow \begin{cases} \Delta z = 0 & \text{on } \Omega \\ z|_{\Gamma\setminus\Gamma_0} = 0 & \text{on } \Gamma\setminus\Gamma_0 \\ \frac{\partial z}{\partial\nu}\Big|_{\Gamma_0} = g & \text{on } \Gamma_0; \end{cases} \tag{6}$$

elliptic theory will then yield that

$$N \in \mathcal{L}(L^2(\Gamma_0), D(A^{\frac{3}{4}-\epsilon})) \;\; \forall \epsilon > 0. \tag{7}$$

- Let $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma_0)$ be the restriction to Γ_0 of the familiar Sobolev trace map; viz.

$$\forall\, z \in H^1(\Omega), \gamma(z) = \begin{cases} z\,|_{\Gamma_0} & \text{on } \Gamma_0 \\ 0 & \text{on } \Gamma\setminus\Gamma_0. \end{cases} \tag{8}$$

Then as is shown in [17], we have

$$N^*A = \gamma(z) \;\; \forall\, z \in D(A^{\frac{1}{2}}), \tag{9}$$

where N^* is the adjoint of $N : L^2(\Gamma) \to L^2(\Omega)$.

- We set $\mathring{\mathbf{A}} : L^2(\Gamma_0) \supset D(\mathring{\mathbf{A}}) \to L^2(\Gamma_0)$ to be

$$\mathring{\mathbf{A}} = \Delta^2, D(\mathring{\mathbf{A}}) = H^4(\Gamma_0) \cap H_0^2(\Gamma_0); \tag{10}$$

$\mathring{\mathbf{A}}$ is also self–adjoint, positive definite, again by [10] we have the characterization

$$\begin{aligned} &D(\mathring{\mathbf{A}}^{\theta}) = H_0^{4\theta}(\Gamma_0)\ , \quad 0 \le \theta \le \tfrac{1}{2}, \\ &\text{with } \left\|\mathring{\mathbf{A}}^{\frac{1}{2}} v\right\|^2_{L^2(\Gamma_0)} = \textstyle\int_{\Gamma_0} |\Delta v|^2\, d\Omega = \|v\|^2_{H_0^2(\Gamma_0)} \;\; \forall\, v \in D(\mathring{\mathbf{A}}^{\frac{1}{2}}). \end{aligned} \tag{11}$$

- We define the energy spaces

$$H_1 \equiv D(A^{\frac{1}{2}}) \times L^2(\Omega)\,; \tag{12}$$

$$H_0 \equiv D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times L^2(\Gamma_0). \tag{13}$$

- We define $A_1 : H_1 \supseteq D(A_1) \to H_1$ and $A_0 : H_0 \supseteq D(A_0) \to H_0$ to be

$$\begin{aligned} &A_1 \equiv \begin{bmatrix} 0 & I \\ -A & -\alpha ANN^*A \end{bmatrix} \text{ with} \\ &D(A_1) = \left\{ [z_1, z_2]^T \in \left[D(A^{\frac{1}{2}})\right]^2 \ni z_1 + \alpha NN^*Az_2 \in D(A) \right\} \\ &\text{(again, with parameter } \alpha \ge 0); \end{aligned} \tag{14}$$

$$\begin{aligned} &A_0 \equiv \begin{bmatrix} 0 & I \\ -\mathring{\mathbf{A}} & -\mathring{\mathbf{A}} \end{bmatrix} \text{ with} \\ &D(A_0) = \left\{ [v_1, v_2]^T \in \left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]^2 \ni v_1 + v_2 \in D(\mathring{\mathbf{A}}) \right\}. \end{aligned} \tag{15}$$

With the above operator definitions, we set

$$\mathcal{A} = \left[\begin{array}{cc|cc} \multicolumn{2}{c}{A_1} & 0 & 0 \\ & & 0 & AN \\ 0 & 0 & \multicolumn{2}{c}{A_0} \\ 0 & -N^*A & & \end{array}\right] \text{ with}$$

$$D(\mathcal{A}) = \left\{[z_1, z_2, v, v_2]^T \in \left[D(A^{\frac{1}{2}})\right]^2 \times \left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]^2 \right. \quad (16)$$
$$\left. \text{such that } - z_1 - \alpha N N^* A z_2 + N v_2 \in D(A) \text{ and } v_1 + v_2 \in D(\mathring{\mathbf{A}})\right\}.$$

If we take the initial data $[z^0, z^1, v^0, v^1]^T$ to be in $H_1 \times H_0$ (as defined in (12)-(13)), we can use the definitions above to rewrite (1)–(2) abstractly as

$$\frac{d}{dt}\begin{bmatrix} z \\ z_t \\ v \\ v_t \end{bmatrix} = \mathcal{A}\begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} \quad (17)$$
$$[z(0), z_t(0), v(0), v_t(0)]^T = [z^0, z^1, v^0, v^1]^T.$$

Remark 1 *The structure of $\mathcal{A}$, given in (16), clearly reflects the coupled nature of this particular system; The operator A_1 which models hyperbolic dynamics is linked via an unbounded coupling with the "elastic" operator A_0 which exhibits parabolic characteristics, and where the coupling is accomplished by the "trace" operators C and C^*.*

We show below that $\mathcal{A}$ generates a C_0–semigroup of contractions $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$, which establishes the well–posedness of the system (17), and our study of the energy decay of the system (1)–(2) will thus be tantamount to investigating the question of *strong stability* of $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$; that is, does

$$\lim_{t\to\infty} e^{\mathcal{A}t}\begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} \to 0 \quad \forall \begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} \in H_1 \times H_0 \ ?$$

1.3 Literature

There is a large quantity of recent results in regards to the well–posedness and stability of the individual components A_1 and A_0 of the dynamics $\mathcal{A}$. [15], [16], [17], and [18] all give results pertaining to the existence and uniqueness and/or strong stability of the wave equation modelled by A_1 (with $\alpha > 0$ only). Also, for the wave equation modeled by this same operator, [7], [11] and [17] show the uniform stabilization of solutions, under suitable geometric conditions on the region Ω. The main results pertaining to the operator A_0 are those provided by [8], which include the information that A_0 generates an analytic semigroup, which subsequently gives the well–posedness and exponential decay on $H_1 \times H_0$ for solutions of

$$v_{tt} = -\Delta^2 v - \Delta^2 v_t \text{ on } (0,\infty) \times \Gamma_0$$

$$v(t) = v(t) = \frac{\partial v(t)}{\partial \nu} = \frac{\partial v(t)}{\partial v} = 0 \text{ on } \partial\Gamma_0$$

$$[v(0), v_t(0)]^T = [v^0, v^1]^T \in H_0.$$

Our main result will reveal that stability properties generated by the analytic part of the system (A_0) is passed onto the entire structure $\mathcal{A}$, even though the hyperbolic part A_1 may be unstable (as is the case when $\alpha = 0$).

We note here that the techniques used in proving the strong stability of the individual components A_1 (with $\alpha > 0$) and A_0 *are not* applicable for the present task at hand. Indeed, in [17], the Nagy–Foias–Fogel decomposition is used to show the weak stability of the semigroup $\{e^{A_1 t}\}_{t\geq 0}$, and its strong stability follows from the fact that A_1 *has compact resolvent.* An analogous argument for our operator $\mathcal{A}$ *is not* available, as for $\lambda \in \rho(\mathcal{A})$, $(\lambda - \mathcal{A})^{-1}$ is not compact. Also, as we have said above, uniform stability for A_0 follows automatically from the discovery that A_0 generates analytic dynamics; $\mathcal{A}$, as it stands here, *is not* an analytic generator; it is a coupling of hyperbolic and parabolic dynamics.

1.4 Statement of Main Results

Theorem 1 *$\mathcal{A}$ given by (16) generates a C_0–semigroup of contractions $\{e^{\mathcal{A}\,t}\}_{t\geq 0}$ on the energy space $H_1 \times H_0$.*

Theorem 2 *The semigroup $\{e^{\mathcal{A}\,t}\}_{t\geq 0}$ is strongly stable; that is, $\forall\ [z^0, z^1, v^0, v^1]^T \in H_1 \times H_0$, one has*

$$\lim_{t \to \infty} e^{\mathcal{A}\,t} \begin{bmatrix} z^0 \\ z^1 \\ v^0 \\ v^1 \end{bmatrix} \to 0; \tag{18}$$

consequently from (3), $E(\overrightarrow{z}, \overrightarrow{v}, t) \to 0$ as $t \to \infty$.

Remark 2 **Theorem 1.2** *reveals that the coupled system is strongly stable, even in the absence of the "damping " boundary term z_t in (1) (corresponding to $\alpha \equiv 0$).*

The proof for strong stability here depends upon showing that certain spectral conditions are satisfied which will allow for the application of the following intrinsically interesting result quoted here from the authors in [1]:

Stability Theorem. *Let X be reflexive. Assume that the C_0–semigroup $\{T(t)\}_{t\geq 0}$ generated by a linear operator L is bounded and no eigenvalue of L lies on the imaginary axis. If $\sigma(L) \cap i\mathbb{R}$ is countable, then $\{T(t)\}_{t\geq 0}$ is strongly stable.*

From **Theorem 1.1,** $\|e^{\mathcal{A}\,t}\| \leq 1\ \forall t$, and so to prove **Theorem 1.2**, it will suffice to show that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

Remark 3 *Our result/proof applies to a more general model than (1)-(2), by allowing (2) to be replaced by any structurally damped problem*

$$v_{tt} + \tilde{A}v + \tilde{A}^{\rho}v_t - z_t = 0,$$

where $\tilde{A}$ is self-adjoint, positive definite, and $\rho \geq 1/2$. Thus other beam and plate models, as well as those of shells, can be considered here.

Remark 4 *The same stability result also holds for the system (1)-(2) with "pure" Neumann boundary data, i.e.*

$$\frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma \setminus \Gamma_0,$$

$$\frac{\partial z}{\partial \nu} + \alpha z_t + \beta z = v_t \text{ on } \Gamma_0,\ \beta \in \mathbb{R} \setminus \{0\}.$$

2 Proofs of the Main Theorems

2.1 The Proof of Theorem 1.1 (Semigroup generation)

In order to show the semigroup generation for the dynamics $\mathcal{A}$, we wish to use the Lumer–Phillips theorem, and hence we must show the maximal dissipativity of $\mathcal{A}$.

The dissipativity of $\mathcal{A}$ is straightforward: Indeed, we have $\forall\ [\overrightarrow{z}, \overrightarrow{v}]^T \in D(\mathcal{A})$,

$$\left(\mathcal{A}\begin{bmatrix} z_1 \\ z_2 \\ v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ v_1 \\ v_2 \end{bmatrix}\right)_{H_1\times H_0} = \left(Az_2^{\frac{1}{2}}, A^{\frac{1}{2}}z_1\right)_{L^2(\Omega)} - \langle Az_1, z_2\rangle_{\left[D(A^{\frac{1}{2}})\right]'\times D(A^{\frac{1}{2}})}$$

$$-\alpha\langle ANN^*Az_2, z_2\rangle_{\left[D(A^{\frac{1}{2}})\right]'\times D(A^{\frac{1}{2}})} + \langle ANv_2, z_2\rangle_{\left[D(A^{\frac{1}{2}})\right]'\times D(A^{\frac{1}{2}})}$$

$$+\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_2, \mathring{\mathbf{A}}^{\frac{1}{2}}v_1\right)_{L^2(\Gamma_0)} - (N^*Az_2, v_2)_{L^2(\Gamma_0)} - \left\langle\mathring{\mathbf{A}}v_1, v_2\right\rangle_{\left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]'\times D(\mathring{\mathbf{A}}^{\frac{1}{2}})}$$

$$-\left\langle\mathring{\mathbf{A}}v_2, v_2\right\rangle_{\left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]'\times D(\mathring{\mathbf{A}}^{\frac{1}{2}})} \Rightarrow$$

$$\mathbf{Re}\left(\mathcal{A}\begin{bmatrix} z_1 \\ z_2 \\ v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ v_1 \\ v_2 \end{bmatrix}\right)_{H_1\times H_0} = -\alpha\,\|N^*Az_2\|^2_{L^2(\Gamma_0)} - \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}v_2\right\|^2_{L^2(\Gamma_0)} \leq 0;$$

and so $\mathcal{A}$ is dissipative.

To show the maximality of $\mathcal{A}$: For $\lambda > 0$, if $[z_1, z_2, v_1, v_2]^T \in D(\mathcal{A})$ solves

$$(\lambda - \mathcal{A})\begin{bmatrix} z_1 \\ z_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} \in H_1 \times H_0, \tag{19}$$

then *a fortiori*, (19) is equivalent to solving

$$\Lambda_\lambda\begin{bmatrix} z_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} f_2 + \lambda f_1 + \alpha ANN^*Af_1 - ANg_1 \\ g_2 + \lambda g_1 + N^*Af_1 + \mathring{\mathbf{A}}g_1 \end{bmatrix}, \tag{20}$$

where Λ_λ is defined as

$$\Lambda_\lambda = \begin{bmatrix} \lambda^2 + A + \alpha\lambda ANN^*A & -\lambda AN \\ \lambda N^*A & \lambda^2 + (1+\lambda)\mathring{\mathbf{A}} \end{bmatrix}. \tag{21}$$

Considering its components, Λ_λ is easily seen to be an element of $\mathcal{L}\left(D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}}), \left[D(A^{\frac{1}{2}})\right]' \times \left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]'\right)$, and if Λ_λ is $D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}})$–elliptic, then its inverse $\Lambda_\lambda^{-1} \in \mathcal{L}\left(\left[D(A^{\frac{1}{2}})\right]' \times \left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]', D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right)$ will exist from Lax–Milgram; and as the RHS of (20) is in the dual $\left[D(A^{\frac{1}{2}})\right]' \times \left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]'$, we can solve for z_1 and

v_1, and subsequently for z_2 and v_2 (after using the relations given by (19)), hence establishing the maximality of $\mathcal{A}$. In fact, letting $\langle \cdot, \cdot \rangle$ denote the pairing between $D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}})$ and its dual, we have from (21) that $\forall\ [z, v] \in D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}})$,

$$\left\langle \Lambda_\lambda \begin{bmatrix} z \\ v \end{bmatrix}, \begin{bmatrix} z \\ v \end{bmatrix} \right\rangle = \lambda^2 \|z\|^2_{L^2(\Omega)}$$

$$+ \left\| A^{\frac{1}{2}} z \right\|^2_{L^2(\Omega)} + \alpha\lambda \|N^* A z\|^2_{L^2(\Gamma_0)}$$

$$-\lambda \langle ANv, z \rangle_{\left[D(A^{\frac{1}{2}})\right]' \times D(A^{\frac{1}{2}})} + \lambda (N^* A z, v)_{L^2(\Gamma_0)} \tag{22}$$

$$+\lambda^2 \|v\|^2_{L^2(\Gamma_0)} + (1+\lambda) \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} v \right\|^2_{L^2(\Gamma_0)}$$

$$\geq c(\lambda) \left[\|z\|^2_{D(A^{\frac{1}{2}})} + \|v\|^2_{D(\mathring{\mathbf{A}}^{\frac{1}{2}})} \right];$$

i.e. Λ_λ is coercive. As mentioned above, the maximality of $\mathcal{A}$ is consequently deduced, and so by Lumer–Phillips $\mathcal{A}$ generates a C_0−semigroup of contractions $\left\{ e^{\mathcal{A}t} \right\}_{t \geq 0}$.

2.2 The Proof of Theorem 1.2 (Strong Stability)

In proving **Theorem 1.2**, we will have need of the following two lemmas.

Lemma 1 *With $\mathcal{A}$ as defined in (16), the (Hilbert space) adjoint $\mathcal{A}^*$ is given to be*

$$\mathcal{A}^* = \begin{bmatrix} 0 & -\mathbf{I} & 0 & 0 \\ A & -\alpha ANN^*A & 0 & -AN \\ 0 & 0 & 0 & -\mathbf{I} \\ 0 & N^*A & \mathring{\mathbf{A}} & -\mathring{\mathbf{A}} \end{bmatrix},$$

with $D(\mathcal{A}^*) = \left\{ [z_1 z_2, v_1, v_2]^T \in D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \right.$
such that $\left. z_1 - \alpha NN^*Az_2 - Nv_2 \in D(A) \text{ and } v_1 - v_2 \in D(\mathring{\mathbf{A}}) \right\}$.

Proof: If we denote $\mathbf{S}$ to be

$$\mathbf{S} \equiv \left\{ [z_1^*, z_2^*, v_1^*, v_2^*]^T \in D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \right.$$
$$\left. \text{such that } z_1^* - \alpha NN^*Az_2^* - Nv_2^* \in D(A) \text{ and } v_1^* - v_2^* \in D(\mathring{\mathbf{A}}) \right\},$$

then, $\forall\ [\overrightarrow{z}, \overrightarrow{v}]^T \in D(\mathcal{A})$, $\left[\overrightarrow{z^*}, \overrightarrow{v^*}\right]^T \in \mathbf{S}$, we have

$$\left(\mathcal{A}\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix}, \begin{bmatrix} \overrightarrow{z^*} \\ \overrightarrow{v^*} \end{bmatrix}\right)_{H_1\times H_0} =$$

$$\langle z_2, Az_1^*\rangle_{D(A^{\frac{1}{2}})\times\left[D(A^{\frac{1}{2}})\right]'} - \left(A^{\frac{1}{2}}z_1, A^{\frac{1}{2}}z_2^*\right)_{L^2(\Omega)}$$

$$-\alpha\,\langle z_2, ANN^*Az_2^*\rangle_{D(A^{\frac{1}{2}})\times\left[D(A^{\frac{1}{2}})\right]'} + (v_2, N^*Az_2^*)_{L^2(\Gamma_0)}$$

$$+\left\langle v_2, \mathring{\mathbf{A}}v_1^*\right\rangle_{D(\mathring{\mathbf{A}}^{\frac{1}{2}})\times\left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]'} - \langle z_2, ANv_2^*\rangle_{D(A^{\frac{1}{2}})\times\left[D(A^{\frac{1}{2}})\right]'}$$

$$-\left(\mathring{\mathbf{A}}^{\frac{1}{2}}v_1, \mathring{\mathbf{A}}^{\frac{1}{2}}v_2^*\right)_{L^2(\Gamma_0)} - \left\langle v_2, \mathring{\mathbf{A}}v_2^*\right\rangle_{D(\mathring{\mathbf{A}}^{\frac{1}{2}})\times\left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]'}$$

$$= \left(\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix}, \Lambda\begin{bmatrix} \overrightarrow{z^*} \\ \overrightarrow{v^*} \end{bmatrix}\right)_{H_1\times H_0},$$

where $\Lambda = \begin{bmatrix} 0 & -I & 0 & 0 \\ A & -\alpha ANN^*A & 0 & -AN \\ 0 & 0 & 0 & -\mathbf{I} \\ 0 & N^*A & \mathring{\mathbf{A}} & -\mathring{\mathbf{A}} \end{bmatrix}$.

Hence $\mathbf{S} \subseteq D(\mathcal{A}^*)$ and $\mathcal{A}^*|_{\mathbf{S}} = \Lambda$. (23)

To show the opposite containment, we can straightforwardly calculate the inverse of $\mathcal{A}$, and subsequently that of $\mathcal{A}^*$ (as $\mathcal{A}$ is closed) to have

$$(\mathcal{A}^*)^{-1} = \left(\mathcal{A}^{-1}\right)^* = \begin{bmatrix} -\alpha NN^*A & A^{-1} & -N & 0 \\ -\mathbf{I} & 0 & 0 & 0 \\ \mathring{\mathbf{A}}^{-1}N^*A & 0 & -\mathbf{I} & \mathring{\mathbf{A}}^{-1} \\ 0 & 0 & -\mathbf{I} & 0 \end{bmatrix}. \tag{24}$$

So if arbitrary $\left[\overrightarrow{z^*}, \overrightarrow{v^*}\right]^T \in D(\mathcal{A}^*)$ satisfies

$$\mathcal{A}^*\begin{bmatrix} \overrightarrow{z^*} \\ \overrightarrow{v^*} \end{bmatrix} = \begin{bmatrix} \overrightarrow{f} \\ \overrightarrow{g} \end{bmatrix} \in H_1 \times H_0, \tag{25}$$

then upon applying $(\mathcal{A}^*)^{-1}$ to (25), we have equivalently that

$$\begin{cases} z_1^* = -\alpha NN^*Af_1 + A^{-1}f_2 - Ng_1 \in D(A^{\frac{1}{2}}) \\ z_2^* = -f_1 \in D(A^{\frac{1}{2}}) \\ v_1^* = \mathring{\mathbf{A}}^{-1}N^*Af_1 - g_1 + \mathring{\mathbf{A}}^{-1}g_2 \in D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \\ v_2^* = -g_1 \in D(\mathring{\mathbf{A}}^{\frac{1}{2}}). \end{cases} \tag{26}$$

Further, using the relations (26), we have that $z_1^* - \alpha NN^*Az_2^* - Nv_2^* \in D(A)$ and $v_1^* - v_2^* \in D\left(\mathring{\mathbf{A}}\right)$, and from (23) we conclude that $D(\mathcal{A}^*) = \mathbf{S}$ and $\mathcal{A}^* \equiv \Lambda$.

Lemma 2 *$\forall\, r, \xi \in \mathbb{R}$ with $\xi \neq 0$, the operator $\mathbf{K}_\xi(r) \in \mathcal{L}\left(D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right)$, defined by*

$$\mathbf{K}_\xi(r) \equiv \begin{bmatrix} \mathbf{I} & 0 \\ 0 & (1+ir)\mathbf{I} \end{bmatrix} + \begin{bmatrix} i\alpha r N N^* A - r^2 A^{-1} & -\xi i r N \\ \xi i r \mathring{\mathbf{A}}^{-1} N^* A & -r^2 \mathring{\mathbf{A}}^{-1} \end{bmatrix}, \tag{27}$$

is boundedy invertible.

Proof: It will suffice here to consider $r \neq 0$, as the result is trivial for $r = 0$. From elliptic theory, $A^{-\frac{1}{2}}$ and $\mathring{\mathbf{A}}^{-\frac{1}{2}}$are compact operators on $L^2(\Omega)$ and $L^2(\Gamma_0)$ respectively, and as $N \in \mathcal{L}\left(L^2(\Gamma_0), H^{\frac{3}{2}}(\Omega)\right)$ with the inclusion $H^{\frac{3}{2}}(\Omega) \hookrightarrow H^1(\Omega)$ being compact (see [12]), we then deduce (by noting its components and subsequently applying the trace theorem together with (9)) that

$$\widetilde{\mathbf{K}}_\xi(r) \equiv \begin{bmatrix} i\alpha r N N^* A - r^2 A^{-1} & -\xi i r N \\ \xi i r \mathring{\mathbf{A}}^{-1} N^* A & -r^2 \mathring{\mathbf{A}}^{-1} \end{bmatrix} \tag{28}$$

is a compact operator on $D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}})$; so by the Fredholm Alternative Theorem, we can deduce the asserted invertibility if $\mathbf{K}_\xi(r)$ is injective. Indeed, if $[z, v]^T \in D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}})$satisfies

$$\mathbf{K}_\xi(r) \begin{bmatrix} z \\ v \end{bmatrix} = 0, \tag{29}$$

then necessarily

$$\begin{cases} z + i\alpha r N N^* A z - r^2 A^{-1} z - \xi i r N v = 0 \\ \xi i r \mathring{\mathbf{A}}^{-1} N^* A z + v + i r v - r^2 \mathring{\mathbf{A}}^{-1} v = 0. \end{cases} \tag{30}$$

Taking the duality pairing in (30) with respect to the elements $[Az, \mathring{\mathbf{A}}v]^T \in \left[D(A^{\frac{1}{2}})\right]' \times \left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]'$ yields

$$\begin{cases} \left\|A^{\frac{1}{2}} z\right\|^2_{L^2(\Omega)} + i\alpha r \left\|N^* A z\right\|^2_{L^2(\Gamma_0)} - r^2 \|z\|^2_{L^2(\Omega)} - \xi i r \overline{(N^* A z, v)}_{L^2(\Gamma_0)} = 0 \\ \xi i r \left(N^* A z, v\right)_{L^2(\Gamma_0)} + \left\|\mathring{\mathbf{A}}^{\frac{1}{2}} v\right\|^2_{L^2(\Gamma_0)} + i r \left\|\mathring{\mathbf{A}}^{\frac{1}{2}} v\right\|^2_{L^2(\Gamma_0)} - r^2 \|v\|^2_{L^2(\Gamma_0)} = 0 \end{cases} \tag{31}$$

(where in (31) $(u, w) = \int u\overline{w}$). Adding the two equations in (31) induces a cancellation of imaginary parts which yields that $\mathring{\mathbf{A}}^{\frac{1}{2}} v = 0$; so by (11) $v = 0$. This in turn, from the use of the second equation in (30) and (9), provides that $z|_{\Gamma_0} = 0$. Thus, z satisfies

$$\Delta z = r^2 z \quad \text{in } \Omega$$

$$\left.\frac{\partial z}{\partial \nu}\right|_{\Gamma_0} = i\alpha r z|_{\Gamma_0} - \xi i r v = 0$$

$$z|_\Gamma = 0;$$

consequently, by Holmgren's uniqueness theorem, $z = 0$ which concludes the proof.

As we said at the end of **Section 1**, we wish to satisy the conditions of Arendt–Batty's **Stability Theorem**; as we are dealing with a contraction semigroup, we need only show that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

Throughout, we will denote $\sigma_c(\mathcal{A})$ as the continuous spectrum of $\mathcal{A}$, $\sigma_p(\mathcal{A})$ as its discrete spectrum, and $\sigma_r(\mathcal{A})$ as its residual spectrum; hence $\sigma(\mathcal{A}) = \sigma_c(\mathcal{A}) \cup \sigma_p(\mathcal{A}) \cup \sigma_r(\mathcal{A})$, with the three sets being disjoint. We now proceed to show that $i\mathbb{R} \setminus \{0\}$ does not intersect with any of these sets (we previously noted that $\mathcal{A}^{-1}$ exists, so $0 \notin \sigma(\mathcal{A})$).

2.2.1 Show $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$

With $\mathcal{A}$ as given in (16), if for $r \in \mathbb{R}$ and $r \neq 0$, there exists $[z_1, z_2, v_1, v_2]^T \equiv [\overrightarrow{z}, \overrightarrow{v}]^T \in D(\mathcal{A})$ such that

$$\mathcal{A}\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix} = ir\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix}; \tag{32}$$

this then is equivalent to

$$\begin{cases} z_2 = irz_1 \\ v_2 = irv_1 \\ z_1 + i\alpha rNN^*Az_1 - irNv_1 - r^2A^{-1}z_1 = 0 \\ ir\mathbf{\mathring{A}}^{-1}N^*Az_1 + v_1 + irv_1 - r^2\mathbf{\mathring{A}}^{-1}v_1 = 0 \end{cases} \tag{33}$$

$\Rightarrow$

$$\mathbf{K}_1(r)\begin{bmatrix} z_1 \\ v_1 \end{bmatrix} = 0, \tag{34}$$

where $\mathbf{K}_1(r)$ is as defined in (27). From **Lemma 2.2**, we then have $z_1 = 0$, and $v_1 = 0$, and consequently from (33) $[\overrightarrow{z}, \overrightarrow{v}] = 0$, and so $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

2.2.2 Show $\sigma_r(\mathcal{A}) \cap i\mathbb{R} = \emptyset$

We will use the fact here that *if $\lambda \in \mathbb{C}$ is in the residual spectrum of $\mathcal{A}$, then λ is in the discrete spectrum of $\mathcal{A}^*$* (see [9],p.127). We recall from **Lemma 2.1** that the adjoint $\mathcal{A}^*$ was computed to be

$$\mathcal{A}^* = \begin{bmatrix} 0 & -I & 0 & 0 \\ A & -\alpha ANN^*A & 0 & -AN \\ 0 & 0 & 0 & -I \\ 0 & N^*A & \mathbf{\mathring{A}} & -\mathbf{\mathring{A}} \end{bmatrix} \tag{35}$$

with $D(\mathcal{A}^*) = \Big\{ [z_1, z_2, v_1, v_2]^T \in \left[D(A^{\frac{1}{2}})\right]^2 \times \left[D(\mathbf{\mathring{A}}^{\frac{1}{2}})\right]^2$
such that $z_1 - \alpha NN^*Az_2 - Nv_2 \in D(A)$ and $v_1 - v_2 \in D(\mathbf{\mathring{A}}) \Big\}$.

So with $r \in \mathbb{R}$ and $r \neq 0$, if there exists $[\overrightarrow{z}, \overrightarrow{v}]^T \in D(\mathcal{A}^*)$ such that

$$\mathcal{A}^*\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix} = ir\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{v} \end{bmatrix},$$

this is then equivalent to

$$\begin{cases} -z_2 = irz_1 \\ -v_2 = irv_1 \\ z_1 + i\alpha r N N^* A z_1 + irNv_1 - r^2 A^{-1} z_1 = 0 \\ -ir\mathring{\mathbf{A}}^{-1} N^* A z_1 + v_1 + irv_1 - r^2 \mathring{\mathbf{A}}^{-1} v_1 = 0 \end{cases}$$

$$\Rightarrow$$

$$\mathbf{K}_{-1}(r) \begin{bmatrix} z_1 \\ v_1 \end{bmatrix} = 0, \tag{36}$$

where again, $\mathbf{K}_{-1}(r)$ is defined by (27). Another application of **Lemma 2.2**, this time to (36), results in $[\vec{z}, \vec{v}]^T = 0$; thus $\sigma_r(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

2.2.3 Show $\sigma_c(\mathcal{A}) \cap i\mathbb{R} = \emptyset$

For arbitrary $\left[\vec{f}, \vec{g}\right]^T \in H_1 \times H_0$, suppose $[\vec{z}, \vec{v}]^T \in D(\mathcal{A})$ solves the equation

$$(ir - \mathcal{A}) \begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} = \begin{bmatrix} \vec{f} \\ \vec{g} \end{bmatrix}, \tag{37}$$

or

$$\begin{cases} irz_1 - z_2 = f_1 & \text{in } D(A^{\frac{1}{2}}) \\ irz_2 + Az_1 + \alpha A N N^* A z_2 - ANv_2 = f_2 & \text{in } L^2(\Omega) \\ irv_1 - v_2 = g_1 & \text{in } D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \\ irv_2 + N^* A z_2 + \mathring{\mathbf{A}} v_1 + \mathring{\mathbf{A}} v_2 = g_2 & \text{in } L^2(\Gamma_0) \end{cases} \tag{38}$$

(again with $r \in \mathbb{R}$ and $r \neq 0$). Defining the operator $\mathbf{F}$ as

$$\mathbf{F} \equiv \begin{bmatrix} irA^{-1} + \alpha N N^* A & A^{-1} & -N & 0 \\ \mathring{\mathbf{A}}^{-1} N^* A & 0 & ir\mathring{\mathbf{A}}^{-1} + \mathbf{I} & \mathring{\mathbf{A}}^{-1} \end{bmatrix}, \tag{39}$$

we have from the trace theorem and (9) that $\mathbf{F} \in \mathcal{L}(H_1 \times H_0, D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}}))$; subsequently, (27), **Lemma 2.2** and (39) provide that the solution $[\vec{z}, \vec{v}]^T$ of (38) satisfies

$$\begin{cases} \begin{bmatrix} z_1 \\ v_1 \end{bmatrix} = \mathbf{K}_1(r)^{-1} \mathbf{F} \begin{bmatrix} \vec{f} \\ \vec{g} \end{bmatrix} & \text{in } D(A^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \\ -z_2 = f_1 - irz_1 & \text{in } D(A^{\frac{1}{2}}) \\ -v_2 = g_1 - irv_1 & \text{in } D(\mathring{\mathbf{A}}^{\frac{1}{2}}). \end{cases} \tag{40}$$

Conversely, the solution $[\vec{z}, \vec{v}]^T$ of (40) for arbitrary $\left[\vec{f}, \vec{g}\right]^T \in H_1 \times H_0$ will *a fortiori* have (upon application of $\mathbf{K}_1(r)$) and the use of the second and fourth equations of (38)) that $z_1 +$

$\alpha NN^* Az_2 - Nv_2 \in D(A)$ and $v_1 + v_2 \in D(\mathring{A})$. With the equivalence of the equations (37) and (40), and consideration of the fact that *if $\lambda \in \mathbb{C}$ is in the continuous spectrum of $\mathcal{A}$, then $\mathcal{A} - \lambda$ does not have a closed range* (see [9], p.128), we deduce that $\sigma_c(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

Having thus found that $i\mathbb{R} \subseteq \rho(\mathcal{A})$, we conclude the proof of **Theorem 1.2** by applying the **Stability Theorem** of Arendt-Batty.

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