SHARP REGULARITY ESTIMATES FOR SOLUTIONS OF THE WAVE EQUATION AND THEIR TRACES WITH PRESCRIBED NEUMANN DATA

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Sharp Regularity Estimates for Solutions of the Wave Equation and Their Traces with Prescribed Neumann Data

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Abstract

In this paper, the regularity properties of second-order hyperbolic equations defined over a rectangular domain Ω with boundary Γ under the action of a Neumann boundary forcing term in $L^2\left(0,T;H^{\frac{1}{4}}(\Gamma)\right)$ are investigated. With this given boundary input, we prove by a cosine operator/functional analytical approach that not only is the solution of the wave equation and its derivatives continuous in time, with their pointwise values in a basic energy space (in the interior of Ω), but also that a trace regularity thereof can be assigned for the solution's time derivative in an appropriate (negative) Sobolev space. This newfound information on the solution and its traces is crucial in handling a mathematical model derived for a particular fluid/structure interaction system.

1 Description of the Problem

Let Ω be a rectangular domain contained in $\mathbb{R}^2$ with boundary Γ. We consider the following wave equation with boundary input u:

$$\begin{aligned} & z_{tt}(x,t) = \Delta z(x,t) && \text{on } \Omega \times (0,T); \\ & \frac{\partial z}{\partial \nu} = u && \text{on } \Gamma \times (0,T); \\ & z(x,0) = z_t(x,0) = 0 \text{ on } \Omega, && \end{aligned} \tag{1.1}$$

where, as usual, $\frac{\partial}{\partial \nu}$ denotes the normal derivative. Needless to say, there is a copious supply of theoretical results pertaining to the regularity of solutions of (1.1), and to second-order hyperbolic equations in general, results which specifically deal with the case when the prescribed datum u is not homogeneous. For example, the works in [7], [8], [9], [10], and [11], all deal with nonhomogeneous Neumann problems of a much more general nature than (1.1), yielding the regularity of solutions with the datum u having an appropriate smoothness (in the sense of Sobolev) in time and/or space.

More specifically, with $\Sigma = (0,T) \times \Gamma$, we have the classical result of [8] which yields

$$u \in H^{\frac{1}{2}}(\Sigma) \Rightarrow z \in C([0,T]; H^1(\Omega)). \tag{1.2}$$

Later, in [9], which deals with exterior domain problems, and in [7], which is concerned with solutions of (1.1) in the interior of a region Ω, one has

$$u \in H^{\frac{1}{\alpha}}(\Sigma) \Rightarrow [z, z_t]^T \in C([0,T]; H^1(\Omega) \times L^2(\Omega)), \tag{1.3}$$

where $\alpha = 4$ if Ω is a parallelpiped, and $\alpha = 3$ if Ω is an arbitrary convex domain. However, note that both of these results require some measure of Sobolev regularity in both time and space; moreover, in neither result is there information given on the trace regularity (if such exists) of the time derivative of the solutions. As we are of a mind to obtain a regularity result which can be used in handling the "structural acoustics problem" (see [1]), where the given datum is no more than square integrable in time, and wherein knowledge of a trace of z_t is indispensable, the above results and techniques used to derive them are not directly applicable.

Alternatively, we have the following result in [10]:

$$u \in L^2\left(0,T; H^{\frac{1}{2}}(\Gamma)\right) \Rightarrow [z, z_t]^T \in L^2\left(0,T; H^1(\Omega) \times L^2(\Omega)\right). \tag{1.4}$$

In (1.4), there is the weaker requisite that the input u be simply square integrable in time; unfortunately, the requirement on the spatial derivative is too restrictive for our intended purposes. In the analysis in [1], it is necessary that the datum u have a Sobolev regularity of at best $H^{\frac{1}{2}-\epsilon}(\Gamma)$ (in space) for fixed t, where the constraint $\epsilon > 0$ cannot be lifted. Also, as in the previous results, there is nothing said about the boundary values of the time derivative, and so this particular result is of no benefit to us.

Most recently, we have the result of Tataru in [11] which provides that

$$u \in L^2((0,T) \times \Gamma) \Rightarrow [z, z|_\Gamma]^T \in H^{\frac{3}{4}}((0,T) \times \Omega) \times H^{\frac{1}{2}}((0,T) \times \Gamma). \tag{1.5}$$

The result posted above requires only square integrability in time and space to yield solutions with a certain smoothness in both variables. Our concern here is with developing an "anisotropic" regularity result.

Hence the purpose of this work is to give a regularity theory for solutions z of (1.1) which

(i) provides for the assignment of some meaning to $z_t|_\Gamma$;

(ii) operates under the limiting assumption that the initial datum is L^2 in time only;

(iii) improves the known differentiability of solutions in the interior.

As mentioned earlier, this result, while being of mathematical interest in its own right, is largely motivated by its application to our aforementioned optimization problem which arises within the context of interacting structures.

2 Statement of Main Result

Theorem 1 *Let Ω be a rectangular domain contained in $\mathbb{R}^2$ with boundary Γ; let z be a weak solution to the following equation with boundary datum u in* $L^2\left(0,T;H^{\frac{1}{4}}(\Gamma)\right)$:

$$\begin{aligned} &z_{tt} = \Delta z && \text{on } Q; \\ &\frac{\partial z}{\partial \nu} = u && \text{on } \Gamma \times (0,T); \\ &z(t=0) = z_t(t=0) = 0 \text{ on } \Omega, \end{aligned} \tag{2.1}$$

then

$$[z, z_t, z_t|_\Gamma]^T \in C\left([0,T]; H^1(\Omega)\right) \times C\left([0,T]; L^2(\Omega)\right) \times L^2\left(0,T; H^{-\frac{1}{4}}(\Gamma)\right). \tag{2.2}$$

Moreover, we have continuous dependence on the datum, viz. $\forall\ t \in [0,T]$,

$$\|z(t)\|^2_{H^1(\Omega)} + \|z_t(t)\|^2_{L^2(\Omega)} + \int_0^T \|z_t(s)|_\Gamma\|^2_{H^{-\frac{1}{4}}(\Gamma)} ds \leq C_0 \|u\|^2_{L^2\left(0,T;H^{\frac{1}{4}}(\Gamma)\right)}. \tag{2.3}$$

Remark 1 *Note that this regularity for z does not follow from the classical results of [8] noted in (1.2), nor from the other posted results. This new result, however, is in the "style" of Myatake's in (1.4), but betters it by "$\frac{1}{4}$", in terms of the smoothness of the boundary datum required; the differentiability of $z_t|_\Gamma$ is also improved (formally) by "$\frac{1}{4}$".*

Remark 2 *Mimicking the details of the proof below of* **Theorem 2.1**, *one could also work to show that the outward normal derivative $\frac{\partial z}{\partial \nu}$ exists as an element of $L^2((0,T) \times \Gamma)$.*

In proving **Theorem 1**, we will adopt the cosine operator approach and accompanying techniques used in [5] (Appendix A) to derive the regularity of $[z, z_t]^T$ when u is given only to be square integrable in time and space. More specifically, we will exploit the explicit form of the solution z and the fact that we are working on a rectangular domain, as well as standard Laplace transform theory, to generate the desired estimates.

3 Proof of Theorem 2.1

Without loss of generality, we can take Ω to be $\{(x,y) \in \mathbb{R}^2) \ni 0 \leq x, y \leq \pi\}$ and by the linearity of the operator concerned, $u \equiv 0$ except on the side $\{y = 0,\ 0 \leq x \leq \pi\}$.

We define now the self-adjoint, positive, semidefinite operator A_N, $A_N = -\Delta$, with

$$D(A_N) = \left\{ z \in H^2(\Omega) \ni \frac{\partial z}{\partial \nu} = 0 \right\}, \tag{3.1}$$

and let $\{\lambda_{mn}, \Phi_{mn}\}_{m,n=1}^{\infty}$ denote respectively the eigenvalues and orthonormalized eigenfunctions (neglecting the zero eigenvalue and its constant eigenfunction), viz.

$$\begin{aligned} &\lambda_{mn} = n^2 + m^2 && \text{for } m,n = 1,2,\ldots; \\ &\Phi_{mn} = \tfrac{2}{\pi} \cos nx \cos my && \text{for } m,n = 1,2,\ldots. \end{aligned} \tag{3.2}$$

With these associated eigenvalues and eigenfunctions we can subsequently define the cosine operator $C(t)$ and sine operator $S(t)$ by having $\forall\, t \in [0,T]$ and $z \in L^2(\Omega)$,

$$
\begin{aligned}
C(t)z &= \sum_{n,m=1}^{\infty} \cos\sqrt{\lambda_{mn}}t(z,\Phi_{mn})_{L^2(\Omega)}\Phi_{mn};\\
S(t)z &= \int_0^t C(\tau)z\,d\tau = \sum_{m,n=1}^{\infty} \frac{1}{\sqrt{\lambda_{mn}}}\sin\sqrt{\lambda_{mn}}t(z,\Phi_{mn})_{L^2(\Omega)}\Phi_{mn}
\end{aligned}
\tag{3.3}
$$

(see [5]). If we recall the "Neumann" map $N \in \mathcal{L}\left(L^2(\Gamma), H^{\frac{3}{2}}(\Omega)\right)$ (see [2]), we then have *a fortiori* that the solution z of (2.1) can be written as the image of an "input" map L_N, i.e.

$$
\begin{aligned}
z(t) &\equiv L_N u(t) = A_N \int_0^t S(t-\tau)Nu(\tau)d\tau; \text{ and}\\
z_t(t) &\equiv (L_N u)_t(t) = A_N \int_0^t C(t-\tau)Nu(\tau)d\tau.
\end{aligned}
\tag{3.4}
$$

Making use of the fact that *a fortiori*

$$
N^*\Phi_{mn} = \frac{1}{\lambda_{mn}}\Phi_{mn}|_\Gamma, \tag{3.5}
$$

we obtain that the solution $[z, z_t]^T$ to (2.1) may be written explicitly as

$$
z(t) \equiv L_N u(t) = \sum_{m,n=1}^{\infty}\left\{\frac{1}{\sqrt{n^2+m^2}}\int_0^t \sin\sqrt{n^2+m^2}(t-\tau)\mu_n(\tau)d\tau\right\}\Phi_{mn}; \tag{3.6a}
$$

and

$$
z_t(t) \equiv (L_N u)_t(t) = \sum_{m,n=1}^{\infty}\left\{\int_0^t \cos\sqrt{n^2+m^2}(t-\tau)\mu_n(\tau)d\tau\right\}\Phi_{mn}; \tag{3.6b}
$$

where

$$
\mu_n(\tau) = \frac{2}{\pi}(u(\tau,\cdot),\cos n(\cdot))_{L^2(\bar{\Gamma})}, \tag{3.7}
$$

(and where $\bar{\Gamma} \equiv [0,\pi]$).

Remark 3 *In proving the asserted regularity result for $[z, z_t]^T$, we will look to obtain the following L^2-estimate:*

$$
\|z\|_{L^2(0,T;H^1(\Omega))} + \|z_t\|_{L^2(0,T;L^2(\Omega))} \le C_0\|u\|_{L^2(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma))} \tag{3.8}
$$

for $\epsilon \le \frac{1}{4}$ and $C_0 > 0$, which of course is tantamount to showing that the map

$$
\begin{bmatrix} L_N \\ (L_N)_t \end{bmatrix} : L^2\left(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma)\right) \to L^2\left(0,T;H^1(\Omega)\times L^2(\Omega)\right) \tag{3.9}
$$

is continuous. Having established the boundedness of this map, we can appeal to the "Lifting Theorem" in [6] for hyperbolic equations to ascertain that $z(\cdot)$ and $z_t(\cdot)$ are actually continuous in time into $H^1(\Omega)$ and $L^2(\Omega)$ respectively.

Proof of Theorem 2.1:

Part 1: Show the asserted L^2-regularity for z. For this part of the proof we need to establish desired estimates for $L_N u$ and the spatial derivatives $\frac{d}{dx}L_N u$ and $\frac{d}{dy}L_N u$, which are given *a fortiori* by

$$\frac{d}{dx}L_N u = \sum_{m,n=1}^{\infty} \frac{n}{\sqrt{n^2+m^2}} \left\{ \int_0^t \sin\sqrt{m^2+n^2}(t-\tau)\mu_n(\tau)d\tau \right\} \Phi_{mn}^{(1)}; \tag{3.10a}$$

and

$$\frac{d}{dy}L_N u = \sum_{m,n=1}^{\infty} \frac{m}{\sqrt{n^2+m^2}} \left\{ \int_0^t \sin\sqrt{m^2+n^2}(t-\tau)\mu_n(\tau)d\tau \right\} \Phi_{mn}^{(2)}, \tag{3.10b}$$

where $\Phi_{mn}^{(1)}(x,y) = -\frac{2}{\pi}\sin mx \cos ny$ and $\Phi_{mn}^{(2)}(x,y) = -\frac{2}{\pi}\cos nx \sin my$ (note that $\left\{\Phi_{mn}^{(j)}\right\}_{m,n=1}^{\infty}$ also forms an orthonormal basis in $L^2(\Omega)$, $j=1,2$). In particular, we need to show that there exists a positive constant C_0 such that

$$\|L_N u\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{d}{dx}L_N u\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{d}{dy}L_N u\right\|_{L^2(0,T;L^2(\Omega))}$$
$$\leq C_0\|u\|_{L^2\left(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma)\right)}, \tag{3.11}$$

which we now proceed to do.

Part 1(A) To estimate $\|\frac{d}{dy}L_N u\|_{L^2(0,T;L^2(\Omega))}$: Denoting $\widehat{f}$ as the Laplace transform of arbitrary function f with Laplace variable $\lambda = \gamma + i\omega$ and extending u by zero outside the interval $[0,T]$, we have upon applying the convolution theorem to equation (3.10b)

$$\widehat{\frac{d}{dy}L_N u}(\lambda) = \sum_{m,n=1}^{\infty} \left\{ \frac{m}{\lambda^2+\lambda_{mn}} \widehat{\mu_n}(\lambda) \right\} \Phi_{mn}^{(2)}. \tag{3.12}$$

In estimating the L^2-norm of $\frac{d}{dy}L_N u$, we use its Laplace Transform (3.12) and the generalized Parseval's relation (see [3], p.212)—here we are closely following the approach of [5], Appendix A—to arrive at

$$2\pi\int_0^\infty e^{-2\gamma t}\left\|\frac{d}{dy}L_N u(t)\right\|_{L^2(\Omega)}^2 dt = \int_{-\infty}^{\infty}\left\|\widehat{\frac{d}{dy}L_N u}(\gamma+i\omega)\right\|_{L^2(\Omega)}^2 d\omega$$

$$= \int_{-\infty}^{\infty}\sum_{n=1}^{\infty}|\widehat{\mu_n}(\gamma+i\omega)|^2 \sum_{m=1}^{\infty}\frac{m^2}{|(\gamma+i\omega)^2+\lambda_{mn}|^2}\, d\omega. \tag{3.13}$$

If we can establish that $\exists$ a positive constant C_0 such that

$$\sum_{m=1}^{\infty} \frac{m^2}{|(\gamma+i\omega)^2+\lambda_{mn}|^2} \leq C_0, \tag{3.14}$$

where C_0 is independent of $\omega \in \mathbb{R}$ and $n = 1, 2, 3, \ldots$, then again by Parseval's relation (this time applied to the function $\mu_n(\cdot)$), we will consequently have from (3.13) the inequality

$$\begin{aligned} 2\pi \int_0^T e^{-2\gamma T} \left\| \tfrac{d}{dy} L_N u(t) \right\|_{L^2(\Omega)}^2 dt &\leq C_0 \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} |\widehat{\mu_n}(\gamma+i\omega)|^2 d\omega \\ = C_0 \int_0^T e^{-2\gamma t} \|u(t)\|_{L^2(\Gamma)}^2 dt &\leq C_2 \|u\|_{L^2(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma))}^2, \end{aligned} \tag{3.15}$$

after using (3.13) and the orthonormality of the $\Phi_{mn}^{(2)}$'s; we will hence have the desired result,

$$\left\| \frac{d}{dy} L_N u \right\|_{L^2(0,T;L^2(\Omega))} \leq C_0 \|u\|_{L^2(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma))}. \tag{3.16}$$

So to show the existence of a constant C_0 such that (3.16) holds, it suffices by (3.14) and the definition of Riemann integrability to show that

$$I \equiv \int_1^{\infty} \frac{y^2 dy}{(y^2+\gamma^2+n^2-\omega^2)^2+4\gamma^2\omega^2} \leq C_0, \tag{3.17}$$

where again C_0 is independent of $\omega \in \mathbb{R}$ and $n = 1, 2, \ldots$. In showing (3.17), the two following cases must be considered.

Case 1: $\gamma^2+n^2-\omega^2 \geq 0$: Easily then,

$$I \leq \int_1^{\infty} \frac{dy}{y^2} = 1. \tag{3.18}$$

Case 2: $\gamma^2+n^2-\omega^2 < 0$: We set $a_{n\omega}^2 = \omega^2 - n^2 - \gamma^2$ and split the integral I into the suitable intervals

$$I = \int_1^{\infty} \leq \int_0^{\frac{a_{n\omega}}{2}} + \int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} + \int_{2a_{n\omega}}^{\infty} = I_1 + I_2 + I_3. \tag{3.19}$$

To estimate I_1:

$$I_1 = \int_0^{\frac{a_{n\omega}}{2}} \frac{y^2 dy}{(y^2-a_{n\omega}^2)^2+4\gamma^2\omega^2} \leq \int_0^{\frac{a_{n\omega}}{2}} \frac{y^2 dy}{\left(\frac{3}{4}a_{n\omega}^2\right)^2+4\gamma^2\omega^2}$$

$$= \frac{1}{3} \frac{\left(\frac{1}{2}a_{n\omega}\right)^3}{\left(\frac{3}{4}a_{n\omega}^2\right)^2+4\gamma^2\omega^2} = \frac{1}{3} \frac{\left(\frac{a_{n\omega}}{2}\right)^3}{\left(\frac{3}{4}a_{n\omega}^2\right)^2+4\gamma^2} \leq C_0 \quad (\text{as } \omega^2 \geq \gamma^2). \tag{3.20}$$

To estimate I_3:

$$I_3 = \int_{2a_{n\omega}}^{\infty} \frac{y^2 dy}{(y^2-a_{n\omega}^2)^2+4\gamma^2\omega^2} \leq \int_{2a_{n\omega}}^{\infty} \frac{y^2 dy}{\left(\frac{3}{4}y^2\right)^2+4\gamma^4} \leq C_0. \tag{3.21}$$

To estimate I_2: Setting $t \equiv y^2 - a_{n\omega}^2$, we then have

$$I_2 = \int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} \frac{y^2 dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} = \frac{1}{2}\int_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \frac{\sqrt{t + a_{n\omega}^2}}{t^2 + 4\gamma^2\omega^2} dt$$

$$\leq a_{n\omega} \int_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \frac{dt}{t^2 + 4\gamma^2\omega^2} = \frac{a_{n\omega}}{2|\gamma||\omega|} \arctan(t)\Big|_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \leq \frac{\pi}{2|\gamma|}. \tag{3.22}$$

Thus the estimate (3.17) is proved and we consequently have (3.16) from the inequality (3.15).

Part 1(B) To estimate $\left\|\frac{d}{dx}L_N u\right\|_{L^2(0,T;L^2(\Omega))}$:

For $\frac{d}{dx}L_N u$, we likewise apply the convolution theorem to (3.10a) to obtain

$$\widehat{\frac{d}{dx}L_n u}(\lambda) = \sum_{m,n=1}^{\infty} \left\{ \frac{n}{\lambda^2 + \lambda_{mn}} \widehat{\mu_n}(\lambda) \right\} \Phi_{mn}^{(1)}. \tag{3.23}$$

Again with Parseval's relation we have

$$2\pi \int_0^{\infty} e^{-2\gamma t} \left\| \frac{d}{dx} L_N u(t) \right\|_{L^2(\Omega)}^2 dt = \int_{-\infty}^{\infty} \left\| \widehat{\frac{d}{dx} L_N u}(\gamma + i\omega) \right\|_{L^2(\Omega)}^2 d\omega$$

$$= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} |\widehat{\mu_n}(\gamma + i\omega)|^2 n^{1-2\epsilon} \sum_{m=1}^{\infty} \frac{n^{1+2\epsilon} d\omega}{|(\gamma + i\omega)^2 + \lambda_{mn}|^2}, \tag{3.24}$$

after using (3.23) and the orthonormality of the $\Phi_{mn}^{(1)}$'s. Now to handle the RHS of (3.24), we define the operator

$$A_{\bar{\Gamma}} = -\frac{d^2}{dx^2}, \text{ with } D(A_{\bar{\Gamma}}) = \left\{ u \in H^2(0,\pi) \ni \frac{\partial u(T)}{\partial x} = \frac{\partial u(0)}{\partial x} = 0 \right\}; \tag{3.25}$$

then $\forall\ g \in H^\alpha(\bar{\Gamma}),\ 0 \leq \alpha \leq \frac{3}{2}$, we have from [4] the characterization

$$\|g\|_{H^\alpha(\bar{\Gamma})} = \left\| A_{\bar{\Gamma}}^{\frac{\alpha}{2}} g \right\|_{L^2(\bar{\Gamma})}. \tag{3.26}$$

Furthermore, as $A_{\bar{\Gamma}}$ has the respective eigenvalues and eigenfunctions $\{\lambda_n, \Phi_n\}_{n=0}^{\infty}$ with

$$\lambda_n = n^2 \text{ and } \Phi_n = \frac{2}{\pi} \cos n(\cdot), \tag{3.27}$$

we deduce by Parseval's Relation that for $g \in H^\alpha(\bar{\Gamma}),\ 0 \leq \alpha \leq \frac{3}{2}$,

$$\|g\|_{H^\alpha(\bar{\Gamma})}^2 = \left\| A_{\bar{\Gamma}}^{\frac{\alpha}{2}} g \right\|_{L^2(\bar{\Gamma})}^2 = \sum_{n=1}^{\infty} |(g, \Phi_n)|^2 n^{2\alpha}. \tag{3.28}$$

Hence, if we can show that for $\epsilon > 0$ (and small enough) that

$$\sum_{m=1}^{\infty} \frac{n^{1+2\epsilon}}{|(\gamma + i\omega)^2 + \lambda_{mn}|^2} \le C_0, \tag{3.29}$$

$\forall\, \omega \in \mathbb{R}$ and $n = 1, 2, \ldots$, where the positive constant C_0 is independent of ω and n, we will consequently have by (the generalized) Parseval's relation and (3.29) that

$$\begin{aligned} \textstyle\int_0^T \left\| \frac{d}{dx} L_N u \right\|^2_{L^2(0,T;\, L^2(\Omega)))} dt &\le C_0 \textstyle\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} n^{1-2\epsilon} \left| \widehat{\mu_n}(\gamma + i\omega) \right|^2 d\omega \\ &= C_0 \textstyle\int_0^T \sum_{n=1}^{\infty} e^{-2\gamma t} \|u(t)\|^2_{L^2(\Gamma)} n^{1-2\epsilon} dt \le C_0 \|u\|^2_{L^2\left(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma)\right)} \end{aligned} \tag{3.30}$$

(after using the characterization of the $H^{\frac{1}{2}-\epsilon}(\Gamma)$ norm given in (3.26). Since deriving the estimate (3.29) is tantamount to showing that

$$I \equiv \int_1^{\infty} \frac{n^{1+2\epsilon} dy}{(y^2 + \gamma^2 + n^2 - \omega^2)^2 + 4\gamma^2\omega^2} \le \text{ constant}, \tag{3.31}$$

for arbitrary $\omega \in \mathbb{R}$ and positive integer n (here we again consider the definition of Riemann integrability), we will estimate I instead; again, we will break the situation up into mutually exclusive cases.

Case 1. $\omega^2 < \gamma^2 + \frac{n^2}{2}$:
Then

$$\int_1^{\infty} \frac{dy}{(y^2 + \gamma^2 + n^2 - \omega^2)^2 + 4\gamma^2\omega^2} \le \int_{-\infty}^{\infty} \frac{dy}{(y^2 + \frac{n^2}{2})^2} = \frac{C_0}{n^3}; \tag{3.32}$$

so $I \le$ constant for $\epsilon \le 1$.

Case 2. $\gamma^2 + \frac{n^2}{2} < \omega^2 < \gamma^2 + n^2$:
Then

$$\int_1^{\infty} \frac{dy}{(y^2 + \gamma^2 + n^2 - \omega^2)^2 + 4\gamma^2\omega^2} \le \int_0^{\infty} \frac{dy}{y^4 + 2\gamma^2 n^2} \le \frac{C_0}{n^{\frac{3}{2}}}, \tag{3.33}$$

so $I \le$ constant for $\epsilon \le \frac{1}{4}$.

Case 3. $\gamma^2 + n^2 < \omega^2$:
Setting $a_{n\omega}^2 = \omega^2 - n^2 - \gamma^2 > 0$, we split the integral I into the following appropriate segments,

$$I = \int_1^{\infty} \le \int_0^{\frac{a_{n\omega}}{2}} + \int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} + \int_{2a_{n\omega}}^{\infty} = I_1 + I_2 + I_3, \tag{3.34}$$

and subsequently estimate each segment.

To estimate I_1:

$$I_1 = n^{1+2\epsilon} \int_0^{\frac{a_{n\omega}}{2}} \frac{dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} \le n^{1+2\epsilon} \frac{\frac{a_{n\omega}}{2}}{\frac{9}{16} a_{n\omega}^4 + 4\gamma^2\omega^2}; \tag{3.35}$$

and if we consider the function

$$f(x) \equiv \frac{x}{\frac{9}{16}x^4 + 4\gamma^2\omega^2} \tag{3.36}$$

with global maximizer (on $(0,\infty)$) $\bar{x} = \frac{2\sqrt{2}}{3^{\frac{3}{4}}}\sqrt{|\gamma|}\sqrt{|\omega|}$, then from (3.35)

$$I_1 \le C_0 \frac{n^{1+2\epsilon}}{\omega^2}\sqrt{|\omega|} \le C_0 \frac{n^{1+2\epsilon}}{n^{\frac{3}{2}}} \qquad (\text{as } \omega^2 > n^2),$$

so $I_1 \le C_0$ for $\epsilon \le \frac{1}{4}$.

To estimate I_3:

$$I_3 = n^{1+2\epsilon}\int_{2a_{n\omega}}^{\infty} \frac{dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} \le n^{1+2\epsilon}\int_{2a_{n\omega}}^{\infty} \frac{dy}{(\frac{3}{4}y^2)^2 + 4\gamma^2 n^2}$$

$$(\text{as } n^2 < \omega^2 \text{ and } \frac{y^2}{4} - a_{n\omega}^2 > 0)$$

$$\le 2n^{1+2\epsilon}\int_{-\infty}^{\infty} \frac{dy}{(\frac{3}{4}y^2 + \sqrt{2}\gamma n)^2} = \frac{C_0 n^{1+2\epsilon}}{n^{\frac{3}{2}}}, \tag{3.37}$$

so $I_3 \le$ constant for $\epsilon \le \frac{1}{4}$.

To estimate I_2:
Making the substitution $t = y^2 - a_{n\omega}^2$, we have

$$n^{1+2\epsilon}\int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} \frac{dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} = \frac{1}{2}\int_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \frac{n^{1+2\epsilon}dt}{(t^2 + 4\gamma^2\omega^2)\sqrt{t + a_{n\omega}^2}}$$

$$\le \frac{n^{1+2\epsilon}}{a_{n\omega}}\int_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \frac{dt}{(t^2 + 4\gamma^2\omega^2)} \qquad \left(\text{as } t + a_{n\omega}^2 \ge \frac{a_{n\omega}^2}{4}\right)$$

$$= \frac{n^{1+2\epsilon}}{2|\gamma||\omega|a_{n\omega}}\left[\arctan\left(\frac{3a_{n\omega}^2}{8|\gamma||\omega|}\right) + \arctan\left(\frac{3a_{n\omega}^2}{|\gamma|\ |\omega|}\right)\right]. \tag{3.38}$$

(a) If for arbitrary $\epsilon > 0$, $a_{n\omega} > n^{2\epsilon}$, then

$$(3.38) \le \frac{\pi n^{1+2\epsilon}}{2|\gamma|n^{1+2\epsilon}} = C_0 \tag{3.39}$$

(as $\omega^2 > n^2$); so $I_2 \le$ constant.

(b) If $a_{n\omega} \le n^{2\epsilon}$, then

$$(3.38) \le \frac{C_0 n^{1+2\epsilon}a_{n\omega}}{\omega^2} \le C_0 n^{-1+4\epsilon}; \tag{3.40}$$

so $I_2 \le$ constant for $\epsilon \le \frac{1}{4}$.

Hence we have established the estimate (3.31), and subsequently via (3.30) we obtain the norm bound

$$\left\| \frac{d}{dx} L_N u \right\|_{L^2(0,T;L^2(\Omega))} \le C_0 \|u\|_{L^2(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma))}. \tag{3.41}$$

Part 1(C) To estimate $\|L_N u\|_{L^2(0,T;L^2(\Omega))}$: For $L_N u$, we once more apply the convolution theorem, this time to (3.6a), to have that

$$\widehat{L_N u}(\lambda) = \sum_{m,n=1}^{\infty} \left\{ \frac{1}{\lambda + n^2 + m^2} \widehat{\mu_n}(\lambda) \right\} \Phi_{mn}; \tag{3.42}$$

and analogous to the case for $\frac{d}{dx} L_N u$ and $\frac{d}{dy} L_N u$, we are interested in the quantity

$$\begin{aligned} 2\pi \int_0^\infty e^{-2\gamma t} \|L_N u(t)\|^2_{L^2(\Omega)} dt &= \int_{-\infty}^{\infty} \|\widehat{L_N u}(\gamma + i\omega)\|^2_{L^2(\Omega)} d\omega \\ &= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} |\widehat{\mu_n}(\gamma + i\omega)|^2 \sum_{m=1}^{\infty} \frac{1}{|(\gamma + i\omega)^2 + \lambda_{mn}|^2} d\omega. \end{aligned} \tag{3.43}$$

As

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} |\widehat{\mu_n}(\gamma + i\omega)|^2 d\omega = 2\pi \int_0^T e^{-2\gamma t} \|u(t)\|^2_{L^2(\Gamma)} dt, \tag{3.44}$$

then using the same argument as was employed above in **Part 1(A)** and **(B)**, if we can show that

$$\int_1^\infty \frac{dy}{(y^2 + \gamma^2 + n^2 - \omega^2)^2 + 4\gamma^2\omega^2} < \infty, \tag{3.45}$$

for all $\omega \in \mathbb{R}$ and $n = 1, 2, \ldots$, we can then deduce from (3.44) that

$$\|L_N u\|_{L^2(0,T;L^2(\Omega))} \le C_0 \|u\|_{L^2(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma))}. \tag{3.46}$$

But as we have just shown above,

$$\int_1^\infty \frac{n^{1+2\epsilon} dy}{(y^2 + \gamma^2 + n^2 - \omega^2)^2 + 4\gamma^2\omega^2} \le C_0 \qquad \left(\text{for } \epsilon \le \frac{1}{4} \right), \tag{3.47}$$

and so we trivially have (3.45) and subsequently–via (3.44)–the estimate (3.46). With (3.16), (3.41), and (3.46) we have the final desired L^2-estimate (3.11).

Part 2: Prove the asserted regularity for z_t. Recalling that the time derivative z_t is given explicitly in (3.6b) by

$$z_t = \sum_{m,n=1}^{\infty} \left\{ \int_0^t \cos \sqrt{n^2 + m^2}(t - \tau) \mu_n(\tau) d\tau \right\} \Phi_{mn}; \tag{3.48}$$

we take the Laplace transform in the same fashion as was done in Part 1, to subsequently obtain

$$\widehat{z_t}(\lambda) = (\widehat{L_N u})_t(\lambda) = \sum_{m,n=1}^{\infty} \left\{ \frac{\lambda}{\lambda^2 + \lambda_{mn}} \widehat{\mu_n}(\lambda) \right\} \Phi_{mn}. \tag{3.49}$$

By Parseval's relation we have (with $\lambda = \gamma + i\omega$) the equality

$$2\pi \int_0^{\infty} e^{-2\gamma t} \|(L_N u)_t(t)\|^2_{L^2(\Omega)} dt = \int_{\infty}^{\infty} \|(\widehat{L_N u})_t(\gamma + i\omega)\|^2_{L^2(\Omega)} d\omega$$

$$= \int_{-\infty}^{\infty} \sum_{m,n=1}^{\infty} \frac{\gamma^2 + \omega^2}{|(\gamma + i\omega)^2 + \lambda_{mn}|^2} \cdot |\widehat{\mu_n}(\gamma + i\omega)|^2 d\omega; \tag{3.50}$$

and by a similar line of reasoning as was used in Part 1—when considering the characterization of the $H^{\frac{1}{2}-\epsilon}(\bar{\Gamma})$ norm given in (3.28)—we will have established the estimate

$$\|(L_N u)_t\|_{L^2(0,T;L^2(\Omega))} \le C_0 \|u\|_{L^2(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma))}, \tag{3.51}$$

provided that

$$\frac{1}{n^{1-2\epsilon}} \int_1^{\infty} \frac{(\gamma^2 + \omega^2) dy}{(y^2 + \gamma^2 + n^2 - \omega^2)^2 + 4\gamma^2\omega^2} \le C_0, \tag{3.52}$$

where the constant C_0 is independent of ω and n. Now since we have already obtained (3.45), it remains only to show that

$$I \equiv \frac{1}{n^{1-2\epsilon}} \int_1^{\infty} \frac{\omega^2 dy}{(y^2 + \gamma^2 + n^2 - \omega^2)^2 + 4\gamma^2\omega^2} \le C_0, \tag{3.53}$$

which we now proceed to do.

Case 1. $\gamma^2 + n^2 \ge \omega^2$:
Then

$$I \le \left[n^{1+2\epsilon} + \frac{\gamma^2}{n^{1-2\epsilon}} \right] \int_1^{\infty} \frac{dy}{(y^2 + \gamma^2 + n^2 - \omega^2)^2 + 4\gamma^2\omega^2} \le C_0 \tag{3.54}$$

when $\epsilon \le \frac{1}{4}$, after making use of the previously derived estimates (3.31) and (3.45).

Case 2. $\gamma^2 + n^2 \le \omega^2$:
Then with $a^2_{n\omega} \equiv \omega^2 - n^2 - \gamma^2$, we again as in **Part 1**, split the integral I into the segments

$$I \le \int_0^{\frac{a_{n\omega}}{2}} + \int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} + \int_{2a_{n\omega}}^{\infty} = I_1 + I_2 + I_3. \tag{3.55}$$

To estimate I_1:

$$I_1 = \frac{1}{n^{1-2\epsilon}} \int_0^{\frac{a_{n\omega}}{2}} \frac{a^2_{n\omega} dy}{(y^2 - a^2_{n\omega})^2 + 4\gamma^2\omega^2} + \frac{1}{n^{1-2\epsilon}} \int_0^{\frac{a_{n\omega}}{2}} \frac{[n^2 + \gamma^2] dy}{(y^2 - a^2_{n\omega})^2 + 4\gamma^2\omega^2}, \tag{3.56}$$

now the second integral on the RHS of (3.56) $\leq$ constant for $\epsilon \leq \frac{1}{4}$, after using the work done to estimate I_1 of **Part 1(A)** and **(B)**, and moreover, we obtain

$$a_{n\omega}^2 \int_0^{\frac{a_{n\omega}}{2}} \frac{dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} \leq \frac{1}{2} \frac{a_{n\omega}^3}{(\frac{3}{4}a_{n\omega}^2)^2 + 4\gamma^2 n^2} \leq C_0 \tag{3.57}$$

(as $n^2 < \omega^2$), so $I_1 \leq$ constant for $\epsilon \leq \frac{1}{4}$.

To estimate I_3:

$$I_3 = \frac{1}{n^{1-2\epsilon}} \int_{2a_{n\omega}}^{\infty} \frac{a_{n\omega}^2 dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} + \frac{1}{n^{1-2\epsilon}} \int_{2a_{n\omega}}^{\infty} \frac{[n^2 + \gamma^2] dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2}, \tag{3.58}$$

and again we will have that the second integral on the RHS of (3.58) $\leq$ constant for $\epsilon \leq \frac{1}{4}$, by the previous analysis done for I_3 of **Part1(A)** and **(B)**. Also

$$a_{n\omega}^2 \int_{2a_{n\omega}}^{\infty} \frac{dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} \leq \frac{1}{4} \int_{2a_{n\omega}}^{\infty} \frac{y^2 dy}{(\frac{3}{4}y^2)^2 + 4\gamma^4} \leq C_0; \tag{3.59}$$

thus $I_3 \leq$ constant for $\epsilon \leq \frac{1}{4}$.

To estimate I_2:
Once more, by the work done in **Part 1(A)** and **(B)** for I_2 we have that

$$\frac{1}{n^{1-2\epsilon}} \int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} \frac{[n^2 + \gamma^2] dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} \leq C_0, \tag{3.60}$$

for $\epsilon \leq \frac{1}{4}$; hence we need only show that

$$I_{21} \equiv \int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} \frac{a_{n\omega}^2 dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} \leq \text{ constant.} \tag{3.61}$$

By making the substitution $t = y^2 - a_{n\omega}^2$, we obtain that

$$I_{21} = \frac{a_{n\omega}^2}{2} \int_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \frac{dt}{(t^2 + 4\gamma^2\omega^2)\sqrt{t + a_{n\omega}^2}} \leq a_{n\omega} \int_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \frac{dt}{t^2 + 4\gamma^2\omega^2}$$

$$= \frac{a_{n\omega}}{2|\gamma||\omega|} \arctan t \Big|_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \leq \frac{\pi}{2|\gamma|} \qquad (\text{as } a_{n\omega} \leq |\omega|). \tag{3.62}$$

Thus $I_2 \leq$ constant for $\epsilon \leq \frac{1}{4}$ and we indeed have (3.53).

So, using the equality (3.50) with the now established estimate (3.53), we deduce the L^2-estimate (3.51). As noted in **Remark 3**, the desired continuity of z and z_t, as well as the estimate

$$\|L_N u(t)\|_{C([0,T];H^1(\Omega))} + \|(L_N u)_t(t)\|_{C([0,T];L^2(\Omega))} \leq C_0 \|u\|_{L^2(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma))} \tag{3.63}$$

follows upon the application of [5].

Part 3: Show the asserted regularity for $z_t|_\Gamma$. Recalling the definition of $A_{\bar\Gamma}$ given in (3.25) and the characterization of its fractional powers given in (3.26), we deduce that for $g \in L^2(\bar\Gamma)$ and $0 \le \alpha \le \frac{3}{2}$,

$$\|g\|_{[H^\alpha(\bar\Gamma)]'} = \left\| A_{\bar\Gamma}^{-\frac{\alpha}{2}} g \right\|_{L^2(\bar\Gamma)} \tag{3.64}$$

(where the inverse $A_{\bar\Gamma}^{-\frac{\alpha}{2}}$ is initially well defined for square integrable functions which are L^2-orthogonal to $V_0 \equiv$ space of constant functions, and subsequently for all functions in $L^2(\bar\Gamma)$, albeit nonuniquely). With extension by density we will thus have that the equality (3.64) is valid for all g in $[H^\alpha(\bar\Gamma)]'$. So with $\alpha \equiv \frac{1}{2} - \epsilon$, to show the desired regularity of $z_t|_\Gamma$, it will suffice to derive the estimate

$$\left\| A_{\bar\Gamma}^{-\frac{\alpha}{2}} (L_N u)_t \Big|_\Gamma \right\|_{L^2(0,T;L^2(\bar\Gamma))} \le C_0 \|u\|_{L^2(0,T;H^{\frac{1}{2}-\epsilon}(\bar\Gamma))} \tag{3.65}$$

(where, again, $\epsilon \le \frac{1}{4}$), for some positive C_0. So we will work towards establishing (3.65), using the fact that:

$$(L_N u)_t(t)|_\Gamma = \sum_{m,n=1}^{\infty} \int_0^t \cos\sqrt{\lambda_{mn}}(t-\tau)\mu_n(\tau)d\tau \frac{2}{\pi}\cos n(\cdot), \tag{3.66a}$$

$$A_{\bar\Gamma}^{-\frac{\alpha}{2}} (L_N u)_t(t)\Big|_\Gamma = \sum_{m,n=1}^{\infty} \int_0^t \cos\sqrt{\lambda_{mn}}(t-\tau)\mu_n(\tau)d\tau n^{-\alpha}\frac{2}{\pi}\cos n(\cdot), \tag{3.66b}$$

and with Laplace variable $\lambda = \gamma + i\omega$,

$$\widehat{A_{\bar\Gamma}^{-\frac{\alpha}{2}} (L_N u)_t}(\lambda)\Big|_\Gamma = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\lambda}{\lambda^2+\lambda_{mn}} \widehat{\mu_n}(\lambda) n^{-\alpha}\frac{2}{\pi}\cos n(\cdot). \tag{3.66c}$$

So with the previous train of logic in mind (involving the use of the Generalized Parseval's Inequality) employed in establishing (3.16), (3.41), (3.46), and (3.51), we arrive at for arbitrary γ,

$$2\pi \int_0^T e^{-2\gamma t} \|(L_N u)_t(t)|_{\bar\Gamma}\|^2_{H^{-\alpha}(\bar\Gamma)}\, dt \le 2\pi \int_0^T e^{-2\gamma t} \left\| A_{\bar\Gamma}^{-\frac{\alpha}{2}} (L_N u)_t(t)\, |_{\bar\Gamma} \right\|_{L^2(\bar\Gamma)} dt$$

(using the characterization (3.64))

$$= \int_{-\infty}^{\infty} \left\| \widehat{A_{\bar\Gamma}^{-\frac{\alpha}{2}} (L_N u)_t}(\gamma+i\omega)\, |_{\bar\Gamma} \right\|^2_{L^2(\bar\Gamma)} d\omega = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} \frac{\lambda}{\lambda^2+\lambda_{mn}} \widehat{\mu_n}(\lambda) \right|^2 n^{-2\alpha} d\omega. \tag{3.67}$$

With $\alpha \equiv \frac{1}{2} - \epsilon$ and given that $u \in L^2\left(0,T; H^{\frac{1}{2}-\epsilon}(\bar\Gamma)\right)$, if we can show that $\exists$ a positive constant C_0 such that

$$\frac{1}{n^{1-2\epsilon}} \sum_{m=1}^{\infty} \frac{\lambda}{\lambda^2+\lambda_{mn}} \le C_0, \tag{3.68}$$

where C_0 does not depend on $\omega \in \mathbb{R}$ and $m \in \mathbb{N}$, then (3.67) will become

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} \frac{\lambda}{\lambda^2 + \lambda_{mn}} \widehat{\mu_n}(\lambda) \right|^2 n^{-2\alpha} d\omega = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} \frac{\lambda}{\lambda^2 + \lambda_{mn}} \right|^2 \frac{|\widehat{\mu_n}(\lambda)|^2 n^{1-2\epsilon}}{n^{2-4\epsilon}} d\omega$$
$$\leq C_0 \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} |\widehat{\mu_n}(\lambda)|^2 \, n^{1-2\epsilon} d\omega = 2\pi C_0 \int_0^T e^{-2\gamma t} \|u_n(t)\|^2_{H^{\frac{1}{2}-\epsilon}(\bar{\Gamma})}, \tag{3.69}$$

and we will consequently deduce that

$$\| (L_N u)_t|_\Gamma \|_{L^2(0,T;H^{-\frac{1}{2}+\epsilon}(\bar{\Gamma}))} \leq C_0 \|u\|_{L^2(0,T;H^{\frac{1}{2}-\epsilon}(\Gamma))}. \tag{3.70}$$

To show the convergence of (3.68) it will suffice as before to show that

$$\frac{1}{n^{1-2\epsilon}} \int_1^\infty \frac{(\gamma + i\omega)dy}{(n^2 + y^2 + \gamma^2 - \omega^2) + 2i\gamma\omega} < \infty, \tag{3.71}$$

and having already established estimates (3.17), (3.31), (3.45), and (3.53), it only remains to show that

$$\frac{\omega}{n^{1-2\epsilon}} \int_1^\infty \frac{(n^2 + y^2 + \gamma^2 - \omega^2)dy}{(n^2 + y^2 + \gamma^2 - \omega^2)^2 + 4\gamma^2\omega^2} < C_0, \tag{3.72}$$

which we now proceed to do.

Case 1. $\omega^2 \leq n^2 + \gamma^2$:
Then $\tilde{a}_{n\omega} \equiv \omega^2 - n^2 - \gamma^2 \leq 0$, and

$$(3.72) = \frac{\omega}{n^{1-2\epsilon}} \int_1^\infty \frac{(y^2 - \tilde{a}_{n\omega})dy}{\sqrt{(y^2 - \tilde{a}_{n\omega})^2 + 4\gamma^2\omega^2} \cdot \sqrt{(y^2 - \tilde{a}_{n\omega})^2 + 4\gamma^2\omega^2}}$$

$$\leq \frac{|\omega|}{n^{1-2\epsilon}} \int_1^\infty \frac{dy}{\sqrt{(y^2 - \tilde{a}_{n\omega})^2 + 4\gamma^2\omega^2}} \leq \frac{\sqrt{2}|\omega|}{n^{1-2\epsilon}} \int_{-\infty}^\infty \frac{dy}{y^2 + 2|\gamma||\omega|}$$

$$= \frac{\pi\sqrt{|\omega|}}{\sqrt{|\gamma|} n^{1-2\epsilon}} \leq \frac{C_0(n^2 + \gamma^2)^{\frac{1}{4}}}{n^{1-2\epsilon}} < C_0, \tag{3.73}$$

for $\epsilon \leq \frac{1}{4}$.

Case 2. $\omega^2 > n^2 + \gamma^2$:
Further specifying that $|\gamma| < 1$—we have this degree of freedom from the use of Parseval's relation in (3.67)—then $a_{n\omega}^2 \equiv \omega^2 - n^2 - \gamma^2 \geq 0$, and as we have similarly done before, we split the integral

$$I \equiv (3.72) = \frac{\omega}{n^{1-2\epsilon}} \int_1^\infty \frac{(y^2 - a_{n\omega}^2)dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2}$$

into

$$I = \int_1^\infty \leq \int_0^{\frac{a_{n\omega}}{2}} + \int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} + \int_{2a_{n\omega}}^\infty = I_1 + I_2 + I_3, \tag{3.74}$$

and estimate each segment.

To estimate I_1:

$$I_1 = \frac{\omega}{n^{1-2\epsilon}} \int_0^{\frac{a_{n\omega}}{2}} \frac{(y^2 - a_{n\omega}^2)dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2}$$

$$\leq \frac{|\omega|}{n^{1-2\epsilon}} \int_0^{\frac{a_{n\omega}}{2}} \frac{|y^2 - a_{n\omega}^2| + 2|\gamma||\omega||dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} \leq \frac{2|\omega|}{n^{1-2\epsilon}} \int_0^{\frac{a_{n\omega}}{2}} \frac{dy}{|y^2 - a_{n\omega}^2| + 2|\gamma||\omega|}$$

$$\leq \frac{2|\omega|}{n^{1-2\epsilon}} \int_0^{a_{n\omega}} \frac{dy}{\frac{3}{4}a_{n\omega}^2 + 2|\gamma||\omega|} = \frac{2|\omega|}{n^{1-2\epsilon}} \frac{a_{n\omega}}{\frac{3}{4}a_{n\omega}^2 + 2|\gamma||\omega|}. \tag{3.75}$$

(a) If $n^2 \leq \omega^2 \leq kn^2$, where fixed k is, say, some integer satisfying $k \geq \frac{\omega^2}{\omega^2 - \gamma^2}$; then considering the function $f(x) = \frac{x}{\frac{3}{4}x^2 + 2|\gamma||\omega|}$ with global maximizer $\bar{x} = \frac{2\sqrt{2}}{\sqrt{3}} \sqrt{|\gamma|}\sqrt{|\omega|}$, then (3.75) becomes

$$(3.75) \leq \frac{|\omega|}{n^{1-2\epsilon}} \cdot \frac{\frac{2\sqrt{2}}{\sqrt{3}} \sqrt{|\gamma|}\sqrt{|\omega|}}{2|\gamma||\omega|} \leq \frac{C_0\sqrt{|\omega|}}{n^{1-2\epsilon}} \leq \frac{C_0 k^{\frac{1}{4}} n^{\frac{1}{2}}}{n^{1-2\epsilon}} \leq C_0, \tag{3.76}$$

for $\epsilon \leq \frac{1}{4}$.

(b) If $\omega^2 > kn^2$; then (3.75) becomes

$$(3.75) \leq \frac{8|\omega|}{3n^{1-2\epsilon}\sqrt{\omega^2 - n^2 - \gamma^2}} \leq \frac{8|\omega|}{3n^{1-2\epsilon}\sqrt{\frac{(k-1)}{k}\omega^2 - \gamma^2}} \leq C_0, \tag{3.77}$$

(as $\left(\frac{k-1}{k}\right)\omega^2 \geq \gamma^2$).

To estimate I_3:

$$I_3 = \frac{\omega}{n^{1-2\epsilon}} \int_{2a_{n\omega}}^{\infty} \frac{(y^2 - a_{n\omega}^2)dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2}$$

$$\leq \frac{|\omega|}{n^{1-2\epsilon}} \int_{2a_{n\omega}}^{\infty} \frac{[y^2 - a_{n\omega}^2 + 2|\gamma||\omega|]dy}{\frac{1}{2}[(y^2 - a_{n\omega}^2) + 2|\gamma||\omega|]^2} \leq \frac{8}{3}\frac{|\omega|}{n^{1-2\epsilon}} \int_{2a_{n\omega}}^{\infty} \frac{dy}{[y^2 + \frac{8}{3}|\gamma||\omega|]}$$

$$= \frac{2\sqrt{2}}{\sqrt{3}} \frac{\sqrt{|\omega|}}{\sqrt{|\gamma|}n^{1-2\epsilon}} \left[\frac{\pi}{2} - \arctan\left[\frac{\sqrt{3}a_{n\omega}}{\sqrt{2}\sqrt{|\gamma|}\sqrt{|\omega|}}\right]\right]. \tag{3.78}$$

(a) If $n^2 \leq \omega^2 \leq kn^2$, where again fixed integer $k \geq \frac{\omega^2}{\omega^2 - \gamma^2}$, then

$$(3.78) \leq \frac{C_0\sqrt{|\omega|}}{n^{1-2\epsilon}} \leq \frac{C_0 k^{\frac{1}{4}} n^{\frac{1}{2}}}{n^{1-2\epsilon}} \leq C_0, \tag{3.79}$$

for $\epsilon \leq \frac{1}{4}$.

(b) If $\omega^2 > kn^2$, then

$$(3.78) \leq \frac{2\sqrt{2}}{\sqrt{3}} \frac{\sqrt{|\omega|}}{n^{1-2\epsilon}} \frac{\sqrt{2}}{\sqrt{3}} \frac{\sqrt{|\omega|}}{a_{n\omega}} \leq \frac{C_0|\omega|}{\sqrt{(\frac{k-1}{k})\omega^2 - \gamma^2}} \leq C_0. \tag{3.80}$$

To estimate I_2**:**
(a) If $n^2 \leq \omega^2 \leq kn^2$, where once more $k \geq \frac{\omega^2}{\omega^2-\gamma^2}$, then we consider:
(a)i If $a_{n\omega} \geq n^{1-2\epsilon}$, then making the substitution $y^2 - a_{n\omega}^2 = t$, we arrive at

$$I_2 = \frac{\omega}{n^{1-2\epsilon}} \int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} \frac{(y^2 - a_{n\omega}^2)dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} \leq \frac{|\omega|}{2n^{1-2\epsilon}} \int_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \frac{t\,dt}{(t^2 + 4\gamma^2\omega^2)\sqrt{t + a_{n\omega}^2}}$$

$$\leq \frac{|\omega|}{n^{1-2\epsilon}a_{n\omega}} \ell n \left[\frac{9a_{n\omega}^4 + 4\omega^2\gamma^2}{\frac{9}{16}a_{n\omega}^4 + 4\omega^2\gamma^2}\right] \leq \frac{C_0|\omega|}{n^{1-2\epsilon}a_{n\omega}} \leq \frac{C_0 k^{\frac{1}{2}} n}{n^{2-4\epsilon}} \leq C_0, \tag{3.81}$$

for $\epsilon \leq \frac{1}{4}$.
(a)ii If $a_{n\omega} < n^{1-2\epsilon}$, then again

$$I_2 = \frac{\omega}{n^{1-2\epsilon}} \int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} \frac{(y^2 - a_{n\omega}^2)dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} = \frac{\omega}{2n^{1-2\epsilon}} \int_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \frac{t\,dt}{(t^2 + 4\omega^2\gamma^2)\sqrt{t + a_{n\omega}^2}}$$

$$\leq \frac{6|\omega|a_{n\omega}}{n^{1-2\epsilon}} \int_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \frac{dt}{(t^2 + 4\omega^2\gamma^2)} \qquad \left(\text{as } \frac{|t|}{\sqrt{t + a_{n\omega}^2}} \leq 12a_{n\omega}\right)$$

$$\leq \frac{6|\omega|a_{n\omega}}{n^{1-2\epsilon}} \int_{-\infty}^{\infty} \frac{dt}{(t^2 + 4\omega^2\gamma^2)} \leq \frac{C_0 a_{n\omega}}{n^{1-2\epsilon}} \leq C_0. \tag{3.82}$$

(b) If $\omega^2 > kn^2$, then again with the change of variable $y^2 - a_{n\omega}^2 = t$, we have

$$I_2 = \frac{\omega}{n^{1-2\epsilon}} \int_{\frac{a_{n\omega}}{2}}^{2a_{n\omega}} \frac{(y^2 - a_{n\omega}^2)dy}{(y^2 - a_{n\omega}^2)^2 + 4\gamma^2\omega^2} = \frac{\omega}{2n^{1-2\epsilon}} \int_{-\frac{3}{4}a_{n\omega}^2}^{3a_{n\omega}^2} \frac{t\,dt}{(t^2 + 4\omega^2\gamma^2)\sqrt{t + a_{n\omega}^2}}$$

$$\leq \frac{|\omega|}{n^{1-2\epsilon}a_{n\omega}} \ell n \left[\frac{9a_{n\omega}^4 + 4\omega^2\gamma^2}{\frac{9}{16}a_{n\omega}^4 + 4\omega^2\gamma^2}\right] \leq \frac{C_0|\omega|}{n^{1-2\epsilon}\sqrt{\omega^2 - n^2 - \gamma^2}}$$

$$\leq \frac{C_0|\omega|}{n^{1-2\epsilon}\sqrt{\frac{(k-1)\omega^2}{k} - \gamma^2}} \leq C_0, \tag{3.83}$$

and we have finally have (3.72), and consequently (3.71) and (3.70). The proof of **Theorem 2.1** is now complete.

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