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By

Juan C. Cockburn

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**INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA**

**514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455**

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JUAN C. COCKBURN

In this paper we present a generalization of the classical bitangential Nevanlinna-Pick theory in which one bounds not the norm of the interpolating functions but their *structured singular value*. This work was motivated by some problems arising robust control of systems with structured uncertainty. This approach is based on the commutant lifting theory of Sz.-Nagy and Foias (1968) and extends previous work of Bercovici, Foias and Tannenbaum (1990) on structured matrix Nevanlinna-Pick interpolation.

1. INTRODUCTION

The problem of interpolation with bounded analytic functions has been studied since the turn of the century (Carathéodory 1907, Carathéodory and Féjer 1911, Schur 1912). A classical problem is to find necessary and sufficient conditions for the existence of an analytic function $f : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ such that $f(z_k) = w_k$, $1 \leq k \leq n$ where $z_1, \dots, z_n, w_1, \dots, w_n \in \mathbb{D}$ are given points in the open unit disk. This problem was first solved by Pick (1916) for the case of finitely many points. Later, Nevanlinna (1919) extended this result for countable sets and also presented a recursive algorithm to parameterize all the solutions. The initial approaches to these problems were purely complex analytic.

Several decades later, in a landmark paper, Sarason (1967) developed an operator theoretic framework in which one could deduce many results on interpolation with bounded analytic functions, even in the case of points with infinite multiplicity (which he called *generalized interpolation in H^∞*). Shortly after, these results were significantly broadened by Sz.-Nagy and Foias (1968) with their commutant lifting theorem which allowed interpolation with bounded analytic *operator valued* functions. Since these seminal papers, there has been significant advances on interpolation theory from an operator theoretic point of view (e.g. Ball and Helton 1983, Dym 1989, Foias and Frazho 1990, Ball, Gohberg and Rodman 1990a, Ball, Gohberg and Kaashoek 1992, Nudelman 1993, Frazho and Kherat 1993, Ivanchenko and Sakhnovich 1994).

Besides its intrinsic mathematical value, interpolation by bounded analytic functions has had a major impact in a number of areas of applied sciences such as circuits and systems theory (Delsarte, Genin and Kamp 1984, Helton 1987), control theory (Francis and Tannenbaum 1988), and geophysics (Foias 1978) among others. Indeed, this work on structured interpolation has been motivated by a problem in robust control, e.g., the synthesis of linear feedback systems in the presence of structured uncertainty (Doyle 1982).

In order to put this result in perspective, consider the standard Nevanlinna-Pick theory in the matrix case. Given a set of distinct points $\alpha_i \in \mathbb{D}$, $1 \leq i \leq n$, together with a set of $N \times N$ complex matrices $F_1, \dots, F_n$ we want to find necessary and sufficient conditions for the existence of an analytic (in the disc) matrix valued function $F(z)$ with $F(\alpha_i) = F_i$ ($1 \leq i \leq n$) such that $\|F\| \leq 1$. The existence of such F can be reduced to the determination of the positivity of certain *Nevanlinna-Pick matrix*. This fact can be deduced,

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e.g., from the commutant lifting theorem (Sz.-Nagy and Foias 1968, Sz.-Nagy and Foias 1970). Interesting variations of this problem in which one bounds not the norm of F but its spectral radius or its structured singular values (see section 3.1 for a precise definition) have also been considered in the literature (Bercovici, Foias and Tannenbaum 1991b, Bercovici et al. 1990).

Many problems in control engineering can be formulated not as full matrix interpolation problems but as tangential interpolation problems (Kimura 1987, Limebeer and Anderson 1988). Tangential Nevanlinna-Pick interpolation problems were studied from a mathematical point of view (Fedčina 1972, Fedčina 1975a, Fedčina 1975b) a decade earlier. In its simplest form, the tangential Nevanlinna-Pick (NP) problem is as follows: given $\alpha_i \in \mathbb{D}$ distinct ($1 \leq i \leq n$) and vectors $a_i, a'_i \in \mathbb{C}^N$ with $a_i \neq 0$ ($1 \leq i \leq n$). Find necessary and sufficient conditions for the existence of analytic functions F with $F(\alpha_i)a_i = a'_i$ such that $\|F\| \leq 1$. This question again reduces to the determination of the positivity of certain Hermitian matrix. Similar results have been shown to hold for problems where F has bounded spectral radius (Bercovici, Foias and Tannenbaum 1991a) or bounded structured singular value (Bercovici, Cockburn, Foias and Tannenbaum 1993).

In this paper we will consider the more general bitangential interpolation problem (see section 4) with analytic matrix valued functions of bounded structured singular value. We will derive, as a consequence of a general structured commutant lifting theorem, necessary and sufficient conditions for the existence of such interpolating functions.

This paper is organized as follows. In section 2 briefly introduce the robust control problem that motivated this work. Then we show how this robust control problem leads naturally to a structured bitangential interpolation problem. Section 3 contains the main results of this paper. Here we setup the operator theoretic framework to study these interpolation problems and prove a bitangential structured commutant lifting theorem. This theorem is applied in section 4 to deduce necessary and sufficient conditions for the solution to structured bitangential Nevanlinna-Pick problems.

2. THE ROBUST CONTROL PROBLEM

FIGURE 1. Standard Feedback Structured

Suppose that we are given the control structure of Fig. 1, where G is an operator representing the fixed parts of a system and $\Delta \in \mathcal{D}$ is an operator that belongs to some set $\mathcal{D}$ representing the uncertainty in the system. The objective is to design a controller K so that the system is stable. Typically w represents the exogenous inputs, u the control commands and actuator inputs, z the regulated outputs and y the measurements or sensor outputs. Consider the case when all the input-output maps in Fig. 1 are causal, linear, time invariant and finite dimensional so they can be represented in the frequency domain (after taking Laplace transforms) as multiplication by rational matrix functions. Suppose that our objective is to design

a controller K that robustly stabilizes G , i.e., that stabilizes G for all $\Delta \in \mathcal{D}$ where

$$(1) \quad \mathcal{D} := \{\Delta = \text{diag}\{\Delta_1, \dots, \Delta_q\} : \|\Delta_i\|_\infty \leq 1, 1 \leq i \leq q\}$$

From the block diagram above we can write

$$(2) \quad z = G_{11}w + G_{12}u$$

$$(3) \quad y = G_{21}w + G_{22}u$$

$$(4) \quad u = Ky$$

$$(5) \quad w = \Delta z$$

If the system is well-posed (Vidyasagar 1985), the closed loop transfer matrix T from w to z is given by the following linear fractional transformation (LFT) on K

$$(6) \quad T = G_{11} + G_{22}K(I - G_{22}K)^{-1}G_{21}$$

In this configuration a controller K stabilizes G iff it stabilizes G_{22} (Francis 1987). A controller K stabilizes G for all $\Delta \in \mathcal{D}$ if the following holds.

Theorem 2.1 (Doyle). *The system of Fig. 1 is robustly stable iff*

$$\mu_\Delta(T) := \sup_\omega \mu(T(i\omega)) < 1$$

where

$$(7) \quad \mu(M) := \left[\min_{\Delta \in \mathcal{D}} \{\|\Delta\| : \det(I - M\Delta) = 0\} \right]^{-1}$$

$\mu(M)$ is called the *structured singular value* of the matrix M (with respect to the structure $\mathcal{D}$) and represents the size of the smallest perturbation $\Delta \in \mathcal{D}$ that makes $I - M\Delta$ singular.

2.1. Interpolation and Stabilization. The connection between stability and interpolation has been a recurring theme in the control systems literature (Youla, Bongiorno and Jabr 1976, Vidyasagar 1985). A classical approach to this problem is to use the parametrization of Youla-Jabr-Bongiorno-Kucera (Youla et al. 1976) to rewrite equation (6) as

$$(8) \quad T = T_1 + T_2QT_3$$

where T_1, T_2, T_3 are proper, real-rational matrix functions analytic in the closed right half plane, $\overline{\mathbb{H}}$, which are completely determined by the generalized plant G , and Q is a free stable, proper, real-rational matrix function that plays the role of the controller K . Given T_1, T_2, T_3 as above it suffices to choose any stable Q to stabilize the system. This freedom in Q can be used to satisfy additional performance objectives.

Since T_1, T_2, T_3 and Q are analytic in $\overline{\mathbb{H}}$ so is T . However, not all analytic functions T correspond to stable closed loops. Indeed, T must satisfy certain tangential (or directional) interpolation constraints at the zeros of T_2 and T_3 in $\tilde{\mathbb{H}} := \overline{\mathbb{H}} \cup \infty$ (Kimura 1987, Limebeer and Anderson 1988). For a detailed discussion on zeros and poles of rational matrix functions in the context of interpolation problems see (Ball et al. 1990a).

If we assume that T_2 and T_3 are invertible and all their zeros are in $\mathbb{H}$, then all stable closed loops T can be characterized as solutions to a bitangential Lagrange-Sylvester interpolation problem (Ball, Gohberg and Rodman 1990b). If in addition we require that the closed loop T be robustly stable, then $\mu_\Delta(T) < 1$. Thus, robust stabilization is reduced to the following interpolation problem:

Problem 2.1. Given $\{(\alpha_j; a_i, a'_i)\}_1^{n_\alpha}$ and $\{(\beta_j; b'_j, b_j)\}_1^{n_\beta}$, with $\{\alpha_i\}_1^{n_\alpha} \subset \mathbb{H}$ distinct, $a_i \in \mathbb{C}^N \setminus \{0\}$, $a'_i \in \mathbb{C}^N$ and $\{\beta_j\}_1^{n_\beta} \subset \mathbb{H}$ distinct, $b_j \in \mathbb{C}^N$, $b'_j \in \mathbb{C}^N \setminus \{0\}$. Find a matrix-valued function $T : \mathbb{H} \rightarrow \mathbb{C}^{N \times N}$, analytic and continuous in $\mathbb{H}$, such that

- i) $T(\alpha_i)a_i = a'_i$, $i = 1, \dots, n_\alpha$.
- ii) $b'_j T(\beta_j) = b_j^*$, $j = 1, \dots, n_\beta$.
- iii) $b'_j T^{(1)}(\alpha_i)a_i = \rho_{ij}$, $\alpha_i = \beta_j$.
- iv) $\mu_\Delta(T) < 1$.

where the superscript (n) denotes the n^{th} derivative and we have considered, for simplicity, the case when all the zeros of T_2 and T_3 are simple. In the robust stabilization problem above $\{(\alpha_i; a_i)\}_1^{n_\alpha}$ are the right zero pairs of T_3 in $\mathbb{H}$, $\{(\beta_j; b'_j)\}_1^{n_\beta}$ are the left zero pairs of T_2 in $\mathbb{H}$, $a'_i = T_1(\alpha_i)a_i$ and $b_j^* = b'_j T_1(\beta_j)$.

Remark 2.1. If T_2 and T_3 are not invertible then additional interpolation constraints are required to characterize all analytic T corresponding to stable systems. For example, if T_2 is injective and T_3 surjective, in addition to the finite tangential interpolation constraints (i) – (iii) we will have right and left kernel constraint on T . Problems of this type have been considered in (Ball and Rakowski 1994, Ball and Rakowski 1992) with (iv) replaced by $\|T\|_\infty < 1$.

2.2. Notation. Our notation follows Sz.-Nagy and Foias (1970). L^p denotes the Lebesgue spaces on the unit circle $\mathbb{T}$ and H^p are the Hardy spaces in $\mathbb{D}$. All Hilbert spaces are assumed complex separable. For $\mathcal{E}$ a Hilbert space we denote $L^p(\mathcal{E})$ the space of functions in L^p taking values in $\mathcal{E}$, i.e., $L^p(\mathcal{E}) = L^p \otimes \mathcal{E}$. For $\mathcal{H}, \mathcal{H}'$ two Hilbert spaces, $\mathcal{B}(\mathcal{H}, \mathcal{H}')$ is the Banach space (in the operator norm) of bounded linear operators. When $\mathcal{H} = \mathcal{H}'$ we will use $\mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$, T^* denotes its adjoint. $P_{\mathcal{H}} : \mathcal{K} \rightarrow \mathcal{H}$ denotes the orthogonal projection operator onto $\mathcal{H}$. For $U : \mathcal{K} \rightarrow \mathcal{K}$ and $\mathcal{H} \subset \mathcal{K}$ the compression of U to $\mathcal{H}$, i.e., $P_{\mathcal{H}}U|_{\mathcal{H}}$, will be denoted $U_{\mathcal{H}}$. $\mathcal{I}(T, T')$ denotes the set of all operators A satisfying the intertwining property $AT = T'A$.

Remark 2.2. The theory that we are going to introduce in this paper has been developed in $\mathbb{D}$. In engineering applications, LTI systems are represented as rational matrix functions in the Laplace transform variable s (for continuous time) and in the Z transform variable z (for discrete time). This does not limit our results to discrete time systems since we can use the conformal map $\phi(z) = (1+z)/(1-z)$ with inverse $\phi^{-1}(s) = (s-1)/(s+1)$ to map $\mathbb{D}$ onto $\mathbb{H}$ and vice-versa. Thus, for each function $T(s)$ analytic in $\mathbb{H}$ there is a corresponding $F(z)$ analytic in $\mathbb{D}$, i.e., $F(z) = T(\frac{1+z}{1-z})$ and the conformal map ϕ preserves all the important properties of T as a bounded analytic function.

3. STRUCTURED COMMUTANT LIFTING THEORY

The commutant lifting theorem (CLT) allows us to study many problems of interpolation in an abstract unified manner. It can be regarded as a geometric result involving norm preserving extensions of operators intertwining restrictions of co-isometries. Here we state one version of the general commutant lifting theorem (Sz.-Nagy and Foias 1970).

Theorem 3.1 (Commutant Lifting Theorem). Let $U \in \mathcal{B}(\mathcal{K})$ and $U' \in \mathcal{B}(\mathcal{K}')$ be two isometries. Assume that $\mathcal{H} \subset \mathcal{K}$ and $\mathcal{H}' \subset \mathcal{K}'$ are invariant for U^* and U'^* , respectively. Let $T^* = U^*|_{\mathcal{H}}$, $T'^* = U'^*|_{\mathcal{H}'}$. Given $A \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ a contraction. If $A \in \mathcal{I}(T, T')$, then

$$A^* = B^*|_{\mathcal{H}} \quad \text{and} \quad \|B\| = \|A\|.$$

In this section we will develop special commutant lifting theorems that take into account the special structure of $\mathcal{K}$ inherited from the structured perturbations $\Delta \in \mathcal{D}$. First, let us give a precise mathematical description of structured singular values in this context.

3.1. Structured Singular Values and Dilations. The structured singular value, introduced by Safonov (1980) and Doyle (1982), has proven to be a useful tool for the analysis and synthesis of systems with structured uncertainty. (For a general introduction to structured singular values in control see the tutorial paper Packard and Doyle (1993)). We begin by giving, without proof, a mathematical definition of the structured singular value and some of its basic properties. For details see Bercovici et al. (1990) and citeasnounDoy82.

Let $\mathcal{E}$ be an arbitrary separable complex Hilbert space, and $\mathcal{D} \subset \mathcal{B}(\mathcal{E})$ a C^* -algebra. For $A \in \mathcal{B}(\mathcal{E})$, $A \neq 0$, we define the *structured singular value* of A , $\mu_\Delta(A)$, as

$$(9) \quad \mu_\Delta(A) := [\inf\{\|X\| : X \in \mathcal{D}, -1 \in \sigma(AX)\}]^{-1}$$

A related quantity of engineering interest is

$$(10) \quad \hat{\mu}_\Delta(A) := \inf\{\|XAX^{-1}\| : X \in \mathcal{D}', X \text{ invertible}\}$$

where $\mathcal{D}'$ denotes the commutant of $\mathcal{D}$.

Remark 3.1. For $\mathcal{D} = \mathcal{B}(\mathcal{E})$, $\mu_\Delta(A) = \|A\|$. For $\mathcal{D} = \mathbb{C}I_\mathcal{E}$, $\mu_\Delta(A) = \|A\|_\rho$, the spectral radius of A . Moreover, $\mu_\Delta(A) = \mu_\Delta(A^*)$.

Let $\mu_1(\cdot)$ denote either $\mu_\Delta(\cdot)$ or $\hat{\mu}_\Delta(\cdot)$. Then $\mu_1(\cdot)$ has the following three basic properties:

- i) $\mu_1(\lambda A) = |\lambda| \mu_1(A)$.
- ii) $\mu_1(AX) \leq \|X\| \mu_1(A)$.
- iii) $\|A\|_\rho \leq \mu_1(A) \leq \|A\|$.

Other useful properties of $\mu_\Delta(\cdot)$ are summarized in the following lemma:

Lemma 3.1 (Bercovici, Foias and Tannenbaum). *Notation as above,*

- i) $\mu_\Delta(A) = \sup\{\|AX\|_\rho : X \in \mathcal{D}, \|X\| \leq 1\}$.
- ii) $\mu_\Delta(\cdot)$ is upper semi-continuous.
- iii) If $\mathcal{E}$ is finite dimensional, then $\mu_\Delta(\cdot)$ is continuous.
- iv) $\mu_\Delta(A) \leq \hat{\mu}_\Delta(A)$.
- v) For $\mathcal{D}$, a C^* -algebra with three or fewer irreducible components,

$$\mu_\Delta(A) = \hat{\mu}_\Delta(A).$$

In this work we will specialize to the case when $\mathcal{E}$ is finite dimensional. After an appropriate choice of basis we will have

$$(11) \quad \mathcal{D} = \left\{ \begin{bmatrix} X_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & X_p \end{bmatrix} : X_j \in \mathcal{B}(\mathbb{C}^{n_j}), (1 \leq j \leq p) \right\}$$

Remark 3.2. In general, it is possible that some of the X_j in (11) are constrained to be equal.

To put the structured singular value in the commutant lifting framework we need to introduce some definitions. Let $\mathcal{G}, \mathcal{E}$ be Hilbert spaces, with $\mathcal{E}$ finite dimensional and let $\mathcal{D} \subset \mathcal{B}(\mathcal{E})$ be a C^* -algebra. For

U an isometry on $\mathcal{K} = \mathcal{G} \otimes \mathcal{E}$, let $\mathcal{H} \subset \mathcal{K}$ be invariant for U^* with $T = U|_{\mathcal{H}}$ the compression of U to $\mathcal{H}$ and define

$$\begin{aligned} I_{\mathcal{G}} \otimes \mathcal{D} &:= \{I_{H^2} \otimes X : X \in \mathcal{D}\}, \\ I_{\mathcal{G}} \otimes \mathcal{D}' &:= \{I_{H^2} \otimes X : X \in \mathcal{D}'\}. \end{aligned}$$

The set $I_{\mathcal{G}} \otimes \mathcal{D} \subset \mathcal{B}(\mathcal{K})$ contains all the operators acting on $\mathcal{K}$ with the “structure” of $\mathcal{D}$ and $I_{\mathcal{G}} \otimes \mathcal{D}'$ is its commutant.

Remark 3.3. *The subspace $\mathcal{H}$ defined above reduces both $I_{\mathcal{G}} \otimes \mathcal{D}$ and $I_{\mathcal{G}} \otimes \mathcal{D}'$.*

Denote

$$\begin{aligned} \{U_{\Delta}\}' &:= \{Y : Y \in \{U\}' \cap I_{\mathcal{G}} \otimes \mathcal{D}'\}, \\ \{T_{\Delta}\}' &:= \{X : X \in \{T\}' \cap I_{\mathcal{G}} \otimes \mathcal{D}'|_{\mathcal{H}}\}. \end{aligned}$$

For $B \in \{U\}'$ define

$$\rho_U^{\Delta}(B) := \inf\{\|YBY^{-1}\| : Y \text{ invertible}, Y \in \{U_{\Delta}\}'\}.$$

Similarly, for $A \in \{T\}'$

$$\rho_T^{\Delta}(A) := \inf\{\|XAX^{-1}\| : X \text{ invertible}, X \in \{T_{\Delta}\}'\}.$$

Let $A\mathcal{H} \subset \mathcal{M}'$, where $\mathcal{M}'$ is invariant for T^* . For $A \in \mathcal{J}(T, P_{\mathcal{M}'})$ we define

$$(12) \quad \rho_{T^*, \mathcal{M}'}^{\Delta}(A) := \inf\{\|P_{X^* \mathcal{M}'} X^{-1} AX\| : X \text{ invertible}, X \in \{T_{\Delta}\}'\}.$$

Similarly, for $A^* \mathcal{H} \subset \mathcal{N}$ with $\mathcal{N}$ invariant for T such that $A \in \mathcal{J}(T|_{\mathcal{N}}, T)$

$$(13) \quad \rho_{T, \mathcal{N}}^{\Delta}(A) := \inf\{\|X^{-1} AX|_{X^{-1} \mathcal{N}}\| : X \text{ invertible}, X \in \{T_{\Delta}\}'\}.$$

Let $\mathcal{M}' \subset \mathcal{H}$ invariant for T^* . For any $X \in \{T_{\Delta}\}'$, $\mathcal{M}'_X := X^* \mathcal{M}'$ is invariant for T^* and

$$(14) \quad \begin{aligned} P_{\mathcal{M}'_X} X^{-1} &= P_{\mathcal{M}'_X} X^{-1} P_{\mathcal{M}'}, \\ P_{\mathcal{M}'} X &= P_{\mathcal{M}'} X P_{\mathcal{M}'_X}. \end{aligned}$$

3.2. Structured Commutant Lifting Theorems. In this section we will derive a bitangential structured commutant lifting theorem that arise when $\mathcal{K}$ and $\mathcal{H}$ in the commutant lifting theorem have a special structure. Let $\mathcal{E}$ be a finite dimensional Hilbert and $\mathcal{D} \subset \mathcal{B}(\mathcal{E})$ a given C^* -algebra. Let U be an isometry on $\mathcal{K} := \mathcal{G} \otimes \mathcal{E}$ with $\mathcal{G}$ a Hilbert space, and $T^* = U^*|_{\mathcal{H}}$ the restriction of the co-isometry U^* to $\mathcal{H} \subset \mathcal{K}$, a finite dimensional hyperinvariant subspace for U^* (i.e., invariant for $\{U\}'$).

We begin with some theorems introduced for the study of one-sided structured tangential problems (Bercovici et al. 1993).

Theorem 3.2 (Structured Left Tangential Commutant-Lifting Theorem). *Let $\mathcal{M}' \subset \mathcal{H}$ invariant for T^* . Given $A \in \mathcal{B}(\mathcal{H})$ such that $A\mathcal{H} \subset \mathcal{M}'$. If $A \in \mathcal{J}(T, P_{\mathcal{M}'} T)$, then*

$$\gamma_{opt} = \rho_{T^*, \mathcal{M}'}^{\Delta}(A) = \inf\{\rho_U^{\Delta}(B) : B \in \text{Dil}(A)\}$$

where

$$\text{Dil}(A) = \{B : BU = UB, A^* = B^*|_{\mathcal{M}'}\}.$$

The bitangential version of the commutant lifting theorem above theorem above requires a combination of Theorem 3.2 and its dual which is given below.

Theorem 3.3 (Structured Right Tangential Commutant-Lifting Theorem). *Let $\mathcal{N} \subset \mathcal{H}$ invariant for T . Given $A \in \mathcal{B}(\mathcal{H})$ such that $A^*\mathcal{H} \subset \mathcal{N}$. If $A \in \mathcal{J}(T|\mathcal{N}, T)$, then there exists $B \in \mathcal{B}(\mathcal{K})$ such that*

$$\gamma_{opt} = \rho_{T, \mathcal{N}}^\Delta(A) = \inf\{\rho_U^\Delta(B) : B \in \text{Dil}(A)\}.$$

where $\text{Dil}(A) = \{B : \{U\}', A = P_{\mathcal{H}}B|\mathcal{N}\}$.

Now we are ready to address the more general bitangential interpolation problem. To do that we will combine theorems 3.2 and 3.3. Again, let $\mathcal{E}$ be a finite dimensional Hilbert and $\mathcal{D} \subset \mathcal{B}(\mathcal{E})$ a given C^* -algebra. Let U be an isometry on $\mathcal{K} := \mathcal{G} \otimes \mathcal{E}$ with $\mathcal{G}$ a Hilbert space, and $T^* = U^*|_{\mathcal{H}}$ the restriction of the co-isometry U^* to a finite dimensional hyperinvariant subspace $\mathcal{H} \subset \mathcal{K}$ for U^* . In addition, suppose that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with $\mathcal{H}_0$ hyperinvariant for T with $\mathcal{M}' \subset \mathcal{H}_0$ and $\mathcal{N} \subset \mathcal{H}_1$ invariant for T^* and T , respectively.

For $X \in \{T_\Delta\}'$ invertible write

$$X = \begin{bmatrix} X_{00} & X_{01} \\ 0 & X_{11} \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} X_{00}^{-1} & -X_{00}^{-1}X_{01}X_{11}^{-1} \\ 0 & X_{11}^{-1} \end{bmatrix}.$$

and denote $\mathcal{M}'_X = X_{00}^*\mathcal{M}'$ and $\mathcal{N}_X = X_{11}^{-1}\mathcal{N}$. Then, it is easy to check that

$$(15) \quad P_{\mathcal{M}' \oplus \mathcal{H}_1} X P_{\mathcal{M}'_X \oplus \mathcal{H}_1} = P_{\mathcal{M}' \oplus \mathcal{H}_1} X,$$

$$(16) \quad P_{\mathcal{H}_0 \oplus \mathcal{N}_X} X^{-1} P_{\mathcal{H}_0 \oplus \mathcal{N}} = X^{-1} P_{\mathcal{H}_0 \oplus \mathcal{N}}.$$

Theorem 3.4 (Structured Bitangential Commutant Lifting Theorem). *Notation as above. Given $A \in \mathcal{B}(\mathcal{H})$ such that $A^*\mathcal{M}' \subset \mathcal{H}_0$ and $A\mathcal{N} \subset \mathcal{H}_1$. If $A \in \mathcal{J}(T|\mathcal{H}_0 \oplus \mathcal{N}, P_{\mathcal{M}' \oplus \mathcal{H}_1}T)$, then*

$$\gamma_{opt} := \rho_{T, \mathcal{M}', \mathcal{N}}^\Delta(A) = \inf\{\rho_U^\Delta(B) : B \in \text{Dil}(A)\},$$

where

$$(17) \quad \rho_{T, \mathcal{M}', \mathcal{N}}^\Delta(A) = \inf\{\|P_{\mathcal{M}'_X \oplus \mathcal{H}_1} X^{-1} A X|_{\mathcal{H}_0 \oplus \mathcal{N}_X}\| : X \in \{T_\Delta\}' \text{ invertible}\}.$$

Proof a) First we will show that $\rho_{T, \mathcal{M}', \mathcal{N}}^\Delta(A) < \rho_U^\Delta(B)$. Let $B_{opt} \in \text{Dil}(A)$ be such that

$$\rho_U^\Delta(B_{opt}) = \inf\{\rho_U^\Delta(B) : B \in \text{Dil}(A)\}.$$

Given $\epsilon > 0$, fix $B \in \text{Dil}(A)$ such that

$$\rho_U^\Delta(B) = \rho_U^\Delta(B_{opt}) + \epsilon.$$

From (Bercovici et al. 1990), there exists $Y \in \{U_\Delta\}'$ invertible such that

$$\|Y^{-1}BY\| < \rho_U^\Delta(B_{opt}) + \epsilon.$$

Since $\mathcal{H}$ hyperinvariant for U^* , there exists $X \in \{T_\Delta\}'$ invertible such that $X = Y_{\mathcal{H}}$ and $X^{-1} = (Y^{-1})_{\mathcal{H}}$. Define

$$(18) \quad A_X = P_{\mathcal{M}'_X \oplus \mathcal{H}_1} X^{-1} A X|_{\mathcal{H}_0 \oplus \mathcal{N}_X}.$$

Using the fact that $B \in \text{Dil}(A)$ and eqs. (15) and (16)

$$\begin{aligned}
\|A_X\| &= \|P_{\mathcal{M}'_X \oplus \mathcal{H}_1} X^{-1} A X | \mathcal{H}_0 \oplus \mathcal{N}_X \| \\
&= \|P_{\mathcal{M}'_X \oplus \mathcal{H}_1} X^{-1} P_{\mathcal{M}' \oplus \mathcal{H}_1} B P_{\mathcal{H}_0 \oplus \mathcal{N}} X P_{\mathcal{H}_0 \oplus \mathcal{N}_X} \| \\
&= \|P_{\mathcal{M}'_X \oplus \mathcal{H}_1} X^{-1} (P_{\mathcal{H}} B | \mathcal{H}) X P_{\mathcal{H}_0 \oplus \mathcal{N}_X} \| \\
&= \|P_{\mathcal{M}'_X \oplus \mathcal{H}_1} (Y^{-1})_{\mathcal{H}} B_{\mathcal{H}} Y_{\mathcal{H}} P_{\mathcal{H}_0 \oplus \mathcal{N}_X} \| \\
&= \|P_{\mathcal{M}'_X \oplus \mathcal{H}_1} (Y^{-1} B Y)_{\mathcal{H}} P_{\mathcal{H}_0 \oplus \mathcal{N}_X} \| \\
&\leq \|Y^{-1} B Y\| \\
\rho_{T, \mathcal{M}', \mathcal{N}}^{\Delta}(A) &< \inf\{\rho_U^{\Delta}(B) : B \in \text{Dil}(A)\} + \epsilon.
\end{aligned}$$

b) To show that $\rho_{T, \mathcal{M}', \mathcal{N}}^{\Delta}(A) \geq \rho_U^{\Delta}(B)$, let $\epsilon > 0$, and choose $X \in \{T_{\Delta}\}'$ invertible such that

$$\|P_{\mathcal{M}'_X \oplus \mathcal{H}_1} X^{-1} A X | \mathcal{H}_0 \oplus \mathcal{N}_X\| < \rho_{T, \mathcal{M}', \mathcal{N}}^{\Delta}(A) + \epsilon.$$

Then there exists $Y \in \{U_{\Delta}\}'$ invertible such that $X = Y_{\mathcal{H}}$. If we set

$$A_1 = P_{\mathcal{M}'_X \oplus \mathcal{H}_1} X^{-1} A X | \mathcal{H}_0 \oplus \mathcal{N}_X,$$

then $A_1 \in \mathcal{J}(T | \mathcal{H}_0 \oplus \mathcal{N}_X, P_{\mathcal{M}'_X \oplus \mathcal{H}_1} T)$. Thus by the commutant lifting theorem there exists $B_1 \in \text{Dil}(A_1)$ such that $\|B_1\| = \|A_1\|$. Set $B = Y B_1 Y^{-1}$ and note that

$$\rho_U^{\Delta}(B) \leq \|B_1\| < \rho_{T, \mathcal{M}', \mathcal{N}}^{\Delta}(A) + \epsilon.$$

But $B \in \text{Dil}(A)$. To see this write

$$\begin{aligned}
A P_{\mathcal{H}_0 \oplus \mathcal{N}} &= P_{\mathcal{M}' \oplus \mathcal{H}_1} X A_1 X^{-1} P_{\mathcal{H}_0 \oplus \mathcal{N}} \\
&= P_{\mathcal{M}' \oplus \mathcal{H}_1} X P_{\mathcal{M}'_X \oplus \mathcal{H}_1} B_1 P_{\mathcal{H}_0 \oplus \mathcal{N}_X} X^{-1} P_{\mathcal{H}_0 \oplus \mathcal{N}} \\
&= P_{\mathcal{M}' \oplus \mathcal{H}_1} X P_{\mathcal{H}} B_1 P_{\mathcal{H}} X^{-1} P_{\mathcal{H}_0 \oplus \mathcal{N}} \\
&= P_{\mathcal{M}' \oplus \mathcal{H}_1} (Y B_1 Y^{-1})_{\mathcal{H}} P_{\mathcal{H}_0 \oplus \mathcal{N}} \\
&= P_{\mathcal{M}' \oplus \mathcal{H}_1} B.
\end{aligned}$$

Thus

$$\inf\{\rho_U^{\Delta}(B) : B \in \text{Dil}(A)\} < \rho_{T, \mathcal{M}', \mathcal{N}}^{\Delta}(A) + \epsilon.$$

and since $\epsilon > 0$ arbitrary we have the desired result. $\square$

4. STRUCTURED NEVANLINNA-PICK INTERPOLATION

In this section we will show how the structured commutant lifting theorems can be applied to deduce necessary and sufficient conditions for the existence of solutions to structured bitangential Nevanlinna-Pick problems. Here $\mathcal{G} = H^2$, $\mathcal{E}$ a finite dimensional Hilbert space. The model for this scenario is $\mathcal{K} = H^2(\mathcal{E})$, $\mathcal{H} = \mathcal{K} \otimes m\mathcal{K}$ with m a finite Blaschke product and U is the operator defined by multiplication by z .

Let $A^{\infty}(\mathcal{B}(\mathcal{E}))$ denote the space of functions $F : \mathbb{D} \rightarrow \mathcal{B}(\mathcal{E})$, analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. The following corollaries (Bercovici et al. 1990) reveal some useful relations between $\rho_U^{\Delta}(\cdot)$ and $\hat{\mu}_{\Delta}(\cdot)$.

Corollary 4.1. *Let $B \in A^{\infty}(\mathcal{B}(\mathcal{E}))$. Suppose that $\hat{\mu}_{\Delta}(B) < 1$, $\forall z \in \overline{\mathbb{D}}$. Then there exists an analytic map $X(z) : \mathbb{D} \rightarrow \mathcal{D}'$ with $X(z)^{-1}$ analytic, such that*

$$\|X(z) B(z) X(z)^{-1}\| < 1, \forall z \in \mathbb{D}$$

Corollary 4.2. *Let $B \in \{U\}'$, and suppose that B is continuous on $\overline{\mathbb{D}}$. Then*

$$\rho_U^\Delta(B) = \sup_{z \in \mathbb{D}} \hat{\mu}_\Delta(B(z)).$$

where $B(z)$ is the symbol of B .

The structured bitangential Nevanlinna-Pick interpolation problem can be stated as follows: Given interpolation data consisting of a right interpolation data set, denoted $\{(\alpha_j; a_i, a'_i)\}_1^{n_\alpha}$ and a left interpolation data set $\{(\beta_j; b'_j, b_j)\}_1^{n_\beta}$, where $\alpha_i \in \mathbb{D}$, $a_i \in \mathbb{C}^N \setminus \{0\}$, $a'_i \in \mathbb{C}^N$ and $\beta_j \in \mathbb{D}$, $b_j \in \mathbb{C}^N$, $b'_j \in \mathbb{C}^N \setminus \{0\}$. Furthermore, assume that all the interpolation points $\{\alpha_i\}_1^{n_\alpha}$ and $\{\beta_j\}_1^{n_\beta}$ are distinct. Find $F \in A^\infty(\mathbb{C}^{N \times N})$ such that

- i) $F(\alpha_i)a_i = a'_i$, $i = 1, \dots, n_\alpha$.
- ii) $b'_j F(\beta_j) = b_j^*$, $j = 1, \dots, n_\beta$.
- iii) $b'_j F^{(1)}(\alpha_i)a_i = \rho_{ij}$, $\alpha_i = \beta_j$.
- iv) $\hat{\mu}_\Delta(F) < \gamma$.

To simplify the notation let

$$\begin{aligned} \mathcal{I} &:= \{F \in A^\infty(\mathbb{C}^{N \times M}) : F \text{ satisfies i), ii) and iii})\}, \\ \mathcal{I}_r &:= \{F \in A^\infty(\mathbb{C}^{N \times M}) : F \text{ satisfies i})\}, \\ \mathcal{I}_l &:= \{F \in A^\infty(\mathbb{C}^{N \times M}) : F \text{ satisfies ii})\}. \end{aligned}$$

Using this notation, the structured left tangential Nevanlinna-Pick interpolation can be stated as follows:

Problem 4.1 (Structured Left Tangential Nevanlinna-Pick). *Find necessary and sufficient conditions for the existence of an interpolating function $F \in \mathcal{I}_l$ such that $\hat{\mu}_\Delta(F) < \gamma$.*

To solve this problem we will put it in the framework of the structured commutant lifting theory, set

$$\begin{aligned} m_\beta(z) &= \prod_{j=1}^{n_\beta} \frac{z - \beta_j}{1 - \overline{\beta_j}z} \quad ; \quad g_{\beta_j}(z) = \frac{1}{1 - \overline{\beta_j}z} \quad ; \quad j = 1, \dots, n_\beta, \\ \mathcal{K} &= H^2(\mathbb{C}^N) \quad ; \quad \mathcal{H} = \mathcal{K} \ominus m_\beta \mathcal{K} \quad ; \quad \mathcal{M}' = \sum_{j=1}^{n_\beta} \mathbb{C} g_{\beta_j} \otimes b_i. \end{aligned}$$

Thus we have

$$T^* g_{\beta_j} \otimes v = \overline{\beta_j} g_{\beta_j} \otimes v, \quad v \in \mathbb{C}^N, \quad j = 1, \dots, n_\beta.$$

Define the operator A by linearity and via its adjoint as

$$A^* g_{\beta_j} \otimes b'_j = g_{\beta_j} \otimes b_j, \quad j = 1, \dots, n_\beta.$$

and denote such operator by $A^*(\beta_j; b_j; b'_j)$. Then by the left tangential commutant lifting theorem, there exists $M_F \in \text{Dil}(A)$, such that $F \in \mathcal{I}_l$. Now fix $\mathcal{D} \subset \mathcal{B}(\mathbb{C}^N)$ and consider $X \in \{T_\Delta\}'$ invertible. Then necessarily

$$X^* g_{\beta_j} \otimes v = X_j g_{\beta_j} \otimes v, \quad X_j \in \mathcal{D}' \text{ invertible}, \quad v \in \mathbb{C}^N, \quad 1 \leq j \leq n_\beta.$$

Moreover, we have that

$$X^* A^*(\beta_j; b_j; b'_j) X^{-*} | \mathcal{M}'_X = A^*(\beta_j; X_j b_j; X_j b'_j).$$

where $\mathcal{M}'_X = |X^* \mathcal{M}'$. Thus,

$$\rho_{T^*, \mathcal{M}'}^\Delta(A) = \inf\{||A(\beta_j; X_j b_j; X_j b'_j)|| : X_j, X_j^{-1} \in \mathcal{D}'\} = \hat{\mu}_\Delta(F).$$

Clearly a solution to this problem exists iff $||A(\beta_j; X_j b_j; X_j b'_j)|| < \gamma$. This condition is equivalent to the positivity of certain *Structured Nevanlinna-Pick Matrix* as the following theorem shows.

Theorem 4.1 (Structured Left Tangential Nevanlinna–Pick Theorem). *There exists $F \in \mathcal{I}_1$ iff there exists invertible matrices $X_j \in \mathcal{D}'$, $1 \leq j \leq n_\beta$ such that $\mathcal{N}_\gamma(\beta_j; X_j b'_j, X_j b_j) > 0$, where*

$$\mathcal{N}_\gamma(\beta_j; X_j b'_j, X_j b_j) := \left[\frac{\langle X_j b'_j, X_i b'_i \rangle - \gamma^{-2} \langle X_j b_j, X_i b_i \rangle}{1 - \bar{\beta}_j \beta_i} \right]_{1 \leq i, j \leq n_\beta}.$$

Proof We have that

$$\begin{aligned} \|A(\beta_j; X_j b'_j, X_j b_j)\| < \gamma &\Leftrightarrow \gamma^2 I - A(\beta_j; X_j b'_j, X_j b_j) A^*(\beta_j; X_j b'_j, X_j b_j) > 0 \\ &\Leftrightarrow \mathcal{N}_\gamma(\beta_j; X_j b'_j, X_j b_j) > 0. \end{aligned}$$

□

Dual of the above problem is the structured right tangential Nevanlinna–Pick problem presented below

Problem 4.2 (Structured Right Tangential Nevanlinna–Pick). *Find necessary and sufficient conditions for the existence of an interpolating function $F \in \mathcal{I}_r$ such that $\hat{\mu}_\Delta(F) < \gamma$.*

In this case we will use Theorem 3.3. Set

$$m_\alpha(z) = \prod_{i=1}^{n_\alpha} \frac{z - \alpha_i}{1 - \bar{\alpha}_i z} \quad ; \quad f_{\alpha_i}(z) = m_\alpha \overline{z g_{\alpha_i}(z)} \quad ; \quad j = 1, \dots, n_\alpha,$$

$$\mathcal{K} = H^2(\mathbb{C}^N) \quad ; \quad \mathcal{H} = \mathcal{K} \ominus m_\alpha \mathcal{K} \quad ; \quad \mathcal{N} = \sum_{i=1}^{n_\alpha} \mathbb{C} f_{\alpha_i} \otimes a_i.$$

Define the operator A by linearity and via its adjoint as

$$A f_{\alpha_i} \otimes a_i = f_{\alpha_i} \otimes a_i', \quad i = 1, \dots, n_\alpha.$$

and denote such operator by $A(\alpha_i; a_i; a_i')$. Then, by Theorem 3.3 there exists $M_F \in \text{Dil}(A)$ such that $F \in \mathcal{I}_r$. Fix $\mathcal{D} \subset \mathcal{B}(\mathbb{C}^N)$. Then, there exists $X \in \{T_\Delta\}'$ invertible such that

$$X^{-1} A X | X^{-1} \mathcal{N} = A(\alpha_i; X_i' a_i, X_i' a_i').$$

Finally, we conclude that

$$(19) \quad \rho_{T, \mathcal{N}}^\Delta(A) = \inf\{\rho_U^\Delta(B) : B \in \text{Dil}(A)\} = \hat{\mu}_\Delta(F).$$

The Nevanlinna–Pick matrix associated with this problem is

$$\mathcal{N}_\gamma(\alpha_i; X_i a_i, X_i a_i') := \left[\frac{\langle X_j a_j, X_i a_i \rangle - \gamma^{-2} \langle X_j a_j', X_i a_i' \rangle}{1 - \bar{\alpha}_i \alpha_j} \right]_{1 \leq i, j \leq n_\alpha},$$

where $X_i \in \mathcal{D}'$ invertible.

Finally, the structured bitangential Nevanlinna–Pick problem is

Problem 4.3 (Structured Bitangential Nevanlinna–Pick). *Find necessary and sufficient conditions for the existence of an interpolating function $F \in \mathcal{I}$ such that $\hat{\mu}_\Delta(F) < \gamma$.*

To solve the structured bitangential Nevanlinna–Pick problem by combining the results above. Set $m = m_\alpha m_\beta$ where

$$m_\alpha(z) = \prod_{i=1}^{n_\alpha} \frac{z - \alpha_i}{1 - \bar{\alpha}_i z}, \quad m_\beta(z) = \prod_{j=1}^{n_\beta} \frac{z - \beta_j}{1 - \bar{\beta}_j z},$$

Set the spaces and operators as follows:

$$\begin{aligned} \mathcal{K} &= H^2(\mathbb{C}^N), \mathcal{H} = \mathcal{H}(m) = \mathcal{H}_0 \oplus \mathcal{H}_1, \\ \mathcal{H}_0 &= m_\alpha \mathcal{H}(m_\beta) := m_\alpha(\mathcal{H}_0 \ominus m_\beta \mathcal{H}_0), \quad \mathcal{H}_1' = \mathcal{H}'(m_\alpha) := \mathcal{H}_1' \ominus m_\alpha \mathcal{H}_1', \end{aligned}$$

$$\mathcal{M}' = \sum_{j=1}^{n_\beta} \mathbb{C} m_\alpha g_{\beta_j} \otimes b'_j, \quad \mathcal{N} = \sum_{i=1}^{n_\alpha} \mathbb{C} f_{\alpha_i} \otimes a_i.$$

Choose U as the canonical unilateral shift on $H^2(\mathbb{C}^N)$ and define $A : \mathcal{H}_0 \oplus \mathcal{N} \rightarrow \mathcal{M}' \oplus \mathcal{H}_1$ as follows:

$$(20) \quad (P_{\mathcal{H}_0} A^* | \mathcal{M}') m_\alpha g_{\beta_j} \otimes b'_j := m_\alpha g_{\beta_j} \otimes b_j, \quad 1 \leq j \leq n_\beta.$$

$$(21) \quad (P_{\mathcal{H}_1} A | \mathcal{N}) f_{\alpha_i} \otimes a_i := f_{\alpha_i} \otimes a'_i, \quad 1 \leq i \leq n_\alpha.$$

$$(22) \quad \langle (P_{\mathcal{M}'} A | \mathcal{N}) (f_{\alpha_i} \otimes a_i), m_\alpha g_{\beta_j} \otimes b'_j \rangle := \begin{cases} \frac{\langle a_i, b_j \rangle - \langle a'_i, b'_j \rangle}{\beta_j - \alpha_i} & , \alpha_i \neq \beta_j \\ \rho_{ij} & , \alpha_i = \beta_j \end{cases}.$$

Then, A intertwines $T|_{\mathcal{H}_0 \oplus \mathcal{N}}$ and $P_{\mathcal{M}' \oplus \mathcal{H}_1} T$ and by the CLT there exists interpolating functions $F \in \mathcal{I}$. From Theorem 3.4 there exists invertible $M_X \in \{U_\Delta\}'$, with $X \in A^\infty(\mathbb{C}^{N \times N})$ and $B = M_F$, with $F \in A^\infty(\mathbb{C}^{N \times N})$ such that

$$(23) \quad X(\alpha_i) = X_i'^{-1}, \text{ where } X_i', X_i'^{-1} \in \mathcal{D}', \quad i = 1, \dots, n_\alpha,$$

$$(24) \quad X^*(\beta_j) = X_j, \text{ where } X_j, X_j^{-1} \in \mathcal{D}' \quad j = 1, \dots, n_\beta,$$

and $\rho_{T, \mathcal{M}', \mathcal{N}}^\Delta(A) = \hat{\mu}_\Delta(F)$.

The Hermitian Nevanlinna-Pick matrix associated to this problem is given in the following theorem.

Theorem 4.2 (Structured Bitangential Nevanlinna-Pick Theorem). *There exists $F \in \mathcal{I}$ with $\hat{\mu}_\Delta(F) < \gamma$ iff there exist invertible matrices $\{X_i'\}_1^{n_\alpha} \subset \mathcal{D}'$ and $\{X_j\}_1^{n_\beta} \subset \mathcal{D}'$, such that*

$$\mathcal{N}_\gamma(\alpha_i; X_i' a_i, X_i' a'_i; \beta_j; X_j b_j, X_j b'_j) := \begin{bmatrix} N_\alpha & \gamma^{-1} N_{\alpha\beta}^* \\ \gamma^{-1} N_{\alpha\beta} & N_\beta \end{bmatrix} > 0,$$

where

$$(25) \quad \begin{aligned} [N_\alpha]_{ij} &= \frac{\langle X_j' a_j, X_i' a_i \rangle - \gamma^{-2} \langle X_j' a'_j, X_i' a'_i \rangle}{1 - \bar{\alpha}_i \alpha_j}, \quad 1 \leq i, j \leq n_\alpha. \\ [N_\beta]_{ij} &= \frac{\langle X_j b_j, X_i b_i \rangle - \gamma^{-2} \langle X_j b'_j, X_i b'_i \rangle}{1 - \bar{\beta}_i \beta_j}, \quad 1 \leq i, j \leq n_\beta. \\ [N_{\alpha\beta}]_{ij} &= \begin{cases} \frac{\langle X_j' a_j, X_i b_i \rangle - \langle X_j' a'_j, X_i b'_i \rangle}{\beta_i - \alpha_j} & , \alpha_j \neq \beta_i, \\ \hat{\rho}_{ij} & , \alpha_j = \beta_i. \end{cases} \quad 1 \leq i \leq n_\beta, 1 \leq j \leq n_\alpha. \end{aligned}$$

and $\hat{\rho}_{ij}$ is defined as

$$(26) \quad \hat{\rho}_{ij} := b_j'^* X_j^* [(X^{-1}(z) F(z) X(z))^{(1)}]_{z=\xi_{ij}} X_i' a_i.$$

Proof We want to show that $\mathcal{N}_\gamma > 0$, or equivalently that $\|A_X\| < 1$, where A_X is defined by (18) and

$$(27) \quad A_X = \begin{pmatrix} A_{00} & A_{01} \\ 0 & A_{11} \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{N}_X \rightarrow \mathcal{M}'_X \oplus \mathcal{H}_1.$$

Since $\|A_X\| < 1 \Leftrightarrow I - A_X^* A_X > 0$. Using (27) we can write

$$I - A_X^* A_X = \begin{bmatrix} I - A_{00}^* A_{00} & -A_{00}^* A_{01} \\ -A_{01}^* A_{00} & I - A_{11}^* A_{11} - A_{01}^* A_{01} \end{bmatrix} > 0.$$

The assumptions on the interpolation data guarantee that A_{00} is a strict contraction (and so is A_{00}^*). Thus both $(I - A_{00}^* A_{00})^{-1}$ and $(I - A_{00} A_{00}^*)^{-1}$ exist. Using a Schur complement argument one can show that

$$(28) \quad I - A^* A > 0 \Leftrightarrow (I - A_{11}^* A_{11}) - A_{01}^* (I - A_{00} A_{00}^*)^{-1} A_{01} > 0.$$

Thus, $\|A\| < 1$ iff

$$(29) \quad \hat{A} := \begin{bmatrix} I - A_{11}^* A_{11} & A_{01}^* \\ A_{01} & I - A_{00} A_{00}^* \end{bmatrix} > 0.$$

The positive operator $\hat{A} : \mathcal{N}_X \oplus \mathcal{M}'_X \rightarrow \mathcal{N}_X \oplus \mathcal{M}'_X$ defined by (29) leads naturally to the Nevanlinna-Pick matrix $\mathcal{N}_\gamma$. To see this, it remains to show that $\langle \hat{A}x, x \rangle > 0$ for all x in $\mathcal{N}_X \oplus \mathcal{M}'_X$. Let $x \in \mathcal{N}_X \oplus \mathcal{M}'_X$. Any such x can be written as $x = x_1 + x_2$ where

$$x_1 = \sum_{i=1}^{n_\alpha} \eta_i f_{\alpha_i} \otimes X_i' a_i, \quad x_2 = \sum_{j=1}^{n_\beta} \xi_j m_{\alpha} g_{\beta_j} \otimes X_j b_j'.$$

Therefore,

$$\begin{aligned} \langle \hat{A}x, x \rangle &= \langle (I - A_{11}^* A_{11})x_1, x_1 \rangle + \langle A_{01}^* x_2, x_1 \rangle + \langle A_{01} x_1, x_2 \rangle + \langle (I - A_{00} A_{00}^*)x_2, x_2 \rangle \\ &= (\langle x_1, x_1 \rangle - \langle A_{11} x_1, A_{11} x_1 \rangle) + \langle A_{01}^* x_2, x_1 \rangle + \langle A_{01} x_1, x_2 \rangle + \\ &\quad (\langle x_2, x_2 \rangle - \langle A_{00}^* x_2, A_{00}^* x_2 \rangle). \end{aligned}$$

Finally, from the definition of the operator A and the scaling γ we obtain:

$$\begin{aligned} (\langle x_1, x_1 \rangle - \langle A_{11} x_1, A_{11} x_1 \rangle) &= \sum_{i,j=1}^{n_\alpha} \eta_j \bar{\eta}_i [\langle f_{\alpha_j} \otimes X_j' a_j, f_{\alpha_i} \otimes X_i' a_i \rangle - \\ &\quad \langle f_{\alpha_j} \otimes \gamma^{-1} X_j' a_j', f_{\alpha_i} \otimes \gamma^{-1} X_i' a_i' \rangle] \\ &= \sum_{i,j=1}^{n_\alpha} \bar{\eta}_i \left[\frac{\langle X_j' a_j, X_i' a_i \rangle - \gamma^{-2} \langle X_j' a_j', X_i' a_i' \rangle}{1 - \bar{\alpha}_i \alpha_j} \right] \eta_j. \\ (\langle x_2, x_2 \rangle - \langle A_{00}^* x_2, A_{00}^* x_2 \rangle) &= \sum_{i,j=1}^{n_\beta} \xi_j \bar{\xi}_i [\langle m_{\alpha} g_{\beta_j} \otimes X_j b_j', m_{\alpha} g_{\beta_i} \otimes X_i b_i' \rangle - \\ &\quad \langle m_{\alpha} g_{\beta_j} \otimes \gamma^{-1} X_j b_j, m_{\alpha} g_{\beta_i} \otimes \gamma^{-1} X_i b_i \rangle] \\ &= \sum_{i,j=1}^{n_\beta} \bar{\xi}_i \left[\frac{\langle X_j b_j', X_i b_i' \rangle - \gamma^{-2} \langle X_j b_j, X_i b_i \rangle}{1 - \beta_i \bar{\beta}_j} \right] \xi_j. \\ \langle A_{01} x_1, x_2 \rangle &= \sum_{i=1}^{n_\beta} \sum_{j=1}^{n_\alpha} \eta_j \bar{\xi}_i [\langle A_{01}(f_{\alpha_j} \otimes X_j' a_j), m_{\alpha} g_{\beta_i} \otimes X_i b_i' \rangle] \\ &= \sum_{i=1}^{n_\beta} \sum_{j=1}^{n_\alpha} \bar{\xi}_i [n_{ij}] \eta_j. \end{aligned}$$

with $n_{ij} = [N_{\alpha\beta}]_{ij}$ as required. $\square$

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REFERENCES

- Ball, J., Gohberg, I. and Kaashoek, M.: 1992, Nevanlinna-pick interpolation for time-varying input-output maps: The discrete case, in I. Gohberg (ed.), *Time-variant systems and interpolation*, OT 56, Birkhäuser Verlag, pp. 1–89.
- Ball, J., Gohberg, I. and Rodman, L.: 1990a, *Interpolation of Rational Matrix Functions*, OT 45, Birkhäuser Verlag, Basel.
- Ball, J., Gohberg, I. and Rodman, L.: 1990b, Two-sided lagrange-sylvester interpolation problems for rational matrix functions, *Proc. of Symposia in Pure Mathematics*, Vol. 51, AMS, pp. 17–83.
- Ball, J. and Helton, J.: 1983, A beurling-lax theorem for the lie group $u(m, n)$ which contains most classical interpolation theory, *J. Operator Theory* 9, 107–142.
- Ball, J. and Rakowski, M.: 1992, Interpolation by rational matrix functions and stability of feedback systems: The 4-block case, in T. Ando and I. Gohberg (eds), *Operator Theory and Complex Analysis*, OT 59, Birkhäuser Verlag, pp. 96–142.
- Ball, J. and Rakowski, M.: 1994, Interpolation by rational matrix functions and stability of feedback systems: The 2-block case, *J. Math. Systems, Estimation, and Control* 4, 261–318.
- Bercovici, H., Cockburn, J., Foias, C. and Tannenbaum, A.: 1993, On structured tangential nevanlinna-pick interpolation, *IEEE Proc. 32nd CDC* 4, 3020–3021.
- Bercovici, H., Foias, C. and Tannenbaum, A.: 1990, Structured interpolation theory, *Extension and Interpolation of Linear Operators and Matrix Functions*, OT 47, Birkhäuser Verlag, Basel, pp. 195–220.
- Bercovici, H., Foias, C. and Tannenbaum, A.: 1991a, On spectral tangential nevanlinna-pick interpolation, *J. Math. Anal. and Appl.* 155, 156–176.
- Bercovici, H., Foias, C. and Tannenbaum, A.: 1991b, A spectral commutant lifting theorem, *Trans. Amer. Math. Society* 352(2), 741–762.
- Carathéodory, C.: 1907, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen, *Math. Ann.* 64, 95–115.
- Carathéodory, C. and Féjer, L.: 1911, Über den Zusammenhang der Extremen von harmonischen Funktionen mit ihren Koeffizienten und über den Picard-Landau'schen Satz, *Rend. Circ. Mat.* 32, 193–217.
- Delsarte, P., Genin, Y. and Kamp, Y.: 1984, On the role of the the nevanlinna-pick problem in circuit and systems theory, *Circuit Theory Appl.* 9, 1115–1193.
- Doyle, J.: 1982, Analysis of feedback systems with structured uncertainties, *IEE Proc.* 129(6-D), 242–250.
- Dym, H.: 1989, *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation*, CBMS No. 71, Amer. Math. Society, Providence, Rhode Island.
- Fedčina, I.: 1972, A criterion for the solvability of the nevanlinna-pick tangent problem, *Mat. Issled* 7, 213–227. in Russian.
- Fedčina, I.: 1975a, Description of solutions of the tangential nevanlinna-pick problem, *Akad. Nauk Armjan.* 60, 37–42. in Russian.
- Fedčina, I.: 1975b, Tangential nevanlinna-pick with multiple points, *Akad. Nauk Armjan.* 61, 214–218. in Russian.
- Foias, C.: 1978, Contractive intertwining dilations and waves in layered media, *Proc. International Congress of Mathematics* 2, 605–613.
- Foias, C. and Frazho, A.: 1990, *The Commutant Lifting Approach to Interpolation Problems*, OT 44: Adv. and Appl., Birkhäuser Verlag Basel.
- Francis, B.: 1987, *A Course in H^∞ Control Theory*, LNCIS Vol. 88, Springer Verlag.
- Francis, B. and Tannenbaum, A.: 1988, Generalized interpolation in control theory, *Mathematical Intelligencer* 10, 48–53.
- Frazho, A. and Kherat, S.: 1993, On mixed $h^2 - h^\infty$ tangential interpolation, in I. Gohberg (ed.), *New Aspects in Interpolation and Completion Theories*, OT 64, Birkhäuser Verlag, Basel, pp. 157–202.
- Helton, J.: 1987, *Operator Theory, Analytic Functions, Matrices and Electrical Engineering*, CBMS No. 68, Amer. Math. Society, Providence.
- Ivanchenko, T. and Sakhnovich, L.: 1994, An operator approach to the potapov scheme for the solution of interpolation problems, in I. Gohberg and L. Sakhnovich (eds), *Matrix and Operator Valued Functions*, OT 72, Birkhäuser Verlag, Basel, pp. 48–86.
- Kimura, H.: 1987, Directional interpolation approach to H^∞ -optimization and robust stabilization, *IEEE Transactions on Automatic Control* 32(12), 1085–1093.
- Limebeer, D. and Anderson, B.: 1988, An interpolation theory approach to H^∞ controller degree bounds, *Linear Algebra and Its Applications* 98, 347–386.
- Nevanlinna, R.: 1919, Über Beschränkte Funktionen, die in Gegebenen Punkten Vorgeschriebene Werte Annehmen, *Ann. Acad. Sci. Fenn.* 13, 27–43.
- Nudelman, A.: 1993, Some generalizations of classical interpolation problems, in A. Gheondea, D. Timotin and F.-H. Vasilescu (eds), *Operator Extensions, Interpolation of Functions and Related Topics*, OT 61, Birkhäuser Verlag, Basel, pp. 171–188.
- Packard, A. and Doyle, J.: 1993, “The Complex Structured Singular Value”, *Automatica* 29(1), 71–109.
- Pick, G.: 1916, Über die Beschränkungen Analytischer Funktionen, Welche Durch Vorgegebenen Funktionswerte Bewiskt Sind, *Math. Ann.* 77, 7–23.
- Safonov, M.: 1980, *Stability and Robustness of Multivariable Systems*, MIT Press, Cambridge, Massachusetts.
- Sarason, D.: 1967, Generalized interpolation in H^∞ , *Trans. Amer. Math. Soc.* 127, 179–203.
- Schur, I.: 1912, Über ein Satz von C. Carathéodory, *S.-B. Preuss. Akad. Wiss.* pp. 4–15.
- Sz.-Nagy, B. and Foias, C.: 1968, Dilatation des commutants, *C.R. Acad. Sci. Paris, Série A* 266, 493–496.
- Sz.-Nagy, B. and Foias, C.: 1970, *Harmonic Analysis of Operators on Hilbert Space*, North Holland.
- Vidyasagar, M.: 1985, *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, Massachusetts.
- Youla, D., Bongiorno, J. and Jabr, H.: 1976, Modern wiener-hopf design of optimal controllers, part i: The single-input case,

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