HOMOGENIZATION OF BIHARMONIC EQUATIONS IN
DOMAINS PERFORATED WITH TINY HOLES

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HOMOGENIZATION OF BIHARMONIC EQUATIONS IN DOMAINS PERFORATED WITH TINY HOLES

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Abstract. This paper deals with the homogenization of the biharmonic equation $\Delta^2 u = f$ in a domain containing randomly distributed tiny holes, with the Dirichlet boundary conditions. The size σ of the holes is assumed to be much smaller compared to the average distance ε between any two adjacent holes. We prove that as $\varepsilon, \sigma \rightarrow 0$, the solutions the biharmonic equation converges to the solution of $\Delta^2 u + \kappa u = f$, where κ depends on the shape of the holes and relative order of σ with respect to ε .

1. Introduction. Let Ω be a bounded domain in R^N . For any $\varepsilon > 0$, Ω^ε denotes a perforated domain formed by randomly removing many tiny holes (depending on ε) from Ω . Consider the biharmonic equation

$$(1.1) \quad \Delta^2 u^\varepsilon = f \quad \text{in } \Omega^\varepsilon,$$

with the Dirichlet boundary conditions

$$(1.2) \quad u^\varepsilon = \frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{in } \partial\Omega^\varepsilon \cap \Omega.$$

$$(1.3) \quad u^\varepsilon = g, \quad \frac{\partial u^\varepsilon}{\partial n} = h \quad \text{in } \partial\Omega,$$

We are concerned with the limiting behavior of the solution u^ε of problem (1.1)–(1.3) as $\varepsilon \rightarrow 0$. Many authors studied homogenization of the biharmonic equations (see [3], [10] and the references cited there). All the previous results were established for the case that the holes are distributed periodically and that the size of the holes is of the

same order as that of the period. It was shown (*cf.* [10]) that as $\varepsilon \rightarrow 0$, the solution u^ε converges to 0 to order ε^4 ; the higher order asymptotic expansion was also studied in [10].

In this paper, we are interested in the case when the holes are distributed randomly and when the volume fraction of the holes approaches zero as the number increases to infinity. Our objective is to determine the order of the volume fraction of the holes in order to obtain a non-zero limit of the solution u^ε of problem (1.1)–(1.3) as the number of the holes approaches ∞ . Let ε be the “average” distance between two adjacent holes (the precise definition will be stated in the next section), σ (depending on ε) the size of the holes. We shall consider the case that $\frac{\sigma}{\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$. The main results are the follows.

(i) If $N \leq 3$, then the solution u^ε of (1.1)–(1.3) always converges to 0, no matter how small the volume fraction of the holes is; (ii) if $N > 3$ and the limit $\mu^\varepsilon \rightarrow \mu$ exists and is finite, where

$$\begin{aligned}\mu^\varepsilon &= \frac{\sigma^{N-4}}{\varepsilon^N} \quad \text{for } N \geq 5, \\ \mu^\varepsilon &= \varepsilon^4 \ln \frac{1}{\sigma} \quad \text{for } N = 4,\end{aligned}$$

then the solution u^ε converges to the solution of

$$\begin{aligned}\Delta^2 u + \kappa \mu u &= f \quad \text{in } \Omega, \\ u &= g, \quad \frac{\partial u}{\partial n} = \frac{\partial g}{\partial n} \quad \text{in } \partial\Omega,\end{aligned}$$

where $\kappa \geq 0$ depends on the shape and distribution of the holes; (iii) if $N > 3$ and $\mu = \infty, g = 0$, then the limit of the solution u^ε is 0.

The results show that the critical order of the size of holes is $\sigma = \varepsilon^{\frac{N}{N-4}}$ for $N > 4$, and $\sigma = \exp\left(-\frac{1}{\varepsilon^4}\right)$ for $N = 4$.

For the analogous problems of the second order equations with periodically distributed holes, the case when the size of the holes is much smaller than the period has been studied extensively in literatures. In [6], [11] and [8] the authors considered the problem for the Laplace equation. The homogenization for the wave equation in that case was studied in [5]. The problem for the Navier-Stokes equation was analyzed in [1], [2]. These papers showed that the critical order for the second order equations is $\sigma = \varepsilon^{\frac{N}{N-2}}$ for $N > 2$, and $\sigma = \exp\left(-\frac{1}{\varepsilon^2}\right)$ for $N = 2$. For homogenization of other equations and general references, we refer to [1], [2], [3], [6], [7], [10], [11].

The proof of our results is based on the energy method. The main idea follows those in [1], [6] and [8]. Since the different natures between the fourth order equations and the second order equations, the treatment here technically is different from the papers mentioned above. In contrast with [1] and [6] where certain periodic structures are assumed, we consider the more general situation that holes are randomly distributed with a density function. Our paper is also the first approach to explore the relationship between the orders of the equations and the critical orders of the holes that will produce non-trivial homogenized equations. The paper is organized as follows. In §2, we shall state the problem more precisely and introduce some notations. The case when $N \leq 3$ is also studied in that section. In §3 we study the problem in a ball containing one hole and derive some estimates. Section 4 is devoted to the study of the homogenized equations and the results of convergence under various norms for $N > 4$. The case $N = 4$ is treated in §5.

2. Problem Setting and Preliminaries. Let Ω and T be two bounded $C^{1,1}$ domains in R^N , locally lying in one side of their boundaries. We assume that

$$(A1) \quad B_1 \subset T \subset B_2$$

where $B_r = B_r(o)$, the ball centered at origin with radius r . For any $\varepsilon > \sigma > 0$ and a set of finite points $\{x_i^\varepsilon : i = 1, \dots, m_\varepsilon\} \subset \Omega$ such that

$$(A2) \quad |x_i^\varepsilon - x_j^\varepsilon| \geq 2\varepsilon \quad \text{for } i \neq j.$$

We define the holes T_i^ε by

$$(A3) \quad T_i^\varepsilon = x_i^\varepsilon + \sigma T.$$

Set

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{i \in \Lambda^\varepsilon} T_i^\varepsilon,$$

where $\Lambda^\varepsilon = \{i : T_i^\varepsilon \subset \Omega\}$. This means that we exclude the situation that the holes intersect with the boundary $\partial\Omega$. We point out that this condition for Ω^ε is set up only for simplicity. All the results can be extended to the case that the holes touching the boundary. We also assume that there exists a bounded density function $P(x) > 0$ such that

$$(A4) \quad \sum_{i=1}^{m_\varepsilon} \varepsilon^N \xi(x_i^\varepsilon) \longrightarrow \iint_{\Omega} P(x) \xi(x) dx$$

for any $\xi \in C^0(\Omega)$, the set of all continuous functions with compact supports in Ω . Note that under the above assumptions, the number of the holes m_ε is approximately $c\varepsilon^{-N}$ for some constants c .

One example that satisfies the above assumptions is the case that the holes are distributed periodically. In that case, we may take

$$x_i^\varepsilon = (\varepsilon k_1, \varepsilon k_2, \dots, \varepsilon k_N), \quad k_j \in \mathbb{Z}$$

and $P(x)$ is a constant.

Consider the Dirichlet problem

$$(2.1) \quad \Delta^2 u^\varepsilon = f \quad \text{in } \Omega^\varepsilon,$$

$$(2.2) \quad u^\varepsilon = \frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on} \quad \partial\Omega^\varepsilon \cap \Omega,$$

$$(2.3) \quad u^\varepsilon = g, \quad \frac{\partial u^\varepsilon}{\partial n} = \frac{\partial g}{\partial n} \quad \text{on} \quad \partial\Omega,$$

where n is the exterior normal to the boundary. To define a weak solution of (2.1)–(2.3), we need to introduce some function spaces. Let $H^m(\Omega)$ be the usual Sobolev space of all functions that have square-integrable derivatives up to m th order, and $H_{loc}^m(\Omega)$ the set of all functions that belong to $H^m(\Omega')$ for any Ω' such that $\bar{\Omega}' \subset \Omega$. The norm in $H^m(\Omega)$ is denoted by $\|\cdot\|_{H^m(\Omega)}$. Denote by $H_0^m(\Omega)$ the closure of $C_0^\infty(\Omega)$, the space of all smooth functions with compact supports in Ω . Let

$$H^2(\Omega; \Delta) = \{v \in H^1(\Omega) : \Delta v \in L^2(\Omega)\}$$

with the norm

$$\|v\|_{H^2(\Omega; \Delta)} = \|v\|_{H^1(\Omega)} + \|\Delta v\|_{L^2(\Omega)}.$$

We know that $H^2(\Omega; \Delta)$ is a Hilbert space and that the closure of $C_0^\infty(\Omega)$, with the norm $\|\cdot\|_{H^2(\Omega; \Delta)}$ coincides with $H_0^2(\Omega)$. For any boundary portion $\Gamma \subset \partial\Omega$, denote by $H_0^m(\Omega; \Gamma)$ the $\|\cdot\|_{H^2(\Omega; \Delta)}$ -closure of all smooth functions that have the compact supports in $\Omega \cup \Gamma$. A weak solution of (2.1)–(2.3) is defined as a function $u^\varepsilon \in H_0^2(\Omega^\varepsilon; \partial\Omega^\varepsilon \cap \Omega)$ such that $u^\varepsilon - g \in H_0^2(\Omega^\varepsilon; \partial\Omega)$ and

$$(2.4) \quad \iint_{\Omega^\varepsilon} \Delta u^\varepsilon \cdot \Delta v dx = \iint_{\Omega^\varepsilon} f v dx$$

for any $v \in H_0^2(\Omega^\varepsilon)$.

LEMMA 2.1. *Under the assumption (A1)–(A4), if $f \in L^2(\Omega)$ and $g \in H^2(\Omega)$, then there exists a unique weak solution u^ε of problem (2.1)–(2.3). Furthermore, for any subdomain $\Omega_0 \subset \bar{\Omega}_0 \subset \Omega$, the following estimates hold with a constant $C(\Omega_0)$*

independent of ε (but depending on Ω_0)

$$(2.5) \quad \|u^\varepsilon\|_{H^2(\Omega^\varepsilon \cap \Omega_0; \Delta)} \leq C(\Omega_0) \|f\|_{L^2(\Omega)}.$$

When $g = 0$, (2.5) holds for $\Omega_0 = \Omega$.

Proof: Existence and uniqueness are well-known. Assume first that $g = 0$.

Taking $v = u^\varepsilon$ in (2.4), we obtain

$$\|\Delta u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \leq \|f\|_{L^2(\Omega)} \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)}.$$

Using the relation

$$\iint_{\Omega^\varepsilon} \|\nabla u^\varepsilon\|^2 dx = - \iint_{\Omega^\varepsilon} u^\varepsilon \Delta u^\varepsilon dx,$$

it follows that

$$\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^4 \leq \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^3 \|f\|_{L^2(\Omega)}.$$

Inequality (2.5) then follows from the Poincaré's inequality. For the general case that $g \not\equiv 0$, we choose a cut-off function ξ such that $\xi = 0$ in a neighborhood of $\partial\Omega$ and $\xi = 1$ in Ω_0 . The desired estimates follow by choosing the test function $v = \xi^2 u^\varepsilon$ in (2.4).

Note that since the weak solution $u^\varepsilon \in H_0^2(\Omega^\varepsilon, \partial\Omega^\varepsilon)$, the extension

$$(2.6) \quad \hat{u}^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & \text{for } x \in \Omega^\varepsilon, \\ 0 & \text{for } x \in T_i^\varepsilon \end{cases}$$

is in $H^2(\Omega)$. By the L^p -estimates of the harmonic equations, we obtain

COROLLARY 2.1. *Let u^ε be the solution of problem (2.1)–(2.3). Then for any $\Omega_0 \subset \bar{\Omega}_0 \subset \Omega$,*

$$(2.7) \quad \|\hat{u}^\varepsilon\|_{H^2(\Omega_0)} = \|u^\varepsilon\|_{H^2(\Omega^\varepsilon \cap \Omega_0)} \leq c(\Omega_0) \|f\|_{L^2(\Omega)}.$$

If $g = 0$, then the inequality holds for $\Omega_0 = \Omega$.

It follows that there exists a subsequence of $\{\hat{u}^\varepsilon(x)\}_{\varepsilon>0}$ that converges weakly in $H^2(\Omega_0)$ to a function, say u . Another direct consequence of (2.7) is the following result.

COROLLARY 2.2. *Let $N \leq 3$ and $\hat{u}^\varepsilon$ be the extended weak solution defined in (2.6). Then as $\varepsilon \rightarrow 0$, $\hat{u}^\varepsilon \rightarrow 0$ uniformly in any $\Omega_0 \subset \bar{\Omega}_0 \subset \Omega$. If in addition, $g = 0$, then the convergence is uniform in Ω .*

Proof: By (2.7) and the Sobolev embedding, it follows that $\hat{u}^\varepsilon$ is α -Hölder continuous for some $\alpha > 0$, and that its α -Hölder norm is uniformly bounded. Hence

$$(2.8) \quad |\hat{u}^\varepsilon(x)| = |\hat{u}^\varepsilon(x) - \hat{u}^\varepsilon(x_i^\varepsilon)| \leq c\varepsilon^\alpha,$$

for any $x \in B_\varepsilon(x_i^\varepsilon) \cap \Omega_0$. Let χ^ε be the characteristic function of $\bigcup_{i=1}^{m_\varepsilon} B_\varepsilon(x_i^\varepsilon)$. Then (2.8) means that $|\hat{u}^\varepsilon \chi^\varepsilon(x)| \leq c\varepsilon^\alpha$ for any $x \in \Omega_0$. Since $\chi^\varepsilon \rightarrow P$ weakly in L^2 , we obtain $\hat{u}^\varepsilon \chi^\varepsilon \rightarrow uP$. Therefore $u \equiv 0$ since $P > 0$.

In sequel, we shall consider the case $N \geq 4$.

3. Local Problems. Let T be a domain satisfying (A1). For any $\rho \geq 3$, consider the following problem:

$$(3.1) \quad \Delta^2 W = 0 \quad \text{in } B_\rho \setminus T,$$

$$(3.2) \quad W = \frac{\partial W}{\partial n} = 0 \quad \text{on } \partial T,$$

$$(3.3) \quad W = 1, \quad \frac{\partial W}{\partial n} = 0 \quad \text{on } \partial B_\rho.$$

By Lemma 2.1, there exists a unique weak solution to the above problem. We extend this weak solution by 0 in T . Denote by W_ρ the extension of this solution. From (3.2),

it is easy to see that $W_\rho \in H^2(B_\rho)$. Note that we are interested in W_ρ only for large ρ . We begin with the estimates for H^2 - norm.

LEMMA 3.1. *Let W_ρ be the extension of the solution of (3.1)-(3.3). Then there exists a constant $c > 0$ independent of $\rho \geq 3$ such that the following estimates hold for W_ρ :*

$$\begin{aligned} (i) \quad & \|\nabla^2 W_\rho\|_{L^2(B_\rho)} \leq c; \\ (ii) \quad & \|\nabla W_\rho\|_{L^2(B_\rho)} \leq c\rho; \\ (iii) \quad & \|W_\rho - 1\|_{L^2(B_\rho)} \leq c\rho^2. \end{aligned}$$

Proof: For simplicity, we take $\rho \geq 5$. Let $\xi \in C^\infty(R^N)$ such that $\xi = 0$ in B_2 , $\xi = 1$ in $R^N \setminus B_3$, $0 \leq \xi \leq 1$. By taking $v = W_\rho - \xi$ in (2.4), we obtain

$$\iint_{B_\rho \setminus T} |\Delta W_\rho|^2 dx = \iint_{B_\rho \setminus T} \Delta W_\rho \cdot \Delta \xi dx.$$

It follows that

$$\iint_{B_\rho \setminus T} |\Delta W_\rho|^2 dx \leq c \iint_{B_\rho \setminus T} |\Delta \xi|^2 dx = c \iint_{B_3 \setminus B_2} |\Delta \xi|^2 dx \leq c.$$

By rescaling, we set $V(x) = W_\rho(\rho x)$ for x in B_1 . The above estimate implies that

$$\iint_{B_1} |\Delta V|^2 dx = \rho^{4-N} \iint_{B_\rho \setminus T} |\Delta W_\rho|^2 dx \leq c\rho^{4-N}.$$

Since $V = 1$ on ∂B_1 , by L^2 - Estimates for the second order elliptic equations, we obtain that

$$\|V - 1\|_{H^2(B_1)}^2 \leq c\rho^{4-N}.$$

The assertions follow by rescaling back.

We next study the behavior of W_ρ near ∂B_ρ . By the regularity theory for the higher order elliptic equations [9], we know that W_ρ is smooth up to the boundary

portion ∂B_ρ (this is also a consequence of (3.14) later in this section). We shall frequently use the following particular form of Rayleigh-Green's identity

$$(3.4) \quad \iint_{B_\rho \setminus T} \Delta W_\rho \cdot \Delta v dx = \int_{\partial B_\rho} \Delta W_\rho \frac{\partial v}{\partial n} ds - \int_{\partial B_\rho} \frac{\partial \Delta W_\rho}{\partial n} v ds,$$

for any function $v \in H^2(B_\rho \setminus T)$ such that $v = \frac{\partial v}{\partial n} = 0$ on ∂T . To show (3.4), we take a smooth function ξ such that $\xi = 0$ in B_2 , $\xi = 1$ outside B_3 . Then $v - v\xi \in H_0^2(B_\rho \setminus T)$.

Hence by (2.4) we obtain

$$\iint_{B_\rho \setminus T} \Delta W_\rho \cdot \Delta v dx = \iint_{B_\rho \setminus T} \Delta W_\rho \cdot \Delta(v\xi) dx = \iint_{B_\rho \setminus B_2} \Delta W_\rho \cdot \Delta(v\xi) dx.$$

From the Rayleigh-Green's identity

$$(3.5) \quad \iint_{\Omega_1} \Delta w \cdot \Delta \tilde{v} dx = \int_{\partial \Omega_1} \Delta w \frac{\partial \tilde{v}}{\partial n} ds - \int_{\partial \Omega_1} \frac{\partial \Delta w}{\partial n} \tilde{v} ds + \iint_{\Omega_1} \Delta^2 w \cdot \tilde{v} dx$$

that holds for any function $w \in H^4(\Omega_1)$, $\tilde{v} \in H^2(\Omega_1)$, where Ω_1 is any Lipschitz domain, the identity (3.4) follows by taking $w = W_\rho$, $\tilde{v} = v\xi$, and $\Omega_1 = B_\rho \setminus B_2$.

To proceed studying the behavior of the function W_ρ near the boundary ∂B_ρ , we need the Green's function G_N for the biharmonic equation in B_ρ with the Dirichlet boundary condition. For $N \geq 5$, the Green's function is

$$(3.6) \quad \begin{aligned} G_N(x, y) = & \frac{1}{2(4-N)(2-N)\omega_N} (|x-y|^{4-N} - (\rho^{-1}|y||x-\bar{y}|)^{4-N}) \\ & + \frac{(4-N)}{2\omega_N\rho^2} (|x|^2 - \rho^2)(|y|^2 - \rho^2) \left(\frac{|y|}{\rho} |x-\bar{y}| \right)^{2-N}. \end{aligned}$$

For $N = 4$, we have

$$(3.7) \quad \begin{aligned} G_4(x, y) = & -\frac{1}{8\omega_4\rho^2} (|x|^2 - \rho^2)(|y|^2 - \rho^2) \left(\frac{|y|}{\rho} |x-\bar{y}| \right)^{-2} \\ & + \frac{1}{4\omega_4} \ln \left(\frac{|y||x-\bar{y}|}{\rho|x-y|} \right). \end{aligned}$$

Here in (3.6) and (3.7), $\bar{y} = \rho^2|y|^{-2}y$ is the reflection of y with respect to B_ρ , and ω_N is the surface area of the unit sphere in R^N . Note that the Green's function $G_N(x, y)$ satisfies the Dirichlet boundary condition

$$G_N(x, y) = \frac{\partial G_N(x, y)}{\partial n(x)} = 0, \quad x \in \partial B_\rho,$$

for any $y \in B_\rho$. By the Rayleigh-Green's identity (3.5), we can represent any smooth function w that has the compact support as the integral

$$(3.8) \quad w(x) = \iint_{B_\rho} G_N(x, y) \Delta^2 w(y) dy.$$

LEMMA 3.2. *Let $N \geq 4$, $\rho \geq 5$ and W_ρ be the solution of (3.1)-(3.3). Then the following estimates hold for any $x \in \bar{B}_\rho \setminus (B_4 \cup B_{\frac{\rho}{2}})$:*

(i) *there exists a constant c independent of ρ such that*

$$(3.9) \quad |\Delta W_\rho(x)| \leq \frac{c}{\rho^{N-2}};$$

(ii) *there exists a constant c independent of ρ and a positive function $\delta(\rho)$ that decreases in ρ such that*

$$(3.10) \quad \left| \nabla(\Delta W_\rho)(x) + \frac{\gamma_\rho(T)}{\omega_N} \frac{x}{|x|^N} \right| \leq \frac{\delta(\rho)}{|x|^{N-1}} + \frac{c}{|x|^N},$$

where

$$(3.11) \quad \gamma_\rho(T) = \iint_{B_\rho \setminus T} |\Delta W_\rho(y)|^2 dx.$$

Proof: Choose a cut-off function $0 \leq \xi \leq 1$ such that $\xi = 0$ in B_2 , $\xi = 1$ in $R^N \setminus B_3$. Set $v = (W_\rho - 1)\xi$. Then $v \in C^\infty(B_\rho) \cap H_0^2$, since $W_\rho(x)$ is smooth for $x \in \bar{B}_\rho \setminus \bar{T}$, and v satisfies

$$(3.12) \quad \Delta^2 v = F \quad \text{in } B_\rho,$$

where

$$\begin{aligned}
F &= 2\Delta W_\rho \cdot \Delta \xi + 4 \nabla (\Delta W_\rho) \cdot \nabla \xi + 4 \nabla W_\rho \cdot \nabla (\Delta \xi) \\
(3.13) \quad &+ (W_\rho - 1)\Delta^2 \xi + 4 \operatorname{trace} (\nabla^2 W_\rho \cdot \nabla^2 \xi).
\end{aligned}$$

Since $\nabla \xi(x) = 0$ for $x \notin B_3 \setminus B_2$, it is easy to see that the function F is supported only in $B_3 \setminus B_2$. By the interior estimates for the higher order elliptic equations, Lemma 3.1, and by the Sobolev embedding, we know that in $B_3 \setminus B_2$, all the derivatives of W_ρ up to the third order are bounded by a constant independent of ρ . Therefore, $|F(x)| \leq c$ for any x . For convenience, here and in what follows, c is always reserved as a universal constant. Using the Green's representation formula (3.8), we can write

$$v(x) = \iint_{B_\rho} G_N(x, y) F(y) dy.$$

In particular, for $x \in B_\rho \setminus B_3$, the above equality reads as

$$(3.14) \quad W_\rho(x) = 1 + \iint_{B_3 \setminus B_2} G_N(x, y) F(y) dy,$$

since $\operatorname{supp} F \subset B_3 \setminus B_2$. Hence, for $x \in B_\rho \setminus B_3$,

$$(3.15) \quad \Delta W_\rho(x) = \iint_{B_3 \setminus B_2} \Delta_x G_N(x, y) F(y) dy,$$

$$(3.16) \quad \nabla(\Delta W_\rho)(x) = \iint_{B_3 \setminus B_2} \nabla_x(\Delta_x G_N(x, y)) F(y) dy.$$

From the explicit formulas (3.6) and (3.7), we can compute ΔG_N directly. It follows that for $N \geq 4$,

$$\Delta_x G_N(x, y) = \frac{|x - y|^{2-N}}{\omega_N(2 - N)} + \theta_1(x, y),$$

where

$$\begin{aligned}\theta_1 &= -\frac{|x - \bar{y}|^{2-N}}{\omega_N(2-N)} \left(\frac{|y|}{\rho}\right)^{4-N} + \frac{(|y|^2 - \rho^2)}{\omega_N \rho^2} \left(\frac{|y|}{\rho}\right)^{2-N} |x - \bar{y}|^{-N} (x_i - \bar{y}_i) \bar{y}_i \\ &\quad + \frac{(4-N)(|y|^2 - \rho^2)}{2\omega_N(2-N)} \frac{1}{\rho^2} \left(\frac{|y|}{\rho}\right)^{2-N} |x - \bar{y}|^{2-N},\end{aligned}$$

and the convention of summation of the repeated indices is applied. For $2 \leq |y| \leq 3, \max(\frac{\rho}{2}, 4) \leq |x| \leq \rho$ we can bound $\Delta_x G_N$ by

$$(3.17) \quad |\Delta_x G_N(x, y)| \leq \frac{c}{\rho^{N-2}},$$

since

$$|x - \bar{y}| = \left| x - \frac{\rho^2}{|y|^2} y \right| \geq c\rho^2.$$

The assertion (i) follows from (3.15) and (3.17). Next, by direct computation, we obtain

$$|\nabla_x \theta_1(x, y)| \leq \frac{c}{\rho^N},$$

for $2 \leq |y| \leq 3, \max(\frac{\rho}{2}, 4) \leq |x| \leq \rho$. Therefore, from (3.16) and the boundedness of F , we deduce that

$$\begin{aligned}(3.18) \quad \nabla(\Delta W_\rho)(x) &= \frac{1}{\omega_N} \iint_{B_3 \setminus B_2} \frac{x - y}{|x - y|^N} F(y) dy + \theta_2(x, y) \\ &= \frac{1}{\omega_N} \frac{x}{|x|^N} \iint_{B_3 \setminus B_2} F(y) dy + \theta_2(x, y) \\ &\quad + \frac{1}{\omega_N} \iint_{B_3 \setminus B_2} \left(\frac{x - y}{|x - y|^N} - \frac{x}{|x|^N} \right) F(y) dy,\end{aligned}$$

where

$$|\theta_2(x, y)| \leq \iint_{B_3 \setminus B_2} |\nabla \theta_1(x, y) F(y)| dy \leq \frac{c}{\rho^N},$$

for $2 \leq |y| \leq 3$, $\max(\frac{\rho}{2}, 4) \leq |x| \leq \rho$. The third term in the last equality of (3.18) can be estimated by

$$\left| \frac{1}{\omega_N} \iint_{B_3 \setminus B_2} \left(\frac{x-y}{|x-y|^N} - \frac{x}{|x|^N} \right) F(y) dy \right| \leq \frac{\delta(\rho)}{|x|^N},$$

for $2 \leq |y| \leq 3$, $\max(\frac{\rho}{2}, 4) \leq |x| \leq \rho$, where $\delta(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$. Finally, from (3.12), we obtain

$$\begin{aligned} \iint_{B_3 \setminus B_2} F(y) dy &= \iint_{B_\rho} \Delta^2 v(y) dy = \int_{\partial B_\rho} \frac{\partial \Delta v}{\partial n} ds \\ &= \int_{\partial B_\rho} \frac{\partial \Delta W_\rho}{\partial n} ds = \int_{\partial B_\rho} \frac{\partial \Delta W_\rho}{\partial n} W_\rho ds, \end{aligned}$$

where we used the fact that $W_\rho = 1$ on ∂B_ρ . It follows from the Rayleigh-Green's identity (3.4) that, noting that $\frac{\partial \Delta W_\rho}{\partial n} = 0$ on ∂B_ρ ,

$$\iint_{B_3 \setminus B_2} F(y) dy = - \iint_{B_\rho \setminus T} |\Delta W_\rho(y)|^2 dy = -\gamma_\rho(T).$$

The proof is complete.

From Lemma 3.1, $\gamma_\rho(T)$ (defined in (3.11)) is uniformly bounded. We next show that the limit of $\gamma_\rho(T)$ exists as $\rho \rightarrow \infty$.

LEMMA 3.3. *Let $N \geq 5$. Then $\gamma_\rho(T) \rightarrow \gamma(T)$ as $\rho \rightarrow \infty$, where $0 < \gamma(T) < \infty$ depends only on the domain T .*

Proof: Let W_ρ be the solution of (3.1)-(3.3) for large ρ . By (ii) of Lemma 3.1, $\|\nabla^2 W_\rho\|_{L^2(B_\rho)} \leq c$. Since on ∂T , $W_\rho = \frac{\partial \Delta W_\rho}{\partial n} = 0$, by the Poincaré inequality we obtain that for any $0 < r < \rho$

$$\|W_\rho\|_{H^2(B_r)} \leq C(r),$$

where $C(r)$ is independent of ρ (but depends on r). By the diagonal argument, we may select a subsequence $\rho_k \rightarrow \infty$ and a function $W \in H_{loc}^2(\mathbb{R}^N \setminus T)$ such that

$$W_{\rho_k} \rightharpoonup W \quad \text{as } \rho_k \rightarrow \infty,$$

where the convergence is interpreted in the sense that for any $r > 0$, W_{ρ_k} weakly converges to W in $H^2(B_r \setminus T)$. Since W_ρ is biharmonic in $B_\rho \setminus T$, one can see that W is also biharmonic outside T . Furthermore, $W_\rho = \frac{\partial \Delta W_\rho}{\partial n} = 0$ on ∂T , and $\nabla^2 W$ is bounded in $L^2(R^N \setminus T)$ since $\|\nabla^2 W_\rho\|_{L^2(B_\rho)} \leq c$.

Now for any $\xi \in H_{loc}^2(R^N \setminus T)$ such that $\xi = \frac{\partial \xi}{\partial n} = 0$ on ∂T , $\xi = 1$ outside B_r , by the definition (2.4), we have, for $\rho_k > r$,

$$\iint_{B_{\rho_k} \setminus T} |\Delta W_{\rho_k}(x)|^2 dx = \iint_{B_{\rho_k} \setminus T} \Delta W_{\rho_k} \cdot \Delta \xi.$$

It follows that for large ρ_k

$$\begin{aligned} \gamma_{\rho_k}(T) &= \iint_{B_{\rho_k} \setminus T} \Delta W_{\rho_k} \cdot \Delta \xi dx = \iint_{B_r \setminus T} \Delta W_{\rho_k} \cdot \Delta \xi \\ &\xrightarrow{\rho_k \rightarrow \infty} \iint_{B_r \setminus T} \Delta W \cdot \Delta \xi = \iint_{R^N \setminus T} \Delta W \cdot \Delta \xi. \end{aligned}$$

We now take, for any fixed ρ_l

$$\xi(x) = \begin{cases} W_{\rho_l}(x), & \text{if } x \in B_{\rho_l} \setminus T, \\ 1, & \text{if } x \in R^N \setminus B_{\rho_l}. \end{cases}$$

Then by the “matching Lemma”, $\xi \in H_{loc}^2$. Hence

$$\begin{aligned} \gamma_{\rho_k}(T) &\xrightarrow{\rho_k \rightarrow \infty} \iint_{R^N \setminus T} \Delta W \cdot \Delta W_{\rho_l} dx \\ &\xrightarrow{\rho_l \rightarrow \infty} \iint_{R^N \setminus T} |\Delta W|^2 dx = \gamma(T). \end{aligned}$$

It remains to show that $\gamma(T) > 0$ and that the whole sequence $\gamma_{\rho_k}(T) \rightarrow \gamma(T)$. To this end, it suffices to show that the weak limit W is unique.

We observe that the Green's functions (3.6), (3.7) and their gradients satisfy the following estimates for $x \notin B_4$, $y \in B_3 \setminus B_2$:

$$\begin{aligned} |G_N(x, y)| &\leq \frac{c}{|x|^{N-4}} + \frac{c}{\rho^{N-4}}, \\ |\nabla_x G_N(x, y)| &\leq \frac{c}{|x|^{N-3}} + \frac{c}{\rho^{N-3}}, \end{aligned}$$

for $N \geq 5$. It follows from the expressions (3.14) that for $x \notin B_3$,

$$\begin{aligned} |W_\rho(x) - 1| &\leq \frac{c}{|x|^{N-4}} + \frac{c}{\rho^{N-4}}, \\ |\nabla W_\rho(x)| &\leq \frac{c}{|x|^{N-3}} + \frac{c}{\rho^{N-3}}, \end{aligned}$$

for $N \geq 5$. Since $W_{\rho_k} \rightarrow W$ strongly in $H^1(B_r \setminus T)$ for any r , it follows that for almost all $x \notin B_3$,

$$(3.19) \quad |W(x) - 1| \leq \frac{c}{|x|^{N-4}},$$

$$(3.20) \quad |\nabla W(x)| \leq \frac{c}{|x|^{N-3}}.$$

We now claim that W is the only biharmonic function in $R^N \setminus T$ that satisfies $W = \frac{\partial W}{\partial n} = 0$ on ∂T and (3.19), (3.20). Otherwise, suppose that $\hat{W}$ is another biharmonic function having the above properties. Then $V = W - \hat{W}$ satisfies

$$(3.21) \quad |V(x)| \leq \frac{c}{|x|^{N-4}}, \quad |\nabla V(x)| \leq \frac{c}{|x|^{N-3}},$$

for large x . Then for any smooth function ξ that has the compact support in R^N ,

$$\iint_{R^N \setminus T} \Delta V \cdot \Delta(V\xi^4) = 0.$$

It follows that

$$\iint_{R^N \setminus T} |\Delta V|^2 \xi^4 \leq c \iint_{R^N \setminus T} (|V \Delta \xi^4| + |\nabla V \cdot \nabla \xi^4|) |\Delta V| dx.$$

Therefore, by the Cauchy's inequality,

$$\iint_{R^N \setminus T} |\Delta V|^2 \xi^2 dx \leq c \iint_{R^N \setminus T} (|V \xi| |\nabla^2 \xi| + |V| |\nabla \xi|^2 + |\nabla V| |\nabla \xi| |\xi|)^2 dx.$$

We choose ξ such that

$$0 \leq \xi \leq 1, \quad \xi = 1 \text{ in } B_r, \quad \xi = 0 \text{ in } R^N \setminus B_{2r},$$

$$|\nabla \xi| \leq \frac{c}{r}, \quad |\nabla^2 \xi| \leq \frac{c}{r^2}.$$

From (3.21), it follows that

$$\iint_{B_r \setminus T} |\Delta V|^2 dx \leq \iint_{R^N \setminus T} |\Delta V|^2 \xi^2 dx \leq \frac{c}{r^{N-4}}.$$

Letting $r \rightarrow \infty$, we obtain $\Delta V = 0$ in $R^N \setminus T$. Consequently, $V = 0$ since $V = 0$ on the boundary. Hence the whole sequence $\{W_\rho\}$ converges to W (in the sense we explained before). It immediately follows that $\gamma_\rho(T) \rightarrow \gamma(T)$. Finally, if $\gamma(T) = 0$, then $\Delta W = 0$. Since $W = \frac{\partial W}{\partial n} = 0$ on ∂T , we can extend it by 0 into T . The extended function is harmonic in R^N , and is bounded by (3.19). Hence $W \equiv \text{constant}$. A contradiction to (3.19) and the fact $W = 0$ on ∂T .

4. Homogenized Problem I: $N \geq 5$. Let u^ε be the solution of (2.1)-(2.3), and u be a weak limit of a subsequence of $\hat{u}^\varepsilon$, the extension of u^ε by 0 in the holes T_i^ε . In this section we shall study the homogenized problem for $N \geq 5$. Set

$$\mu^\varepsilon = \frac{\sigma^{N-4}}{\varepsilon^N}.$$

THEOREM 4.1. *Assume (A1)-(A4), $N \geq 5$, $g \in C^2$, and that the limit*

$$(4.1) \quad \mu = \lim_{\varepsilon \rightarrow 0} \frac{\sigma^{N-4}}{\varepsilon^N}$$

exists. If $\mu < \infty$, then the whole sequence $\hat{u}^\varepsilon$ converges weakly to u in $H^2(\Omega)$. Furthermore, u satisfies

$$(4.2) \quad \Delta^2 u + \mu \gamma(T) P u = f \quad \text{in } \Omega,$$

$$(4.3) \quad u = g, \quad \frac{\partial u}{\partial n} = \frac{\partial g}{\partial n} \quad \text{on } \partial \Omega,$$

If $\mu = \infty$, then for any subdomain $\Omega_0 \subseteq \bar{\Omega}_0 \subseteq \Omega$, the whole sequence $\hat{u}^\varepsilon$ converges to 0 weakly in $H^2(\Omega_0)$.

To prove the theorem, we need to construct a test function for any $\varepsilon > 0$. Let W_ρ be the solution of (3.1)-(3.3). Under the assumption (A2), the balls $\{B_\varepsilon(x_i^\varepsilon)\}$ are disjoint. Define a function w^ε in Ω by

$$(4.4) \quad w^\varepsilon(x) = \begin{cases} W_\rho\left(\frac{x - x_i^\varepsilon}{\sigma}\right) & \text{for } x \in B_\varepsilon(x_i^\varepsilon) \setminus T_i^\varepsilon, \\ 0 & \text{for } x \in T_i^\varepsilon, \\ 1 & \text{otherwise,} \end{cases}$$

where $\rho = \frac{\varepsilon}{\sigma} \rightarrow \infty$ as $\varepsilon \rightarrow \infty$. One can see easily that $w^\varepsilon \in H^2(\Omega)$.

LEMMA 4.1. *Let $N \geq 5$, $\mu \leq \infty$ and w^ε be the function defined in (4.4). Then*

$$(4.5) \quad \|w^\varepsilon - 1\|_{L^2}^2 \leq c\mu^\varepsilon \varepsilon^4,$$

$$(4.6) \quad \|\nabla w^\varepsilon\|_{L^2}^2 \leq c\mu^\varepsilon \varepsilon^2,$$

$$(4.7) \quad \|\nabla^2 w^\varepsilon\|_{L^2}^2 \leq c\mu^\varepsilon.$$

Consequently, $w^\varepsilon \rightarrow 1$ weakly in $H^2(\Omega)$.

Proof: By Lemma 3.1, we obtain, noting that $|m_\varepsilon| \leq c\varepsilon^{-N}$,

$$(4.8) \quad \begin{aligned} \|w^\varepsilon - 1\|_{L^2}^2 &\leq \sum_{i=1}^{m_\varepsilon} \iint_{B_\varepsilon(x_i^\varepsilon) \setminus T_i^\varepsilon} \left| W_\rho\left(\frac{x - x_i^\varepsilon}{\sigma}\right) - 1 \right|^2 dx + \sum_{i=1}^{m_\varepsilon} |T_i^\varepsilon| \\ &= c m_\varepsilon \sigma^N \|W_\rho - 1\|_{L^2(B_\rho)}^2 + c m_\varepsilon \sigma^4 \leq c\mu^\varepsilon \varepsilon^4, \end{aligned}$$

$$\begin{aligned}
\|\nabla w^\varepsilon\|_{L^2}^2 &\leq \sum_{i=1}^{m_\varepsilon} \frac{1}{\sigma^2} \iint_{B_\varepsilon(x_i^\varepsilon) \setminus T_i^\varepsilon} \left| \nabla W_\rho \left(\frac{x - x_i^\varepsilon}{\sigma} \right) \right|^2 dx \\
(4.9) \qquad &= m_\varepsilon \sigma^{N-2} \|\nabla W_\rho\|_{L^2(B_\rho)}^2 \leq c \mu^\varepsilon \varepsilon^2,
\end{aligned}$$

and

$$\begin{aligned}
\|\Delta w^\varepsilon\|_{L^2}^2 &\leq \sum_{i=1}^{m_\varepsilon} \frac{1}{\sigma^4} \iint_{B_\varepsilon(x_i^\varepsilon) \setminus T_i^\varepsilon} \left| \Delta W_\rho \left(\frac{x - x_i^\varepsilon}{\sigma} \right) \right|^2 dx \\
(4.10) \qquad &= m_\varepsilon \sigma^{N-4} \|\Delta W_\rho\|_{L^2(B_\rho)}^2 \leq c \mu^\varepsilon.
\end{aligned}$$

The assertions (4.5) and (4.6) follow immediately. Again from (4.8)-(4.10), we have $\|w^\varepsilon - 1\|_{H_0^2(\Omega; \Delta)}^2 \leq c \mu^\varepsilon$. The assertion (4.7) follows from the equivalence of two norms $\|\cdot\|_{H_0^2(\Omega; \Delta)}$ and $\|\cdot\|_{H_0^2(\Omega)}$, since $(w^\varepsilon - 1) \in H_0^2(\Omega; \Delta)$.

Proof of Theorem 4.1: Consider first the case that $\mu < \infty$. In this case we claim that (as claimed in Lemma 2.1 in the case $g = 0$) $\|\hat{u}^\varepsilon\|_{H^2(\Omega)}$ is uniformly bounded, and

$$(4.11) \qquad \|u^\varepsilon\|_{H^2(\Omega^\varepsilon)} \leq c \left(\|f\|_{L^2(\Omega)} + \|g\|_{c^2} \right).$$

Indeed, we choose the function $v = u^\varepsilon - w^\varepsilon g$. It is easy to verify that v is eligible as a test function in (2.4). It follows from (2.4) that

$$\iint_{\Omega^\varepsilon} |\Delta u^\varepsilon|^2 dx = \iint_{\Omega^\varepsilon} \Delta u^\varepsilon \cdot \Delta (w^\varepsilon g) dx + \iint_{\Omega^\varepsilon} f \cdot (u^\varepsilon - w^\varepsilon g) dx.$$

By Lemma 4.1, we obtain

$$(4.12) \qquad \|\Delta u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \leq c \left(\|f\|_{L^2(\Omega)} + \|g\|_{c^2} \right) \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)}.$$

Next, from the identity

$$\iint_{\Omega^\varepsilon} v \Delta v dx = - \iint_{\Omega^\varepsilon} |\nabla v|^2 dx,$$

and using (4.12), we have

$$(4.13) \quad \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^4 \leq c \left(\|f\|_{L^2(\Omega)} + \|g\|_{c^2} \right) \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^3 + c \|g\|_{c^2}^4.$$

Since $\hat{u}^\varepsilon - g \in H_0^2(\Omega)$, by applying Poincaré's inequality to $\hat{u}^\varepsilon - g$, we derive from (4.12) and (4.13) that

$$\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)} + \|\Delta u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq c \left(\|f\|_{L^2(\Omega)} + \|g\|_{c^2} \right).$$

(4.11) follows from the equivalence between $H_0^2(\Omega)$ -norm and $H_0^2(\Omega; \Delta)$ -norm for $\hat{u}^\varepsilon - g$.

By (4.11), we know that there exists a subsequence of $\hat{u}^\varepsilon$ that weakly converges to a function u . It now suffices to show that any weak limit u satisfies (4.2). By (2.4), for any $\xi \in C_0^\infty(\Omega)$, we have

$$\iint_{\Omega} \Delta \hat{u}^\varepsilon \cdot \Delta(w^\varepsilon \xi) dx = \iint_{\Omega} f w^\varepsilon \xi dx.$$

This equality can be rewritten as

$$(4.14) \quad \iint_{\Omega} w^\varepsilon \Delta \hat{u}^\varepsilon \cdot \Delta \xi dx + \iint_{\Omega} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta w^\varepsilon dx = \iint_{\Omega} f w^\varepsilon \xi dx + \theta,$$

where

$$\theta = \iint_{\Omega} (\hat{u} \Delta w^\varepsilon \cdot \Delta \xi + 2 \Delta w^\varepsilon \nabla \hat{u} \cdot \nabla \xi - 2 \Delta \hat{u}^\varepsilon \nabla w^\varepsilon \cdot \nabla \xi) dx.$$

Since $\hat{u}^\varepsilon$ is uniformly bounded in $H^2(\Omega)$ and $w^\varepsilon \rightarrow 1$ weakly in $H^2(\Omega)$, we know that $\theta \rightarrow 0$ as $\varepsilon \rightarrow 0$, and that

$$(4.15) \quad \iint_{\Omega} w^\varepsilon \Delta \hat{u}^\varepsilon \cdot \Delta \xi dx \longrightarrow \iint_{\Omega} \Delta u \cdot \Delta \xi dx,$$

$$(4.16) \quad \iint_{\Omega} f w^\varepsilon \xi dx \longrightarrow \iint_{\Omega} f \xi dx.$$

It remains to study the limit of the second term on the left-hand side of (4.14). We now apply the identity (3.4) with $v = \hat{u}^\varepsilon \xi$. It follows that

$$(4.17) \quad \begin{aligned} & \iint_{B_\varepsilon(x_i^\varepsilon) \setminus T_i^\varepsilon} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta w^\varepsilon dx \\ &= \int_{\partial B_\varepsilon(x_i^\varepsilon)} \Delta w^\varepsilon \frac{\partial(\hat{u}^\varepsilon \xi)}{\partial n} ds - \int_{\partial B_\varepsilon(x_i^\varepsilon)} \frac{\partial \Delta w^\varepsilon}{\partial n} \hat{u}^\varepsilon \xi ds. \end{aligned}$$

For any $x \in \partial B_\rho(x_i^\varepsilon)$, by Lemma 3.2 we have

$$(4.18) \quad |\Delta w^\varepsilon(x)| = \frac{1}{\sigma^2} \left| \Delta W_\rho \left(\frac{x - x_i^\varepsilon}{\sigma} \right) \right| \leq c \mu^\varepsilon \varepsilon^2,$$

where $\rho = \frac{\varepsilon}{\sigma}$. Next we rescaling $\hat{u}^\varepsilon \xi$ by setting $v(x) = (\hat{u}^\varepsilon \xi)(x_i^\varepsilon + \varepsilon x)$ for $x \in B_1$. By applying the trace inequality

$$\int_{\partial B_1} |\nabla v|^2 ds \leq c \iint_{B_1} (|\nabla v|^2 + |\nabla^2 v|^2) dx,$$

we derive that

$$(4.19) \quad \int_{\partial B_\varepsilon(x_i^\varepsilon)} |\nabla(\hat{u}^\varepsilon \xi)|^2 ds \leq \frac{c}{\varepsilon} \iint_{B_\varepsilon(x_i^\varepsilon)} (|\nabla(\hat{u}^\varepsilon \xi)|^2 + \varepsilon^2 |\nabla^2(\hat{u}^\varepsilon \xi)|^2) dx.$$

Combining (4.18) and (4.19), the first term on the right-hand side of (4.17) can be estimated by

$$(4.20) \quad \begin{aligned} & \left| \int_{\partial B_\varepsilon(x_i^\varepsilon)} \Delta w^\varepsilon \frac{\partial(\hat{u}^\varepsilon \xi)}{\partial n} ds \right| \leq c \varepsilon^2 \mu^\varepsilon \int_{\partial B_\varepsilon(x_i^\varepsilon)} \left| \frac{\partial(\hat{u}^\varepsilon \xi)}{\partial n} \right| ds \\ & \leq c \varepsilon^{\frac{N+3}{2}} \mu^\varepsilon \left(\int_{\partial B_\varepsilon(x_i^\varepsilon)} \left| \frac{\partial(\hat{u}^\varepsilon \xi)}{\partial n} \right|^2 ds \right)^{\frac{1}{2}} \\ & \leq c \varepsilon^{\frac{N+2}{2}} \mu^\varepsilon \left(\iint_{B_\varepsilon(x_i^\varepsilon)} (|\nabla(\hat{u}^\varepsilon \xi)|^2 + \varepsilon^2 |\nabla^2(\hat{u}^\varepsilon \xi)|^2) dx \right)^{\frac{1}{2}} \\ & \leq c \varepsilon \mu^\varepsilon \left(\varepsilon^N + \iint_{B_\varepsilon(x_i^\varepsilon)} (|\nabla(\hat{u}^\varepsilon \xi)|^2 + \varepsilon^2 |\nabla^2(\hat{u}^\varepsilon \xi)|^2) dx \right). \end{aligned}$$

We now evaluate the second integral on the right-hand side of (4.17). By (ii) of Lemma 3.2, for $x \in \partial B_\varepsilon(x_i^\varepsilon)$,

$$\begin{aligned}\nabla(\Delta w^\varepsilon)(x) &= \frac{1}{\sigma^3} \nabla(\Delta W_\rho)\left(\frac{x - x_i^\varepsilon}{\sigma}\right) \\ &= -\frac{\gamma_\rho(T)}{\omega_N} \sigma^{N-4} \frac{x - x_i^\varepsilon}{|x - x_i^\varepsilon|^N} + \theta_1,\end{aligned}$$

where

$$|\theta_1| \leq \frac{c}{\sigma^3} \left(\frac{\delta(\rho)}{\rho^{N-1}} + \frac{1}{\rho^N} \right) \leq c\varepsilon\mu^\varepsilon \left(\delta(\rho) + \frac{1}{\rho} \right).$$

It follows that

$$(4.21) \quad \int_{\partial B_\varepsilon(x_i^\varepsilon)} \frac{\partial \Delta w^\varepsilon}{\partial n} \hat{u}^\varepsilon \xi ds = -\frac{\gamma_\rho(T)}{\omega_N} \mu^\varepsilon \varepsilon \int_{\partial B_\varepsilon(x_i^\varepsilon)} \hat{u}^\varepsilon \xi ds + \theta_2(i),$$

where

$$\begin{aligned}(4.22) \quad |\theta_2(i)| &\leq \int_{\partial B_\varepsilon(x_i^\varepsilon)} |\theta_1 \hat{u}^\varepsilon \xi| ds \\ &\leq c\mu^\varepsilon \varepsilon \left(\delta(\rho) + \frac{1}{\rho} \right) \int_{\partial B_\varepsilon(x_i^\varepsilon)} |\hat{u}^\varepsilon \xi| ds.\end{aligned}$$

Combining (4.17), (4.20)-(4.22), we derive that

$$\begin{aligned}(4.23) \quad \iint_{\Omega} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta w^\varepsilon dx &= \sum_{i=1}^{m_\varepsilon} \iint_{B_\varepsilon(x_i^\varepsilon) \setminus T_i^\varepsilon} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta w^\varepsilon dx \\ &= \frac{\gamma_\rho(T)}{\omega_N} \mu^\varepsilon \varepsilon \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(x_i^\varepsilon)} \hat{u}^\varepsilon \xi ds + \theta_3,\end{aligned}$$

where

$$\begin{aligned}(4.24) \quad |\theta_3| &\leq \varepsilon \left(\varepsilon^N + \iint_{B_\varepsilon(x_i^\varepsilon)} (|\nabla(\hat{u}^\varepsilon \xi)|^2 + \varepsilon^2 |\nabla^2(\hat{u}^\varepsilon \xi)|^2) dx \right) + \sum_{i=1}^{m_\varepsilon} \theta_2(i) \\ &\leq c\varepsilon\mu^\varepsilon \|\hat{u}^\varepsilon \xi\|_{H^2} + \sum_{i=1}^{m_\varepsilon} \theta_2(i).\end{aligned}$$

By assumption (A4), for any smooth function ϕ ,

$$\begin{aligned} \varepsilon \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(x_i^\varepsilon)} \phi \, ds &= \omega_N \sum_{i=1}^{m_\varepsilon} \varepsilon^N \phi(x_i^\varepsilon) + \sum_{i=1}^{m_\varepsilon} \varepsilon \int_{\partial B_\varepsilon(x_i^\varepsilon)} (\phi(x) - \phi(x_i^\varepsilon)) \, ds \\ &\longrightarrow \omega_N \iint_{\Omega} P(x) \phi(x) \, dx \quad (\text{as } \varepsilon \longrightarrow 0). \end{aligned}$$

Approximating $\hat{u}^\varepsilon \xi$ by smooth functions, we thus obtain

$$(4.25) \quad \varepsilon \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(x_i^\varepsilon)} \hat{u}^\varepsilon \xi \, ds \longrightarrow \omega_N \iint_{\Omega} P u \xi \, dx \quad (\text{as } \varepsilon \longrightarrow 0).$$

From (4.22), (4.25) also implies that

$$(4.26) \quad \sum_{i=1}^{m_\varepsilon} \theta_2(i) \leq c \left(\delta(\rho) + \frac{1}{\rho} \right).$$

It follows from (4.23)-(4.26) and Lemma 3.3 that

$$(4.27) \quad \iint_{\Omega} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta w^\varepsilon \, dx \longrightarrow \mu \gamma(T) \iint_{\Omega} P u \xi \, dx.$$

Letting $\varepsilon \rightarrow 0$ in (4.14), we obtain from (4.15), (4.16), and (4.27) that

$$\iint_{\Omega} \Delta u \cdot \Delta \xi \, dx + \mu \gamma(T) \iint_{\Omega} P u \xi \, dx = \iint_{\Omega} f \xi,$$

for any $\xi \in C_0^\infty(\Omega)$. This is the weak formulation of (4.2), (4.3).

It remains to prove the case that $\mu = \infty$. In this case, we define

$$\tilde{w}^\varepsilon(x) = \frac{1}{\mu^\varepsilon} w^\varepsilon(x),$$

where w^ε is defined in (4.4). It follows from (4.8)-(4.10) that

$$\|\tilde{w}^\varepsilon\|_{L^2} + \|\nabla \tilde{w}^\varepsilon\|_{L^2} \longrightarrow 0 \quad (\text{as } \varepsilon \longrightarrow 0),$$

$$\|\nabla^2 \tilde{w}^\varepsilon\|_{L^2} \leq c.$$

We note that when $g \not\equiv 0$, $\hat{u}^\varepsilon$ in general is not uniformly bounded in $H^2(\Omega)$. By Lemma 2.1, however, it is locally uniformly bounded. Let $\Omega_0 \subseteq \bar{\Omega}_0 \subseteq \Omega$ is any subdomain of Ω and $\xi \in C_0^\infty(\Omega_0)$. By replacing w^ε with $\tilde{w}^\varepsilon$ in (4.14), it follows that

$$\iint_{\Omega} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta \tilde{w}^\varepsilon dx \longrightarrow 0.$$

Since $\hat{u}^\varepsilon$ is uniformly bounded in $H^2(\Omega_0)$, there exists a subsequence that converges weakly (in $H^2(\Omega_0)$) to a function $u \in H^2(\Omega_0)$. Moreover, the estimates (4.17)-(4.26) remain true except that the constant c must be replaced by $c(\Omega_0)$ which depends on Ω_0 . Therefore we can evaluate the above integral by the same methods as in deriving (4.27). By modifying (4.17)-(4.26), we obtain

$$\iint_{\Omega} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta \tilde{w}^\varepsilon dx \longrightarrow \mu \gamma(T) \iint_{\Omega} P u \xi dx.$$

Hence

$$\iint_{\Omega} P u \xi dx = 0$$

for any $\xi \in C_0^\infty(\Omega_0)$. It follows that $u = 0$ since $P > 0$ and Ω_0 can be chosen arbitrarily. The proof is complete.

From the above results, we know that $\hat{u}^\varepsilon - u \rightarrow 0$ weakly in H^2 if μ is finite. One may ask if the convergence is also strong. In general, it cannot be strongly convergent. But if we add a corrector $(1 - w^\varepsilon)u$, the convergence is strong.

THEOREM 4.2. *Let $N \geq 5$, $g \in C^2$, and u be the solution of (4.2), (4.3). Suppose that $\mu < \infty$, where μ is defined in (4.1). Then*

$$(4.28) \quad \iint_{\Omega} |\nabla^2(\hat{u}^\varepsilon - w^\varepsilon u)|^p dx \longrightarrow 0,$$

where $p = \frac{N}{N-2}$.

Proof: For any $\xi \in C^\infty(\Omega)$ such that $\xi = g$ on $\partial\Omega$, we write, by using the weak formulation (2.4) for u^ε ,

$$\begin{aligned}
(4.29) \quad & \iint_{\Omega} |\Delta(\hat{u}^\varepsilon - w^\varepsilon \xi)|^2 dx \\
&= \iint_{\Omega} \Delta \hat{u}^\varepsilon \cdot \Delta(\hat{u}^\varepsilon - w^\varepsilon \xi) dx - \iint_{\Omega} \Delta(w^\varepsilon \xi) \cdot \Delta(\hat{u}^\varepsilon - w^\varepsilon \xi) dx \\
&= \iint_{\Omega} f(\hat{u}^\varepsilon - w^\varepsilon \xi) dx - \iint_{\Omega} \Delta \xi \cdot \Delta(w^\varepsilon(\hat{u}^\varepsilon - w^\varepsilon \xi)) dx \\
&\quad - \iint_{\Omega} \Delta w^\varepsilon \cdot \Delta(u(\hat{u}^\varepsilon - w^\varepsilon \xi)) dx + \theta,
\end{aligned}$$

where

$$\begin{aligned}
(4.30) \quad \theta &= \iint_{\Omega} ((\hat{u}^\varepsilon - w^\varepsilon \xi) \Delta w^\varepsilon \cdot \Delta \xi + 2 \Delta w^\varepsilon \cdot \nabla \xi \cdot \nabla(\hat{u}^\varepsilon - w^\varepsilon \xi)) dx \\
&\quad + \iint_{\Omega} (w^\varepsilon - 1) \Delta \xi \cdot \Delta(\hat{u}^\varepsilon - w^\varepsilon \xi) dx \\
&\quad - \iint_{\Omega} 2(\nabla w^\varepsilon \cdot \nabla \xi) \Delta(\hat{u}^\varepsilon - w^\varepsilon \xi) dx.
\end{aligned}$$

Since $w^\varepsilon \rightarrow 0$, it is easy to see that $\theta \rightarrow 0$ as $\varepsilon \rightarrow 0$. By repeating the process of deriving (4.27), we obtain

$$(4.31) \quad \iint_{\Omega} \Delta w^\varepsilon \cdot \Delta(\xi(\hat{u}^\varepsilon - w^\varepsilon \xi)) dx \longrightarrow \mu \gamma(T) \iint_{\Omega} P \xi(u - \xi) dx,$$

since $\xi(\hat{u}^\varepsilon - w^\varepsilon \xi) \rightarrow \xi(u - \xi)$ weakly in H^2 . It follows from (4.29)-(4.31) that

$$\begin{aligned}
(4.32) \quad & \iint_{\Omega} |\Delta(\hat{u}^\varepsilon - w^\varepsilon \xi)|^2 dx \longrightarrow \iint_{\Omega} f(u - \xi) dx - \iint_{\Omega} \Delta \xi \cdot \Delta(u - \xi) dx \\
&\quad - \mu \gamma(T) \iint_{\Omega} P \xi(u - \xi) dx \\
&= \iint_{\Omega} |\Delta(u - \xi)|^2 dx + \mu \gamma(T) \iint_{\Omega} P(u - \xi)^2 dx.
\end{aligned}$$

From the Sobolev imbedding, for any $\phi \in C^\infty$,

$$\|\phi\|_{L^{p_1}} \leq c \|\phi\|_{H^2}, \quad p_1 = \frac{2N}{N-4},$$

$$\|\nabla \phi\|_{L^{p_2}} \leq c \|\phi\|_{H^2}, \quad p_2 = \frac{2N}{N-2}.$$

Therefore,

$$\begin{aligned}
(4.33) \quad & \iint_{\Omega} |\Delta(w^\varepsilon(u - \xi))|^p dx \leq c \iint_{\Omega} |\Delta w^\varepsilon|^p |\xi - u|^p dx \\
& + c \iint_{\Omega} (|\nabla w^\varepsilon|^p |\nabla(\xi - u)|^p + |w^\varepsilon|^p |\Delta(\xi - u)|^p) dx \\
& \leq c \|\xi - u\|_{L^{pq}}^p + c \|\nabla(\xi - u)\|_{L^{pq_1}}^p + c \|\Delta(\xi - u)\|_{L^{pq_2}}^p,
\end{aligned}$$

where

$$q = \frac{2(N-2)}{(N-4)}, \quad q_1 = 2, \quad q_2 = \frac{2(N-2)}{N}.$$

We thus obtain, since $pq = p_1$, $pq_1 = p_2$, $pq_2 = 2$,

$$\begin{aligned}
\iint_{\Omega} |\Delta(\hat{u}^\varepsilon - w^\varepsilon u)|^p dx & \leq \iint_{\Omega} |\Delta(\hat{u}^\varepsilon - w^\varepsilon \xi)|^p dx + \iint_{\Omega} |\Delta(w^\varepsilon(u - \xi))|^p dx \\
& \leq c \left(\iint_{\Omega} |\Delta(\hat{u}^\varepsilon - w^\varepsilon \xi)|^2 dx \right)^{\frac{p}{2}} + c \|\xi - u\|_{H^2}^p.
\end{aligned}$$

Letting $\varepsilon \rightarrow 0$ and using (4.32), we derive that

$$\overline{\lim}_{\varepsilon \rightarrow 0} \iint_{\Omega} |\Delta(\hat{u}^\varepsilon - w^\varepsilon u)|^p \leq c \|\xi - u\|_{H^2}^p.$$

The assertion follows by letting $\xi \rightarrow u$ and the L^p -Estimates for the second order elliptic equations.

In the special case that u is a C^2 function (this is true, for example, if Ω , T , f are regular enough), the Theorem 4.2 can be strengthened such that (4.28) holds for $p = 2$.

THEOREM 4.3. *Let $N \geq 5$, $\mu < \infty$, $g \in C^2$, and u be the C^2 solution of (4.2), (4.3). Then*

$$\|\hat{u}^\varepsilon - w^\varepsilon u\|_{H^2(\Omega)}^2 \leq c \|\hat{u}^\varepsilon - w^\varepsilon u\|_{H^1(\Omega)} + c\varepsilon^2 \longrightarrow 0.$$

Proof: Taking $\xi = u$ in (4.29) and using the weak formulation for u , we have

$$(4.34) \quad \begin{aligned} \iint_{\Omega} |\Delta(\hat{u}^\varepsilon - w^\varepsilon u)|^2 dx &= \mu\gamma(T) \iint_{\Omega} uP(\hat{u}^\varepsilon - w^\varepsilon u) dx \\ &\quad + \iint_{\Omega} \Delta w^\varepsilon \cdot \Delta(u(\hat{u}^\varepsilon - w^\varepsilon u)) dx + \theta. \end{aligned}$$

It is easy to see that θ (defined in (4.30)) can be estimated by

$$(4.35) \quad |\theta| \leq c \|\hat{u}^\varepsilon - w^\varepsilon u\|_{H^1} + c \|w^\varepsilon - 1\|_{L^2}^2 + c \|\nabla w^\varepsilon\|_{L^2}^2 + \frac{1}{4} \|\Delta(\hat{u}^\varepsilon - w^\varepsilon u)\|_{L^2}^2.$$

We now estimate the second integral in the right-hand side of (4.34). By (3.4), in each $B_\varepsilon(x_i^\varepsilon) \setminus T_i^\varepsilon$, we have

$$(4.36) \quad \begin{aligned} &\iint_{B_\varepsilon(x_i^\varepsilon) \setminus T_i^\varepsilon} \Delta w^\varepsilon \cdot \Delta(u(\hat{u}^\varepsilon - w^\varepsilon u)) dx \\ &= \int_{\partial B_\varepsilon(x_i^\varepsilon)} \Delta w^\varepsilon \cdot \frac{\partial}{\partial n} (u(\hat{u}^\varepsilon - w^\varepsilon u)) ds - \int_{\partial B_\varepsilon(x_i^\varepsilon)} \frac{\partial \Delta w^\varepsilon}{\partial n} \cdot (u(\hat{u}^\varepsilon - w^\varepsilon u)) ds. \end{aligned}$$

By (4.18) and the trace inequality analog to (4.19), we obtain, similar to (4.20),

$$(4.37) \quad \begin{aligned} &\left| \int_{\partial B_\varepsilon(x_i^\varepsilon)} \Delta w^\varepsilon \cdot \frac{\partial}{\partial n} (u(\hat{u}^\varepsilon - w^\varepsilon u)) ds \right| \\ &\leq c\mu^\varepsilon \varepsilon^2 \int_{\partial B_\varepsilon(x_i^\varepsilon)} |\nabla(u(\hat{u}^\varepsilon - w^\varepsilon u))| ds \\ &\leq c\mu^\varepsilon \varepsilon^{\frac{N+3}{2}} \left(\int_{\partial B_\varepsilon(x_i^\varepsilon)} |\nabla(u(\hat{u}^\varepsilon - w^\varepsilon u))|^2 ds \right)^{\frac{1}{2}} \\ &\leq c\mu^\varepsilon \varepsilon^{\frac{N+2}{2}} \left(\iint_{B_\varepsilon(x_i^\varepsilon)} |\nabla(u(\hat{u}^\varepsilon - w^\varepsilon u))|^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

$$+c\mu^\varepsilon \varepsilon^{\frac{N+4}{2}} \left(\iint_{B_\varepsilon(x_i^\varepsilon)} |\nabla^2 (u(\hat{u}^\varepsilon - w^\varepsilon u))|^2 ds \right)^{\frac{1}{2}}.$$

By the estimates analog to (4.21) and (4.22), the second integral in the right-hand side of (4.36) can be bounded by

$$\begin{aligned} & \left| \int_{\partial B_\varepsilon(x_i^\varepsilon)} \frac{\partial \Delta w^\varepsilon}{\partial n} \cdot (u(\hat{u}^\varepsilon - w^\varepsilon u)) ds \right| \leq c\mu^\varepsilon \varepsilon \int_{\partial B_\varepsilon(x_i^\varepsilon)} |u(\hat{u}^\varepsilon - w^\varepsilon u)| ds \\ (4.38) \quad & \leq c\mu^\varepsilon \varepsilon^{\frac{N}{2}} \left(\iint_{B_\varepsilon(x_i^\varepsilon)} |(u(\hat{u}^\varepsilon - w^\varepsilon u))|^2 dx \right)^{\frac{1}{2}} \\ & \quad + c\mu^\varepsilon \varepsilon^{\frac{N+2}{2}} \left(\iint_{B_\varepsilon(x_i^\varepsilon)} |\nabla (u(\hat{u}^\varepsilon - w^\varepsilon u))|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

where we also used the trace formula like (4.19) and the fact that $\delta(\rho)$ is bounded.

Substituting (4.38) and (4.38) into (4.36), we derive that

$$\begin{aligned} (4.39) \quad & \left| \iint_{\Omega} \Delta w^\varepsilon \cdot \Delta (u(\hat{u}^\varepsilon - w^\varepsilon u)) dx \right| \leq \sum_{i=1}^{m_\varepsilon} \left| \iint_{B_\varepsilon(x_i^\varepsilon) \setminus T_i^\varepsilon} \Delta w^\varepsilon \cdot \Delta (u(\hat{u}^\varepsilon - w^\varepsilon u)) dx \right| \\ & \leq c\mu^\varepsilon \left(\|u(\hat{u}^\varepsilon - w^\varepsilon u)\|_{H^1(\Omega)} + \varepsilon^2 \|\nabla^2 (u(\hat{u}^\varepsilon - w^\varepsilon u))\|_{L^2(\Omega)} \right), \end{aligned}$$

where we have used the inequality

$$(4.40) \quad \left(\sum_{i=1}^{m_\varepsilon} a_i \right)^2 \leq m_\varepsilon \sum_{i=1}^{m_\varepsilon} a_i^2.$$

Note that $\|u(\hat{u}^\varepsilon - w^\varepsilon u)\|_{H^2(\Omega)}$ is uniformly bounded. Combining (4.34), (4.35) and (4.40), we find

$$\|\Delta (u(\hat{u}^\varepsilon - w^\varepsilon u))\|_{L^2(\Omega)}^2 \leq c \|\hat{u}^\varepsilon - w^\varepsilon u\|_{H^1(\Omega)} + c \|w^\varepsilon - 1\|_{H^1(\Omega)}^2 + c\varepsilon^2.$$

By (4.5), (4.6) and the fact that $H_0^2(\Omega; \Delta)$ is isometric to $H_0^2(\Omega)$, it follows that

$$\|u(\hat{u}^\varepsilon - w^\varepsilon u)\|_{H^2(\Omega)}^2 \leq c \|\hat{u}^\varepsilon - w^\varepsilon u\|_{H^1(\Omega)} + c\varepsilon^2.$$

5. Homogenized Problem II: $N = 4$. In this section we study the case $N = 4$. In this case, the Green's function (3.7) behaves like $\frac{1}{4\omega_4} \ln \rho - \frac{1}{8\omega_4}$ as $\rho \rightarrow \infty$. Hence we need different approaches to treat the limiting equation. We first look for a biharmonic function V_ρ in $B_\rho \setminus B_2$ (for $\rho > 5$) such that

$$\begin{aligned}\Delta^2 V_\rho &= 0 \quad \text{in } B_\rho \setminus B_2, \\ V_\rho &= \frac{\partial V_\rho}{\partial n} = 0 \quad \text{on } \partial B_2, \\ V_\rho &= 1, \quad \frac{\partial V_\rho}{\partial n} = 0 \quad \text{on } \partial B_\rho.\end{aligned}$$

Existence and uniqueness follows from Lemma 2.1. In fact, such a solution can be constructed explicitly. One may verify that for $|x| = r$, $2 < r < \rho$,

$$(5.1) \quad V_\rho(x) = \frac{1}{\lambda_\rho} \left(r^2 + (\rho^2 - 4) - 2(\rho^2 + 4) \ln \left(\frac{r}{2} \right) - \frac{4\rho^2}{r^2} \right),$$

where

$$\lambda_\rho = (\rho^2 - 4) - (\rho^2 + 4) \ln \left(\frac{\rho}{2} \right).$$

We observe that for large ρ , $c_1 \rho^2 \ln \rho \leq |\lambda_\rho| \leq c_2 \rho^2 \ln \rho$, for some constant $c_2 > c_1 > 0$.

By direct computation, we obtain

$$(5.2) \quad \Delta V_\rho(x) = \frac{2}{\lambda_\rho} \left(2 - \frac{\rho^2 + 4}{r^2} \right),$$

$$\begin{aligned}|\nabla V_\rho(x)| &\leq \frac{c\rho}{\lambda_\rho} \left(1 + \frac{\rho}{r} + \frac{\rho}{r^3} \right), \\ |\Delta V_\rho(x)| &\leq \frac{c}{\lambda_\rho} \left(1 + \frac{\rho^2}{r^2} + \frac{\rho^2}{r^4} \right).\end{aligned}$$

We now, for convenience, extend the function V_ρ into B_ρ by setting $V_\rho = 0$ in B_2 . The extended function is still in $H^2(B_\rho)$. One can see that for large ρ , the following

estimates hold:

$$(5.3) \quad \|\nabla V_\rho\|_{L^2(B_\rho)} \leq \frac{c\rho^3}{\lambda_\rho} \leq \frac{c\rho}{\ln \rho},$$

$$(5.4) \quad \|\Delta V_\rho\|_{L^2(B_\rho)} \leq \frac{c\rho^2\sqrt{1+\ln \rho}}{\lambda_\rho} \leq \frac{c}{\sqrt{\ln \rho}}.$$

We next define a test function w^ε as in (4.4) by

$$(5.5) \quad w^\varepsilon(x) = \begin{cases} V_\rho\left(\frac{x-x_i^\varepsilon}{\sigma}\right) & \text{for } x \in B_\varepsilon(x_i^\varepsilon), \\ 1 & \text{otherwise in } \Omega, \end{cases}$$

where $\rho = \frac{\varepsilon}{\sigma}$. Set

$$(5.6) \quad \mu_4^\varepsilon = -\frac{1}{\varepsilon^4 \ln \sigma}.$$

LEMMA 5.1. *Let $\rho = \frac{\varepsilon}{\sigma} \rightarrow 0$. Then*

$$(5.7) \quad \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 \leq c\varepsilon^6 \mu_4^\varepsilon \left|1 - \frac{\ln \varepsilon}{\ln \sigma}\right|^{-1},$$

$$(5.8) \quad \|\Delta w^\varepsilon\|_{L^2(\Omega)}^2 \leq c\mu_4^\varepsilon \left|1 - \frac{\ln \varepsilon}{\ln \sigma}\right|^{-1}.$$

Consequently, if μ_4^ε is uniformly bounded, then $w^\varepsilon \rightarrow 1$ weakly in $H^2(\Omega)$.

Proof: The proof is based on the estimates similar to (4.9) and (4.10). It follows from (5.3) and (5.4) that

$$\begin{aligned} \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 &\leq cm_\varepsilon \sigma^2 \|\nabla V_\rho(x)\|_{L^2(B_\rho)}^2 \\ &\leq \frac{\sigma^2}{\varepsilon^4} \frac{c\rho^2}{|\ln \rho|^2} = \frac{c}{\varepsilon^2 |\ln \varepsilon - \ln \sigma|^2} \\ \|\nabla^2 w^\varepsilon\|_{L^2(\Omega)}^2 &\leq cm_\varepsilon \|\nabla^2 V_\rho(x)\|_{L^2(B_\rho)}^2 \leq \frac{c}{\varepsilon^4 |\ln \varepsilon - \ln \sigma|^2}. \end{aligned}$$

The assertion follows immediately.

We now in the position to prove the main result in the case $N = 4$.

THEOREM 5.1. *Let $N = 4$, u^ε be the solution of (2.1)-(2.3) with $g \in C^2$, and $\hat{u}^\varepsilon$ be the extension of u^ε by 0 in T_i^ε . If*

$$\mu = \lim_{\varepsilon \rightarrow 0} \mu_4^\varepsilon < \infty,$$

then the whole sequence $\hat{u}^\varepsilon \rightarrow u$ weakly in $H^2(\Omega)$ as $\varepsilon \rightarrow 0$. Moreover, u satisfies the following homogenized problem:

$$(5.9) \quad \Delta^2 u + 4\mu\omega_4 Pu = f \quad \text{in } \Omega,$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

If $\mu = \infty$, then $u = 0$.

Proof: Consider first the case that $\mu < \infty$. As in the proof of Theorem 4.1, we can show that $\hat{u}^\varepsilon$ is uniformly bounded in H^2 . Since the rest of the proof is local, we may assume, without loss of generality, that $g = 0$. For any $\xi \in C_0^\infty(\Omega)$, we take $w^\varepsilon \xi$ as a test function. Note that $B_\sigma(x_i^\varepsilon) \subset T_i^\varepsilon \subset B_{2\sigma}(x_i^\varepsilon)$, and that the function w^ε is biharmonic only in $B_{2\sigma}(x_i^\varepsilon)$. Hence in this time we need to handle some extra terms. As in (4.14), we can write the variational formulation (2.4) as

$$(5.10) \quad \iint_{\Omega} w^\varepsilon \Delta \hat{u}^\varepsilon \cdot \Delta \xi dx + \iint_{\Omega} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta w^\varepsilon dx = \iint_{\Omega} f w^\varepsilon \xi dx + \theta,$$

where

$$\theta = \iint_{\Omega} (\hat{u} \Delta w^\varepsilon \cdot \Delta \xi + 2 \Delta w^\varepsilon \nabla \hat{u} \cdot \nabla \xi - 2 \Delta \hat{u}^\varepsilon \nabla w^\varepsilon \cdot \nabla \xi) dx.$$

Consider first the case that $\mu < \infty$. Then by Lemma 5.1, $\theta \rightarrow 0$ as $\varepsilon \rightarrow 0$. Also (4.15) and (4.16) hold. By (3.5), note that $w^\varepsilon = 0$ in $B_{2\sigma}(x_i^\varepsilon)$, we have

$$\iint_{B_\varepsilon(x_i^\varepsilon)} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta w^\varepsilon dx = \iint_{B_\varepsilon(x_i^\varepsilon) \setminus B_{2\sigma}(x_i^\varepsilon)} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta w^\varepsilon dx$$

$$\begin{aligned}
&= \int_{\partial B_\varepsilon(x_i^\varepsilon)} \Delta w^\varepsilon \frac{\partial(\hat{u}^\varepsilon \xi)}{\partial n} ds - \int_{\partial B_\varepsilon(x_i^\varepsilon)} \frac{\partial \Delta w^\varepsilon}{\partial n} \hat{u}^\varepsilon \xi ds \\
&\quad + \int_{\partial B_{2\sigma}(x_i^\varepsilon)} \Delta w^\varepsilon \frac{\partial(\hat{u}^\varepsilon \xi)}{\partial n} ds - \int_{\partial B_{2\sigma}(x_i^\varepsilon)} \frac{\partial \Delta w^\varepsilon}{\partial n} \hat{u}^\varepsilon \xi ds \\
(5.11) \quad &= I_i^1 + I_i^2 + I_i^3 + I_i^4.
\end{aligned}$$

We now proceed to estimate each I_i^k . By (5.2), for any $x \in \partial B_\varepsilon(x_i^\varepsilon)$,

$$\Delta w^\varepsilon(x) = \frac{1}{\sigma^2} \Delta V_\rho \left(\frac{x - x_i^\varepsilon}{\sigma} \right) = \frac{2}{\sigma^2 \lambda_\rho} \left(2 - \frac{\rho^2 + 4}{\rho^2} \right) \stackrel{def}{=} \beta_\rho,$$

where

$$(5.12) \quad |\beta_\rho| \leq \frac{c}{\sigma^2 \rho^2 \ln \rho} \leq \frac{c \mu_4^\varepsilon \varepsilon^2}{\left| 1 - \frac{\ln \varepsilon}{\ln \sigma} \right|} \leq c \mu_4^\varepsilon \varepsilon^2.$$

It follows that

$$\begin{aligned}
\sum_{i=1}^{m_\varepsilon} I_i^1 &= \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(x_i^\varepsilon)} \Delta w^\varepsilon \frac{\partial(\hat{u}^\varepsilon \xi)}{\partial n} ds \\
&= \beta_\rho \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(x_i^\varepsilon)} \frac{\partial(\hat{u}^\varepsilon \xi)}{\partial n} ds = \beta_\rho \sum_{i=1}^{m_\varepsilon} \iint_{B_\varepsilon(x_i^\varepsilon)} \Delta(\hat{u}^\varepsilon \xi) dx.
\end{aligned}$$

Combining with (5.12), we obtain

$$(5.13) \quad \left| \sum_{i=1}^{m_\varepsilon} I_i^1 \right| \leq c \mu_4^\varepsilon \varepsilon^2 \iint_{\Omega} |\Delta(\hat{u}^\varepsilon \xi)| dx \leq c \varepsilon^2 \|\hat{u}^\varepsilon \xi\|_{H^2(\Omega)}.$$

To estimate I_i^3 , we observe that for $x \in \partial B_{2\sigma}(x_i^\varepsilon)$, from (5.2),

$$\Delta w^\varepsilon(x) = \frac{1}{\sigma^2} \Delta V_\rho \left(\frac{x - x_i^\varepsilon}{\sigma} \right) = \frac{2}{\sigma^2 \lambda_\rho} \left(2 - \frac{\rho^2 + 4}{4} \right) \stackrel{def}{=} \bar{\beta}_\rho,$$

while

$$(5.14) \quad |\bar{\beta}_\rho| \leq \frac{c}{\sigma^2 \ln \rho}.$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{m_\varepsilon} I_i^3 &= \sum_{i=1}^{m_\varepsilon} \int_{\partial B_{2\sigma}(x_i^\varepsilon)} \Delta w^\varepsilon \frac{\partial(\hat{u}^\varepsilon \xi)}{\partial n} ds = \bar{\beta}_\rho \sum_{i=1}^{m_\varepsilon} \int_{\partial B_{2\sigma}(x_i^\varepsilon)} \frac{\partial(\hat{u}^\varepsilon \xi)}{\partial n} ds \\
&= \bar{\beta}_\rho \sum_{i=1}^{m_\varepsilon} \iint_{B_{2\sigma}(x_i^\varepsilon)} \Delta(\hat{u}^\varepsilon \xi) dx = \bar{\beta}_\rho \iint_{\bigcup_{i=1}^{m_\varepsilon} B_{2\sigma}(x_i^\varepsilon)} \Delta(\hat{u}^\varepsilon \xi) dx.
\end{aligned}$$

It follows from the Cauchy inequality and (5.14) that

$$\begin{aligned}
\left| \sum_{i=1}^{m_\varepsilon} I_i^3 \right| &\leq \frac{c}{\sigma^2 \ln \rho} \iint_{\bigcup_{i=1}^{m_\varepsilon} B_{2\sigma}(x_i^\varepsilon)} |\Delta(\hat{u}^\varepsilon \xi)| dx \\
&\leq \frac{c}{\sigma^2 \ln \rho} \left| \bigcup_{i=1}^{m_\varepsilon} B_{2\sigma}(x_i^\varepsilon) \right|^{\frac{1}{2}} \left(\iint_{\Omega} |\Delta(\hat{u}^\varepsilon \xi)|^2 dx \right)^{\frac{1}{2}} \\
(5.15) \quad &\leq \frac{c(m_\varepsilon \sigma^4)^{\frac{1}{2}}}{\sigma^2 \ln \rho} \|\hat{u}^\varepsilon \xi\|_{H^2(\Omega)} \leq c\mu_4^\varepsilon \varepsilon^2 \|\hat{u}^\varepsilon \xi\|_{H^2(\Omega)}.
\end{aligned}$$

We next estimate I_i^4 . Again from (5.2), for $x \in \partial B_{2\sigma}(x_i^\varepsilon)$,

$$\begin{aligned}
\frac{\partial \Delta w^\varepsilon}{\partial n} &= \frac{\partial}{\partial n} \left(\frac{1}{\sigma^2} \Delta V_\rho \left(\frac{x - x_i^\varepsilon}{\sigma} \right) \right) = \frac{2}{\sigma^2 \lambda_\rho} \frac{\partial}{\partial n} \left(2 - \frac{(\rho^2 + 4)\sigma^2}{|x - x_i^\varepsilon|^2} \right) \\
&= \frac{-2}{\sigma^2 \lambda_\rho} \frac{2(\rho^2 + 4)\sigma^2}{8\sigma^3} \stackrel{def}{=} \hat{\beta}_\rho,
\end{aligned}$$

while

$$(5.16) \quad |\hat{\beta}_\rho| \leq \frac{c}{\sigma^3 \ln \rho}.$$

It follows that

$$(5.17) \quad \sum_{i=1}^{m_\varepsilon} I_i^4 = - \sum_{i=1}^{m_\varepsilon} \int_{\partial B_{2\sigma}(x_i^\varepsilon)} \frac{\partial \Delta w^\varepsilon}{\partial n} \hat{u}^\varepsilon \xi ds = -\hat{\beta}_\rho \sum_{i=1}^{m_\varepsilon} \int_{\partial B_{2\sigma}(x_i^\varepsilon)} \hat{u}^\varepsilon \xi ds.$$

Consider the following Neumann problem

$$\begin{aligned}
\Delta^2 \phi &= q \quad \text{in } B_1, \\
\Delta \phi &= 0, \quad \frac{\partial \Delta \phi}{\partial n} = 1 \quad \text{on } \partial B_1,
\end{aligned}$$

where $q = \frac{\omega_4}{|B_1|} = 4$. Existence of a smooth solution for general Neumann problem is followed by variation inequality. But in this particular case, one can easily verify that the function

$$\phi(x) = \frac{r^4 - 3r^2}{48}, \quad r = |x|$$

provide a smooth solution. We now set, for $x \in B_{2\sigma}$

$$\phi^\varepsilon(x) = (2\sigma)^3 \phi\left(\frac{x}{2\sigma}\right).$$

Then ϕ^ε satisfies

$$(5.18) \quad \Delta^2 \phi^\varepsilon = \frac{q}{2\sigma} \quad \text{in } B_{2\sigma},$$

$$(5.19) \quad \Delta \phi^\varepsilon = 0, \quad \frac{\partial \Delta \phi^\varepsilon}{\partial n} = 1 \quad \text{on } \partial B_{2\sigma}.$$

Using the identity (3.5) with $w = \phi^\varepsilon$, $v = \hat{u}^\varepsilon \xi$, we obtain

$$\int_{\partial B_{2\sigma}(x_i^\varepsilon)} \hat{u}^\varepsilon \xi ds = \frac{q}{2\sigma} \iint_{B_{2\sigma}(x_i^\varepsilon)} \hat{u}^\varepsilon \xi dx - \iint_{B_{2\sigma}(x_i^\varepsilon)} \Delta \phi^\varepsilon \cdot \Delta (\hat{u}^\varepsilon \xi) dx.$$

Substituting this into (5.17), we derive

$$(5.20) \quad \begin{aligned} \sum_{i=1}^{m_\varepsilon} I_i^4 &= -\frac{q\hat{\beta}_\rho}{2\sigma} \sum_{i=1}^{m_\varepsilon} \iint_{B_{2\sigma}(x_i^\varepsilon)} \hat{u}^\varepsilon \xi dx + \hat{\beta}_\rho \sum_{i=1}^{m_\varepsilon} \iint_{B_{2\sigma}(x_i^\varepsilon)} \Delta \phi^\varepsilon \cdot \Delta (\hat{u}^\varepsilon \xi) dx \\ &= J_1 + J_2. \end{aligned}$$

Since $\Delta \phi^\varepsilon(x) = 2\sigma \Delta \phi\left(\frac{x}{2\sigma}\right)$, by (5.16), we obtain

$$(5.21) \quad \begin{aligned} |J_2| &\leq |\hat{\beta}_\rho| \sum_{i=1}^{m_\varepsilon} \left(\iint_{B_{2\sigma}(x_i^\varepsilon)} |\Delta \phi^\varepsilon|^2 dx \right)^{\frac{1}{2}} \left(\iint_{B_{2\sigma}(x_i^\varepsilon)} |\Delta (\hat{u}^\varepsilon \xi)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{c}{\sigma^3 \ln \rho} \sum_{i=1}^{m_\varepsilon} \sigma^3 \|\Delta \phi\|_{L^2(B_1)} \left(\iint_{B_{2\sigma}(x_i^\varepsilon)} |\Delta (\hat{u}^\varepsilon \xi)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{c}{\ln \rho} \frac{1}{\varepsilon^2} \|\Delta (\hat{u}^\varepsilon \xi)\|_{L^2(B_1)}, \end{aligned}$$

where we used (4.40). To estimate J_1 , we observe that for any function $\zeta \in H^2(B_{2\sigma})$ such that $\zeta = 0$ in B_σ , by rescaling the following Poincaré's inequality holds:

$$\iint_{B_{2\sigma}(x_i^\varepsilon)} \zeta^2 dx \leq c\sigma^2 \iint_{B_{2\sigma}(x_i^\varepsilon)} |\nabla \zeta|^2 dx.$$

Using this inequality twice we deduce that

$$\iint_{B_{2\sigma}(x_i^\varepsilon)} |\hat{u}^\varepsilon \xi|^2 dx \leq c\sigma^2 \iint_{B_{2\sigma}(x_i^\varepsilon)} |\nabla(\hat{u}^\varepsilon \xi)|^2 dx \leq c\sigma^4 \iint_{B_{2\sigma}(x_i^\varepsilon)} |\nabla^2(\hat{u}^\varepsilon \xi)|^2 dx.$$

Therefore

$$\begin{aligned} |J_1| &\leq \frac{c}{\sigma^4 \ln \rho} \sum_{i=1}^{m_\varepsilon} \iint_{B_{2\sigma}(x_i^\varepsilon)} |\hat{u}^\varepsilon \xi| dx \\ (5.22) \quad &\leq \frac{c}{\sigma^4 \ln \rho} \sum_{i=1}^{m_\varepsilon} \sigma^2 \left(\iint_{B_{2\sigma}(x_i^\varepsilon)} |\hat{u}^\varepsilon \xi|^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{c}{\ln \rho} \sum_{i=1}^{m_\varepsilon} \left(\iint_{B_{2\sigma}(x_i^\varepsilon)} |\nabla^2(\hat{u}^\varepsilon \xi)|^2 dx \right)^{\frac{1}{2}} \leq \frac{c}{\varepsilon^2 \ln \rho} \|\hat{u}^\varepsilon \xi\|_{H^2(\Omega)}. \end{aligned}$$

Combining (5.13), (5.15), (5.21) and (5.22), we find

$$(5.23) \quad \left| \sum_{i=1}^{m_\varepsilon} (I_i^1 + I_i^3 + I_i^4) \right| \leq \frac{c}{\varepsilon^2 \ln \rho} \|\hat{u}^\varepsilon \xi\|_{H^2(\Omega)} \longrightarrow 0.$$

Finally, we evaluate I_i^2 . For $x \in \partial B_\varepsilon(x_i^\varepsilon)$, we have, by (5.2),

$$\frac{\partial \Delta w^\varepsilon}{\partial n} = \frac{1}{\sigma^2} \frac{\partial}{\partial n} V_\rho \left(\frac{x - x_i^\varepsilon}{\sigma} \right) = \frac{4(\rho^2 + 1)}{\lambda_\rho \varepsilon^3}.$$

Hence

$$(5.24) \quad \sum_{i=1}^{m_\varepsilon} I_i^2 = - \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(x_i^\varepsilon)} \frac{\partial \Delta w^\varepsilon}{\partial n} \hat{u}^\varepsilon \xi ds = - \frac{4(\rho^2 + 1)}{\lambda_\rho \varepsilon^3} \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(x_i^\varepsilon)} \hat{u}^\varepsilon \xi ds.$$

Using the arguments as we did in showing (4.25), we can easily obtain

$$\varepsilon \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(x_i^\varepsilon)} \hat{u}^\varepsilon \xi \, ds \longrightarrow \omega_4 \iint_{\Omega} Pu \xi \, dx, \quad \text{as } \varepsilon \longrightarrow 0.$$

Note that

$$-\frac{4(\rho^2 + 1)}{\lambda_\rho \varepsilon^4} = -\frac{4(\rho^2 + 1)}{\left[(\rho^2 - 4) - (\rho^2 + 4) \ln \frac{\rho}{2}\right] \varepsilon^4} \longrightarrow 4\mu.$$

Therefore, it follows from (5.24) that

$$\sum_{i=1}^{m_\varepsilon} I_i^2 \longrightarrow 4\mu\omega_4 \iint_{\Omega} Pu \xi \, dx.$$

Combing this with (5.10) and (5.23), we obtain from (5.11)

$$\iint_{\Omega} \Delta(\hat{u}^\varepsilon \xi) \cdot \Delta w^\varepsilon \, dx \longrightarrow 4\mu\omega_4 \iint_{\Omega} Pu \xi \, dx.$$

The assertion for $\mu < \infty$ follows.

The case $\mu = \infty$ can be handled in the way similar to the proof of Theorem 4.1 (by taking $\tilde{w}^\varepsilon = \frac{w^\varepsilon}{\mu^\varepsilon}$ as the test function in (5.10), instead of w^ε). We skip the details.

We conclude this paper by the corrector results for $N = 4$.

THEOREM 5.2. *Let $N = 4$. Then the assertion (4.28) in Theorem 4.2 remains true for any $p < 2$. Furthermore, if $u \in C^2$, then (4.28) holds for $p = 2$.*

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