MICROLOCAL ANALYSIS ON MORREY SPACES

By

Michael E. Taylor

IMA Preprint Series # 1322

July 1995

**INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA**

**514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455**

Microlocal Analysis on Morrey Spaces

MICHAEL E. TAYLOR¹

*University of North Carolina
Chapel Hill NC 27599*

Contents

1. Morrey scales
 2. A class of second order elliptic systems in divergence form
 3. Connections with Ricci curvature bounds
 4. Commutator estimates on Morrey scales
 5. Operators with vmo coefficients
 6. Morrey space estimates for wave equations
 7. Conormal spaces and variants
-
- A. A Schwartz kernel estimate
 - B. Another proof of Proposition 1.2

¹ Research partially supported by NSF grant.

IMA Workshop: Singularities and Waves, April 9-14, 1995.

Introduction

The spaces now called Morrey spaces were introduced by C.B. Morrey to study regularity properties of solutions to quasilinear elliptic PDE, but since then they have been useful in other areas of PDE. Before saying more on this, let us first define the Morrey spaces $M_q^p(\mathbb{R}^n)$.

If $1 \leq q \leq p < \infty$ and $f \in L_{\text{loc}}^q(\mathbb{R}^n)$, we say $f \in M_q^p(\mathbb{R}^n)$ provided

$$(0.1) \quad R^{-n} \int_{B_R} |f(x)|^q dx \leq C R^{-nq/p},$$

for all balls B_R of radius $R \leq 1$ in $\mathbb{R}^n$. If we set $\delta_R f(x) = f(Rx)$, the left side of (0.1) is equal to $\int_{B_1} |\delta_R f(x)|^q dx$, so an equivalent condition is

$$(0.2) \quad \|\delta_R f\|_{L^q(B_1)} \leq C' R^{-n/p},$$

for all balls B_1 of radius 1, and for all $R \in (0, 1]$. It follows from Hölder's inequality that

$$L_{\text{unif}}^p(\mathbb{R}^n) = M_p^p(\mathbb{R}^n) \subset M_q^p(\mathbb{R}^n) \subset M_1^p(\mathbb{R}^n).$$

We also say $f \in \mathcal{M}_q^p(\mathbb{R}^n)$ provided (0.1) holds, for all $R < \infty$.

Morrey used these spaces to study inhomogeneous equations

$$(0.3) \quad \sum \partial_j a^{jk}(x) \partial_k u = f,$$

on a domain in $\mathbb{R}^n$, when $a^{jk}(x)$ are bounded and measurable and (0.3) is elliptic. Using a clever dilation argument and the DeGiorgi-Nash-Moser estimates on solutions to the homogeneous version of (0.3), Morrey was able to show that, if $p = n + \delta$, with small $\delta > 0$, and $f = \sum \partial_j g_j$, with $g_j \in L^p$, then $\nabla u \in M_2^p$. Hölder continuity of the solution u is then a consequence of Morrey's lemma:

$$(0.4) \quad \nabla u \in M_1^p(\mathbb{R}^n), \quad p > n \implies u \in C^r(\mathbb{R}^n), \quad r = 1 - \frac{n}{p}.$$

In fact, (0.4) is a special case of the following:

$$(0.5) \quad M_1^p(\mathbb{R}^n) \subset C_*^{-n/p}(\mathbb{R}^n).$$

Here, $C_*^r(\mathbb{R}^n)$ is a Zygmund space, which can be defined as follows. Pick $\Psi_0 \in C_0^\infty(\mathbb{R}^n)$, such that $\Psi_0(\xi) = 1$ for $|\xi| \leq 1$, 0 for $|\xi| \geq 2$, set $\Psi_k(\xi) = \Psi_0(2^{-k}\xi)$, and then set $\psi_0 = \Psi_0$, $\psi_k = \Psi_k - \Psi_{k-1}$ for $k \geq 1$. The family

$\{\psi_k\}$ is called a Littlewood-Paley partition of unity. For any $r \in \mathbb{R}$, one defines

$$(0.6) \quad C_*^r(\mathbb{R}^n) = \{u : \|\psi_k(D)u\|_{L^\infty} \leq C2^{-kr}\}.$$

It is not hard to show that, for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $C^r(\mathbb{R}^n) = C_*^r(\mathbb{R}^n)$. To see that (0.5) holds, one can check from the definition (0.1) that

$$(0.7) \quad \begin{aligned} f \in M_1^p(\mathbb{R}^n) &\iff \\ \|e^{t\Delta}|f|\|_{L^\infty} \leq Ct^{-n/2p} &\implies \|e^{t\Delta}f\|_{L^\infty} \leq Ct^{-n/2p}, \end{aligned}$$

for $t \in (0, 1]$. From this one readily deduces that, if $u \in M_1^p(\mathbb{R}^n)$, then (0.6) holds, with $r = -n/p$.

In recent times, Morrey spaces have been incorporated into techniques of microlocal analysis, and it is our purpose to carry out this development further in this article.

In §1 we recall some known results about the action of pseudodifferential operators (ψ DOs) on Morrey spaces. We define ‘Morrey scales,’ spaces $M_q^{p,s}(\mathbb{R}^n)$, for $s \in \mathbb{R}$, and make note of the consequent action of ψ DOs on these spaces. We also extend to Morrey scales E.Stein’s theorem on the action of ψ DOs with symbols in $S_{1,1}^m$.

This is useful for applications of the paradifferential operator calculus of J.M.Bony and Y.Meyer. We recall Meyer’s formula for the action of a smooth function F on a function u (possibly taking values in $\mathbb{R}^\ell$). More details can be found in [Mey], or in [T1]. We have

$$(0.8) \quad F(u) = M(u; x, D)u + F(u_0),$$

where $u_0 = \Psi_0(D)u$ and

$$(0.9) \quad \begin{aligned} M(u; x, \xi) &= \sum_{k \geq 0} m_k(x) \psi_{k+1}(\xi), \\ m_k(x) &= \int_0^1 F'(\Psi_k(D)u + \tau \psi_{k+1}(D)) d\tau. \end{aligned}$$

A straightforward calculation using the chain rule shows that

$$(0.10) \quad u \in L^\infty(\mathbb{R}^n) \implies M(u; x, \xi) \in S_{1,1}^0.$$

We recall that, for $0 \leq \delta \leq 1$, $m \in \mathbb{R}$,

$$(0.11) \quad p(x, \xi) \in S_{1,\delta}^m \iff |D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|+\delta|\beta|},$$

where $\langle \xi \rangle^2 = 1 + |\xi|^2$. If $p(x, \xi) \in S_{1,0}^m$ has an asymptotic expansion in terms homogeneous of degree $m - j$, $j \geq 0$, we say $p(x, \xi) \in S^m$, or sometimes, for emphasis, $p(x, \xi) \in S_{cl}^m$.

A further ingredient in paradifferential operator calculus is the process of ‘symbol smoothing.’ Given a symbol $M(x, \xi) \in S_{1,1}^m$, write

$$(0.12) \quad M(x, \xi) = M^\#(x, \xi) + M^b(x, \xi),$$

where, with $J_k = \Psi_0(\varepsilon_k D_x)$, and $\varepsilon_k \searrow 0$,

$$(0.13) \quad M^\#(x, \xi) = \sum_{k \geq 0} J_k M(x, \xi) \psi_{k+1}(\xi).$$

Choices most frequently made are

$$(0.14) \quad J_k = \Psi_0(2^{-\delta k} D_x), \quad \text{or} \quad J_k = \Psi_{k-3}(D_x).$$

In these respective cases, one gets

$$(0.15) \quad M^\#(x, \xi) \in S_{1,\delta}^m, \quad \text{or} \quad M^\#(x, \xi) \in \mathcal{B}S_{1,1}^m,$$

where

$$(0.16) \quad \mathcal{B}S_{1,1}^m = \{p(x, \xi) \in S_{1,1}^m : \hat{p}(\eta, \xi) \text{ is supported in } |\eta| \leq \rho|\xi|\},$$

for some $\rho < 1$. For fixed $\rho < 1$, the class (0.16) will be denoted $\mathcal{B}_\rho S_{1,1}^m$. One can show that, if $M(x, \xi) = M(u; x, \xi)$ is given by (0.9), then, for $r > 0$,

$$(0.17) \quad u \in C^r \implies M^b(x, \xi) \in S_{1,\delta}^{-r\delta}, \quad \text{or} \quad \mathcal{B}S_{1,1}^{-r},$$

in the two respective cases of (0.14). Thus, the action of ψ DOs with symbols in these various classes are significant for nonlinear analysis. For example, we extend to Morrey scales Moser estimates on nonlinear functions $F(u)$, and also Rauch’s lemma.

In §2 we apply Morrey space analysis in its traditional context: analysis of quasilinear elliptic PDE. We analyze a family of such equations, containing as an important example the system relating the metric tensor of a Riemannian manifold to its Ricci tensor, in harmonic coordinates. The analysis involves a combination of paradifferential operator calculus and integration by parts arguments. The specific application to the Ricci tensor is given in §3.

In §4 we resume the internal development of analysis on Morrey spaces. We extend a commutator estimate of T.Kato and G.Ponce [KP] to the Morrey scale setting. We also extend to ‘microlocal’ Morrey scales a commutator estimate of M.Beals [Be], and we recall some known results on commutators $[P, M_f]$, when $f \in \text{bmo}$, and sketch a proof of this given in [AT]. One ingredient in these commutator estimates is the decomposition

$$(0.18) \quad fv = T_f v + T_v f + R(f, v),$$

where

$$(0.19) \quad T_f v = \sum_{k \geq 4} \Psi_{k-4}(D) f \cdot \psi_k(D) v$$

is Bony's paraproduct. This is an example of (0.8)-(0.14), with $F(u_1, u_2) = u_1 u_2$.

In §5 we recall and extend some work of [CFL1-2] and [DR1-2] on a class of pseudodifferential operators whose symbols $p(x, \xi)$ are bmo in x , and a subalgebra whose symbols have x -dependence in $\text{vmo} \cap L^\infty$. Here, $\text{bmo}(\mathbb{R}^n)$ is the 'local' version of $\text{BMO}(\mathbb{R}^n)$, with norm

$$(0.20) \quad \|u\|_{\text{bmo}} = \|u\|_{\text{BMO}} + \|\Psi_0(D)u\|_{L^\infty}.$$

The seminorm $\|u\|_{\text{BMO}}$ is given by $\sup_r \eta_u(r)$, where

$$(10.21) \quad \eta_u(r) = \sup_{\text{diam } B = \rho \leq r} \rho^{-n} \int_B |u(x) - u_B| dx.$$

Here, B runs over all balls of diameter ρ , and u_B stands for the mean value of u on B . The subspace $\text{VMO}(\mathbb{R}^n)$ consists of $u \in \text{BMO}$ such that $\eta_u(r) \rightarrow 0$ as $r \rightarrow 0$, and vmo consists of $u \in \text{VMO}$ such that $\Psi_0(D)u \in L^\infty(\mathbb{R}^n)$. It is known (cf. [Sar]; see also [CFL]) that VMO is the closure in BMO of the space of uniformly continuous functions on $\mathbb{R}^n$, or equivalently of the space

$$(0.22) \quad \mathcal{B}^\infty = \{u \in L^\infty(\mathbb{R}^n) : D^\alpha u \in L^\infty(\mathbb{R}^n), \forall \alpha\}.$$

Similarly, vmo is the closure of $\mathcal{B}^\infty$ in bmo . Clearly $\text{vmo} \cap L^\infty = \text{VMO} \cap L^\infty$.

In §6 we derive some Morrey space estimates for solutions to wave equations. In §7 we discuss spaces of conormal distributions and variants.

1. Morrey scales

Since the work [P] it has been known that, if $0 \leq \delta < 1$,

$$(1.1) \quad P \in OPS_{1,\delta}^0(\mathbb{R}^n) \implies P : M_q^p(\mathbb{R}^n) \rightarrow M_q^p(\mathbb{R}^n), \quad 1 < q \leq p < \infty.$$

Thus, when $1 < q \leq p < \infty$, it is reasonable to consider the scale of spaces

$$(1.2) \quad M_q^{p,s}(\mathbb{R}^n) = \Lambda^{-s} M_q^p(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \Lambda^s u \in M_q^p(\mathbb{R}^n)\},$$

where

$$(1.3) \quad (\Lambda^s u)^\wedge(\xi) = (1 + |\xi|^2)^s \widehat{u}(\xi).$$

Clearly the standard Sobolev space $H^{s,p}(\mathbb{R}^n) \subset M_q^{p,s}(\mathbb{R}^n)$. It follows from (1.1) that, given $s, m \in \mathbb{R}$, $\delta \in [0, 1)$, $1 < q \leq p < \infty$,

$$(1.4) \quad P \in OPS_{1,\delta}^m(\mathbb{R}^n) \implies P : M_q^{p,s}(\mathbb{R}^n) \rightarrow M_q^{p,s-m}(\mathbb{R}^n).$$

Since such P map C_*^s to C_*^{s-m} for all $s \in \mathbb{R}$, we see that (0.5) implies

$$(1.5) \quad M_q^{p,s}(\mathbb{R}^n) \subset C_*^{s-n/p}(\mathbb{R}^n).$$

Similarly, we can define

$$(1.6) \quad \mathcal{M}_q^{p,s}(\mathbb{R}^n) = \Lambda^{-s} \mathcal{M}_q^p(\mathbb{R}^n),$$

and we have

$$(1.7) \quad P \in OPS_{1,\delta}^m(\mathbb{R}^n) \implies P : \mathcal{M}_q^{p,s-m}(\mathbb{R}^n),$$

provided $1 < q \leq p < \infty$. We will mainly use the spaces $M_q^{p,s}(\mathbb{R}^n)$, and occasionally refer to the fact that analogous results hold for $\mathcal{M}_q^{p,s}(\mathbb{R}^n)$.

We mention some further results, which will be useful in our development. The following proposition was established in Theorem 3.8 of [T2]. A number of cases had appeared earlier, e.g., in [Ad], [CF], [P].

Proposition 1.1. *If $1 < p_1 < p_2 < \infty$ and*

$$(1.8) \quad m = -\beta = -n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) < 0,$$

then, for $1 < q_1 \leq p_1 < \infty$, $0 \leq \delta < 1$,

$$(1.9) \quad P \in OPS_{1,\delta}^m(\mathbb{R}^n) \implies P : M_{q_1}^{p_1}(\mathbb{R}^n) \rightarrow M_{q_2}^{p_2}(\mathbb{R}^n),$$

with

$$(1.10) \quad \frac{q_2}{q_1} = \frac{p_2}{p_1}, \quad \text{if also } p_1 \leq n,$$

and otherwise (1.9) holds provided $q_2/q_1 < p_2/p_1$. Furthermore,

$$(1.11) \quad P \in OPS_{1,\delta}^m(\mathbb{R}^n) \implies P : M_1^{p_1}(\mathbb{R}^n) \rightarrow M_{q_2}^{p_2}(\mathbb{R}^n), \quad \text{for } q_2 < \frac{p_2}{p_1}.$$

In addition, (1.4), (1.9), and (1.11) hold for $P \in OPBS_{1,1}^m(\mathbb{R}^n)$; in particular,

$$(1.12) \quad P \in OPBS_{1,1}^m(\mathbb{R}^n) \implies P : M_q^{p,s}(\mathbb{R}^n) \rightarrow M_q^{p,s-m}(\mathbb{R}^n),$$

for $1 < q \leq p < \infty$, $m, s \in \mathbb{R}$.

It follows that, for p_j and q_j related as above,

$$(1.13) \quad M_{q_1}^{p_1, s+|m|}(\mathbb{R}^n) \subset M_{q_2}^{p_2, s}(\mathbb{R}^n).$$

Another general result established in [T2] is:

Proposition 1.2. *Assume the Schwartz kernel $k(x, y)$ of T satisfies*

$$(1.14) \quad |k(x, y)| \leq C_M |x - y|^{-n} (1 + |x - y|)^{-M}$$

for sufficiently large M . Then, if $1 < q \leq p < \infty$,

$$(1.15) \quad T : L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n) \implies T : M_q^p(\mathbb{R}^n) \rightarrow M_q^p(\mathbb{R}^n).$$

We include another proof of Proposition 1.2 in Appendix B. Proposition 1.2 implies (1.1) and (1.12). Another application of Proposition 1.2 is the following result, noted in [T2]:

Proposition 1.3. *Given $k \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, we have*

$$(1.16) \quad P \in OPS_{1,1}^{-k}(\mathbb{R}^n) \implies D^\alpha P : M_q^p(\mathbb{R}^n) \rightarrow M_q^p(\mathbb{R}^n),$$

for $|\alpha| \leq k$, $1 < q \leq p < \infty$.

That (1.16) holds follows from the fact that $D^\alpha P : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ and that

$$(1.17) \quad D^\alpha P(x, D) = \sum_{\beta+\gamma=\alpha} P_\beta(x, D) D^\gamma \in OPS_{1,1}^0,$$

so $T = D^\alpha P$ has Schwartz kernel satisfying (1.14). This fact, in conjunction with (1.5), applied to $D^\alpha \Lambda^{-k} \in OPS_{1,0}^0$, shows that, for $k \in \mathbb{Z}^+$,

$$(1.18) \quad P \in OPS_{1,1}^{-k}(\mathbb{R}^n) \implies \Lambda^k P : M_q^p(\mathbb{R}^n) \rightarrow M_q^p(\mathbb{R}^n).$$

More generally, one can replace Λ^k in (1.18) by any $A \in OPS_{1,\delta}^k$, $\delta \in [0, 1)$. We can rewrite (1.18) as

$$(1.19) \quad P \in OPS_{1,1}^0(\mathbb{R}^n) \implies \Lambda^k P \Lambda^{-k} : M_q^p(\mathbb{R}^n) \rightarrow M_q^p(\mathbb{R}^n),$$

and more generally

$$(1.20) \quad P \in OPS_{1,1}^0(\mathbb{R}^n) \implies \Lambda^{k+i\sigma} P \Lambda^{-(k+i\sigma)} : M_q^p(\mathbb{R}^n) \rightarrow M_q^p(\mathbb{R}^n),$$

for $k \in \mathbb{Z}^+$, $\sigma \in \mathbb{R}$. For each $k \in \mathbb{Z}^+$, the family of operators has norm polynomially bounded in σ . It follows that

$$(1.21) \quad P \in OPS_{1,1}^0, s \in [1, \infty) \implies \Lambda^s P \Lambda^{-s} : M_q^p(\mathbb{R}^n) \rightarrow M_q^p(\mathbb{R}^n).$$

In fact, we improve Proposition 1.3 to the following.

Proposition 1.4. *Given $s > 0$, $\delta \in (0, 1]$, and $1 < q \leq p < \infty$, we have*

$$(1.22) \quad A \in OPS_{1,\delta}^s, P \in OPS_{1,1}^{-s} \implies AP : M_q^p(\mathbb{R}^n) \rightarrow M_q^p(\mathbb{R}^n).$$

Proof. It suffices to show that, for $s > 0$,

$$(1.23) \quad P \in OPS_{1,1}^{-s} \implies \Lambda^s P : M_q^p(\mathbb{R}^n) \rightarrow M_q^p(\mathbb{R}^n),$$

and, granted (1.21), we need only consider the cases $0 < s < 1$. We want to apply Proposition 1.2 to $T = \Lambda^s P$, and we know that $T : L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ under our hypotheses. Thus we need to verify that the Schwartz kernel of T satisfies (1.14). That this holds for $|x - y| \geq 1$ is easy. The fact that it holds on the region $|x - y| \leq 1$ is proved in Appendix A.

Proposition 1.4 can be rephrased in the language of Morrey scales as

$$(1.24) \quad P \in OPS_{1,1}^m(\mathbb{R}^n) \implies P : M_q^{p,s}(\mathbb{R}^n) \rightarrow M_q^{p,s-m}(\mathbb{R}^n), \quad s - m > 0,$$

assuming $1 < q \leq p < \infty$.

If $s > n/p$ and $1 < q \leq p$, then $M_q^{p,s}(\mathbb{R}^n)$ is an algebra. In fact, one can apply a general smooth nonlinear function F to (a vector-valued) $u \in M_q^{p,s}$, and obtain $F(u) \in M_q^{p,s}$, with Moser-type estimates. To see this, write $F(u)$ in terms of a paradifferential operator:

$$(1.25) \quad F(u) = M(u; x, D)u + R(u),$$

as in (0.8)-(0.9), with $R(u) \in C^\infty$, and, by (0.10),

$$(1.26) \quad u \in C^0 \implies M(u; x, \xi) \in S_{1,1}^0.$$

Using (1.24), we obtain:

Proposition 1.5. *If $u \in M_q^{p,s}(\mathbb{R}^n)$ with $1 < q \leq p$ and $s > n/p$, then, given smooth F , we have $F(u) \in M_q^{p,s}$ and*

$$(1.27) \quad \|F(u)\|_{M_q^{p,s}} \leq C_F(\|u\|_{L^\infty}) \left(1 + \|u\|_{M_q^{p,s}}\right).$$

If also $v \in M_q^{p,s}(\mathbb{R}^n)$, then

$$(1.28) \quad \|uv\|_{M_q^{p,s}} \leq C \left[\|u\|_{L^\infty} \|v\|_{M_q^{p,s}} + \|u\|_{M_q^{p,s}} \|v\|_{L^\infty} \right].$$

If $s \leq n/p$, such estimates fail, unless we also assume that $u \in L^\infty$. If $s = 0$, what we have in place of (1.28) is

$$(1.29) \quad v, w \in M_q^p(\mathbb{R}^n) \implies vw \in M_{q/2}^{p/2}(\mathbb{R}^n),$$

provided $2 \leq q \leq p$. The following, while not sharp, will be useful in §2.

Proposition 1.6. *Let $p > 2$, $q \in (2, p]$, $0 \leq \sigma \leq 1$. Then*

$$(1.30) \quad v, w \in M_q^{p, \sigma} \implies vw \in M_{q/2}^{p/2, \sigma}.$$

Proof. Say $v = \Lambda^{-\sigma} f$, $w = \Lambda^{-\sigma} g$, with $f, g \in M_q^p$. We seek an estimate of the form

$$(1.31) \quad \sup_{z \in \Omega} \|e^{z^2} \Lambda^z (\Lambda^{-z} f \cdot \Lambda^{-z} g)\|_{M_{q/2}^{p/2}} \leq C \|f\|_{M_q^p} \|g\|_{M_q^p},$$

where

$$\Omega = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}.$$

It suffices to establish this estimate for $f, g \in \mathcal{S}(\mathbb{R}^n)$. Note that we are taking the norm of a holomorphic function, so it suffices to check $z = iy$ and $z = 1 + iy$, $y \in \mathbb{R}$. We have

$$(1.32) \quad \|e^{-y^2} \Lambda^{iy} (\Lambda^{-iy} f \cdot \Lambda^{-iy} g)\|_{M_{q/2}^{p/2}} \leq C' \|f \cdot g\|_{M_{q/2}^{p/2}} \leq C \|f\|_{M_q^p} \|g\|_{M_q^p},$$

by the boundedness of $\langle y \rangle^{-K} \Lambda^{iy}$ in $\mathcal{L}(M_{q/2}^{p/2})$ and in $\mathcal{L}(M_q^p)$. Similarly,

$$(1.33) \quad \begin{aligned} & \|e^{1-y^2} \Lambda^{1+iy} (\Lambda^{-1-iy} f \cdot \Lambda^{-1-iy} g)\|_{M_{q/2}^{p/2}} \\ & \leq C \sum_{j=1}^n \|e^{-\frac{y^2}{2}} \partial_j (\Lambda^{-1-iy} f \cdot \Lambda^{-1-iy} g)\|_{M_{q/2}^{p/2}} \\ & \quad + C \|e^{-\frac{y^2}{2}} \Lambda^{-1-iy} f \cdot \Lambda^{-1-iy} g\|_{M_{q/2}^{p/2}}. \end{aligned}$$

Now using

$$(1.34) \quad \begin{aligned} \partial_j (\Lambda^{-1-iy} f \cdot \Lambda^{-1-iy} g) &= (\partial_j \Lambda^{-1-iy} f) \cdot (\Lambda^{-1-iy} g) \\ &\quad + (\Lambda^{-1-iy} f) \cdot (\partial_j \Lambda^{-1-iy} g), \end{aligned}$$

plus $\partial_j \Lambda^{-1} \in OPS^0$, we easily bound (1.33) by $C \|f\|_{M_q^p} \|g\|_{M_q^p}$. This completes the proof.

As in the case of Sobolev spaces, we can define the notion of u belonging microlocally to a space $M_q^{p,s}$. Assume $1 < q \leq p < \infty$, $s \in \mathbb{R}$. Let Γ be a closed conic subset of $T^*\mathbb{R}^n \setminus 0$. We say that

$$(1.35) \quad u \in M_q^{p,s}{}_{ml}(\Gamma) \iff Pu \in M_q^{p,s}(\mathbb{R}^n),$$

for some $P \in OPS^0(\mathbb{R}^n)$ which is elliptic on some conic neighborhood of Γ . There is the following variant of Rauch's lemma:

Proposition 1.7. Assume $u \in C^r \cap M_q^{p,s}$, with $r, s > 0$, $1 < q \leq p < \infty$. If F is smooth, then

$$(1.36) \quad u \in M_q^{p,\sigma}{}_{ml}(\Gamma) \implies F(u) \in M_q^{p,\sigma}{}_{ml}(\Gamma), \quad \text{provided } s \leq \sigma < s + r.$$

Proof. As in (0.12)-(0.17), write, mod C^∞ ,

$$(1.37) \quad F(u) = M^\# u + M^b u, \quad M^\# \in OPS_{1,\delta}^0, \quad M^b \in OPS_{1,1}^{-r\delta},$$

for any $\delta < 1$. Then $M^\# u \in M_q^{p,\sigma}{}_{ml}(\Gamma)$, by (1.1) and symbol calculus, while, by (1.24), $M^b u \in M_q^{p,s+r\delta}$. This proves (1.36).

This result can be sharpened, in a way parallel to the treatment of [Mey] for Sobolev spaces. In the decomposition $M = M^\# + M^b$, choose the second method of (0.14); then $M^b \in OPS_{1,1}^{-r}$. Furthermore, if $r > 0$, $u \in C^r \implies M^\# \in OPB^r S_{1,1}^0$, where $B^r S_{1,1}^m$ consists of $p(x, \xi) \in \mathcal{BS}_{1,1}^m$ satisfying the additional conditions

$$(1.38) \quad \begin{aligned} & \|D_\xi^\alpha p(\cdot, \xi)\|_{C^r} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|}, \\ & |D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|+\delta(|\beta|-r)}, \quad \text{for } |\beta| > r. \end{aligned}$$

Operator calculus then yields the following:

Lemma 1.8. If $p(x, \xi) \in B^r S_{1,1}^m$ and $u \in M_q^{p,s}$, with $1 < q \leq p < \infty$, $s > 0$, then

$$(1.39) \quad u \in M_q^{p,\sigma}{}_{ml}(\Gamma) \implies p(x, D)u \in M_q^{p,\sigma-m}{}_{ml}(\Gamma), \quad s \leq \sigma \leq s + r.$$

The proof is parallel to that in [Mey]; see also Proposition 3.4.D in [T1]. With this in hand, one can now replace the condition $s \leq \sigma < s + r$ in (1.36) by $s \leq \sigma \leq s + r$.

2. A class of second order elliptic systems in divergence form

Here we study regularity of solutions to elliptic equations of the form

$$(2.1) \quad \sum \partial_j a_{jk}(x, u) \partial_k u + B(x, u, \nabla u) = f.$$

This can be an $M \times M$ system, with u taking values in $\mathbb{R}^M$. We assume that $a_{jk}(x, u)$ and $B(x, u, \zeta)$ are smooth in (x, u) , and that $B(x, u, \zeta)$ is a quadratic form in ζ , or more generally satisfies

$$|B(x, u, \zeta)| \leq C \langle \zeta \rangle^2.$$

Proposition 2.1. *Assume that a solution u to (2.1) satisfies*

$$(2.2) \quad \nabla u \in M_2^q, \text{ for some } q > n, \text{ hence } u \in C^r,$$

for some $r \in (0, 1)$, and

$$(2.3) \quad f \in M_s^{p,-1},$$

for some $p \in (q, \infty)$, $s \in [2, p]$. Then

$$(2.4) \quad \nabla u \in M_s^p,$$

If $s = p$, then (2.4) is the conclusion of Proposition 2.2.I of [T1], but the hypothesis (2.2) above is weaker than the corresponding hypothesis made in [T1]. The case $f = 0$ of Proposition 2.1 is also contained in Theorem 4.1 of [Sch], when $\dim \Omega = 2$.

To begin the proof of Proposition 2.1, we write

$$(2.5) \quad \sum_k a_{jk}(x, u) \partial_k u = A_j(u; x, D)u,$$

mod C^∞ , with

$$(2.6) \quad A_j(u; x, \xi) \in C^r S_{1,0}^1 \cap S_{1,1}^1 + S_{1,1}^{1-r},$$

as established in (3.3.23) of [T1], and hence, by (3.3.25) of [T1], given $\delta \in (0, 1)$,

$$(2.7) \quad \begin{aligned} A_j(u; x, \xi) &= A_j^\#(x, \xi) + A_j^b(x, \xi), \\ A_j^\#(x, \xi) &\in S_{1,\delta}^1, \quad A_j^b(x, \xi) \in S_{1,1}^{1-r\delta}. \end{aligned}$$

It follows that we can write

$$(2.8) \quad \sum \partial_j a_{jk}(x, u) \partial_k u = P^\# u + P^b u,$$

with

$$(2.9) \quad P^\# = \sum \partial_j A_j^\#(x, D) \in OPS_{1,\delta}^2, \quad \text{elliptic},$$

and

$$(2.10) \quad P^b = \sum \partial_j A_j^b(x, D).$$

By Proposition 1.4, we have

$$(2.11) \quad \Lambda^{r\delta-1} P^b \Lambda^{-1} : M_{q'}^{p'} \longrightarrow M_{q'}^{p'}, \quad 1 < q' \leq p' < \infty.$$

In particular,

$$(2.12) \quad \nabla u \in M_2^q \implies P^b u \in M_2^{q, -1+r\delta}.$$

Now, if

$$(2.13) \quad E^\# \in OPS_{1,\delta}^{-2}$$

denotes a parametrix of $P^\#$, we have, mod C^∞ ,

$$(2.14) \quad u = E^\# f - E^\# B(x, u, \nabla u) - E^\# P^b u,$$

and we see that, under the hypothesis (2.2), we have some control over the last term:

$$(2.15) \quad E^\# P^b u \in M_2^{q, 1+r\delta}.$$

Note also that, under our hypothesis on $B(x, u, \zeta)$,

$$(2.16) \quad \nabla u \in M_2^q \implies B(x, u, \nabla u) \in M_1^{q/2}.$$

Now, by (1.12),

$$(2.17) \quad \Lambda^{-1} : M_1^{q/2} \longrightarrow M_2^{\tilde{p}}, \quad \tilde{p} = \frac{q}{2 - \frac{q}{n}}, \quad \text{if } n < q < 2n,$$

while the range is contained in C^σ for some $\sigma > 0$ if $q > 2n$, by Morrey's Lemma, and the range is contained in BMO if $q = 2n$. Thus

$$(2.18) \quad E^\# B(x, u, \nabla u) \in M_2^{\tilde{p}, 1},$$

with $\tilde{p} = q/(2 - \frac{q}{n})$ if $q < 2n$ and for all $\tilde{p} < \infty$ if $q \geq 2n$. Note that $\tilde{p} > q(1 + \frac{a}{n})$ if $q = n + a$. This treats the middle term on the right side of (2.14). Of course, the hypothesis (2.3) yields

$$(2.19) \quad E^\# f \in M_s^{p, 1},$$

which is just where we want to place u .

We can draw from (2.15) a conclusion parallel to (2.18)-(2.19), using

$$(2.20) \quad \Lambda^{-r\delta} : M_2^q \longrightarrow M_2^{\tilde{q}}, \quad \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{r\delta}{n},$$

which follows from (1.9). We then have

$$(2.21) \quad E^\# P^b u \in M_2^{\tilde{q}, 1}.$$

Having thus analyzed the three terms on the right side of (2.14), we have

$$(2.22) \quad u \in M_2^{q^\#, 1}, \quad q^\# = \min(\tilde{p}, p, \tilde{q}).$$

Iterating this argument a finite number of times, we get

$$(2.23) \quad u \in M_2^{p, 1}.$$

If $s = 2$ in (2.3), our work is done.

If $s \in (2, p]$, we can proceed with an argument similar to that above. Details are omitted.

We next establish the following generalization of Proposition 2.1.

Proposition 2.2. Assume that $\nabla u \in M_2^q$ for some $q > n$, that u satisfies (2.1), and that

$$(2.24) \quad f \in M_s^{p,\tau-1},$$

for some $p \in (q, \infty)$, $s \in [2, p]$, $\tau \geq 0$. Then

$$(2.25) \quad \nabla u \in M_s^{p,\tau}.$$

Proof. Note that Proposition 2.1 handles the case $\tau = 0$. Thus we can assume

$$(2.26) \quad u \in M_s^{p,\rho},$$

with $\rho = 1$. We want to show that (2.26) holds with $\rho = 1 + \tau$. As before, we make use of (2.14). The hypothesis (2.24) yields

$$(2.27) \quad E^\# f \in M_s^{p,\tau+1},$$

which is where we want to place u . Whenever (2.26) holds, with $\rho \geq 1$, we have

$$(2.28) \quad E^\# P^b u \in M_s^{p,\rho+\tau\delta},$$

parallel to (2.15). This is a desirable gain in regularity. It remains to examine the term $E^\# B(x, u, \nabla u)$ in (2.14).

To begin,

$$(2.29) \quad u \in M_s^{p,1} \implies B(x, u, \nabla u) \in M_{s/2}^{p/2}.$$

Thus, by Proposition 1.1, for arbitrarily small $\varepsilon > 0$,

$$(2.30) \quad \Lambda^{-\mu} B(x, u, \nabla u) \in M_s^p, \quad \mu = \frac{n}{p} + \varepsilon.$$

Since $p > n$, we can take $\mu < 1$. Hence

$$(2.31) \quad u \in M_s^{p,1} \implies E^\# B(x, u, \nabla u) \in M_s^{p,1+\sigma}, \quad \forall \sigma < 1 - \frac{n}{p}.$$

We now prove Proposition 2.2 for $0 < \tau \leq 1$. First assume $s > 2$; use Proposition 1.6 to get, for any $\beta \in (0, 1]$,

$$(2.32) \quad u \in M_s^{p,1+\beta} \implies B(x, u, \nabla u) \in M_{s/2}^{p/2,\beta},$$

given that $B(x, u, \nabla u)$ is a quadratic form in ∇u . This time, an application of Proposition 1.1 to the analysis of $E^\# B(x, u, \nabla u)$ yields

$$(2.33) \quad u \in M_s^{p,1+\beta} \implies E^\# B(x, u, \nabla u) \in M_s^{p,1+\beta+\sigma}, \quad \forall \sigma < 1 - \frac{n}{p},$$

given $\beta \in (0, 1]$, $p > n$, provided $s > 2$. On the other hand, if $s = 2$, the arguments (2.27)-(2.31) yield $u \in M_s^{p, 1+\beta}$ for $\beta = \min \{\tau, 1 - \frac{n}{p} - \varepsilon\}$, $\forall \varepsilon > 0$. Then, use

$$M_2^{p, 1+\beta} \subset M_{2+\varepsilon}^{p, 1+\beta-\delta}$$

for some small positive ε , δ , and again apply the argument above. Thus we extend the implication (2.33) to the case $s = 2$.

This is a desirable gain in regularity. Thus a finite iteration of the arguments above establishes Proposition 2.2, if $\tau \in [0, 1]$.

On the other hand, by Proposition 1.5, if $s > 2$,

$$(2.34) \quad \begin{aligned} u \in M^{p, 1+\sigma}, \quad \sigma > \frac{n}{p} &\implies B(x, u, \nabla u) \in M_s^{p, \sigma} \\ &\implies E^\# B(x, u, \nabla u) \in M_s^{p, 2+\sigma}. \end{aligned}$$

Thus, if we have $u \in M_s^{p, 1+\sigma}$ for some $\sigma > n/p$, a finite number of iterations of this argument will yield the desired conclusion (2.26), provided $s > 2$. If $s = 2$, use

$$(2.35) \quad M_2^{p, 1+\sigma} \subset M_{2+\varepsilon}^{p, 1+\sigma-\delta}$$

for small $\varepsilon > 0$, $\delta > 0$, and again apply Proposition 1.5 to get

$$(2.36) \quad u \in M_2^{p, 1+\sigma}, \quad \sigma > \frac{n}{p} \implies E^\# B(x, u, \nabla u) \in M_2^{p, 2+\sigma-\delta},$$

and iterate.

Using this, we can establish Proposition 2.2 in the case $\tau > 1$. Indeed, in such a case, we can use the conclusion from the $\tau = 1$ case to deduce that $u \in M_s^{p, 2}$. This is more than enough regularity to apply (2.34)-(2.36), so the proof is complete.

Our next goal is to derive the hypothesis (2.2) on u as a consequence of a weaker hypothesis, at least for an important special case of systems of the form (2.1).

Proposition 2.3. *Let $u \in H^1(\Omega)$ solve (2.1). Assume the very strong ellipticity condition*

$$(2.37) \quad a_{\alpha\beta}^{jk}(x, u) \zeta_{j\alpha} \zeta_{k\beta} \geq \lambda_0 |\zeta|^2, \quad \lambda_0 > 0.$$

Also assume $B(x, u, \nabla u)$ is a quadratic form in ∇u . Assume furthermore that u is continuous on Ω . Then, locally, if $p > n/2$,

$$(2.38) \quad f \in M_2^p \implies \nabla u \in M_2^q, \quad \text{for some } q > n.$$

Hence $u \in C^r$, for some $r > 0$.

To begin, given $x_0 \in \Omega$, shrink Ω down to a smaller neighborhood, on which

$$(2.39) \quad |u(x) - u_0| \leq E,$$

for some $u_0 \in \mathbb{R}^M$ (if (2.1) is an $M \times M$ system). We will specify E below. Write

$$(2.40) \quad (\partial_j a^{jk}(x, u) \partial_k u, w)_{L^2} = - \int \langle \nabla u, \nabla w \rangle dx,$$

where $a_{\alpha\beta}^{jk}(x, u)$ determines an inner product on $T_x^* \otimes \mathbb{R}^M$ for each $x \in \Omega$, in a fashion that depends on u , perhaps, but one has bounds on the set of inner products so arising. Now, if we let $\psi \in C_0^\infty(\Omega)$ and $w = \psi(x)^2(u - u_0)$, and take the inner product of (2.1) with w , we have

$$(2.41) \quad \begin{aligned} & \int \psi^2 |\nabla u|^2 dx + 2 \int \psi (\nabla u) (\nabla \psi) (u - u_0) dx \\ & - \int \psi^2 (u - u_0) B(x, u, \nabla u) dx \\ & = - \int \psi^2 f(u - u_0) dx. \end{aligned}$$

Hence we obtain the inequality

$$(2.42) \quad \begin{aligned} & \int \psi^2 [|\nabla u|^2 - |u - u_0| \cdot |B(x, u, \nabla u)| - \delta^2 |\nabla u|^2] dx \\ & \leq \frac{1}{\delta^2} \int |\nabla \psi|^2 |u - u_0|^2 dx + \int \psi^2 |f| \cdot |u - u_0| dx, \end{aligned}$$

for any $\delta \in (0, 1)$. Now, for some $A < \infty$, we have

$$(2.43) \quad |B(x, u, \nabla u)| \leq A |\nabla u|^2.$$

Then we choose E in (2.39) so that

$$(2.44) \quad EA \leq 1 - a < 1.$$

Then take $\delta^2 = a/2$, and we have

$$(2.45) \quad \frac{a}{2} \int \psi^2 |\nabla u|^2 dx \leq \frac{2}{a} \int |\nabla \psi|^2 \cdot |u - u_0|^2 dx + \int \psi^2 |f| \cdot |u - u_0| dx.$$

Now, given $x \in \Omega$, for $r < \text{dist}(x, \partial\Omega)$ define $U(x, R)$ by

$$(2.46) \quad U(x, R) = R^{-n} \int_{B_R(x)} |u(y) - u_{x,R}|^2 dy,$$

where, as before, $u_{x,R}$ is the mean value of $u|_{B_R(x)}$.

Lemma 2.4. *Let $\overline{\mathcal{O}} \subset\subset \Omega$. There exist $R_0 > 0$, $\rho \in (0, 1)$, $\vartheta < 1$, and $C_0 < \infty$ such that, if $x \in \overline{\mathcal{O}}$ and $r \leq R_0$, then either*

$$(2.47) \quad U(x, r) \leq C_0 r^{2(2-\frac{n}{p})},$$

or

$$(2.48) \quad U(x, \rho r) \leq \vartheta U(x, r).$$

We first describe how to pick ρ , using the following; compare [Gia], pp.91-92.

Lemma 2.5. *There is a constant $A_0 = A_0(n, M, \lambda_1/\lambda_0)$ such that, whenever $b_{\alpha\beta}^{jk}$ are constants satisfying*

$$(2.49) \quad \lambda_1 |\zeta|^2 \geq \sum b_{\alpha\beta}^{jk} \zeta_{j\alpha} \zeta_{k\beta} \geq \lambda_0 |\zeta|^2, \quad \lambda_0 > 0,$$

the following holds. If $u \in H^1(B_1(0), \mathbb{R}^M)$ solves

$$(2.50) \quad \partial_j b_{\alpha\beta}^{jk} \partial_k u^\beta = 0 \quad \text{on } B_1(0),$$

then, for all $\rho \in (0, 1)$,

$$(2.51) \quad U(0, \rho) \leq A_0 \rho^2 U(0, 1).$$

Proof. For $\rho \in (0, \frac{1}{2}]$, we have

$$(2.52) \quad U(0, \rho) \leq \rho^{2-n} \int_{B_\rho(0)} |\nabla u(y)|^2 dy \leq C_n \rho^2 \|\nabla u\|_{L^\infty(B_{\frac{1}{2}}(0))}^2.$$

On the other hand, regularity for the constant coefficient elliptic PDE (2.50) readily yields an estimate

$$(2.53) \quad \|\nabla u\|_{L^\infty(B_{\frac{1}{2}}(0))}^2 \leq B_0 \|\nabla u\|_{L^2(B_{\frac{3}{4}}(0))}^2 \leq B_1 \|u - u_{0,1}\|_{L^2(B_1(0))}^2,$$

with $B_j = B_j(n, M, \lambda_1/\lambda_0)$, from which (2.51) easily follows.

Now, to pick ρ for Lemma 2.4, we assume (2.49) holds for all frozen coefficient principal parts of (2.1), take the A_0 given by Lemma 2.5, and then pick ρ so that $A_0 \rho^2 \leq \frac{1}{2}$.

Having picked ρ , we proceed to prove Lemma 2.4 by contradiction. If the result is false, there exist $x_\nu \in \overline{\mathcal{O}}$, $R_\nu \rightarrow 0$, $\vartheta_\nu \rightarrow 1$, and $u_\nu \in H^1(\Omega, \mathbb{R}^M)$ solving (2.1) such that

$$(2.54) \quad U_\nu(x_\nu, R_\nu) = \varepsilon_\nu^2 > C_0 R_\nu^{2(2-\frac{n}{p})}$$

and

$$(2.55) \quad U_\nu(x_\nu, \rho R_\nu) > \vartheta_\nu U_\nu(x_\nu, R_\nu).$$

The hypothesis that u is continuous implies $\varepsilon_\nu \rightarrow 0$. We want to obtain a contradiction.

We next set

$$(2.56) \quad v_\nu(x) = \varepsilon_\nu^{-1} [u_\nu(x_\nu + R_\nu x) - u_{\nu x_\nu, R_\nu}].$$

Then v_ν solves

$$(2.57) \quad \begin{aligned} & \partial_j a_{\alpha\beta}^{jk}(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}) \partial_k v_\nu^\beta \\ & + \varepsilon_\nu B(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}, \nabla v_\nu(x)) = \frac{R_\nu^2}{\varepsilon_\nu} f. \end{aligned}$$

Note that, by the hypothesis (2.54),

$$(2.58) \quad \frac{R_\nu^2}{\varepsilon_\nu} < \frac{1}{C_0} R_\nu^{n/p}.$$

Now set

$$(2.59) \quad V_\nu(0, r) = r^{-n} \int_{B_r(0)} |v_\nu(y) - v_{\nu 0, r}|^2 dy.$$

Then, since $v_{\nu 0, 1} = 0$, we have

$$(2.60) \quad V_\nu(0, 1) = \|v_\nu\|_{L^2(B_1(0))}^2 = 1, \quad V_\nu(0, \rho) > \vartheta_\nu.$$

Passing to a subsequence, we can assume that

$$(2.61) \quad v_\nu \rightarrow v \text{ weakly in } L^2(B_1(0), \mathbb{R}^M), \quad \varepsilon_\nu v_\nu \rightarrow 0 \text{ a.e. in } B_1(0).$$

Also

$$(2.62) \quad a_{\alpha\beta}^{jk}(x_\nu, u_{\nu x_\nu, R_\nu}) \longrightarrow b_{\alpha\beta}^{jk},$$

An array of constants satisfying (2.49). Boundedness of $\varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}$ plus continuity of $a_{\alpha\beta}^{jk}$ imply

$$(2.63) \quad a_{\alpha\beta}^{jk}(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}) \longrightarrow b_{\alpha\beta}^{jk} \text{ a.e. in } B_1(0),$$

and this is bounded convergence.

We next need to estimate the L^2 -norm of ∇v_ν . Substituting the quantity $\varepsilon_\nu v_\nu \left(\frac{x-x_\nu}{R_\nu} \right) + u_{\nu x_\nu, R_\nu}$ for $u_\nu(x)$ in (2.45), and replacing u_0 by $u_{\nu x_\nu, R_\nu}$, we have

$$\begin{aligned}
 (2.64) \quad & \frac{a}{2} \int \psi^2 \left| \nabla v_\nu \left(\frac{x-x_\nu}{R_\nu} \right) \right|^2 dx \\
 & \leq \frac{2}{a} \int R_\nu^2 |\nabla \psi|^2 \left| v_\nu \left(\frac{x-x_\nu}{R_\nu} \right) \right|^2 dx \\
 & \quad + \frac{R_\nu^2}{\varepsilon_\nu} \int \psi^2 |f| \cdot \left| v_\nu \left(\frac{x-x_\nu}{R_\nu} \right) \right| dx,
 \end{aligned}$$

for $\psi \in C_0^\infty(B_{R_\nu}(x_\nu))$. Actually, for this new value of u_0 , the estimate (2.39) might change to $|u(x) - u_0| \leq 2E$, so at this point we strengthen the hypothesis (2.44) to

$$(2.65) \quad 2EA \leq 1 - a < 1,$$

in order to get (2.59). Since $R_\nu^2/\varepsilon_\nu \leq R_\nu^{n/p}/C_0$, we have, for $\Psi(x) = \psi(x_\nu + R_\nu x) \in C_0^\infty(B_1(0))$,

$$(2.66) \quad \frac{a}{2} \int \Psi^2 |\nabla v_\nu|^2 dx \leq \frac{2}{a} \int |\nabla \Psi|^2 |v_\nu|^2 dx + \frac{R_\nu^{n/p}}{C_0} \int \Psi^2 |F| \cdot |v_\nu| dx,$$

where $F(x) = f(x_\nu + R_\nu x)$.

Since $\|v_\nu\|_{L^2(B_1(0))} = 1$, if $\Psi \leq 1$ we have

$$(2.67) \quad \int \Psi^2 |F| \cdot |v_\nu| dx \leq \left(\int_{B_1(0)} |F|^2 dx \right)^{\frac{1}{2}} \leq C_1 R_\nu^{-n/p},$$

if $f \in M_2^p$, so we have

$$(2.68) \quad \frac{a}{2} \int \Psi^2 |\nabla v_\nu|^2 dx \leq \frac{2}{a} \int |\nabla \Psi|^2 |v_\nu|^2 dx + \frac{C_1}{C_0} \|f\|_{M_2^p}.$$

This implies that v_ν is bounded in $H^1(B_\rho(0))$ for each $\rho < 1$. Now, we can pass to a further subsequence and obtain

$$\begin{aligned}
 (2.69) \quad & v_\nu \longrightarrow v \text{ strongly in } L_{\text{loc}}^2(B_1(0)), \\
 & \nabla v_\nu \longrightarrow \nabla v \text{ weakly in } L_{\text{loc}}^2(B_1(0)).
 \end{aligned}$$

Thus, we can pass to the limit in (2.57), to obtain

$$(2.70) \quad \partial_j b_{\alpha\beta}^{jk} \partial_k v^\beta = 0, \quad \text{on } B_1(0).$$

Also, by (2.60),

$$(2.71) \quad V(0, 1) = \|v\|_{L^2(B_1(0))} \leq 1, \quad V(0, \rho) \geq 1.$$

This contradicts Lemma 2.5, which requires $V(0, \rho) \leq \frac{1}{2} V(0, 1)$.

Now that we have Lemma 2.4, the proof of Proposition 2.3 is easily completed. From (2.47)-(2.48) we have

$$(2.72) \quad U(x, r) \leq Cr^{2\alpha}$$

for some $\alpha > 0$. In other words

$$(2.73) \quad \int_{B_r(x)} |u(y) - u_{x,r}|^2 dy \leq Cr^{n+2\alpha},$$

uniformly for $x \in \overline{\mathcal{O}} \subset \subset \Omega$. This in itself implies $u \in C^\alpha(\overline{\mathcal{O}})$. Furthermore, by (2.45), we have

$$(2.74) \quad \int_{B_r(x)} |\nabla u|^2 dy \leq Cr^{n-2(1-\alpha)},$$

which implies

$$(2.75) \quad \nabla u|_{\overline{\mathcal{O}}} \in M_2^q, \quad q = \frac{n}{1-\alpha}.$$

Thus Proposition 2.3 is proved.

We can extend Proposition 2.3 to the following result, which interfaces most conveniently with Propositions 2.1-2.2.

Proposition 2.6. *Under the hypotheses of Proposition 2.3, if $p > n$,*

$$(2.76) \quad f \in M_2^{p,-1} \implies u \in M_2^{q,1}, \quad \text{for some } q > n.$$

Proof. Writing $f = \sum \partial_j g_j$, $g_j \in M_2^p$, we replace the right side of (2.41) by (the sum over j of)

$$(2.77) \quad \begin{aligned} & - \int \psi^2 (\partial_j g_j) (u - u_0) dx = \\ & \int \psi^2 g_j (\partial_j u) dx + 2 \int \psi (\partial_j \psi) g_j (u - u_0) dx. \end{aligned}$$

Thus, in place of (2.42), we have the inequality

$$(2.78) \quad \begin{aligned} & \int \psi^2 [|\nabla u|^2 - |u - u_0| \cdot |B(x, u, \nabla u)| - 2\delta^2 |\nabla u|^2] dx \leq \\ & \frac{1}{\delta^2} \int \{ |\nabla \psi|^2 |u - u_0|^2 + \psi^2 |g|^2 \} dx + 2 \int |\psi| \cdot |\nabla \psi| \cdot |g| \cdot |u - u_0| dx, \end{aligned}$$

where $|g|^2 = \sum |g_j|^2$. The estimates (2.43)-(2.75) proceed essentially as before, with a few minor changes, resulting from replacing the estimate for $F(x) = f(x_\nu + R_\nu x)$ by the following estimate for $G_j(x) = g_j(x_\nu + R_\nu x)$:

$$(2.79) \quad \left(\int_{B_1(0)} |G_j|^2 dx \right)^{\frac{1}{2}} \leq C'_1 R^{-n/p},$$

if $g_j \in M_2^p$. Details are left to the reader.

Combining Propositions 2.2 and 2.6, we have:

Proposition 2.7. *Assume $u \in H^1(\Omega) \cap C(\Omega)$ solves (2.1), that the very strong ellipticity condition (2.37) holds, and that $B(x, u, \nabla u)$ is a quadratic form in ∇u . If $p > n$, $\tau \geq 0$, $2 \leq s \leq p$, then*

$$(2.80) \quad f \in M_s^{p, \tau-1} \implies u \in M_s^{p, \tau+1}.$$

3. Connections with Ricci curvature bounds

Consider a Riemannian metric g_{jk} defined on the unit ball $B_1 \subset \mathbb{R}^n$. We will work under the following hypotheses:

(i) For some constants $a_j \in (0, \infty)$, there are estimates

$$(3.1) \quad 0 < a_0 I \leq (g_{jk}(x)) \leq a_1 I.$$

(ii) The coordinates $x_1, \dots, x_n$ are harmonic, i.e.,

$$(3.2) \quad \Delta x_\ell = 0.$$

Here, Δ is the Laplace operator determined by the metric g_{jk} . In general,

$$(3.3) \quad \Delta v = g^{jk} \partial_j \partial_k v - \lambda^\ell \partial_\ell v, \quad \lambda^\ell = g^{jk} \Gamma_{jk}^\ell.$$

Note that $\Delta x_\ell = \lambda^\ell$, so the coordinates are harmonic if and only if $\lambda^\ell = 0$. Thus, in harmonic coordinates,

$$(3.4) \quad \Delta v = g^{jk} \partial_j \partial_k v.$$

We will also assume some bounds on the Ricci tensor, and desire to see how this influences the regularity of g_{jk} in these coordinates. Generally, the Ricci tensor is given by

$$(3.5) \quad \begin{aligned} \text{Ric}_{jk} &= \frac{1}{2} g^{\ell m} [-\partial_\ell \partial_m g_{jk} - \partial_j \partial_k g_{\ell m} \\ &\quad + \partial_k \partial_m g_{\ell j} + \partial_\ell \partial_j g_{km}] + M_{jk}(g, \nabla g) \\ &= -\frac{1}{2} g^{\ell m} \partial_\ell \partial_m g_{jk} + \frac{1}{2} g_{j\ell} \partial_k \lambda^\ell + \frac{1}{2} g_{k\ell} \partial_j \lambda^\ell + H_{jk}(g, \nabla g), \end{aligned}$$

with λ^ℓ as in (3.3). In harmonic coordinates, we obtain

$$(3.6) \quad \sum \partial_j g^{jk}(x) \partial_k g_{\ell m} + Q_{\ell m}(g, \nabla g) = \text{Ric}_{\ell m},$$

and $Q_{\ell m}(g, \nabla g)$ is a quadratic form in ∇g , with coefficients which are smooth functions of g , as long as (3.1) holds. Also, when (3.1) holds, the equation (3.6) is elliptic, of the form (2.1). Thus Proposition 2.7 directly implies the following.

Proposition 3.1. *Assume the metric tensor satisfies hypotheses (i) and (ii). Also assume that*

$$(3.7) \quad g_{jk} \in H^1(B_1) \cap C(B_1),$$

and

$$(3.8) \quad \text{Ric}_{\ell m} \in M_s^{p, r-1},$$

for some $p \in (n, \infty)$, $2 \leq s \leq p$, $r \geq 0$. Then, on the ball $B_{\frac{9}{10}}$,

$$(3.9) \quad g_{jk} \in M_s^{p, r+1}.$$

Geometrical consequences of estimates on the Ricci tensor can be found in [An], [AC], and references given in these papers.

4. Commutator estimates on Morrey scales

In this section we establish a number of commutator estimates, starting with the following variant of an estimate of T.Kato and G.Ponce [KP]:

Theorem 4.1. *If $P \in OPBS_{1,1}^m$ and $m > 0$, then*

$$(4.1) \quad \|P(fu) - fPu\|_{\mathcal{M}_q^{p,s}} \leq C\|f\|_{\text{Lip}^1}\|u\|_{\mathcal{M}_q^{p,m-1+s}} + C\|f\|_{\mathcal{M}_q^{p,m+s}}\|u\|_{L^\infty},$$

provided $s \geq 0$, $1 < q \leq p < \infty$.

Proof. We start with

$$(4.2) \quad \begin{aligned} f(Pu) &= T_f Pu + T_{Pu} f + R(f, Pu), \\ P(fu) &= PT_f u + PT_u f + PR(f, u). \end{aligned}$$

As shown in Proposition 4.2 of [AT], possibly replacing the ‘4’ in (0.19) by a larger number, we have $[T_f, P] \in OPBS_{1,1}^{m-1}$ when $f \in \text{Lip}^1(\mathbb{R}^n)$. Hence

$$(4.3) \quad \|[T_f, P]u\|_{\mathcal{M}_q^{p,s}} \leq C\|f\|_{\text{Lip}^1}\|u\|_{\mathcal{M}_q^{p,m-1+s}}.$$

Next, $u \in L^\infty \Rightarrow T_u \in OPBS_{1,1}^0$, so

$$(4.4) \quad \|PT_u f\|_{\mathcal{M}_q^{p,s}} \leq C \|u\|_{L^\infty} \|f\|_{\mathcal{M}_q^{p,m+s}}.$$

Furthermore,

$$(4.5) \quad u \in L^\infty \implies Pu \in C_*^{-m} \implies T_{Pu} \in OPBS_{1,1}^m, \text{ if } m > 0,$$

so

$$(4.6) \quad \|T_{Pu} f\|_{\mathcal{M}_q^{p,s}} \leq C \|u\|_{L^\infty} \|f\|_{\mathcal{M}_q^{p,m+s}}.$$

It remains to estimate $R(f, Pu)$ and $PR(f, u)$.

First, we mention that ρ_f , given by $\rho_f u = R(f, u)$, is a Calderon-Zygmund operator, for any $f \in \text{bmo}$, satisfying

$$(4.7) \quad \|\rho_f u\|_{L^q} \leq C_q \|f\|_{\text{BMO}} \|u\|_{L^q}, \quad 1 < q < \infty,$$

and with Schwartz kernel K_f satisfying (cf. Lemma 3.5.E of [T1])

$$(4.8) \quad |K_f(x, y)| \leq C \|f\|_{C_*^0} |x - y|^{-n},$$

as well as

$$(4.9) \quad |\nabla_{x,y} K_f(x, y)| \leq C \|f\|_{C_*^0} |x - y|^{-n-1}.$$

As shown in Appendix B, (4.7)-(4.8) lead to $\mathcal{M}_q^p$ boundedness. Since $\text{bmo} \subset \text{BMO} \cap C_*^0$, we have

$$(4.10) \quad \|R(f, u)\|_{\mathcal{M}_q^p} \leq C_{pq} \|f\|_{\text{bmo}} \|u\|_{\mathcal{M}_q^p},$$

for $1 < q \leq p < \infty$. Now we establish a variant of Proposition 3.5.D in [T1]:

Lemma 4.2. *Let X^r be a Banach space with the property that*

$$(4.11) \quad P \in OPS_{1,0}^r \implies P : X^r \longrightarrow \text{bmo}.$$

Then

$$(4.12) \quad \|R(f, u)\|_{\mathcal{M}_q^{p,s}} \leq C_{prs} \|f\|_{X^r} \|u\|_{\mathcal{M}_q^{p,s-r}}, \quad s \geq 0, \quad 1 < q \leq p < \infty.$$

Proof. First we treat the case $s = 0$. Decompose f into $\sum_{\ell=1}^{20} f_\ell$, via operators in $OPS_{1,0}^0$, so that

$$\text{supp } \hat{f} \subset \bigcup \left\{ 2^k \leq |\xi| \leq 2^{k+2} : k = \ell \bmod 20 \right\}.$$

Similarly decompose u . (We needn't worry about pieces left over with spectrum contained in, say, $|\xi| \leq 3$.) It suffices to estimate such $R(f_\ell, u_m)$. In such a case, we can find

$$(4.13) \quad F_\ell = Q_+ f_\ell \in \text{bmo}, \quad V_m = Q_- u_m, \quad Q_\pm \in OPS_{1,0}^{\pm r}$$

such that, for each k ,

$$(4.14) \quad \psi_k^a(D) f_\ell = 2^{-kr} \psi_k^a(D) F_\ell, \quad \psi_k(D) u_m = 2^{kr} \psi_k(D) V_m.$$

Here, $\{\psi_k\}$ is a Littlewood-Paley partition of unity, such as used in (0.6),

and $\psi_k^a(\xi) = \sum_{\ell=k-5}^{k+5} \psi_\ell(\xi)$, so that

$$(4.15) \quad R(f, u) = \sum_k (\psi_k^a(D) f) \cdot \psi_k(D) u.$$

Hence

$$(4.16) \quad R(f, u) = R(F_\ell, V_m),$$

so the $s = 0$ case of (4.12) follows from the estimate

$$(4.17) \quad \|V_m\|_{\mathcal{M}_q^p} \leq C \|u_m\|_{\mathcal{M}_q^{p,-r}},$$

plus (4.10).

So far, we have

$$(4.18) \quad \rho_f : \mathcal{M}_q^{p,-r} \longrightarrow \mathcal{M}_q^p, \quad \text{for } f \in X^r,$$

under the hypothesis (4.11). Next, we claim $\rho_f : \mathcal{M}_q^{p,1-r} \rightarrow \mathcal{M}_q^{p,1}$, for $f \in X^r$. This follows from

$$(4.19) \quad \partial_j(\rho_f u) = \rho_{(\partial_j f)} u + \rho_f(\partial_j u),$$

plus the fact that, if $f \in X^r$, then $P \in OPS_{1,0}^{r-1} \Rightarrow P(\partial_j f) \in \text{bmo}$, and hence the argument above shows that $\rho_{(\partial_j f)} : \mathcal{M}_q^{p,1-r} \rightarrow \mathcal{M}_q^p$. Once we have (4.19), then by induction we obtain

$$(4.20) \quad \rho_f : \mathcal{M}_q^{p,j-r} \longrightarrow \mathcal{M}_q^{p,j}, \quad j = 0, 1, 2, \dots$$

for $f \in X^r$, and then (4.12) follows by interpolation.

Our application of Lemma 4.2 to estimation of $R(f, Pu)$ and $PR(f, u)$ in (4.2) is the following:

$$(4.21) \quad \|R(f, u)\|_{\mathcal{M}_q^{p,\sigma}} \leq C \|f\|_{\text{Lip}^1} \|u\|_{\mathcal{M}_q^{p,\sigma-1}}, \quad \sigma \geq 0, \quad 1 < q \leq p < \infty.$$

Hence, given $P \in OPBS_{1,1}^m$, we have, taking $\sigma = s$,

$$(4.22) \quad \|R(f, Pu)\|_{\mathcal{M}_q^{p,s}} \leq C \|f\|_{\text{Lip}^1} \|u\|_{\mathcal{M}_q^{p,m-1+s}}, \quad s \geq 0,$$

and, taking $\sigma = s + m$,

$$(4.23) \quad \|PR(f, u)\|_{\mathcal{M}_q^{p,s}} \leq C \|f\|_{\text{Lip}^1} \|u\|_{\mathcal{M}_q^{p,m-1+s}}, \quad s + m \geq 0.$$

The proof of Theorem 4.1 is complete.

We next establish a commutator result along the lines of Lemma 1.13 in [Be]. Set $M_b u = bu$.

Proposition 4.3. *Let $1 < p \leq q < \infty$; consider*

$$(4.24) \quad v \in M_q^{p,s}(\mathbb{R}^n) \cap M_q^{p,r}{}_{ml}(\Gamma), \quad b \in M_q^{p,s+1}(\mathbb{R}^n) \cap M_q^{p,r+1}{}_{ml}(\Gamma).$$

Assume

$$(4.25) \quad \frac{n}{p} < s \leq r < 2s - \frac{n}{p}.$$

Then

$$(4.26) \quad P \in OPS_{1,0}^1 \implies [P, M_b]v \in M_q^{p,s}(\mathbb{R}^n) \cap M_q^{p,r}{}_{ml}(\Gamma).$$

Proof. Write

$$(4.27) \quad M_b v = T_b v + T_v b + R_b v.$$

Then, as in (4.2),

$$(4.28) \quad [P, M_b]v = [P, T_b]v + PT_v b - T_{Pv} b + PR_b v - R_b P v.$$

The hypotheses imply $b \in C_*^{s+1-\frac{n}{p}}$, hence, if $s > n/p$, the $OPBS_{1,1}^m$ calculus gives

$$(4.29) \quad T_b \in OPB^{\sigma+1}S_{1,1}^0, \quad [P, T_b] \in OPB^\sigma S_{1,1}^0, \quad \sigma = s - \frac{n}{p},$$

where $B^\sigma S_{1,1}^m$ is the subspace of $BS_{1,1}^m$ defined by (1.38).

Also, $v \in C_*^\sigma$, hence

$$(4.30) \quad T_v \in OPB^\sigma S_{1,1}^0, \quad T_{Pv} \in OPB^\sigma S_{1,1}^1.$$

Now (4.29) plus the hypothesis (4.24) on v gives

$$(4.31) \quad [P, T_b]v \in M_q^{p,s}(\mathbb{R}^n) \cap M_q^{p,r}{}_{ml}(\Gamma), \quad r < s + \sigma.$$

Also, (4.30) implies that, for $r < s + \sigma$,

$$(4.32) \quad \begin{aligned} T_v b &\in M_q^{p,s+1}(\mathbb{R}^n) \cap M_q^{p,r+1}{}_{ml}(\Gamma), \\ T_{Pv} b &\in M_q^{p,s}(\mathbb{R}^n) \cap M_q^{p,r}{}_{ml}(\Gamma). \end{aligned}$$

Finally, we have

$$(4.33) \quad R_b \in OPS_{1,1}^{-\sigma-1}, \quad \text{hence } PR_b v, R_b P v \in M_q^{p,s+\sigma}(\mathbb{R}^n).$$

It follows from (4.31)-(4.33) that

$$(4.34) \quad [P, M_b]v \in M_q^{p,s}(\mathbb{R}^n) \cap M_q^{p,r}{}_{ml}(\Gamma) + M_q^{p,2s-\frac{n}{p}}(\mathbb{R}^n),$$

which gives (4.25).

The next result was proved for $P \in OPS_{cl}^0$ in [DR1], following the seminal L^p estimate of [CRW]. This estimate will be useful in §5. We sketch a proof of an extension given in [AT].

Proposition 4.4. *If $P \in OPBS_{1,1}^0$, $f \in bmo(\mathbb{R}^n)$, and $1 < q \leq p < \infty$, then*

$$(4.35) \quad \|fPu - P(fu)\|_{M_q^p} \leq C\|f\|_{bmo}\|u\|_{M_q^p}.$$

Sketch of proof. As before, we use (4.2). We have (4.10), and similarly

$$(4.36) \quad \|T_u f\|_{M_q^p} \leq C\|f\|_{bmo}\|u\|_{M_q^p}.$$

Hence

$$(4.37) \quad \begin{aligned} & \|T_{Pu} f\|_{M_q^p} + \|PT_u f\|_{M_q^p} + \|R(f, Pu)\|_{M_q^p} + \|PR(f, u)\|_{M_q^p} \\ & \leq C\|f\|_{bmo}\|u\|_{M_q^p}. \end{aligned}$$

On the other hand, $bmo(\mathbb{R}^n) \subset C_*^0(\mathbb{R}^n)$, and, as shown in [AT],

$$(4.38) \quad f \in C_*^0(\mathbb{R}^n) \implies [T_f, P] \in OPBS_{1,1}^0,$$

so

$$(4.39) \quad \|[T_f, P]u\|_{M_q^p} \leq C\|f\|_{bmo}\|u\|_{M_q^p}.$$

This gives (4.35).

5. Operators with vmo coefficients

Consider a symbol

$$(5.1) \quad p(x, \xi) \in bmoS_{1,0}^m,$$

i.e., such that

$$\|D_\xi^\alpha p(\cdot, \xi)\|_{bmo} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|}.$$

Let us assume furthermore that

$$(5.2) \quad p(x, r\xi) = r^m p(x, \xi), \quad r \geq 1, \quad |\xi| \geq 1.$$

so $p(x, \xi) \in bmoS_{cl}^m$. Thus, if $\{w_j : j \geq 1\}$ is an orthonormal basis of $L^2(S^{n-1})$ consisting of eigenfunctions of the Laplace operator Δ_S on S^{n-1} , we can write

$$(5.3) \quad p(x, \xi) = p_0(x, \xi) + \sum_j f_j(x) w_j\left(\frac{\xi}{|\xi|}\right) |\xi|^m (1 - \varphi(\xi)),$$

where $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$, and $p_0(x, \xi)$ is supported on $|\xi| \leq 1$. Furthermore,

$$(5.4) \quad \|f_j\|_{\text{bmo}} \leq C_N \langle j \rangle^{-N}.$$

Write

$$(5.5) \quad p_j(x, \xi) = f_j(x) w_j\left(\frac{\xi}{|\xi|}\right) |\xi|^m (1 - \varphi(\xi)) = f_j(x) a_{jm}(\xi),$$

so we have

$$(5.6) \quad p(x, \xi) = \sum_{j \geq 0} p_j(x, \xi).$$

The operator $p_0(x, D)$ has a simple analysis. One can write

$$(5.7) \quad p_0(x, \xi) = \sum_{\ell} p_{\ell}(x) e^{i\ell \cdot \xi} \varphi(\xi/2),$$

with

$$(5.8) \quad \|p_{\ell}\|_{\text{bmo}} \leq C'_N \langle \ell \rangle^{-N}.$$

Thus

$$(5.9) \quad p_0(x, D)u = \sum p_{\ell}(x) \psi_{\ell}(D)u = \sum p_{\ell}(x) \hat{\psi}_{\ell} * u,$$

where

$$(5.10) \quad \hat{\psi}_{\ell}(x) = C \hat{\varphi}(2x + \ell).$$

Hence

$$(5.11) \quad p_0(x, D) : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \cap \text{bmo}(\mathbb{R}^n), \quad \forall p \in [1, \infty).$$

We now establish some commutator estimates. First, suppose

$$(5.12) \quad B = b(x, D) \in OPS_{1, \delta}^0, \quad 0 \leq \delta < 1.$$

Take $m = 0$ above, and use the notation $a_j(\xi)$ instead of $a_{j0}(\xi)$. Then

$$(5.13) \quad \begin{aligned} [B, p_j(x, D)]u &= [B, M_{f_j} a_j(D)]u \\ &= f_j(x) [B, a_j(D)]u + [B, M_{f_j}] a_j(D)u. \end{aligned}$$

Since

$$(5.14) \quad C_j = [B, a_j(D)] \in OPS_{1, \delta}^{-(1-\delta)},$$

and there are polynomial bounds (in j) on the relevant seminorms of the symbols, we have

$$(5.15) \quad \|f_j [B, a_j(D)]u\|_{M_q^p} \leq C_{N, K} \langle j \rangle^{-N} \|u\|_{M_q^p}, \quad 1 < q \leq p < \infty,$$

given $\text{supp } u \subset K$, compact. Also, by Proposition 4.4

$$(5.16) \quad \|[B, M_{f_j}]v\|_{M_q^p} \leq C \|f_j\|_{\text{bmo}} \|v\|_{M_q^p},$$

so we have

$$(5.17) \quad \|[B, p_j(x, D)]u\|_{M_q^p} \leq C_N \langle j \rangle^{-N} \|u\|_{M_q^p},$$

for $j \geq 1$. Summing over j , we have:

Proposition 5.1. *If $p(x, \xi) \in bmoS_{cl}^0$ and $B \in OPS_{1,\delta}^0$, $\delta < 1$, then, for $K \subset \mathbb{R}^n$ compact,*

$$(5.18) \quad [B, p(x, D)] : M_q^p(K) \longrightarrow M_q^p, \quad 1 < q \leq p < \infty.$$

If $p(x, \xi) \in vmoS_{cl}^0$, this comutator is compact.

Next, we consider the commutator $[M_g, p(x, D)]$. We have

$$(5.19) \quad [M_g, p(x, D)] = [M_g, p_0(x, D)] + \sum_{j \geq 1} [M_g, p_j(x, D)].$$

Clearly, for $g \in bmo$,

$$(5.20) \quad [M_g, p_0(x, D)] : M_q^p(K) \longrightarrow M_q^p, \quad 1 < q \leq p < \infty.$$

Next, for $j \geq 1$,

$$(5.21) \quad [M_g, p_j(x, D)] = M_{f_j} [M_g, a_j(D)].$$

Thus, applying Proposition 4.4 to the commutator $[M_g, a_j(D)]$, we have

$$(5.22) \quad \|[M_g, p_j(x, D)]u\|_{M_q^p} \leq C \|f_j\|_{L^\infty} \|g\|_{bmo} \|u\|_{M_q^p},$$

and, if $g \in vmo$, this commutator is compact. Summing, we have:

Proposition 5.2. *If $p(x, \xi) \in L^\infty S_{cl}^0$ and $g \in bmo$, then*

$$(5.23) \quad [M_g, p(x, D)] : M_q^p(K) \longrightarrow M_q^p, \quad 1 < q \leq p < \infty.$$

If also $g \in vmo$ and $p(x, \xi)$ is supported on $x \in K$ compact, then this commutator is compact.

For the spaces L^p , this result was proved in [CFL]; see Theorem 2.11 there. Furthermore, weighted L^p estimates are obtained in Theorem 2.1 of [DR2], and the Morrey space estimate (5.23) is contained in Theorems 2.2-2.3 of [DR2].

Now suppose $q(x, \xi)$ has the form

$$(5.24) \quad \begin{aligned} q(x, \xi) &= \sum_{j \geq 0} q_j(x, \xi), \quad q_0(x, \xi) = \sum_{\ell} q_\ell(x) \psi_\ell(\xi), \\ q_j(x, \xi) &= g_j(x) a_j(\xi), \quad j \geq 1, \end{aligned}$$

with ψ_ℓ as in (5.9)-(5.10). Then

$$(5.25) \quad [q(x, D), p(x, D)] = [q_0(x, D), p(x, D)] + \sum_{j \geq 1} [q_j(x, D), p(x, D)].$$

Clearly, if $p(x, \xi)$ and $q(x, \xi)$ have compact x -support, then

$$(5.26) \quad p(x, \xi), q(x, \xi) \in L^\infty S_{cl}^0 \implies [q_0(x, D), p(x, D)] \text{ compact on } M_q^p.$$

Next,

$$(5.27) \quad [q_j(x, D), p(x, D)] = M_{g_j} [a_j(D), p(x, D)] + a_j(D) [M_{g_j}, p(x, D)].$$

Now we have, for some $M < \infty$,

$$(5.28) \quad \|M_{g_j} [a_j(D), p(x, D)] u\|_{M_q^p} \leq C \|g_j\|_{L^\infty} \langle j \rangle^M \|u\|_{M_q^p},$$

if $p(x, \xi) \in \text{bmo} S_{cl}^0$, and compactness if $p(x, \xi) \in \text{vmo} S_{cl}^0$. Also, we have

$$(5.29) \quad \|a_j(D) [M_{g_j}, p(x, D)] u\|_{M_q^p} \leq C \langle j \rangle^M \|g\|_{\text{bmo}} \|u\|_{M_q^p},$$

if $p(x, \xi) \in L^\infty S_{cl}^0$, and compactness if $g_j \in \text{vmo}$. This proves:

Proposition 5.3. Assume $p(x, \xi), q(x, \xi) \in (L^\infty \cap \text{vmo}) S_{cl}^0$, with compact x -support. Then

$$(5.30) \quad [p(x, D), q(x, D)] \text{ is compact on } M_q^p(\mathbb{R}^n), \quad 1 < q \leq p < \infty.$$

In fact, we have the following result, which is more precise than Proposition 5.3.

Theorem 5.4. Assume $p(x, \xi), q(x, \xi) \in (L^\infty \cap \text{vmo}) S_{cl}^0$, with compact x -support. Then

$$(5.31) \quad p(x, D)q(x, D) = a(x, D) + K, \quad a(x, \xi) = p(x, \xi)q(x, \xi),$$

where K is compact on M_q^p .

Proof. The argument is similar to that given above. We have

$$(5.32) \quad \begin{aligned} p(x, D)q(x, D) &= p_0(x, D)q(x, D) + p(x, D)q_0(x, D) \\ &\quad + \sum_{j,k} M_{f_j} a_j(D) M_{g_k} a_k(D), \end{aligned}$$

and the first two terms on the right are compact. The double sum is equal to

$$(5.33) \quad \sum_{j,k} M_{f_j g_k} a_j(D) a_k(D) + \sum_{j,k} M_{f_j} [a_j(D), M_{g_k}] a_k(D).$$

The first sum in (5.33) differs from $a(x, D)$ by a compact operator, and the second sum is equal to

$$(5.34) \quad \sum_j M_{f_j} [a_j(D), \tilde{q}(x, D)],$$

Thus E is a left Fredholm inverse of L .

On the other hand,

$$(5.48) \quad LE = L(1 - \Delta)^{-1}B(x, D)(1 - \Delta) = P(x, D)Q(x, D),$$

with

$$(5.49) \quad P(x, \xi) = - \sum a_{jk}(x) \xi_j \xi_k \langle \xi \rangle^{-2}, \quad Q(x, \xi) = P(x, \xi)^{-1} (1 - \varphi(\xi)),$$

both symbols belonging to $(L^\infty \cap \text{vmo})S_{cl}^0$. Thus Theorem 5.4 implies $P(x, D)Q(x, D) = I + \mathcal{K}_2$, with $\mathcal{K}_2$ compact. Hence E is a two-sided Fredholm inverse of L .

6. Morrey-space estimates for wave equations

Proposition 6.1. *Assume n is odd. Let $w(t, x)$ solve the Cauchy problem*

$$(6.1) \quad (\partial_t^2 - \Delta)w = 0, \quad w(0) = f, \quad w_t(0) = 0$$

on $\mathbb{R} \times \mathbb{R}^n$. If $f \in L^\infty(\mathbb{R}^n)$, then, for $z \in \mathbb{R}^n$, $\rho \in (0, 1]$,

$$(6.2) \quad \|w(t, \cdot)\|_{L^2(B_\rho(z))} \leq C \langle t \rangle \|f\|_{L^\infty} \rho^{\frac{1}{2}}.$$

Proof. By the strong Huygens principle, the value of $w(t, x)$ for $x \in B_\rho(z)$ is unaffected if f is replaced by

$$(6.3) \quad f^\#(x) = \begin{cases} f(x) & \text{if } |t| - 2\rho \leq |x - z| \leq |t| + 2\rho, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$(6.4) \quad \|f^\#\|_{L^2(\mathbb{R}^n)} \leq C \langle t \rangle \rho^{\frac{1}{2}} \|f\|_{L^\infty},$$

so

$$(6.5) \quad w^\#(t) = \cos t \sqrt{-\Delta} f^\# \implies \|w^\#(t)\|_{L^2} \leq C \langle t \rangle \rho^{\frac{1}{2}} \|f\|_{L^\infty}.$$

Since $w(t, x) = w^\#(t, x)$ for $x \in B_\rho(z)$, we have (6.2).

Note that Proposition 6.1 can be stated in terms of a Morrey space:

$$(6.6) \quad \cos t \sqrt{-\Delta} : L^\infty(\mathbb{R}^n) \longrightarrow M_2^{2n/(n-1)}(\mathbb{R}^n).$$

More generally, we can replace L^∞ by L^p , $2 \leq p \leq \infty$, obtaining

$$(6.7) \quad \|w(t, \cdot)\|_{L^2(B_\rho(z))} \leq C(t) \|f\|_{L^p} \rho^{\frac{1}{2} - \frac{1}{p}},$$

so, for $p \geq 2$, n odd,

$$(6.8) \quad \cos t\sqrt{-\Delta} : L^p(\mathbb{R}^n) \longrightarrow M_2^q(\mathbb{R}^n), \quad q = \frac{2n}{n-1 + \frac{2}{p}}.$$

We now extend Proposition 6.1, to the case where Δ is the Laplace operator on a complete Riemannian manifold M , with bounded geometry, whose dimension n is odd. In such a case, there exists $\tau \in (0, \infty]$ such that the solution to (6.1) can be written

$$(6.9) \quad w(t) = R'(t)f + R(t)g,$$

and, for $|t| < \tau$,

$$(6.10) \quad R(t) = R_0(t) + B(t),$$

where $R_0(t)$ and $B(t)$ have the following properties. First, the Schwartz kernel of $R_0(t)$ is supported on the ‘light cone’ $\{(t, x, y) : \text{dist}(x, y) = |t|\}$. Next, for $|t| < \tau$, $B(t)$ is a family of FIOPs of order -2 , and $B'(t)$ is a family of FIOPS of order -1 , having the mapping properties

$$(6.11) \quad B(t) : H^s(M) \rightarrow H^{s+2}(M), \quad B'(t) : H^s(M) \rightarrow H^{s+1}(M).$$

We can now prove the following extension of Proposition 6.1.

Proposition 6.2. *Let $w(t, x)$ solve the Cauchy problem (6.1) on $\mathbb{R} \times M$, where M is a complete Riemannian manifold with bounded geometry whose dimension n is odd. There exists $\tau \in (0, \infty]$ such that, if $p \in [2, \infty]$, and $f \in L^p(M)$, then, for $|t| < \tau$, $z \in M$, and $\rho \in (0, 1]$, we have*

$$(6.12) \quad \|w(t, \cdot)\|_{L^2(B_\rho(z))} \leq C(t) \|f\|_{L^p} \cdot \rho^{\frac{1}{2} - \frac{1}{p}}.$$

Proof. Defining $f^\#$ as in (6.3), but using $\text{dist}(x, z)$, we see that

$$(6.13) \quad R'_0(t)f = R'_0(t)f^\# \quad \text{for } x \in B_\rho(z),$$

and since $R'_0(t) : L^2(M) \rightarrow L^2(M)$, we obtain

$$(6.14) \quad \|R'_0(t)f\|_{L^2(B_\rho(z))} \leq C(t) \|f^\#\|_{L^2} \leq C(t) \|f\|_{L^p} \cdot \rho^{\frac{1}{2} - \frac{1}{p}}.$$

Meanwhile, by finite propagation speed and (6.11) we have

$$(6.15) \quad \|B'(t)f\|_{H^1(B_1(z))} \leq C(t) \|f\|_{L^p},$$

and (6.12) follows from (6.14)-(6.15), since

$$(6.16) \quad H^1(B_1(z)) \subset L^{2n/(n-2)}(B_1(z)) \subset M_2^{2n/(n-2)}(B_1(z)).$$

A statement equivalent to (6.12) is

$$(6.17) \quad \|w(t)\|_{M_2^q} \leq C(t)\|f\|_{L^p}, \quad q = \frac{2n}{n-1+\frac{2}{p}}.$$

7. Conormal spaces and variants

We now define a class of spaces that include ‘conormal spaces.’ Let $\mathcal{M}$ be a collection of vector fields in $\mathbb{R}^n$ (which may or may not be smooth everywhere). If $J = (j_\ell, \dots, j_1)$, we set $X^J = X_{j_\ell} \cdots X_{j_1}$, and we set $|J| = \ell$. By convention, $X^\emptyset u = u$. Assume $1 < q \leq p < \infty$, $k \in \mathbb{Z}^+$, $s \in \mathbb{R}$. We say $u \in N^k(M_q^{p,s}, \mathcal{M})$ if $u \in M_q^{p,s}$ and

$$(7.1) \quad X_{j_\nu} \in \mathcal{M}, |J| < k \implies X^J u \in M_q^{p,s}.$$

Important special cases include the following. Suppose $\Sigma \subset \mathbb{R}^n$ is a smooth submanifold and $\mathcal{M}$ consists of all smooth vector fields (well-behaved at infinity) which are tangent to Σ . We denote the space defined above by $N^k(M_q^{p,s}, \Sigma)$ in this case. Compare the definition of $N^{k,s}(\Sigma)$ in [Be], p.52. As another example, $\mathcal{F}$ could be a smooth foliation of $\mathbb{R}^n$ (by submanifolds of dimension d), and $\mathcal{M}$ could consist of all smooth vector fields tangent to $\mathcal{F}$. We denote the resulting space by $N^k(M_q^{p,s}, \mathcal{F})$. An example of a collection $\mathcal{M}$ of vector fields X_j which are smooth on $\mathbb{R}^n \setminus 0$ is given in [Be], p.119.

Let f be smooth; we want to estimate $X^J f(u)$, for $u \in N^k(M_q^{p,s}, \mathcal{M})$. Repeated application of the chain rule gives

$$(7.2) \quad X^J f(u) = \sum_{I_1 + \cdots + I_\ell = J} C_I(X^{I_1} u) \cdots (X^{I_\ell} u) f^{(\ell)}(u).$$

If we set

$$(7.3) \quad g(u, X^{I_1} u, \dots, X^{I_\ell} u) = (X^{I_1} u) \cdots (X^{I_\ell} u) f^{(\ell)}(u),$$

then we know that, for $s > 0$,

$$(7.4) \quad \begin{aligned} & \|g(u, v_{I_1}, \dots, v_{I_\ell})\|_{M_q^{p,s}} \\ & \leq C(\|u\|_{L^\infty}, \|v_{I_1}\|_{L^\infty}, \dots, \|v_{I_\ell}\|_{L^\infty}) \\ & \quad \cdot \left(1 + \|u\|_{M_q^{p,s}} + \|v_{I_1}\|_{M_q^{p,s}} + \cdots + \|v_{I_\ell}\|_{M_q^{p,s}}\right). \end{aligned}$$

Hence we have the estimate:

$$(7.5) \quad \|X^J f(u)\|_{M_q^{p,s}} \leq C(\|X^I u\|_{L^\infty} : I \leq J) \cdot \left(1 + \sum_{I \leq J} \|X^I u\|_{M_q^{p,s}}\right).$$

While we have briefly allowed the possibility that $\mathcal{M}$ contains nonsmooth vector fields, we will henceforth assume that all the vector fields in $\mathcal{M}$ are smooth, with coefficients which are bounded on $\mathbb{R}^n$, together with all their derivatives. We will also adopt the standing assumption that

$$(7.6) \quad 1 < q \leq p < \infty.$$

Proposition 7.1. *Given $P \in OPS_{1,0}^m$, $s, m \in \mathbb{R}$,*

$$(7.7) \quad P : N^k(M_q^{p,s}, \mathcal{M}) \longrightarrow N^k(M_q^{p,s-m}, \mathcal{M}).$$

Proof. For any $X \in \mathcal{M}$, we have

$$(7.8) \quad XPu = PXu + P_X u,$$

where $P_X = [X, P] \in OPS_{1,0}^m$; in fact, if $X = \sum a_j(x) \partial / \partial x_j$,

$$(7.9) \quad P_X(x, \xi) = X \cdot P(x, \xi) + P_X^b(x, \xi),$$

where

$$(7.10) \quad \begin{aligned} X \cdot P(x, \xi) &= \sum a_j(x) \partial_{x_j} P(x, \xi), \\ P_X^b(x, \xi) &\sim \sum_{|\alpha| \geq 1} \frac{i^{|\alpha|}}{\alpha!} P^{(\alpha)}(x, \xi) \partial_x^\alpha X(x, \xi). \end{aligned}$$

Inductively, we obtain

$$(7.11) \quad X^J Pu = \sum_{I \leq J} P_{J \setminus I} X^I u, \quad P_{J \setminus I} \in OPS_{1,0}^m.$$

Given this, (7.7) follows from (1.4).

In fact, using (7.8)-(7.9), we can say more. Let us say that

$$(7.12) \quad p(x, \xi) \in (\mathcal{M}^k) S_{1,\delta}^m$$

provided $p(x, \xi) \in S_{1,\delta}^m$ and

$$(7.13) \quad X^J \cdot p(x, \xi) \in S_{1,\delta}^m, \quad \forall |J| \leq k.$$

Similarly define $(\mathcal{M}^k) \mathcal{B}S_{1,1}^m$ to consist of $p(x, \xi) \in \mathcal{B}S_{1,1}^m$ such that $X^J \cdot p(x, \xi) \in \mathcal{B}S_{1,1}^m$ whenever $|J| \leq k$. Replacing the use of (1.4) by that of (1.12) and (1.24), and noting that (7.9)-(7.10) is valid even for $P \in OPS_{1,1}^m$, we have:

Proposition 7.2. *Given $P \in OP(\mathcal{M}^k)\mathcal{BS}_{1,1}^m$, the property (7.7) holds. Furthermore, given $P \in OP(\mathcal{M}^k)\mathcal{S}_{1,1}^m$, (7.7) holds provided $s - m > 0$.*

Note that we can substitute other spaces for $M_q^{p,s}$ in (7.1), producing such spaces as $N^k(C_*^s, \mathcal{M})$, for which we have analogues of (7.5) and of Propositions 7.1-7.2.

Next, given a smooth function F , write

$$(7.14) \quad F(u) = M_F(u; x, D)u + F(u_0),$$

as in (0.8)-(0.9).

Proposition 7.3. *If $u \in N^k(C_*^r, \mathcal{M})$, $r > 0$, then*

$$(7.15) \quad M_F(u; x, \xi) \in (\mathcal{M}^k)\mathcal{A}_*^r\mathcal{S}_{1,1}^m.$$

Here, $\mathcal{A}_*^r\mathcal{S}_{1,\delta}^m \subset \mathcal{S}_{1,\delta}^m$ consists of symbols satisfying

$$(7.16) \quad \|D_\xi^\alpha p(\cdot, \xi)\|_{C_*^s} \leq C_s \langle \xi \rangle^{m-|\alpha|+\delta(s-r)}, \quad s \geq r.$$

Proof. Using (0.9), we need to estimate

$$(7.17) \quad m_\ell(x) = \int_0^1 F'(u_{\ell,\tau}) d\tau, \quad u_{\ell,\tau} = \Psi_\ell(D)u + \tau\psi_{\ell+1}(D)u.$$

The analogue of (7.5), with $M_q^{p,s}$ replaced by C_*^s , is

$$(7.18) \quad \|X^J F'(u_{\ell,\tau})\|_{C_*^s} \leq C(\|X^I u_{\ell,\tau}\|_{L^\infty} : I \leq J) \cdot \left(1 + \sum_{I \leq J} \|X^I u_{\ell,\tau}\|_{C_*^s}\right).$$

To proceed, we use the following:

Lemma 7.4. *If $u \in N^k(C_*^r, \mathcal{M})$, $r > 0$, then*

$$(7.19) \quad \|X^I \Psi_\ell(D)u\|_{C_*^s} \leq C_{s,I} \cdot 2^{\ell(s-r)}, \quad s \geq r, \quad |I| \leq k.$$

We will establish this after using it to prove the proposition. In fact, we now deduce from (7.18) that

$$(7.20) \quad \|X^J m_\ell\|_{C_*^s} \leq C_{J,s} \cdot 2^{\ell(s-r)}, \quad s \geq r, \quad |I| \leq k.$$

Since $\psi_{\ell+1}(\xi)$ in (0.9) is supported on $\langle \xi \rangle \sim 2^\ell$, we have (7.16), and Proposition 7.3 is established, modulo a proof of Lemma 7.4.

To prove Lemma 7.4, we can treat $X^I \Psi_\ell(D)$ as in (7.8)-(7.11), obtaining

$$(7.21) \quad X^I \Psi_\ell(D) = \sum_{K \leq I} \psi_{I \setminus K, \ell}(x, D) X^K.$$

Furthermore,

$$(7.22) \quad \psi_{I \setminus K, \ell}(x, D) - \psi_{I \setminus K, \ell}(x, D) \Psi_{\ell+3}(D)$$

is bounded in $OPS_{1,0}^{-\infty}$. Since $X^K u \in C_*^r$ for $|K| \leq k$, (7.19) follows from:

$$(7.23) \quad v \in C_*^r \implies \|\Psi_{\ell+3}(D)v\|_{C_*^s} \leq C_s \cdot 2^{\ell(s-r)}, \quad s \geq r,$$

which is elementary.

A. A Schwartz kernel estimate

Here we want to prove that, if $p(x, \xi) \in S_{1,1}^0(\mathbb{R}^n)$, then the operator product

$$(A.1) \quad \Lambda^s p(x, D) \Lambda^{-s} = P_s$$

has Schwartz kernel K_s satisfying

$$(A.2) \quad |K_s(x, y)| \leq C_s |x - y|^{-n},$$

for $0 < s < 1$. Note that P_s defines a bounded linear operator on $H^{\sigma,p}(\mathbb{R}^n)$ for all $\sigma > -s$, $p \in (1, \infty)$. However, P_s is perhaps not an element of $OPS_{1,1}^0(\mathbb{R}^n)$. Of course, (A.2) clearly holds for $s = 0$, as a consequence of the implication

$$(A.3) \quad |D_\xi^\alpha A(\xi)| \leq C_\alpha |\xi|^{\tau-|\alpha|} \implies |\widehat{A}(x)| \leq C_\tau |x|^{-n-\tau}, \quad \tau > -n.$$

Recall that Λ^s is Fourier multiplication by $\langle \xi \rangle^s$. It will be convenient for the dilation argument we intend to apply, to replace Λ^s by λ^s , Fourier multiplication by $|\xi|^s$. We will show that

$$(A.4) \quad \tilde{P}_s = \lambda^s p(x, D) \lambda^{-s}$$

has Schwartz kernel $\tilde{K}_s$ satisfying

$$(A.5) \quad |\tilde{K}_s(x, y)| \leq C_s |x - y|^{-n}.$$

It is clear that

$$(A.6) \quad \tilde{P}_s - P_s : H_{\text{comp}}^{\sigma,p}(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n), \quad \sigma > -s, \quad 1 < p < \infty,$$

provided $0 < s < n$, and hence (A.5) readily implies (A.2).

To prove (A.5), we will examine

$$(A.7) \quad \vartheta_r K_s(x, y) = K_s(rx, ry),$$

which is the Schwartz kernel of

$$(A.8) \quad \tilde{P}_{sr} = r^{-n} \delta_r \tilde{P}_s \delta_r^{-1}; \quad \delta_r f(x) = f(rx).$$

We will show that, for $r \in (0, 1]$,

$$(A.9) \quad |\vartheta_r K_s(x, y)| \leq C r^{-n} \quad \text{on } \Omega = \{(x, y) : 1 \leq |x - y| \leq 2\},$$

which implies (A.5) for $|x - y| \leq 1$, hence (A.2) for $|x - y| \leq 1$. It is relatively easy to show that $K_s(x, y)$ is rapidly decreasing as $|x - y| \rightarrow \infty$, so this will suffice. Now, since $\lambda^s \delta_r^{-1} = r^{-s} \delta_r^{-1} \lambda^s$, we have

$$(A.10) \quad \vartheta_r K_s(x, y) = r^{-n} \tilde{K}_{s,r}(x, y)$$

where

$$(A.11) \quad \tilde{K}_{s,r}(x, y) = \text{Schwartz kernel of } \lambda^s \delta_r p(x, D) \delta_r^{-1} \lambda^{-s},$$

or, setting

$$(A.12) \quad \kappa_r(x, y) = \text{Schwartz kernel of } p_r(x, D) = \delta_r p(x, D) \delta_r^{-1},$$

we have

$$(A.13) \quad \tilde{K}_{s,r}(x, y) = \lambda_x^s \lambda_y^{-s} \kappa_r(x, y),$$

and we want to show that

$$(A.14) \quad |\lambda_x^s \lambda_y^{-s} \kappa_r(x, y)| \leq C_s \quad \text{on } \Omega, \quad 0 < r \leq 1,$$

with C_s independent of r .

Note that the symbol of $p_r(x, D)$ is

$$(A.15) \quad p_r(x, \xi) = p(rx, \xi/r),$$

which satisfies

$$(A.16) \quad \begin{aligned} |D_x^\beta D_\xi^\alpha p_r(x, \xi)| &\leq C_{\alpha\beta} r^{|\beta| - |\alpha|} \langle \xi/r \rangle^{|\beta| - |\alpha|} \\ &= C_{\alpha\beta} (r^2 + |\xi|^2)^{(|\beta| - |\alpha|)/2} \\ &\leq C_{\alpha\beta} \langle \xi \rangle^{|\beta|} |\xi|^{-|\alpha|}, \quad \text{for } 0 < r \leq 1. \end{aligned}$$

Hence, by (A.3),

$$(A.17) \quad |\kappa_r(x, y)| \leq C|x - y|^{-n},$$

with C independent of $r \in (0, 1]$.

Similarly, $\lambda_y^{-s} \kappa_r(x, y)$ is the Schwartz kernel of $q_r(x, D) = p_r(x, D)\lambda^{-s}$, with symbol

$$(A.18) \quad q_r(x, \xi) = p(rx, \xi/r)|\xi|^{-s}$$

satisfying

$$(A.19) \quad |D_x^\beta D_\xi^\alpha q_r(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{|\beta|} |\xi|^{-s-|\alpha|},$$

which implies

$$(A.20) \quad |\lambda_y^{-s} \kappa_r(x, y)| \leq C_s |x - y|^{s-n},$$

and more generally

$$(A.21) \quad |D_x^\beta D_y^\gamma \lambda_y^{-s} \kappa_r(x, y)| \leq C_{s\alpha\beta} |x - y|^{s-n-|\beta|-|\gamma|},$$

provided $s < n$. The estimate (A.14) is a simple consequence of this.

B. Another proof of Proposition 1.2

Here we include a self-contained proof of:

Proposition B.1. *Assume the Schwartz kernel $k(x, y)$ of T satisfies*

$$(B.1) \quad |k(x, y)| \leq C|x - y|^{-n}(1 + |x - y|)^{-M}$$

for some $M > 0$. Then, if $1 < q \leq p < \infty$,

$$(B.2) \quad T : L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n) \implies T : M_q^p(\mathbb{R}^n) \rightarrow M_q^p(\mathbb{R}^n).$$

Proof. Let $f \in M_q^p(\mathbb{R}^n)$. Pick $z \in \mathbb{R}^n$, $r \in (0, 1]$, and write

$$(B.3) \quad f = f_0 + \sum_{2^j r \leq 1} g_j + h,$$

where

$$(B.4) \quad f_0 = \chi_{B_{2r}(z)} f, \quad g_j = \chi_{A_{r,j}} f, \quad A_{r,j} = \{x : |x - z| \in [2^j r, 2^{j+1} r]\},$$

and $j \geq 1$ in the sum. We want to estimate Tf on $B_r(z)$. Clearly

$$(B.5) \quad \|Tf_0\|_{L^q(\mathbb{R}^n)} \leq C\|f_0\|_{L^q(\mathbb{R}^n)} \leq Cr^a, \quad a = \frac{n}{q} - \frac{n}{p},$$

and the estimate (B.1) for $|x - y| \geq 1$ implies

$$(B.6) \quad \|Th\|_{L^\infty(B_{\frac{1}{2}}(z))} \leq C\|h\|_{M_q^p(\mathbb{R}^n)}.$$

It remains to estimate $\sum Tg_j$ on $B_r(z)$. To do this, write

$$(B.7) \quad \chi_{B_r(z)} Tg_j = T_j(\chi_{A_{rj}} f),$$

where T_j has integral kernel

$$(B.8) \quad k_j(x, y) = \chi_{B_r(z)} k(x, y) \chi_{A_{rj}}(y).$$

Now, using (B.1) for $|x - y| \leq 1$, we have

$$(B.9) \quad \begin{aligned} \int |k_j(x, y)| \, dx &= \int_{B_r(z)} |k(x, y)| \chi_{A_{rj}}(y) \, dx \\ &\leq C(2^j r)^{-n} \cdot \text{vol } B_r(z) \\ &\leq C2^{-jn}, \end{aligned}$$

and

$$(B.10) \quad \begin{aligned} \int |k_j(x, y)| \, dy &= \int_{A_{rj}} \chi_{B_r(z)} |k(x, y)| \, dy \\ &\leq C(2^j r)^{-n} \cdot \text{vol } A_{rj} \\ &\leq C. \end{aligned}$$

Hence, if $f \in M_q^p(\mathbb{R}^n)$,

$$(B.11) \quad \begin{aligned} \|T_j(\chi_{A_{rj}} f)\|_{L^q} &\leq C2^{-jn/q} \|\chi_{A_{rj}} f\|_{L^q} \\ &\leq C2^{-jn/q} (2^j r)^a \\ &\leq C2^{-jn/p} r^a, \end{aligned}$$

so, if $p < \infty$,

$$(B.12) \quad \sum \|Tg_j\|_{L^q(B_r(z))} \leq C \left(\sum_{j \geq 1} 2^{-jn/p} \right) r^a \leq C' r^a,$$

as desired. This completes the proof.

If $f \in \mathcal{M}_q^p(\mathbb{R}^n)$, one can replace (B.3) by

$$f = f_0 + \sum_{j=1}^{\infty} g_j$$

and repeat the estimates above, obtaining:

Proposition B.2. *Assume the Schwartz kernel $k(x, y)$ of T satisfies*

$$(B.13) \quad |k(x, y)| \leq C|x - y|^{-n}.$$

Then, if $1 < q \leq p < \infty$,

$$(B.14) \quad T : L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n) \implies T : \mathcal{M}_q^p(\mathbb{R}^n) \rightarrow \mathcal{M}_q^p(\mathbb{R}^n).$$

References

- [Ad] D.Adams, A note on Riesz potentials, Duke Math. J. 42(1975), 765-778.
- [An] M.Anderson, Convergence and rigidity of manifolds under Ricci curvature bounds, Invent. Math. 102(1990), 429-445.
- [AC] M.Anderson and J.Cheeger, C^α -compactness for manifolds with Ricci curvature and injectivity radius bounded below, J.Diff. Geom. 35(1992), 265-281.
- [AT] P.Auscher and M.Taylor, Paradifferential operators and commutator estimates, Preprint, 1994.
- [Be] M.Beals, Propagation and Interaction of Singularities in Nonlinear Hyperbolic Problems, Birkhauser, Boston, 1989.
- [BL] J.Bergh and J.Lofström, Interpolation Spaces, Springer, New York, 1976.
- [Bo] J.Bony, Calcul symbolique et propagation des singularities pour les equations aux derivees nonlineaires, Ann. Sci. Ecole Norm. Sup. 14(1981), 209-246.
- [Caf] L.Caffarelli, Elliptic second order equations, Rend. Sem. Mat. Fis. Milano 58(1988), 253-284.
- [C] S.S.Chern (ed.), Seminar on Nonlinear Partial Differential Equations, MSRI Publ.#2, Springer, New York, 1984.
- [CF] F.Chiarenza and M.Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat. S. 7(1987), 273-279.
- [CFL1] F.Chiarenza, M.Frasca, and P.Longo, Interior $W^{2,p}$ estimates for non divergence elliptic equations with discontinuous coefficients, Ricerche Mat. 40(1991), 149-168.
- [CFL2] F.Chiarenza, M.Frasca, and P.Longo, $W^{2,p}$ -solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, Trans. AMS 336(1993), 841-853.
- [Chr] M.Christ, Lectures on Singular Integral Operators, CBMS Reg. Conf. Ser. in Math. #77, AMS, Providence, RI, 1990.
- [CLMS] R.Coifman, P.Lions, Y.Meyer, and S.Semmes, Compensated compactness and Hardy spaces, J. Math. Pure Appl. 72(1993), 247-286.

- [CM] R.Coifman and Y.Meyer, Au-dela des Operateurs Pseudo-differentiels. Asterisque, #57, Soc. Math. de France, 1978.
- [CRW] R.Coifman, R.Rochberg, and G.Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. 103(1976), 611-635.
- [CH] H.O.Cordes and E.Herman, Gelfand theory of pseudo-differential operators, Amer. J. Math. 90(1968), 681-717.
- [DK] D.DeTurck and J.Kazdan, Some regularity theorems in Riemannian geometry, Ann. Sci. Ecole Norm. Sup. 14(1980), 249-260.
- [DR1] G.DiFazio and M.Ragusa, Commutators and Morrey spaces, Boll. Un. Mat. Ital. 5-A(1991), 323-332.
- [DR2] G.DiFazio and M.Ragusa, Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, Jour. Funct. Anal. 112(1993), 241-256.
- [Dou] R.Douglas, On the spectrum of Toeplitz and Wiener-Hopf operators, Abstract Spaces and Approximation (Proc. Conf. Oberwolfach 1968), pp.53-66, Birkhauser, Basel, 1969.
- [Fed] P.Federbush, Navier and Stokes meet the wavelet, Commun. Math. Phys. 155(1993), 219-248.
- [FS] C.Fefferman and E.Stein, H^p spaces of several variables, Acta Math. 129(1972), 137-193.
- [Fre] J.Frehse, A discontinuous solution to a mildly nonlinear elliptic system, Math. Zeit. 134(1973), 229-230.
- [Gia] M.Giaquinta, Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Princeton Univ. Press, 1983.
- [Gia2] M.Giaquinta, Nonlinear elliptic systems with quadratic growth, Manuscripta Math. 24(1978), 323-349.
- [GH] M.Giaquinta and S.Hildebrandt, A priori estimates for harmonic mappings, J. Reine Angew. Math. 336(1982), 124-164.
- [GM] Y.Giga and T.Miyakawa, Navier-Stokes flows in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces, Comm. PDE 14(1989), 577-618.
- [JR] J.Joly and J.Rauch, Justification of multidimensional single phase semilinear geometrical optics, Trans. AMS 330(1992), 599-623.
- [K] T.Kato, Strong solutions of the Navier-Stokes equations in Morrey spaces, Bol. Soc. Brasil. Math. 22(1992), 127-155.
- [KP] T.Kato and G.Ponce, Commutator estimates and the Euler and Navier-Stokes equations, CPAM 41(1988), 891-907.
- [KY] H.Kozono and M.Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data, Preprint, 1993.
- [Mey] Y.Meyer, Regularite des solutions des equations aux derivees partielles nonlineaires, Springer LNM #842(1980), 293-302.
- [P] J.Peetre, On the theory of $\mathcal{L}_{p,\lambda}$ spaces, J. Funct. Anal. 4(1969),

71-87.

- [RR] J.Rauch and M.Reed, Bounded, stratified, and striated solutions of hyperbolic systems, *Nonlinear Partial Differential Equations and their Applications*, Vol.9 (H.Brezis and J.Lions, eds.), Research Notes in Math. #181, Pitman, New York, 1989.
- [Sar] D.Sarason, Functions of vanishing mean oscillation, *Trans. AMS* 207(1975), 391-405.
- [Sch] R.Schoen, Analytic aspects of the harmonic map problem, pp.321-358 in [C].
- [St] E.Stein, *Singular Integrals and Pseudo-Differential Operators*, Graduate Lecture Notes, Princeton Univ., 1972.
- [T1] M.Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Birkhauser, Boston, 1991.
- [T2] M.Taylor, Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, *Comm. PDE* 17(1992), 1407-1456.

Recent IMA Preprints

#	Author/s	Title
1229	Zhangxin Chen,	Large-scale averaging analysis of multiphase flow in fractured reservoirs
1230	Bruce Hajek & Babu Narayanan,	Multigraphs with the most edge covers
1231	K.B. Athreya,	Entropy maximization
1232	F.I. Karpelevich & Yu.M. Suhov,	Functional equations in the problem of boundedness of stochastic branching dynamics
1233	E. Dibenedetto & V. Vespri,	On the singular equation $\beta(u)_t = \Delta u$
1234	M.Ya. Kelbert & Yu.M. Suhov,	The Markov branching random walk and systems of reaction-diffusion (Kolmogorov-Petrovskii-Piskunov) equations
1235	M. Hildebrand,	Random walks on random regular simple graphs
1236	W.S. Don & A. Solomonoff,	Accuracy enhancement for higher derivatives using Chebyshev collocation and a mapping technique
1237	D. Gurarie,	Symmetries and conservation laws of two-dimensional hydrodynamics
1238	Z. Chen,	Finite element methods for the black oil model in petroleum reservoirs
1239	G. Bao & A. Friedman,	Inverse problems for scattering by periodic structure
1240	G. Bao,	Some inverse problems in partial differential equations
1241	G. Bao,	Diffraction optics in periodic structures: The TM polarization
1242	C.C. Lim & D.A. Schmidt,	On noneven digraphs and symplectic pairs
1243	H.M. Soner, S.E. Shreve & J. Cvitanić,	There is no nontrivial hedging portfolio for option pricing with transaction costs
1244	D.L. Russell & B-Yu Zhang,	Exact controllability and stabilizability of the Korteweg-de Vries equation
1245	B. Morton, D. Enns & B-Yu Zhang,	Stability of dynamic inversion control laws applied to nonlinear aircraft pitch-axis models
1246	S. Hansen & G. Weiss,	New results on the operator Carleson measure criterion
1247	V.A. Malyshev & F.M. Spieksma,	Intrinsic convergence rate of countable Markov chains
1248	G. Bao, D.C. Dobson & J.A. Cox,	Mathematical studies in rigorous grating theory
1249	G. Bao & W.W. Symes,	On the sensitivity of solutions of hyperbolic equations to the coefficients
1250	D.A. Huntley & S.H. Davis,	Oscillatory and cellular mode coupling in rapid directional solidification
1251	M.J. Donahue, L. Gurvits, C. Darken & E. Sontag,	Rates of convex approximation in non-Hilbert spaces
1252	A. Friedman & B. Hu,	A Stefan problem for multi-dimensional reaction diffusion systems
1253	J.L. Bona & B-Y. Zhang,	The initial-value problem for the forced Korteweg-de Vries equation
1254	A. Friedman & R. Gulliver,	Organizers, Mathematical modeling for instructors
1255	S. Kichenassamy,	The prolongation formula for tensor fields
1256	S. Kichenassamy,	Fuchsian equations in Sobolev spaces and blow-up
1257	H.S. Dumas, L. Dumas, & F. Golse,	On the mean free path for a periodic array of spherical obstacles
1258	C. Liu,	Global estimates for solutions of partial differential equations
1259	C. Liu,	Exponentially growing solutions for inverse problems in PDE
1260	Mary Ann Horn & I. Lasiecka,	Nonlinear boundary stabilization of parallelly connected Kirchhoff plates
1261	B. Cockburn & H. Gau,	A posteriori error estimates for general numerical methods for scalar conservation laws
1262	B. Cockburn & P-A. Gremaud,	A priori error estimates for numerical methods for scalar conservation laws. Part I: The general approach
1263	R. Spigler & M. Vianello,	Convergence analysis of the semi-implicit euler method for abstract evolution equations
1264	R. Spigler & M. Vianello,	WKB-type approximation for second-order differential equations in C^* -algebras
1265	M. Menshikov & R.J. Williams,	Passage-time moments for continuous non-negative stochastic processes and applications
1266	C. Mazza,	On the storage capacity of nonlinear neural networks
1267	Z. Chen, R.E. Ewing & R. Lazarov,	Domain decomposition algorithms for mixed methods for second order elliptic problems
1268	Z. Chen, M. Espedal & R.E. Ewing,	Finite element analysis of multiphase flow in groundwater hydrology
1269	Z. Chen, R.E. Ewing, Y.A. Kuznetsov, R.D. Lazarov & S. Maliassov,	Multilevel preconditioners for mixed methods for second order elliptic problems
1270	S. Kichenassamy & G.K. Srinivasan,	The structure of WTC expansions and applications
1271	A. Zinger,	Positiveness of Wigner quasi-probability density and characterization of Gaussian distribution
1272	V. Malkin & G. Papanicolaou,	On self-focusing of short laser pulses
1273	J.N. Kutz & W.L. Kath,	Stability of pulses in nonlinear optical fibers using phase-sensitive amplifiers
1274	S.K. Patch,	Recursive recovery of a family of Markov transition probabilities from boundary value data
1275	C. Liu,	The completeness of plane waves
1276	Z. Chen & R.E. Ewing,	Stability and convergence of a finite element method for reactive transport in ground water

- 1277 **Z. Chen & Do Y. Kwak**, The analysis of multigrid algorithms for nonconforming and mixed methods for second order elliptic problems
- 1278 **Z. Chen**, Expanded mixed finite element methods for quasilinear second order elliptic problems II
- 1279 **M.A. Horn & W. Littman**, Boundary control of a Schrödinger equation with nonconstant principal part
- 1280 **R.E. Ewing, Yu.A. Kuznetsov, R.D. Lazarov & S. Maliassov**, Substructuring preconditioning for finite element approximations of second order elliptic problems. I. Nonconforming linear elements for the Poisson equation in a parallelepiped
- 1281 **S. Maliassov**, Substructuring preconditioning for finite element approximations of second order elliptic problems. II. Mixed method for an elliptic operator with scalar tensor
- 1282 **V. Jakšić & C.-A. Pillet**, On model for quantum friction II. Fermi's golden rule and dynamics at positive temperatures
- 1283 **V. M. Malkin**, Kolmogorov and nonstationary spectra of optical turbulence
- 1284 **E.G. Kalnins, V.B. Kuznetsov & W. Miller, Jr.**, Separation of variables and the XXZ Gaudin magnet
- 1285 **E.G. Kalnins & W. Miller, Jr.**, A note on tensor products of q -algebra representations and orthogonal polynomials
- 1286 **E.G. Kalnins & W. Miller, Jr.**, q -algebra representations of the Euclidean, pseudo-Euclidean and oscillator algebras, and their tensor products
- 1287 **L.A. Pastur**, Spectral and probabilistic aspects of matrix models
- 1288 **K. Kastella**, Discrimination gain to optimize detection and classification
- 1289 **L.A. Peletier & W.C. Troy**, Spatial patterns described by the Extended Fisher-Kolmogorov (EFK) equation: Periodic solutions
- 1290 **A. Friedman & Y. Liu**, Propagation of cracks in elastic media
- 1291 **A. Friedman & C. Huang**, Averaged motion of charged particles in a curved strip
- 1292 **G. R. Sell**, Global attractors for the 3D Navier-Stokes equations
- 1293 **C. Liu**, A uniqueness result for a general class of inverse problems
- 1294 **H-O. Kreiss**, Numerical solution of problems with different time scales II
- 1295 **B. Cockburn, G. Gripenberg, S-O. Londen**, On convergence to entropy solutions of a single conservation law
- 1296 **S-H. Yu**, On stability of discrete shock profiles for conservative finite difference scheme
- 1297 **H. Behncke & P. Rejto**, A limiting absorption principle for separated Dirac operators with Wigner Von Neumann type potentials
- 1298 **R. Lipton B. Vernescu**, Composites with imperfect interface
- 1299 **E. Casas**, Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations
- 1300 **G.R. Sell**, References on dynamical systems
- 1301 **J. Zhang**, Swelling and dissolution of polymer: A free boundary problem
- 1302 **J. Zhang**, A nonlinear nonlocal multi-dimensional conservation law
- 1303 **M.E. Taylor**, Estimates for approximate solutions to acoustic inverse scattering problems
- 1304 **J. Kim & D. Sheen**, A priori estimates for elliptic boundary value problems with nonlinear boundary conditions
- 1305 **B. Engquist & E. Luo**, New coarse grid operators for highly oscillatory coefficient elliptic problems
- 1306 **A. Boutet de Monvel & I. Egorova**, On the almost periodicity of solutions of the nonlinear Schrödinger equation with the cantor type spectrum
- 1307 **A. Boutet de Monvel & V. Georgescu**, Boundary values of the resolvent of a self-adjoint operator: Higher order estimates
- 1308 **S.K. Patch**, Diffuse tomography modulo Graßmann and Laplace
- 1309 **A. Friedman & J.J.L. Velázquez**, Liouville type theorems for fourth order elliptic equations in a half plane
- 1310 **T. Aktosun, M. Klaus & C. van der Mee**, Recovery of discontinuities in a nonhomogeneous medium
- 1311 **V. Bondarevsky**, On the global regularity problem for 3-dimensional Navier-Stokes equations
- 1312 **M. Cheney & D. Isaacson**, Inverse problems for a perturbed dissipative half-space
- 1313 **B. Cockburn, D.A. Jones & E.S. Titi**, Determining degrees of freedom for nonlinear dissipative equations
- 1314 **B. Engquist & E. Luo**, Convergence of a multigrid method for elliptic equations with highly oscillatory coefficients
- 1315 **L. Pastur & M. Shcherbina**, Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles
- 1316 **V. Jakšić, S. Molchanov & L. Pastur**, On the propagation properties of surface waves
- 1317 **J. Nečas, M. Ružička & V. Šverák**, On self-similar solutions of the Navier-Stokes equations
- 1318 **S. Stojanovic**, Remarks on $W^{2,p}$ -solutions of bilateral obstacle problems
- 1319 **E. Luo & H-O. Kreiss**, Pseudospectral vs. Finite difference methods for initial value problems with discontinuous coefficients
- 1320 **V.E. Krikurov**, Soliton's rebuilding in one-dimensional Schrödinger model with polynomial nonlinearity
- 1321 **J.M. Harrison & R.J. Williams**, A multiclass closed queueing network with unconventional heavy traffic behavior
- 1322 **M.E. Taylor**, Microlocal analysis on Morrey spaces
- 1323 **C. Huang**, Homogenization of biharmonic equations in domains perforated with tiny holes
- 1324 **C. Liu**, An inverse obstacle problem: A uniqueness theorem for spheres