DIFFUSE TOMOGRAPHY MODULO GRAßMANN AND LAPLACE

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Abstract. Complicated and unphysical families of modified transition probabilities for the 4×4 diffuse tomographic problem are presented. Graßmann-Plücker identities and Laplace expansions of determinants are used to simplify the initial transition probabilities. Besides restoring self-consistency to the system enforcing range conditions eliminates one half of the parameters.

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1. Introduction. Applications in medical imaging motivated efforts to invert two and three dimensional versions of the Redheffer *-product [1], [2], [3], [4]. Motivation for this work comes from optical imaging. One possible application is monitoring the brains of neonates, which are unable to withstand repeated X ray irradiation. Brain hemorrhage is the leading cause of death among premature infants. The goal is to detect cranial bleeds without overexposing the infant to harmful radiation.

Range conditions were studied in [5], [6] for very general two and three dimensional systems. Because of these range conditions the inverse problem is underdetermined. In [7] a recursive algorithm was sketched for recovering a p -parameter family of solutions to the inverse problem in the plane where the difference between the amount of independent data and number of unknowns is p . The purpose of this paper is to fill in the details.

The first section of this paper is an introduction to the problem, description of its range conditions as well as a $2p$ parameter family of solutions. The heart of this paper is sections 2 and 3, where the first level of the recursive scheme is made practical. Because of the combinatorial nature of the problem several different matrix identities prove useful. Graßmann-Plücker identities are used to derive some of them in section 2.1. The solutions given in section 1.4 are presented modulo the matrix identities in section 2.2. These solutions are functions of the data and 64 parameters. In section 3.3.2,

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Graßmann-Plücker relations in conjunction with range conditions yield even more useful identities. In section 3.3.4 all of these identities are combined with the Laplace expansion of a determinant. (Sections 3.2.2 and 3.3.4 are quite technical and may be omitted without losing sight of the main objective.) Finally, in section 3 range conditions are used to place p independent constraints upon the parameters. Sections 1.3 and 1.4 give insight into the difficulty of enforcing the various range conditions. Those corresponding to travel from the center of the 4×4 system out of the system are much simpler than those corresponding to travel towards the center.

1.1. Description of the model. Consider a $n \times n$ array of pixels in the plane. On each outer face there are two devices. One device shoots photons across the outside edge into the neighboring pixel; the other device detects photons as they leave the system. For each of the $4n$ outside edges $4n$ pieces of data are collected. These data are stored as a $4n \times 4n$ transition matrix, Q . Within the system, photons travel either horizontally or vertically. Label the horizontal and vertical axes x_1 and x_2 , respectively. A photon traveling parallel to the x_i axis in the positive direction moves with velocity x_i^+ . Photons traveling in the opposite direction travel with velocity x_i^- . They do not interact and may be absorbed within a pixel. Photons move according to a Markov process. The probabilities with which a photon moves to a neighboring pixel depend upon its previous, as well as present, location. In this two step formulation the state space consists of locations. The state space may be redefined so that photons move according to a one step Markov process. In the new state space a single state consists of the photon's location and direction of travel.

There are three different types of these Markov states: incoming, outgoing, and hidden. The probabilities with which photons move from one state to another are referred to as transition probabilities. For example, a photon which travels with velocity x_i^+ into pixel O and travels straight through pixel O does so with some probability, denoted by $x_i^+ O x_i^+$. For each pixel O and incident direction the sum of the absorption probability and the four possible transition probabilities must be identically one. The absorption probability is therefore neglected in the rest of this paper. Each pixel corresponds to 16 transition probabilities. The same photon travels to O 's neighbor in the x_j^+ direction with probability $x_i^+ O x_j^+$. These probabilities are the nonzero entries of the Markov transition matrix M . M is sparse and may be written as a block matrix with nontrivial subblocks P_{io} , P_{ih} , P_{ho} , and P_{hh} . P_{io} , for example, contains the probabilities with which photons in incoming states move directly to outgoing states. P_{ih} contains the probabilities with which photons in incoming states move to hidden states. P_{ho} and P_{hh} are the transition matrices for photons starting in hidden states traveling to outgoing and hidden states, respectively. P_{io} and P_{hh} are always square matrices.

1.2. Forward Problem. The forward map takes $16n^2$ transition probabilities to the $4n \times 4n$ data matrix Q . The domain of the forward map lies in the unit cube in $\mathbb{R}^{16n^2}$ and is defined by

$$(1.1) \quad (x_i^\pm O x_1^+ + x_i^\pm O x_1^-) + (x_i^\pm O x_2^+ + x_i^\pm O x_2^-) \leq 1 \quad \pm = +, - \quad i = 1, 2$$

for each pixel O . Furthermore, none of these transition probabilities is permitted to be zero. Q_i^j represents the probability that a photon which enters the system at source i exits the system at detector j . Q provides no time-of-flight information. Because Q is a transition matrix acceptable solutions lie in the unit cube in $\mathbb{R}^{16n^2}$ and satisfy

$$(1.2) \quad 0 \leq \sum_{\lambda=1}^{4n} Q_i^\lambda \leq 1, \quad i = 1, 2, \dots, 4n$$

The forward map is given by the following matrix expression:

$$(1.3) \quad Q = Pio + Pih \sum_{n=0}^{\infty} Phh^n Pho = Pio + Pih (I - Phh)^{-1} Pho$$

1.3. Range Conditions. Range conditions appear as rank deficient submatrices of Q . Each of these rank deficient submatrices represents travel from one “side” of the system to the other “side”. Let b be a (not necessarily straight) barrier of $\#b$ hidden states separating the “sides”. The Markovian nature of the system can be used to show that the corresponding submatrix is generically of rank $\#b$ [5]. For the purposes of this paper it is only necessary to consider straight barriers.

Notation: Let Q_r^c denote the submatrix of Q taken from rows r and columns c . Let dQ_r^c denote the determinant of this submatrix. Furthermore, let $a - b$ denote $a, a + 1, \dots, b$ where $a, b \in \mathbb{N}$.

For example, the data matrix for a 2×2 system has many rank deficient submatrices. See figure 1. The 4×4 submatrix representing travel from left to right, $Q_{1,2,3,4}^{5,6,7,8}$, is generically rank two, as is $Q_{5,6,7,8}^{1,2,3,4}$. Similarly, the submatrices $Q_{3,4,5,6}^{1,2,7,8}$ and $Q_{1,2,7,8}^{3,4,5,6}$ are generically of rank two as well.

1.4. Modified Transition Matrices in terms of A . Although the final goal is to recover the microscopic transition probabilities for each pixel from boundary value data, the purpose of this section is more modest. A 32 parameter family of data sets for the subsystems shown in figure 3 are computed from the original data. Once the data sets for each of the subsystems are found the 2×2 subsystems can be tackled in parallel.

The sparse block structures of Pio , Pho , Pih , and Phh were used in [7] to solve for nonzero blocks of Phh , Pio , and Pih in terms of $A \equiv Pho^{-1}$. This result is used here and requires the following

Notation: $[M : N]$ denotes the concatenation of matrices M and N (where M and N have the same number of rows). Furthermore, $a - b \equiv a, a + 1, \dots, b$ and let r_{a-b} denote any choice of one half of the natural numbers between a and b inclusive. ($b - a$ is always odd.) Finally, define $r_1 \equiv r_{1-8}$, $r_5 \equiv r_{5-12}$, $r_9 \equiv r_{9-16}$, and $r_s \equiv r_{1-4,13-16}$.

The following solutions will be used later to eliminate some of the A_i^j s.

$$(1.4) \quad Phh_{1-4}^{5,6} = Ph_{1-4}^{1-4} (Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6}$$

$$(1.5) \quad Phh_{1-4}^{15,16} = Ph_{1-4}^{1-4} (Q_{r_5}^{1-4})^{-1} Q_{r_5}^{13-16} A_{13-16}^{15,16}$$

$$(1.6) \quad Pih_{1-4}^{5,6} = \left[Q_{1-4}^{5-8} - (Q_{1-4}^{1-4} - Pio_{1-4}^{1-4}) (Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8} \right] A_{5-8}^{5,6}$$

$$(1.7) \quad Pih_{1-4}^{15,16} = \left[Q_{1-4}^{13-16} - (Q_{1-4}^{1-4} - Pio_{1-4}^{1-4}) (Q_{r_5}^{1-4})^{-1} Q_{r_5}^{13-16} \right] A_{13-16}^{15,16} \quad \text{where}$$

$$Pio_{1-4}^{1-4} = Q_{1-4}^{1-4} - \left[Q_{1-4}^{13-16} (Q_{r_5}^{13-16})^{-1} Q_{r_5}^{1-4} A_{1-4}^{1,2} : Q_{1-4}^{5-8} (Q_{r_9}^{5-8})^{-1} Q_{r_9}^{1-4} A_{1-4}^{3,4} \right] Ph_{1-4}^{1-4} \quad (1.8)$$

Solutions for blocks within the same transition matrix have a common structure. Hidden-outgoing and hidden-hidden transitions can be expressed in terms of Q and A much more simply than incoming-outgoing and incoming-hidden transitions.

2. Simplifying the Solutions. Although the above solutions are relatively compact, the solutions for Pio and Pih can be simplified. The Graßmann-Plücker identities found in classical algebraic geometry are used to derive matrix identities in the next section. The results are used to simplify the solutions for Pio and Pih in section 2.2.

2.1. Matrix truisms. Since the inverse problem involves linear systems, it is not surprising that Graßmannians and the Graßmann-Plücker embedding come into play. The identities which embed Graßmannians $G(k, n)$ in $\mathbb{P}^{\binom{n}{k}-1}$ are derived below. A cursory explanation of the embedding can be found in [8, 9]. For a more thorough exposition see [10, 11]. Let $\mathbf{\Lambda}$ be any rectangular matrix with k rows and n columns where $k < n - 1$ and $\mathbf{\Lambda} = (a)_{ij}$. Let $I = (i_1, i_2, i_3, \dots, i_{(k-1)})$ index $(k-1)$ distinct columns of $\mathbf{\Lambda}$. Let $J = (j_1, j_2, j_3, \dots, j_{(k+1)})$ index $(k+1)$ distinct columns of $\mathbf{\Lambda}$. Then,

$$(2.9) \quad \sum_{\lambda=1}^{k+1} \pi_{(i_1, i_2, \dots, i_{k-1}, j_\lambda)} \pi_{(j_1, j_2, \dots, j_{\lambda-1}, j_{\lambda+1}, \dots, j_{k+1})} = 0$$

Equation 2.9 defines the Graßmann relations. Let $\alpha, \beta, \gamma, \eta, \kappa \in \mathbb{N}^k$. For any matrix Q basic matrix properties imply

$$(2.10) \quad (Q_\alpha^\gamma)^{-1} Q_\alpha^\eta = \left(\frac{dQ_\alpha^{\gamma_1, \gamma_2, \dots, \gamma_{i-1}, \eta_j, \gamma_{i+1}, \dots, \gamma_n}}{dQ_\alpha^\gamma} \right)_{i,j}$$

$$(2.11) \quad Q_\kappa^\gamma (Q_\alpha^\gamma)^{-1} Q_\alpha^\eta = Q_\kappa^\eta - (1/dQ_\alpha^\gamma) (dQ_{\kappa_i, \alpha}^{\eta_j, \gamma})_{i,j}$$

The identity 2.10 combines with Graßmann-Plücker relations to imply

$$\begin{aligned} (dQ_{\kappa_i, \alpha}^{\gamma_k, \beta})_{i,k} (Q_\alpha^\gamma)^{-1} Q_\alpha^\eta &= (dQ_{\kappa_i, \alpha}^{\gamma_k, \beta})_{i,k} \left(\frac{dQ_\alpha^{\gamma_1, \gamma_2, \dots, \gamma_{k-1}, \eta_j, \gamma_{k+1}, \dots, \gamma_n}}{dQ_\alpha^\gamma} \right)_{k,j} \\ &= \frac{-1}{dQ_\alpha^\gamma} \sum_{k=1}^n (-1)^k \left(dQ_{\kappa_i, \alpha}^{\gamma_k, \beta} dQ_\alpha^{\eta_j, \gamma_1, \gamma_2, \dots, \gamma_{k-1}, \gamma_{k+1}, \dots, \gamma_n} \right)_{i,j} \\ (2.12) \quad &= \left(dQ_{\kappa_i, \alpha}^{\eta_j, \beta} - \frac{dQ_\alpha^\beta}{dQ_\alpha^\gamma} dQ_{\kappa_i, \alpha}^{\eta_j, \gamma} \right)_{i,j} \end{aligned}$$

These matrix identities can be used to simplify 1.6 through 1.8. Furthermore, in this case where Pho is block diagonal and $A = Pho^{-1}$,

$$(2.13) \quad I = A_{1-4}^{1-4} Pho_{1-4}^{1-4} = A_{1-4}^{1,2} Pho_{1,2}^{1-4} + A_{1-4}^{3,4} Pho_{3,4}^{1-4}$$

This argument applies to other blocks on the diagonal, as well.

2.2. Matrix Solutions Revisited. The solution for Pio_{1-4}^{1-4} can be simplified using 2.11 as follows

$$\begin{aligned}
(2.14) \quad Pio_{1-4}^{1-4} &= Q_{1-4}^{1-4} - \left[\left(Q_i^j - \frac{dQ_{i,r_5}^{j,13-16}}{Q_{r_5}^{13-16}} \right)_{i,j} A_{1-4}^{1,2} : \left(Q_i^j - \frac{dQ_{i,r_9}^{j,5-8}}{Q_{r_9}^{5-8}} \right)_{i,j} A_{1-4}^{3,4} \right] Pho_{1-4}^{1-4} \\
&= Q_{1-4}^{1-4} - \left[Q_{1-4}^{1-4} A_{1-4}^{1,2} : Q_{1-4}^{1-4} A_{1-4}^{3,4} \right] Pho_{1-4}^{1-4} + \\
&\quad \left[\left(\frac{dQ_{i,r_5}^{j,13-16}}{Q_{r_5}^{13-16}} \right)_{i,j} A_{1-4}^{1,2} : \left(\frac{dQ_{i,r_9}^{j,5-8}}{Q_{r_9}^{5-8}} \right)_{i,j} A_{1-4}^{3,4} \right] Pho_{1-4}^{1-4} \\
&= \left[\left(\frac{dQ_{i,r_5}^{j,13-16}}{Q_{r_5}^{13-16}} \right)_{i,j} A_{1-4}^{1,2} : \left(\frac{dQ_{i,r_9}^{j,5-8}}{Q_{r_9}^{5-8}} \right)_{i,j} A_{1-4}^{3,4} \right] Pho_{1-4}^{1-4} \\
&= \left(\frac{dQ_{i,r_5}^{j,13-16}}{Q_{r_5}^{13-16}} \right)_{i,j} A_{1-4}^{1,2} Pho_{1,2}^{1-4} + \left(\frac{dQ_{i,r_9}^{j,5-8}}{Q_{r_9}^{5-8}} \right)_{i,j} A_{1-4}^{3,4} Pho_{3,4}^{1-4} \\
&= \left(\frac{dQ_{i,r_5}^{j,13-16}}{Q_{r_5}^{13-16}} - \frac{dQ_{i,r_9}^{j,5-8}}{Q_{r_9}^{5-8}} \right)_{i,j} A_{1-4}^{1,2} Pho_{1,2}^{1-4} + \left(\frac{dQ_{i,r_9}^{j,5-8}}{Q_{r_9}^{5-8}} \right)_{i,j} A_{1-4}^{3,4} Pho_{3,4}^{1-4}
\end{aligned}$$

Define

$$(2.15) \quad M^1 = \left(\frac{dQ_{i,r_5}^{j,13-16}}{Q_{r_5}^{13-16}} - \frac{dQ_{i,r_9}^{j,5-8}}{Q_{r_9}^{5-8}} \right)_{i,j} A_{1-4}^{1,2} Pho_{1,2}^{1-4}$$

Then the nonzero entries of Pio can be expressed (fairly) compactly, albeit nonuniquely:

$$(2.16) \quad Pio_{1-4}^{1-4} = M^1 + \left(\frac{dQ_{i,r_9}^{j,5-8}}{Q_{r_9}^{5-8}} \right)_{i,j} A_{1-4}^{3,4} Pho_{3,4}^{1-4}$$

Simpler solutions for Pio can be used to express Pih more succinctly. For instance, substituting the matrix identity 2.10 as well as 2.16 and

$$(2.17) \quad \left(\frac{dQ_{i+12,5-8}^{j+12,9-12}}{Q_{5-8}^{9-12}} \right)_{i,j} (Q_{5-8}^{13-16})^{-1} Q_{5-8}^{1-4} = \left(\frac{dQ_{i+12,5-8}^{j+12,9-12}}{dQ_{5-8}^{9-12}} - \frac{dQ_{i+12,5-8}^{j+12,13-16}}{dQ_{5-8}^{13-16}} \right)_{i,j}$$

$$(2.18) \quad \left(\frac{dQ_{i+12,r_5}^{j+12,1-4}}{Q_{r_5}^{1-4}} \right)_{i,j} (Q_{r_5}^{13-16})^{-1} Q_{r_5}^{1-4} = \left(\frac{dQ_{i+12,r_5}^{j+12,1-4}}{dQ_{r_5}^{1-4}} - \frac{dQ_{i+12,r_5}^{j+12,13-16}}{dQ_{r_5}^{13-16}} \right)_{i,j} \equiv - \left(\frac{dQ_{i+12,r_5}^{j+12,13-16}}{dQ_{r_5}^{13-16}} \right)_{i,j}$$

into equations 1.6 and 1.7 yields

$$(2.19) \quad Pih_{1-4}^{5,6} = M^1 (Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6}$$

$$(2.20) \quad Pih_{1-4}^{15,16} = \left[\left(\frac{dQ_{i,9-12}^{j+12,5-8}}{dQ_{9-12}^{5-8}} \right)_{i,j} + M^1 (Q_{9-12}^{1-4})^{-1} Q_{9-12}^{13-16} \right] A_{13-16}^{15,16}$$

In 2.17 and 2.20 the choice of rows is dictated by the matrix identity 2.12. Similar computations give simple expressions for each block of Pio and Pih in terms of Q and the 64 nonzero entries of A .

3. Removing A_i^j s. In this section range conditions are used to derive 32 independent conditions upon the A_i^j s. The entries of Pho , Phh , Pio , and Pih are (almost) the data for each of the 2×2 subsystems. They do not obey the range conditions mentioned in section 1.3, however. Range conditions for the 2×2 subsystems are zero valued 3×3 minors of the subsystems' data sets. Enforcing these conditions yields polynomials in the A_i^j s. These polynomials factor into products of several terms; the relevant term is always linear. Although there are many such conditions, only 32 of them are independent. Only half, therefore, of the parameters may be eliminated by virtue of range conditions upon the 2×2 subsystems.

Each of the four subsystems has an 8×8 data matrix. The data matrix for the 1,1 subsystem is shown below:

$$(3.21) \quad Q_{11} = \begin{bmatrix} Pio_2^2 & Pio_2^3 & Pio_2^4 & Pih_2^5 & Pih_2^6 & Pih_2^{15} & Pih_2^{16} & Pio_2^1 \\ Pio_3^2 & Pio_3^3 & Pio_3^4 & Pih_3^5 & Pih_3^6 & Pih_3^{15} & Pih_3^{16} & Pio_3^1 \\ Pio_4^2 & Pio_4^3 & Pio_4^4 & Pih_4^5 & Pih_4^6 & Pih_4^{15} & Pih_4^{16} & Pio_4^1 \\ Pho_4^2 & Pho_4^3 & Pho_4^4 & Phh_4^5 & Phh_4^6 & Phh_4^{15} & Phh_4^{16} & Pho_4^1 \\ Pho_3^2 & Pho_3^3 & Pho_3^4 & Phh_3^5 & Phh_3^6 & Phh_3^{15} & Phh_3^{16} & Pho_3^1 \\ Pho_2^2 & Pho_2^3 & Pho_2^4 & Phh_2^5 & Phh_2^6 & Phh_2^{15} & Phh_2^{16} & Pho_2^1 \\ Pho_1^2 & Pho_1^3 & Pho_1^4 & Phh_1^5 & Phh_1^6 & Phh_1^{15} & Phh_1^{16} & Pho_1^1 \\ Pio_1^2 & Pio_1^3 & Pio_1^4 & Pih_1^5 & Pih_1^6 & Pih_1^{15} & Pih_1^{16} & Pio_1^1 \end{bmatrix}$$

Q_{11} has four rank deficient submatrices. They are 4×4 submatrices of rank two (or less). Two constraints are required to force a generic vector in $\mathbb{R}^4$ to lie in a given two dimensional subspace. Four conditions are required, therefore, to force a generic 4×4 matrix to be of rank two. These conditions will be studied in order of increasing complexity. (Clearly, the conditions which involve variables from Pih are bound to be horrendous, so they are not considered until much later.) Eight of the conditions are identities of the form $A_i^j = 0$. The rest reduce (at a generic point) to four term linear equations. Right-left, left-right, top-bottom, and bottom-top rank deficient submatrices are labeled as Qij_{rl} , Qij_{lr} , Qij_{tb} , and Qij_{bt} where $i, j = 1, 2$. One such rank deficient submatrix whose entries are relatively simple expressions in A and Q is

$$(3.22) \quad Q_{11_{rl}} = \begin{bmatrix} Q_{11}_5^1 & Q_{11}_5^2 & Q_{11}_5^3 & Q_{11}_5^4 \\ Q_{11}_6^1 & Q_{11}_6^2 & Q_{11}_6^3 & Q_{11}_6^4 \\ Q_{11}_7^1 & Q_{11}_7^2 & Q_{11}_7^3 & Q_{11}_7^4 \\ Q_{11}_8^1 & Q_{11}_8^2 & Q_{11}_8^3 & Q_{11}_8^4 \end{bmatrix} = \begin{bmatrix} Pho_3^2 & Pho_3^3 & Pho_3^4 & Phh_3^5 \\ Pho_2^2 & Pho_2^3 & Pho_2^4 & Phh_2^5 \\ Pho_1^2 & Pho_1^3 & Pho_1^4 & Phh_1^5 \\ Pio_1^2 & Pio_1^3 & Pio_1^4 & Pih_1^5 \end{bmatrix}$$

Other relatively simple rank deficient submatrices are the “hidden-outgoing” submatrices: $Q_{11_{bt}}$, $Q_{12_{bt}}$, $Q_{12_{lr}}$, $Q_{21_{tb}}$, $Q_{21_{rl}}$, $Q_{22_{lr}}$, and $Q_{22_{tb}}$. The preponderance of their entries represent hidden-outgoing transitions. Entries of “incoming-hidden” submatrices are predominantly incoming-hidden transitions. There are eight “incoming-hidden” submatrices: $Q_{11_{lr}}$, $Q_{21_{lr}}$, $Q_{21_{bt}}$, $Q_{22_{bt}}$, $Q_{22_{rl}}$, $Q_{12_{rl}}$, and $Q_{12_{tb}}$.

3.1. Identities. Since $Q_{11_{rl}}$ is rank two, the determinant of any 3×3 minor is identically zero. Recall that A is a 16×16 block matrix, with 4×4 blocks on the diagonal. The upper left block of A is the inverse of the upper left block of Pho , and so

$$-A_4^1 = \begin{vmatrix} Pho_1^2 & Pho_1^3 & Pho_1^4 \\ Pho_2^2 & Pho_2^3 & Pho_2^4 \\ Pho_3^2 & Pho_3^3 & Pho_3^4 \end{vmatrix} / dPho_{1-4}^1 = \begin{vmatrix} Q_{11}_5^1 & Q_{11}_5^2 & Q_{11}_5^3 \\ Q_{11}_6^1 & Q_{11}_6^2 & Q_{11}_6^3 \\ Q_{11}_7^1 & Q_{11}_7^2 & Q_{11}_7^3 \end{vmatrix} / dPho_{1-4}^1 = 0$$

The same reasoning applies to $Q_{11_{bt}}$ and shows that $A_1^4 = 0$. This argument also applies to the rank-deficient submatrices $Q_{21_{rl}}$, $Q_{21_{tb}}$, $Q_{12_{lr}}$, $Q_{12_{bt}}$, $Q_{22_{tb}}$, and $Q_{22_{lr}}$ and yields the following identities:

$$(3.23) \quad \begin{aligned} A_1^4 &= 0, & A_4^1 &= 0, & A_5^8 &= 0, & A_8^5 &= 0 \\ A_9^{12} &= 0, & A_{12}^9 &= 0, & A_{13}^{16} &= 0, & A_{16}^{13} &= 0 \end{aligned}$$

3.2. “Easy” Conditions. The rest of the equations used to removed parameters are linear in the A_i^j s. The simplest result from forcing the other 3×3 minors of “hidden-outgoing” submatrices to be zero. In this section eight more conditions will be derived by considering the 3×3 minors of “hidden-outgoing” submatrices whose entries are only hidden-outgoing and hidden-hidden transitions. Then it is shown that all other conditions upon the “hidden-outgoing” submatrices are redundant.

3.2.1. Hidden-outgoing and hidden-hidden conditions. Consider the 3×4 submatrix of Q_{11} ,

$$(3.24) \quad \begin{bmatrix} Q_{11}_5^1 & Q_{11}_5^2 & Q_{11}_5^3 & Q_{11}_5^4 \\ Q_{11}_6^1 & Q_{11}_6^2 & Q_{11}_6^3 & Q_{11}_6^4 \\ Q_{11}_7^1 & Q_{11}_7^2 & Q_{11}_7^3 & Q_{11}_7^4 \end{bmatrix} = \begin{bmatrix} Pho_3^2 & Pho_3^3 & Pho_3^4 & Phh_3^5 \\ Pho_2^2 & Pho_2^3 & Pho_2^4 & Phh_2^5 \\ Pho_1^2 & Pho_1^3 & Pho_1^4 & Phh_1^5 \end{bmatrix}$$

Since $Phh_{1-3}^5 = Pho_{1-3}^{1-4} (Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8} A_{5-8}^5$ it helps to define

$$(3.25) \quad \mathbf{v} = (Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8} A_{5-8}^5$$

Then every 3×3 minor of $\begin{bmatrix} Pho_1^2 & Pho_1^3 & Pho_1^4 \\ Pho_2^2 & Pho_2^3 & Pho_2^4 & Pho_{1-3}^{1-4} \mathbf{v} \\ Pho_3^2 & Pho_3^3 & Pho_3^4 \end{bmatrix}$ must be identically zero.

Let $\{\alpha, \beta, \gamma\} = \{2, 3, 4\}$. Then

$$(3.26) \quad \begin{aligned} \begin{vmatrix} Pho_1^\alpha & Pho_1^\beta \\ Pho_2^\alpha & Pho_2^\beta & Pho_{1-3}^{1-4} \mathbf{v} \\ Pho_3^\alpha & Pho_3^\beta \end{vmatrix} &= \begin{vmatrix} Pho_1^\alpha & Pho_1^\beta & Pho_1^1 \\ Pho_2^\alpha & Pho_2^\beta & Pho_2^1 \\ Pho_3^\alpha & Pho_3^\beta & Pho_3^1 \end{vmatrix} v_1 + \begin{vmatrix} Pho_1^\alpha & Pho_1^\beta & Pho_1^\gamma \\ Pho_2^\alpha & Pho_2^\beta & Pho_2^\gamma \\ Pho_3^\alpha & Pho_3^\beta & Pho_3^\gamma \end{vmatrix} v_\gamma \\ &= \pm A_4^\gamma v_1 / dA_{1-4}^{1-4} + \pm A_4^1 v_\gamma / dA_{1-4}^{1-4} \\ &= \pm A_4^\gamma v_1 / dA_{1-4}^{1-4} \end{aligned}$$

since $A_4^1 \equiv 0$. Furthermore, A_4^γ is generically nonzero, forcing $v_1 = 0$. The matrix identity 2.10 says that the first row of $(Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8}$ equals $\left(\frac{dQ_{r_9}^{j+4,2,3,4}}{dQ_{r_9}^{1-4}} \right)_j$ which implies

$$(3.27) \quad 0 = \sum_{j=5}^8 dQ_{r_9}^{j,2,3,4} A_j^5 \equiv \sum_{j=5}^7 dQ_{r_9}^{j,2,3,4} A_j^5$$

where the last equivalence holds by virtue of the fact that $A_8^5 \equiv 0$. The same reasoning applied to the other “hidden-outgoing” submatrices results in

$$(3.28) \quad \begin{aligned} 0 &= \sum_{j=1}^3 dQ_{r_5}^{j,14,15,16} A_j^1 = \sum_{j=2}^4 dQ_{r_9}^{j,5,6,7} A_j^4 = \sum_{j=6}^8 dQ_{r_8}^{j,9,10,11} A_j^8 = \sum_{j=9}^{11} dQ_{r_8}^{j,6,7,8} A_j^9 \\ &= \sum_{j=10}^{12} dQ_{r_1}^{j,13,14,15} A_j^{12} = \sum_{j=13}^{15} dQ_{r_1}^{j,10,11,12} A_j^{13} = \sum_{j=14}^{16} dQ_{r_5}^{j,1,2,3} A_j^{16} \end{aligned}$$

3.2.2. Redundant “Easy” Conditions. Conditions which involve incoming-outgoing as well as hidden-outgoing transitions are equivalent to 3.27 and

$$\begin{bmatrix} Q11_5^1 & Q11_5^2 & Q11_5^3 \\ Q11_6^1 & Q11_6^2 & Q11_6^3 \\ Q11_7^1 & Q11_7^2 & Q11_7^3 \\ Q11_8^1 & Q11_8^2 & Q11_8^3 \end{bmatrix} = \begin{bmatrix} Pho_3^2 & Pho_3^3 & Pho_3^4 \\ Pho_2^2 & Pho_2^3 & Pho_2^4 \\ Pho_1^2 & Pho_1^3 & Pho_1^4 \\ Pio_1^2 & Pio_1^3 & Pio_1^4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \mathbf{x} A_{1-4}^{1,2} Pho_{1,2}^{1-4} + \mathbf{w} \end{bmatrix}$$

where

$$(3.29) \quad \mathbf{x} = \left(\frac{dQ_{1,r_5}^{j,13-16}}{dQ_{r_5}^{13-16}} - \frac{dQ_{1,r_9}^{j,5-8}}{dQ_{r_9}^{5-8}} \right)_{j=1,2,3,4}$$

$$(3.30) \quad \mathbf{w} = \left(\frac{dQ_{1,r_9}^{j,5-8}}{dQ_{r_9}^{5-8}} \right)_{j=2,3,4}$$

Then for $1 \leq \alpha < \beta \leq 3$

$$\begin{aligned}
0 &= \begin{vmatrix} Pho_\alpha^2 & Pho_\alpha^3 & Pho_\alpha^4 \\ Pho_\beta^2 & Pho_\beta^3 & Pho_\beta^4 \\ Pio_1^2 & Pio_1^3 & Pio_1^4 \end{vmatrix} \\
&= \begin{vmatrix} \begin{bmatrix} \mathbf{e}_\alpha \\ \mathbf{e}_\beta \\ \mathbf{x} [A_{1-4}^{1,2} : 0] \end{bmatrix} & \begin{bmatrix} Pho_{1-3}^{2-4} \end{bmatrix} \end{vmatrix} + \begin{vmatrix} \begin{bmatrix} \mathbf{e}_\alpha \\ \mathbf{e}_\beta \end{bmatrix} & \begin{bmatrix} Pho_{1-3}^{2-4} \\ \mathbf{w} \end{bmatrix} \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{e}_\alpha \\ \mathbf{e}_\beta \\ \mathbf{x} [A_{1-4}^{1,2} : 0] \end{vmatrix} \begin{vmatrix} Pho_{1-3}^{2-4} \end{vmatrix} + \begin{vmatrix} Pho_\alpha^2 & Pho_\alpha^3 & Pho_\alpha^4 \\ Pho_\beta^2 & Pho_\beta^3 & Pho_\beta^4 \\ \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 \end{vmatrix} \\
(3.31) \quad &= \mathbf{w}_2 \ dPho_{\alpha,\beta}^{3,4} - \mathbf{w}_3 \ dPho_{\alpha,\beta}^{2,4} + \mathbf{w}_4 \ dPho_{\alpha,\beta}^{2,3}
\end{aligned}$$

The last line follows from the fact that $dPho_{1-3}^{2-4} \equiv 0$. Since $A \equiv Pho^{-1}$, for $\{a, b, c, d\} = \{1, 2, 3, 4\}$ and $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$ $dPho_{\alpha,\beta}^{a,b} = -1^{\alpha+\beta+a+b} \frac{dA_{c,d}^{\gamma,4}}{dA_{1-4}^{1-4}}$. Substituting this and 3.30 into 3.31 implies

$$(3.32) \quad 0 = A_1^\gamma \left(A_2^4 \ dQ_{1,r_9}^{2,5-8} + A_3^4 \ dQ_{1,r_9}^{3,5-8} + A_4^4 \ dQ_{1,r_9}^{4,5-8} \right)$$

Generically $A_1^\gamma \neq 0$. The second term of 3.32 is really equivalent to the second equation in 3.28. Consider the Jacobian of both equations.

$$(3.33) \quad \begin{bmatrix} dQ_{1,r_9}^{2,5-8} & dQ_{1,r_9}^{3,5-8} & dQ_{1,r_9}^{4,5-8} \\ dQ_{r_9}^{2,5-7} & dQ_{r_9}^{3,5-7} & dQ_{r_9}^{4,5-7} \end{bmatrix}$$

Let $j, k \in \{2, 3, 4\}$. Graßmann relations imply

$$(3.34) \quad dQ_{1,r_9}^{j,5-8} \ dQ_{r_9}^{k,5-7} - dQ_{1,r_9}^{k,5-8} \ dQ_{r_9}^{j,5-7} = dQ_{r_9}^{5-8} \ dQ_{1,r_9}^{j,k,5-7} = 0$$

The last equality follows because of range conditions. For any choice of rows in r_9 and any $j, k \in \{2, 3, 4\}$ $Q_{1,r_9}^{j,k,5-7}$ represents travel across the barrier separating incoming states [1,7-16] to outgoing states [2-6]. This matrix is generically of rank four, so $dQ_{1,r_9}^{j,k,5-7} \equiv 0$. Because of 3.34 the Jacobian 3.33 is rank one.

The only minors of Q_{11,r_l} which have not been considered are those which are functions of incoming-hidden transitions.

$$(3.35) \quad \begin{vmatrix} Pho_2^3 & Pho_2^4 & Phh_2^5 \\ Pho_1^3 & Pho_1^4 & Phh_1^5 \\ Pio_1^3 & Pio_1^4 & Pih_1^5 \end{vmatrix}$$

is an example of such a minor. Recall 3.29,

$$\begin{aligned}
Phh_{1-2}^5 &= Ph_{1-2}^{1-4} (Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8} A_{5-8}^5 \\
(3.36) \quad Pih_1^5 &= \mathbf{x} \left(Q_{r_9}^{1-4} \right)^{-1} Q_{r_9}^{5-8} A_{5-8}^5 \\
Pio_1^{3,4} &= \mathbf{x} A_{1-4}^{1,2} Ph_{1,2}^{3,4} + \mathbf{w}^{3,4}
\end{aligned}$$

Define $\mathbf{U}$ and $\mathbf{u}$ as follows

$$\mathbf{U} \equiv (Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8} A_{5-8}^5 = \frac{1}{dQ_{r_9}^{1-4}} \begin{bmatrix} dQ_{r_9}^{2,3,4,5} & dQ_{r_9}^{2,3,4,6} & dQ_{r_9}^{2,3,4,7} \\ -dQ_{r_9}^{1,3,4,5} & -dQ_{r_9}^{1,3,4,6} & -dQ_{r_9}^{1,3,4,7} \\ dQ_{r_9}^{1,2,4,5} & dQ_{r_9}^{1,2,4,6} & dQ_{r_9}^{1,2,4,7} \\ -dQ_{r_9}^{1,2,3,5} & -dQ_{r_9}^{1,2,3,6} & -dQ_{r_9}^{1,2,3,7} \end{bmatrix} A_{5-7}^5 = \frac{1}{dQ_{r_9}^{1-4}} \begin{bmatrix} 0 \\ \mathbf{u} \end{bmatrix}$$

where the last equality is due to 3.27. Recall that 3.33 is rank one. Therefore,

$$\begin{aligned}
\mathbf{w} \cdot \mathbf{u} &= C \left(dQ_{1,r_9}^{k,5-8} \right)_{k=2-4} \cdot \begin{pmatrix} -dQ_{r_9}^{1,3,4,j} \\ dQ_{r_9}^{1,2,4,j} \\ -dQ_{r_9}^{1,2,3,j} \end{pmatrix}_{j=5-7} A_{5-7}^5 \\
&= C' \left(dQ_{r_9}^{k,5-7} \right)_{k=2-4} \cdot \begin{pmatrix} -dQ_{r_9}^{1,3,4,j} \\ dQ_{r_9}^{1,2,4,j} \\ -dQ_{r_9}^{1,2,3,j} \end{pmatrix}_{j=5-7} A_{5-7}^5 \\
&= C' dQ_{r_9}^{1,5-7} \left(dQ_{r_9}^{2,3,4,j} \right)_j \cdot A_{5-7}^5 \\
(3.37) \quad &= 0
\end{aligned}$$

because of range conditions which force $dQ_{r_9}^{1,5-7} \equiv 0$ and more Graßmann relations. Furthermore,

$$\begin{aligned}
\begin{vmatrix} Ph_{1,2}^{2-4} \\ \mathbf{w} \end{vmatrix} &= C \begin{vmatrix} -dA_{1,3,4}^{2,3,4} & dA_{1,2,4}^{2,3,4} & -dA_{1,2,3}^{2,3,4} \\ dA_{1,3,4}^{1,3,4} & -dA_{1,2,4}^{1,3,4} & dA_{1,2,3}^{1,3,4} \\ dQ_{r_9}^{2,5-7} & dQ_{r_9}^{3,5-7} & dQ_{r_9}^{4,5-7} \end{vmatrix} \\
&= C' dQ_{r_9}^{2,5-7} dA_{1,2}^{3,4} + dQ_{r_9}^{3,5-7} dA_{1,3}^{3,4} + dQ_{r_9}^{4,5-7} dA_{1,4}^{3,4} \\
&= C'' \sum_{j=2}^4 dQ_{r_9}^{j,5,6,7} A_j^4 \\
(3.38) \quad &= 0
\end{aligned}$$

thanks to 3.28 and 3.24. Then

$$\begin{vmatrix} Ph_{1,2}^{3,4} & : & Ph_{1,2}^{2-4} \mathbf{u} \\ \mathbf{x} A_{1-4}^{1,2} Ph_{1,2}^{3,4} + \mathbf{w}^{3,4} & : & \mathbf{x} A_{1-4}^{1,2} Ph_{1,2}^{2-4} \mathbf{u} \end{vmatrix}$$

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$$\begin{aligned}
&= \left| \begin{array}{cc} Ph_{1,2}^{3,4} & : & Ph_{1,2}^{2-4} \mathbf{u} \\ \mathbf{x} A_{1-4}^{1,2} Ph_{1,2}^{3,4} & : & \mathbf{x} A_{1-4}^{1,2} Ph_{1,2}^{2-4} \mathbf{u} \end{array} \right| + \left| \begin{array}{cc} Ph_{1,2}^{3,4} & : & Ph_{1,2}^{2-4} \mathbf{u} \\ \mathbf{w}^{3,4} & : & 0 \end{array} \right| \\
&= \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \mathbf{x} A_{1-4}^{1,2} \end{pmatrix} Ph_{1,2}^{2-4} \begin{pmatrix} 0 & 0 & \mathbf{u}_1 \\ 1 & 0 & \mathbf{u}_2 \\ 0 & 1 & \mathbf{u}_3 \end{pmatrix} \right| + \left| \begin{pmatrix} Ph_{1,2}^{2-4} \\ \mathbf{w} \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathbf{u}_1 \\ 1 & 0 & \mathbf{u}_2 \\ 0 & 1 & \mathbf{u}_3 \end{pmatrix} \right|
\end{aligned}$$

$$(3.39) = 0$$

because the first term is the determinant of a the product of a 3×2 matrix with a 2×3 matrix and the second term is a multiple of 3.38. Analogous computations hold for the rest of the range conditions upon $Q11_{rl}$, $Q11_{bt}$, $Q12_{bt}$, $Q12_{lr}$, $Q21_{tb}$, $Q21_{rl}$, $Q22_{lr}$, and $Q22_{tb}$. All of these conditions are equivalent to the 3.24, 3.27, and 3.28, so only 16 of the possible 32 conditions are independent. In the next section, more conditions are found amongst the modified data for $Q11_{lr}$, $Q21_{lr}$, $Q21_{bt}$, $Q22_{bt}$, $Q22_{rl}$, $Q12_{rl}$, and $Q12_{tb}$.

3.3. “Hard” Conditions. Consider now the rank deficient “incoming-hidden” submatrix

$$(3.40) \quad Q11_{tb} = \begin{bmatrix} Pih_1^5 & Pih_1^6 & Pih_1^{15} & Pio_1^4 \\ Pih_2^5 & Pih_2^6 & Pih_2^{15} & Pio_2^4 \\ Pih_3^5 & Pih_3^6 & Pih_3^{15} & Pio_3^4 \\ Phh_1^5 & Phh_1^6 & Phh_1^{15} & Ph_{o1}^4 \end{bmatrix}$$

and define $M_{\mathbf{r}}^j$ to be the submatrix taken from rows $\mathbf{r}$ of M^j . In this section range conditions upon $Q11_{tb}$ are derived. Section 3.3.1 contains a derivation of 16 such conditions. These conditions are simplified in section 3.3.3. Finally, all other conditions upon the “incoming-hidden” submatrices are shown to be redundant in section 3.3.4.

3.3.1. Derive “Hard” Conditions. In order to start with as few incoming-hidden transitions as possible let $1 \leq \alpha < \beta \leq 3$ and consider the minor of 3.40 taken from rows $\alpha, \beta, 4$ and columns 1, 2, 4:

$$(3.41) \quad \left| \begin{array}{cc} M_{\alpha,\beta}^1 (Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6} & : & M_{\alpha,\beta}^1 e_4 + \left(\frac{dQ_{i,r_9}^{4-8}}{dQ_{r_9}^{5-8}} \right)_{i,j} \\ Ph_{o1}^{1-4} (Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6} & : & Ph_{o1}^4 \end{array} \right|$$

Now define the 2×4 and 3×4 matrices

$$(3.42) \quad n \equiv \begin{pmatrix} \frac{dQ_{\alpha,r_5}^{j,13-16}}{dQ_{r_5}^{13-16}} - \frac{dQ_{\alpha,r_9}^{j,5-8}}{dQ_{r_9}^{5-8}} \\ \frac{dQ_{\beta,r_5}^{j,13-16}}{dQ_{r_5}^{13-16}} - \frac{dQ_{\beta,r_9}^{j,5-8}}{dQ_{r_9}^{5-8}} \end{pmatrix}_j \quad \text{for } j \in \{1, 2, 3, 4\}$$

$$(3.43) \quad N \equiv \begin{bmatrix} n A_{1-4}^{1,2} \\ 1 & 0 \end{bmatrix} Pho_{1,2}^{1-4}$$

respectively and note that N is rank two. Substituting 3.42 and 3.43 into 3.41 yields

$$\begin{aligned}
(3.44) \quad & \left| N \left(Q_{r_9}^{1-4} \right)^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6} : Ne_4 + \frac{1}{dQ_{r_9}^{5-8}} \begin{pmatrix} dQ_{\alpha, r_9}^{4-8} \\ dQ_{\beta, r_9}^{4-8} \\ 0 \end{pmatrix} \right| \\
&= \left| N \left[\left(Q_{r_9}^{1-4} \right)^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6} : e_4 \right] \right| + \frac{1}{dQ_{r_9}^{5-8}} \left| N \left(Q_{r_9}^{1-4} \right)^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6} : \begin{matrix} dQ_{\alpha, r_9}^{4-8} \\ dQ_{\beta, r_9}^{4-8} \\ 0 \end{matrix} \right| \\
&= \frac{1}{dQ_{r_9}^{5-8}} \left| \begin{bmatrix} n A_{1-4}^{1,2} \\ 1 & : & 0 \end{bmatrix} Pho_{1,2}^{1-4} \left(Q_{r_9}^{1-4} \right)^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6} : \begin{matrix} dQ_{\alpha, r_9}^{4-8} \\ dQ_{\beta, r_9}^{4-8} \\ 0 \end{matrix} \right| \\
&= \frac{1}{dQ_{r_9}^{5-8}} \left| \begin{matrix} n A_{1-4}^{1,2} & : & dQ_{\alpha, r_9}^{4-8} \\ & & dQ_{\beta, r_9}^{4-8} \\ 1 & 0 & : & 0 \end{matrix} \right| \left| \begin{matrix} Pho_{1,2}^{1-4} \left(Q_{r_9}^{1-4} \right)^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6} & 0 \\ & 0 \\ 0 & 0 & 1 \end{matrix} \right| \\
&= C \left| \begin{matrix} n A_{1-4}^2 & : & dQ_{\alpha, r_9}^{4-8} \\ & & dQ_{\beta, r_9}^{4-8} \end{matrix} \right| \\
&= C \sum_{j=1}^4 \left| \begin{pmatrix} \frac{dQ_{\alpha, r_5}^{j, 13-16}}{dQ_{r_5}^{13-16}} - \frac{dQ_{\alpha, r_9}^{j, 5-8}}{dQ_{r_9}^{5-8}} \end{pmatrix} dQ_{\alpha, r_9}^{4-8} \right| A_j^2 \\
&= \sum_{j=1}^4 \left(\frac{1}{dQ_{r_5}^{13-16}} \left(dQ_{\alpha, r_5}^{j, 13-16} dQ_{\beta, r_9}^{4-8} - dQ_{\beta, r_5}^{j, 13-16} dQ_{\alpha, r_9}^{4-8} \right) - \right. \\
&\quad \left. \frac{1}{dQ_{r_9}^{5-8}} \left(dQ_{\alpha, r_9}^{j, 5-8} dQ_{\beta, r_9}^{4-8} - dQ_{\beta, r_9}^{j, 5-8} dQ_{\alpha, r_9}^{4-8} \right) \right) A_j^2 \\
(3.45) &= \frac{C}{dQ_{r_5}^{13-16}} \sum_{j=1}^4 \left(dQ_{\alpha, r_5}^{j, 13-16} dQ_{\beta, r_9}^{4-8} - dQ_{\beta, r_5}^{j, 13-16} dQ_{\alpha, r_9}^{4-8} - dQ_{r_5}^{13-16} dQ_{\alpha, \beta, r_9}^{j, 4-8} \right) A_j^2
\end{aligned}$$

where $C \equiv \left| Pho_{1,2}^{1-4} \left(Q_{r_9}^{1-4} \right)^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6} \right| / dQ_{r_9}^{5-8}$. The last equality holds because Graßmann relations imply

$$(3.46) \quad dQ_{\alpha, r_9}^{j, 5-8} dQ_{\beta, r_9}^{4-8} - dQ_{\beta, r_9}^{j, 5-8} dQ_{\alpha, r_9}^{4-8} = dQ_{\alpha, \beta, r_9}^{j, 4-8} dQ_{r_9}^{5-8}$$

If $C \neq 0$ then forcing 3.41 to be zero gives three conditions upon the A_j^2 s, since there are three ways to choose α, β in 3.45.

Claim: $C \neq 0$

Proof. Define $\beta \equiv Pih(I - Phh)^{-1}$. Then $Q_{r_9}^{1-4} = \beta_{r_9}^{1-4} Ph o_{1-4}^{1-4}$ and $Q_{r_9}^{5-8} = \beta_{r_9}^{5-8} Ph o_{5-8}^{5-8}$. Since $A \equiv Ph o^{-1}$,

$$(3.47) \quad C \equiv \left| Ph o_{1,2}^{1-4} (Q_{r_9}^{1-4})^{-1} Q_{r_9}^{5-8} A_{5-8}^{5,6} \right| / dQ_{r_9}^{5-8}$$

$$(3.48) \quad = \frac{1}{dQ_{r_9}^{5-8}} \left| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} (\beta_{r_9}^{1-4})^{-1} \beta_{r_9}^{5-8} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right|$$

$$(3.49) \quad = - \left| \begin{matrix} d\beta_{r_9}^{2-5} & d\beta_{r_9}^{2,3,4,6} \\ d\beta_{r_9}^{1,3,4,5} & d\beta_{r_9}^{1,3,4,6} \end{matrix} \right| / dQ_{r_9}^{5-8} (d\beta_{r_9}^{1-4})^2$$

$$(3.50) \quad = d\beta_{r_9}^{3-6} / d\beta_{r_9}^{1-4}$$

Equation 3.48 follows from the definitions of $Q_{r_9}^{1-4}$, $Q_{r_9}^{5-8}$, and A in terms of β and $Ph o$. Equation 2.10 implies 3.49, which combined with Graßmann relations yields 3.50. Since there is no barrier of less than four states separating incoming states 9,10,11, and 12 from hidden states 1,2,3, and 4, $d\beta_{r_9}^{3-6} \neq 0$ in the generic case. $\square$

3.3.2. More Matrix Identities. The major result of the following section requires many matrix identities. First note that Graßmann relations imply

$$(3.51) \quad \begin{vmatrix} dQ_{14,15,5-8}^{i,9-13} & dQ_{14,15,5-8}^{j,9-13} \\ dQ_{14,16,5-8}^{i,9-13} & dQ_{14,16,5-8}^{j,9-13} \end{vmatrix} = dQ_{14,5-8}^{9-13} dQ_{14,15,16,5-8}^{i,j,9-13}$$

$$\text{and} \quad dQ_{14,15,16,5-8}^{i,j,9-13} dQ_{15,16,5-8}^{k,9-13} - dQ_{14,15,16,5-8}^{i,k,9-13} dQ_{15,16,5-8}^{j,9-13} + dQ_{14,15,16,5-8}^{j,k,9-13} dQ_{15,16,5-8}^{i,9-13} = 0$$

Therefore,

$$(3.52) \quad \begin{vmatrix} dQ_{14,15,5-8}^{i,9-13} & dQ_{14,15,5-8}^{j,9-13} & dQ_{14,15,5-8}^{k,9-13} \\ dQ_{14,16,5-8}^{i,9-13} & dQ_{14,16,5-8}^{j,9-13} & dQ_{14,16,5-8}^{k,9-13} \\ dQ_{15,16,5-8}^{i,9-13} & dQ_{15,16,5-8}^{j,9-13} & dQ_{15,16,5-8}^{k,9-13} \end{vmatrix} \\ = dQ_{14,5-8}^{9-13} \left(dQ_{14,15,16,5-8}^{i,j,9-13} dQ_{15,16,5-8}^{k,9-13} \right. \\ \left. - dQ_{14,15,16,5-8}^{i,k,9-13} dQ_{15,16,5-8}^{j,9-13} + dQ_{14,15,16,5-8}^{j,k,9-13} dQ_{15,16,5-8}^{i,9-13} \right) \\ \equiv 0$$

Because Q_{1-8}^{9-16} is generically of rank four,

$$(3.53) \quad Q_{5-8}^{13-16} = Q_{5-8}^{9-12} (Q_{1-4}^{9-12})^{-1} Q_{1-4}^{13-16}$$

This means that for $\alpha \in \{1, 2, 3, 4\}$

$$\begin{aligned}
dQ_{\alpha,5-8}^{n,13-16} &= \begin{vmatrix} Q_{\alpha}^n & \vdots & Q_{\alpha}^{13-16} \\ Q_{5-8}^n & \vdots & Q_{5-8}^{13-16} \end{vmatrix} \\
&= \begin{vmatrix} Q_{\alpha}^n & \vdots & Q_{\alpha}^{13-16} \\ Q_{5-8}^n & \vdots & Q_{5-8}^{9-12} (Q_{1-4}^{9-12})^{-1} Q_{1-4}^{13-16} \end{vmatrix} \\
&= \begin{vmatrix} Q_{\alpha}^n & \vdots & \mathbf{e}_{\alpha} \\ Q_{5-8}^n & \vdots & Q_{5-8}^{9-12} (Q_{1-4}^{9-12})^{-1} \end{vmatrix} \begin{vmatrix} 1 \\ Q_{1-4}^{13-16} \end{vmatrix} \\
&= \begin{vmatrix} Q_{\alpha}^n & \vdots & Q_{\alpha}^{9-12} \\ Q_{5-8}^n & \vdots & Q_{5-8}^{9-12} \end{vmatrix} \begin{vmatrix} 1 \\ (Q_{1-4}^{9-12})^{-1} \end{vmatrix} \begin{vmatrix} 1 \\ Q_{1-4}^{13-16} \end{vmatrix} \\
(3.54) \quad &= dQ_{\alpha,5-8}^{n,9-12} \frac{dQ_{1-4}^{13-16}}{dQ_{1-4}^{9-12}}
\end{aligned}$$

Similarly,

$$(3.55) \quad dQ_{\alpha,\beta,13-16}^{n,4-8} = dQ_{\alpha,\beta,13-16}^{n,4,9-12} \frac{dQ_{1-4}^{5-8}}{dQ_{1-4}^{9-12}} \quad \text{and} \quad dQ_{\alpha,13-16}^{n,5-8} = dQ_{\alpha,13-16}^{n,9-12} \frac{dQ_{1-4}^{5-8}}{dQ_{1-4}^{9-12}}$$

Finally, Graßmann relations, range conditions combine with the above identities in the following:

$$\begin{aligned}
&\sum_{\sigma \in S_3(1,2,3,4)} \frac{\text{sgn } \sigma}{2} dQ_{14,15,16,5-8}^{\sigma_1,\sigma_2,9-13} dQ_{\alpha,5-8}^{\sigma_3,13-16} \\
(3.56) \quad &= \frac{dQ_{1-4}^{13-16}}{dQ_{1-4}^{9-12}} \sum_{\sigma \in S_3(1,2,3,4)} \frac{\text{sgn } \sigma}{2} dQ_{14,15,16,5-8}^{\sigma_1,\sigma_2,9-13} dQ_{\alpha,5-8}^{\sigma_3,9-12}
\end{aligned}$$

$$(3.57) \quad = \frac{dQ_{1-4}^{13-16}}{dQ_{1-4}^{9-12}} \left(\pm dQ_{\alpha,5-8}^{9-13} dQ_{14-6,5-8}^{\sigma_1,\sigma_2,\sigma_3,9-12} + dQ_{5-8}^{9-12} dQ_{\alpha,14-6,5-8}^{\sigma_1,\sigma_2,\sigma_3,9-13} \right)$$

$$(3.58) \quad = \frac{dQ_{1-4}^{13-16}}{dQ_{1-4}^{9-12}} dQ_{5-8}^{9-12} dQ_{\alpha,14-6,5-8}^{\sigma_1,\sigma_2,\sigma_3,9-13}$$

$$(3.59) \quad = dQ_{5-8}^{13-16} dQ_{\alpha,14-6,5-8}^{\sigma_1,\sigma_2,\sigma_3,9-13}$$

3.57 follows from 3.56 by Graßmann relations; 3.58 follows from 3.57 because consistency conditions force $dQ_{\alpha,5-8}^{9-13} \equiv 0$. Finally, 3.53 forces 3.59. The following notation will also be used in the next section.

Notation: Define the matrix $\mathbf{C}$ with columns $\mathbf{c}^j$ and rows $\mathbf{c}_j$, i.e.

$$\mathbf{C} = [\mathbf{c}^1 \ \mathbf{c}^2 \ \mathbf{c}^3 \ \mathbf{c}^4] = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix}$$

3.3.3. Simplification of “Hard” Conditions. This section consists of a single

Claim: The conditions in 3.45 are equivalent to

$$(3.60) \quad 0 = \sum_{n=1}^4 dQ_{\alpha,\beta,5-8}^{n,9-13} A_n^2 \quad \text{for } \alpha, \beta \in \{14, 15, 16\}$$

Furthermore, equations 3.45 and 3.60 constitute only two independent conditions upon the A_i^2 s.

Proof. Due to 3.52, only two independent conditions upon the A_n^2 s are given by 3.60. For convenience consider 3.45 where $r_5 = 5 - 8$ and $r_9 = 13, 14, 15, 16$. To finish the proof of this claim it is necessary to show that the matrix $\mathbf{C}$

$$(3.61) \quad \mathbf{c}^j = \begin{pmatrix} dQ_{\alpha,5-8}^{j,13-16} dQ_{\beta,13-16}^{4-8} - dQ_{\beta,5-8}^{j,13-16} dQ_{\alpha,13-16}^{4-8} - dQ_{5-8}^{13-16} dQ_{\alpha,\beta,13-16}^{j,4-8} \\ dQ_{14,15,5-8}^{j,9-13} \\ dQ_{14,16,5-8}^{j,9-13} \end{pmatrix}_j$$

is also of rank two. 3.51, 3.59, and 3.55 are combined to show that for $i, j, k \in \{1, 2, 3, 4\}$ $\begin{vmatrix} \mathbf{c}^i & \mathbf{c}^j & \mathbf{c}^k \end{vmatrix} = 0$, forcing $\mathbf{C}$ to be rank two:

$$(3.62) \quad \begin{aligned} & \frac{\begin{vmatrix} \mathbf{c}^i & \mathbf{c}^j & \mathbf{c}^k \end{vmatrix}}{dQ_{14,5-8}^{9-13}} \\ &= \left(\left(dQ_{\alpha,5-8}^{i,13-16} dQ_{\beta,13-16}^{4-8} - dQ_{\beta,5-8}^{i,13-16} dQ_{\alpha,13-16}^{4-8} - dQ_{5-8}^{13-16} dQ_{\alpha,\beta,13-16}^{i,4-8} \right) dQ_{14-16,5-8}^{j,k,9-13} - \right. \\ & \quad \left(dQ_{\alpha,5-8}^{j,13-16} dQ_{\beta,13-16}^{4-8} - dQ_{\beta,5-8}^{j,13-16} dQ_{\alpha,13-16}^{4-8} - dQ_{5-8}^{13-16} dQ_{\alpha,\beta,13-16}^{j,4-8} \right) dQ_{14-16,5-8}^{i,k,9-13} + \\ & \quad \left. \left(dQ_{\alpha,5-8}^{k,13-16} dQ_{\beta,13-16}^{4-8} - dQ_{\beta,5-8}^{k,13-16} dQ_{\alpha,13-16}^{4-8} - dQ_{5-8}^{13-16} dQ_{\alpha,\beta,13-16}^{k,4-8} \right) dQ_{14-16,5-8}^{i,j,9-13} \right) \\ &= \sum_{\sigma \in S_3(1,2,3,4)} \frac{\text{sgn } \sigma}{2} dQ_{14,15,16,5-8}^{\sigma_1,\sigma_2,9-13} \left(\left(dQ_{\alpha,5-8}^{\sigma_3,13-16} dQ_{\beta,13-16}^{4-8} - dQ_{\beta,5-8}^{\sigma_3,13-16} dQ_{\alpha,13-16}^{4-8} \right) - \right. \\ & \quad \left. dQ_{5-8}^{13-16} dQ_{\alpha,\beta,13-16}^{\sigma_3,4-8} \right) \end{aligned}$$

$$(3.63) \quad \begin{aligned} &= dQ_{5-8}^{13-16} \left(dQ_{\alpha,14-16,5-8}^{\sigma_1,\sigma_2,\sigma_3,9-13} dQ_{\beta,13-16}^{4-8} - dQ_{\beta,14-16,5-8}^{\sigma_1,\sigma_2,\sigma_3,9-13} dQ_{\alpha,13-16}^{4-8} - \right. \\ & \quad \left. \sum_{\sigma \in S_3(1,2,3,4)} \frac{\text{sgn } \sigma}{2} dQ_{14,15,16,5-8}^{\sigma_1,\sigma_2,9-13} dQ_{\alpha,\beta,13-16}^{\sigma_3,4-8} \right) \\ &= \frac{dQ_{5-8}^{13-16} dQ_{1-4}^{5-8}}{dQ_{1-4}^{9-12}} \left(dQ_{\alpha,14-16,5-8}^{\sigma_1,\sigma_2,\sigma_3,9-13} dQ_{\beta,13-16}^{4,9-12} - dQ_{\beta,14-16,5-8}^{\sigma_1,\sigma_2,\sigma_3,9-13} dQ_{\alpha,13-16}^{4,9-12} - \right. \end{aligned}$$

$$(3.64) \quad \left. \sum_{\sigma \in S_3(1,2,3,4)} \frac{\text{sgn } \sigma}{2} dQ_{14,15,16,5-8}^{\sigma_1,\sigma_2,9-13} dQ_{\alpha,\beta,13-16}^{\sigma_3,4,9-12} \right)$$

$$(3.65) = \frac{dQ_{5-8}^{13-16} dQ_{1-4}^{5-8}}{dQ_{1-4}^{9-12}} \left(\pm dQ_{\alpha,\beta,13-16}^{4,9-13} dQ_{14-6,5-8}^{\sigma_1,\sigma_2,\sigma_3,9-12} \pm dQ_{\alpha,\beta,14-6}^{4,9-12} dQ_{13-16,5-8}^{\sigma_1,\sigma_2,\sigma_3,9-13} \right)$$

$$(3.66) = 0$$

Equation 3.63 follows from 3.62 and 3.59. Equation 3.64 follows from 3.63, 3.54, and 3.55. Equation 3.65 is the result of another Graßmann relation. The last equation, 3.66, holds because consistency conditions force $dQ_{\alpha,\beta,13-16}^{4,9-12} = 0 = dQ_{\alpha,\beta,13-16}^{4,9-13}$. $\square$

Similar calculations hold for $Q11_{lr}$, $Q21_{lr}$, $Q21_{bt}$, $Q22_{bt}$, $Q22_{rl}$, $Q12_{rl}$, and $Q12_{tb}$. The result are sixteen independent conditions upon the A_i^j s:

$$(3.67) \quad 0 = \sum_{n=1}^4 dQ_{\alpha,\beta,5-8}^{n,9-13} A_n^2 \quad \text{for } \alpha, \beta \in \{14, 15, 16\}$$

$$(3.68) \quad 0 = \sum_{n=1}^4 dQ_{\alpha,\beta,13-16}^{n,8-12} A_n^3 \quad \text{for } \alpha, \beta \in \{5, 6, 7\}$$

$$(3.69) \quad 0 = \sum_{n=5}^8 dQ_{\alpha,\beta,9-12}^{n,1,13-16} A_n^6 \quad \text{for } \alpha, \beta \in \{2, 3, 4\}$$

$$(3.70) \quad 0 = \sum_{n=5}^8 dQ_{\alpha,\beta,1-4}^{n,12-6} A_n^7 \quad \text{for } \alpha, \beta \in \{9, 10, 11\}$$

$$(3.71) \quad 0 = \sum_{n=9}^{12} dQ_{\alpha,\beta,13-16}^{n,1-5} A_n^{10} \quad \text{for } \alpha, \beta \in \{6, 7, 8\}$$

$$(3.72) \quad 0 = \sum_{n=9}^{12} dQ_{\alpha,\beta,5-8}^{n,1-4,16} A_n^{11} \quad \text{for } \alpha, \beta \in \{13, 14, 15\}$$

$$(3.73) \quad 0 = \sum_{n=13}^{16} dQ_{\alpha,\beta,1-4}^{n,5-9} A_n^{14} \quad \text{for } \alpha, \beta \in \{10, 11, 12\}$$

$$(3.74) \quad 0 = \sum_{n=13}^{16} dQ_{\alpha,\beta,9-12}^{n,4-8} A_n^{15} \quad \text{for } \alpha, \beta \in \{1, 2, 3\}$$

3.3.4. Redundant “Hard” Conditions. In the previous section two equations were derived which force the first, second, and fourth columns of $Q11_{tb}$ to be rank two. In this section conditions forcing the first three columns of $Q11_{tb}$ to be rank two are considered. These conditions turn out to be equivalent to conditions 3.67 through 3.74.

Much of the notation of the previous section is retained. Here, however, r_5 and r_9 are fixed so that $r_5 \equiv 9 - 12 \equiv r_9$. Define the matrices $\mathbf{L}$ and $\mathbf{D}$ whose columns are given by

$$(3.75) \quad \mathbf{l}^j \equiv \begin{pmatrix} dQ_{\alpha,r_9}^{j+12,5-8} \\ dQ_{\beta,r_9}^{j+12,5-8} \end{pmatrix}$$

$$(3.76) \quad \mathbf{d}^j \equiv \begin{pmatrix} \frac{dQ_{\alpha,r_5}^{j,13-16}}{dQ_{r_5}^{13-16}} - \frac{dQ_{\alpha,r_9}^{j,5-8}}{dQ_{r_9}^{5-8}} \\ \frac{dQ_{\beta,r_5}^{j,13-16}}{dQ_{r_5}^{13-16}} - \frac{dQ_{\beta,r_9}^{j,5-8}}{dQ_{r_9}^{5-8}} \end{pmatrix}$$

The minor taken from rows $\alpha, \beta, 4$ and first three columns of $Q11_{tb}$ can be written as follows:

$$\begin{aligned}
& \begin{vmatrix} Pih_{\alpha}^5 & Pih_{\alpha}^6 & Pih_{\alpha}^{15} \\ Pih_{\beta}^5 & Pih_{\beta}^6 & Pih_{\beta}^{15} \\ Phh_1^5 & Phh_1^6 & Phh_1^{15} \end{vmatrix} \\
&= \left| N \left(Q_{9-12}^{1-4} \right)^{-1} Q_{9-12}^{5-8} A_{5-8}^{5,6} : \left(N \left(Q_{9-12}^{1-4} \right)^{-1} Q_{9-12}^{13-16} + \frac{1}{dQ_{9-12}^{5-8}} \begin{pmatrix} \mathbf{L} \\ 0 \end{pmatrix} \right) A_{13-16}^{15} \right| \\
&= \left| N \left(Q_{9-12}^{1-4} \right)^{-1} \left[Q_{9-12}^{5-8} A_{5-8}^{5,6} \quad : \quad Q_{9-12}^{13-16} A_{13-16}^{15} \right] \right| + \\
&\quad \left| \frac{1}{dQ_{9-12}^{5-8}} \left[N \left(Q_{9-12}^{1-4} \right)^{-1} Q_{9-12}^{5-8} A_{5-8}^{5,6} \quad : \quad \begin{pmatrix} \mathbf{L} \\ 0 \end{pmatrix} A_{13-16}^{15} \right] \right| \\
&= C \left| \begin{array}{cc} \mathbf{D} A_{1-4}^{1,2} & : \quad \begin{pmatrix} \mathbf{L} \\ 0 \end{pmatrix}_j A_{13-16}^{15} \\ 1 & 0 \end{array} \right|, \quad \text{since } N \text{ is rank two} \\
(3.77) &= C \left| \mathbf{D} A_{1-4}^2 : \mathbf{L} A_{13-16}^{15} \right|
\end{aligned}$$

At first blush this appears to be quadratic in A_i^j s. Equation 3.45 can be used to factor the A_i^j s out of 3.77. Now define the matrices $\mathbf{F}$ and $\mathbf{G}$ whose columns are

$$(3.78) \quad \mathbf{f}^j \equiv \begin{pmatrix} dQ_{14,15,5-8}^{j,9-13} \\ dQ_{14,16,5-8}^{j,9-13} \end{pmatrix} \quad \mathbf{g}^j \equiv \begin{pmatrix} dQ_{1,2,9-12}^{j+12,4-8} \\ dQ_{1,3,9-12}^{j+12,4-8} \end{pmatrix}$$

Then 3.67 and 3.74 imply

$$(3.79) \quad 0 = \mathbf{F} \begin{pmatrix} A_1^2 \\ A_2^2 \\ A_3^2 \\ A_4^2 \end{pmatrix} \quad \text{and} \quad 0 = \mathbf{G} \begin{pmatrix} A_{13}^{15} \\ A_{14}^{15} \\ A_{15}^{15} \\ A_{16}^{15} \end{pmatrix}$$

or

$$(3.80) \quad \begin{pmatrix} A_1^2 \\ A_2^2 \\ A_3^2 \\ A_4^2 \end{pmatrix} = \begin{pmatrix} -[\mathbf{f}^1 \quad \mathbf{f}^2]^{-1} [\mathbf{f}^3 \quad \mathbf{f}^4] \\ I \end{pmatrix} \begin{pmatrix} A_3^2 \\ A_4^2 \end{pmatrix}$$

$$(3.81) \quad \begin{pmatrix} A_{13}^{15} \\ A_{14}^{15} \\ A_{15}^{15} \\ A_{16}^{15} \end{pmatrix} = \begin{pmatrix} -[\mathbf{g}^1 \quad \mathbf{g}^2]^{-1} [\mathbf{g}^3 \quad \mathbf{g}^4] \\ I \end{pmatrix} \begin{pmatrix} A_{15}^{15} \\ A_{16}^{15} \end{pmatrix}$$

Plugging 3.76 and 3.80 into 3.77 yields,

$$(3.82)C \left| \mathbf{D} \begin{pmatrix} -[\mathbf{f}^1 \ \mathbf{f}^2]^{-1} [\mathbf{f}^3 \ \mathbf{f}^4] \\ I \end{pmatrix} \begin{pmatrix} A_3^2 \\ A_4^2 \end{pmatrix} : \mathbf{L} \begin{pmatrix} -[\mathbf{g}^1 \ \mathbf{g}^2]^{-1} [\mathbf{g}^3 \ \mathbf{g}^4] \\ I \end{pmatrix} \begin{pmatrix} A_{15}^{15} \\ A_{16}^{15} \end{pmatrix} \right|$$

Factoring A_3^2 and A_4^2 out of 3.77 requires several steps:

1. Show that $\mathbf{D} \begin{pmatrix} -[\mathbf{f}^1 \ \mathbf{f}^2]^{-1} [\mathbf{f}^3 \ \mathbf{f}^4] \\ I \end{pmatrix}$ and $\mathbf{L} \begin{pmatrix} -[\mathbf{g}^1 \ \mathbf{g}^2]^{-1} [\mathbf{g}^3 \ \mathbf{g}^4] \\ I \end{pmatrix}$ are both rank one.
2. Compute column vectors $\mathbf{c}$, $\bar{\mathbf{c}}$, such that $\mathbf{D} \begin{pmatrix} -[\mathbf{f}^1 \ \mathbf{f}^2]^{-1} [\mathbf{f}^3 \ \mathbf{f}^4] \\ I \end{pmatrix} = \mathbf{c} \mathbf{r}^T$ and $\mathbf{L} \begin{pmatrix} -[\mathbf{g}^1 \ \mathbf{g}^2]^{-1} [\mathbf{g}^3 \ \mathbf{g}^4] \\ I \end{pmatrix} = \bar{\mathbf{c}} \bar{\mathbf{r}}^T$ for some column vectors $\mathbf{r}$, and $\bar{\mathbf{r}}$.
3. Note that 3.82 can be written as $C * \left(\mathbf{r}^T \begin{pmatrix} A_3^2 \\ A_4^2 \end{pmatrix} \right) * \left(\bar{\mathbf{r}}^T \begin{pmatrix} A_{15}^{15} \\ A_{16}^{15} \end{pmatrix} \right) * |\mathbf{c} : \bar{\mathbf{c}}|$ and check that $0 \equiv |\mathbf{c} : \bar{\mathbf{c}}|$

Step One. The following identity is helpful:

$$(3.83) \quad \begin{bmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^3 & \mathbf{f}^4 \end{bmatrix} = \frac{1}{\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix}} \begin{bmatrix} -\begin{vmatrix} \mathbf{f}^2 & \mathbf{f}^3 \end{vmatrix} & \begin{vmatrix} \mathbf{f}^2 & \mathbf{f}^4 \end{vmatrix} \\ \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^3 \end{vmatrix} & -\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^4 \end{vmatrix} \end{bmatrix}$$

Therefore,

$$(3.84) \quad \left| -\begin{bmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^3 & \mathbf{f}^4 \end{bmatrix} + \begin{bmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}^3 & \mathbf{d}^4 \end{bmatrix} \right| \\ = \left| \begin{aligned} &+ \left(\frac{\begin{vmatrix} \mathbf{f}^2 & \mathbf{f}^3 \end{vmatrix}}{\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix}} - \frac{\begin{vmatrix} \mathbf{d}^2 & \mathbf{d}^3 \end{vmatrix}}{\begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix}} \right) - \left(\frac{\begin{vmatrix} \mathbf{f}^2 & \mathbf{f}^4 \end{vmatrix}}{\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix}} - \frac{\begin{vmatrix} \mathbf{d}^2 & \mathbf{d}^4 \end{vmatrix}}{\begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix}} \right) \\ &- \left(\frac{\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^3 \end{vmatrix}}{\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix}} - \frac{\begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^3 \end{vmatrix}}{\begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix}} \right) + \left(\frac{\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^4 \end{vmatrix}}{\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix}} - \frac{\begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^4 \end{vmatrix}}{\begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix}} \right) \end{aligned} \right| \\ = \frac{1}{\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix} \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix}} \left(\left(\begin{vmatrix} \mathbf{f}^2 & \mathbf{f}^3 \end{vmatrix} \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix} - \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix} \begin{vmatrix} \mathbf{d}^2 & \mathbf{d}^3 \end{vmatrix} \right) \left(\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^4 \end{vmatrix} \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix} - \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix} \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^4 \end{vmatrix} \right) - \right. \\ \left. \left(\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^3 \end{vmatrix} \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix} - \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix} \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^3 \end{vmatrix} \right) \left(\begin{vmatrix} \mathbf{f}^2 & \mathbf{f}^4 \end{vmatrix} \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix} - \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix} \begin{vmatrix} \mathbf{d}^2 & \mathbf{d}^4 \end{vmatrix} \right) \right) \\ = \frac{1}{\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix} \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix}} \left(\begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix}^2 \left(\begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^4 \end{vmatrix} \begin{vmatrix} \mathbf{f}^2 & \mathbf{f}^3 \end{vmatrix} - \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^3 \end{vmatrix} \begin{vmatrix} \mathbf{f}^2 & \mathbf{f}^4 \end{vmatrix} \right) + \right. \\ \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix}^2 \left(\begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^4 \end{vmatrix} \begin{vmatrix} \mathbf{d}^2 & \mathbf{d}^3 \end{vmatrix} - \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^3 \end{vmatrix} \begin{vmatrix} \mathbf{d}^2 & \mathbf{d}^4 \end{vmatrix} \right) + \\ \left. \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{vmatrix} \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix} \left(-\begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^4 \end{vmatrix} \begin{vmatrix} \mathbf{f}^2 & \mathbf{f}^3 \end{vmatrix} - \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^4 \end{vmatrix} \begin{vmatrix} \mathbf{d}^2 & \mathbf{d}^3 \end{vmatrix} + \right. \right. \\ \left. \left. \begin{vmatrix} \mathbf{d}^1 & \mathbf{d}^4 \end{vmatrix} \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^3 \end{vmatrix} + \begin{vmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{vmatrix} \begin{vmatrix} \mathbf{d}^2 & \mathbf{d}^4 \end{vmatrix} \right) \right) \right)$$

$$(3.85) \quad \left(|\mathbf{f}^1 \mathbf{f}^3| |\mathbf{d}^2 \mathbf{d}^4| + |\mathbf{d}^1 \mathbf{d}^3| |\mathbf{f}^2 \mathbf{f}^4| \right)$$

$$(3.86) \quad = \frac{1}{|\mathbf{f}^1 \mathbf{f}^2| |\mathbf{d}^1 \mathbf{d}^2|} \left(-|\mathbf{d}^1 \mathbf{d}^2|^2 |\mathbf{f}^1 \mathbf{f}^2| |\mathbf{f}^3 \mathbf{f}^4| - |\mathbf{f}^1 \mathbf{f}^2|^2 |\mathbf{d}^1 \mathbf{d}^2| |\mathbf{d}^3 \mathbf{d}^4| + \right. \\ \left. |\mathbf{d}^1 \mathbf{d}^2| |\mathbf{f}^1 \mathbf{f}^2| \left(-|\mathbf{d}^1 \mathbf{d}^4| |\mathbf{f}^2 \mathbf{f}^3| - |\mathbf{f}^1 \mathbf{f}^4| |\mathbf{d}^2 \mathbf{d}^3| \right. \right. \\ \left. \left. + |\mathbf{f}^1 \mathbf{f}^3| |\mathbf{d}^2 \mathbf{d}^4| + |\mathbf{d}^1 \mathbf{d}^3| |\mathbf{f}^2 \mathbf{f}^4| \right) \right)$$

$$(3.87) \quad = - \left(|\mathbf{d}^1 \mathbf{d}^2| |\mathbf{f}^3 \mathbf{f}^4| - |\mathbf{d}^1 \mathbf{d}^3| |\mathbf{f}^2 \mathbf{f}^4| + |\mathbf{d}^1 \mathbf{d}^4| |\mathbf{f}^2 \mathbf{f}^3| \right. \\ \left. + |\mathbf{d}^2 \mathbf{d}^3| |\mathbf{f}^1 \mathbf{f}^4| - |\mathbf{d}^2 \mathbf{d}^4| |\mathbf{f}^1 \mathbf{f}^3| + |\mathbf{d}^3 \mathbf{d}^4| |\mathbf{f}^1 \mathbf{f}^2| \right)$$

$$(3.88) \quad = - \begin{vmatrix} \mathbf{D} \\ \mathbf{F} \end{vmatrix} \\ = - \begin{vmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{F} \end{vmatrix}$$

$$(3.89) \quad = \frac{1}{dQ_{5-8}^{13-16} dQ_{\alpha,13-16}^{4-8}} \begin{vmatrix} \mathbf{d}_1 \\ -dQ_{5-8}^{13-16} dQ_{\alpha,13-16}^{4-8} \mathbf{d}_2 \\ \mathbf{F} \end{vmatrix} \\ = \frac{1}{dQ_{5-8}^{13-16} dQ_{\alpha,13-16}^{4-8}} \begin{vmatrix} \mathbf{d}_1 \\ dQ_{5-8}^{13-16} \left(dQ_{\beta,13-16}^{4-8} \mathbf{d}_1 - dQ_{\alpha,13-16}^{4-8} \mathbf{d}_2 \right) \\ \mathbf{F} \end{vmatrix} \\ = \frac{1}{dQ_{5-8}^{13-16} dQ_{\alpha,13-16}^{4-8}} \begin{vmatrix} \mathbf{d}_1 \\ \mathbf{C} \end{vmatrix} \quad \text{so}$$

$$(3.90) \quad \left| - \begin{bmatrix} \mathbf{f}^1 & \mathbf{f}^2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^3 & \mathbf{f}^4 \end{bmatrix} + \begin{bmatrix} \mathbf{d}^1 & \mathbf{d}^2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}^3 & \mathbf{d}^4 \end{bmatrix} \right| = 0$$

Line 3.85 follows from 3.86 by virtue of Graßmann relations; 3.87 is simply the Laplace expansion of 3.88. In equation 3.89, $\mathbf{C}$ is the same rank two matrix defined in 3.61, so $\begin{vmatrix} \mathbf{d}_1 \\ \mathbf{C} \end{vmatrix}$ is rank deficient. Now,

$$(3.91) \quad \left| \mathbf{D} \begin{pmatrix} -[\mathbf{f}^1 \mathbf{f}^2]^{-1} [\mathbf{f}^3 \mathbf{f}^4] \\ I \end{pmatrix} \right| =$$

$$(3.92) \quad = \left| -[\mathbf{d}^1 \mathbf{d}^2] [\mathbf{f}^1 \mathbf{f}^2]^{-1} [\mathbf{f}^3 \mathbf{f}^4] + [\mathbf{d}^3 \mathbf{d}^4] \right|$$

$$(3.93) \quad = \left| [\mathbf{d}^1 \mathbf{d}^2] \left(-[\mathbf{f}^1 \mathbf{f}^2]^{-1} [\mathbf{f}^3 \mathbf{f}^4] + [\mathbf{d}^1 \mathbf{d}^2]^{-1} [\mathbf{d}^3 \mathbf{d}^4] \right) \right|$$

$$(3.94) \quad = \left| [\mathbf{d}^1 \mathbf{d}^2] \right| \left| \left(-[\mathbf{f}^1 \mathbf{f}^2]^{-1} [\mathbf{f}^3 \mathbf{f}^4] + [\mathbf{d}^1 \mathbf{d}^2]^{-1} [\mathbf{d}^3 \mathbf{d}^4] \right) \right|$$

$$(3.95) \quad = 0$$

so $\mathbf{D} \begin{pmatrix} -[\mathbf{f}^1 \ \mathbf{f}^2]^{-1} [\mathbf{f}^3 \ \mathbf{f}^4] \\ I \end{pmatrix}$ is rank deficient. A similar method works to show that $\mathbf{L} \begin{pmatrix} -[\mathbf{g}^1 \ \mathbf{g}^2]^{-1} [\mathbf{g}^3 \ \mathbf{g}^4] \\ I \end{pmatrix}$ is also rank deficient. Recall 3.46 and the definition of $\mathbf{l}_\alpha$:

$$(3.96) \quad \mathbf{l}_\alpha = \begin{pmatrix} dQ_{\alpha,9-12}^{13,5-8} & dQ_{\alpha,9-12}^{14,5-8} & dQ_{\alpha,9-12}^{15,5-8} & dQ_{\alpha,9-12}^{16,5-8} \end{pmatrix}$$

Furthermore, define

$$(3.97) \quad \mathbf{l}_{\alpha,\beta} = \begin{pmatrix} dQ_{\alpha,\beta,9-12}^{13,4-8} & dQ_{\alpha,\beta,9-12}^{14,4-8} & dQ_{\alpha,\beta,9-12}^{15,4-8} & dQ_{\alpha,\beta,9-12}^{16,4-8} \end{pmatrix}$$

and notice that

$$(3.98) \quad dQ_{\beta,9-12}^{4-8} \mathbf{l}_\alpha - dQ_{\alpha,9-12}^{4-8} \mathbf{l}_\beta = dQ_{9-12}^{5-8} \mathbf{l}_{\alpha,\beta}$$

The argument used to go from 3.84 to 3.88 and the fact that

$$(3.99) \quad \mathbf{L} = \begin{pmatrix} \mathbf{l}_\alpha \\ \mathbf{l}_\beta \end{pmatrix} \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} \mathbf{l}_{1,2} \\ \mathbf{l}_{1,3} \end{pmatrix}$$

lead to the following

$$(3.100) \quad \left| \mathbf{L} \begin{pmatrix} -[\mathbf{g}^1 \ \mathbf{g}^2]^{-1} [\mathbf{g}^3 \ \mathbf{g}^4] \\ I \end{pmatrix} \right| =$$

$$(3.101) \quad = \left| [\mathbf{l}^1 \ \mathbf{l}^2] \right| \left| \begin{pmatrix} -[\mathbf{g}^1 \ \mathbf{g}^2]^{-1} [\mathbf{g}^3 \ \mathbf{g}^4] + [\mathbf{l}^1 \ \mathbf{l}^2]^{-1} [\mathbf{l}^3 \ \mathbf{l}^4] \end{pmatrix} \right|$$

$$(3.102) \quad = \left| [\mathbf{l}^1 \ \mathbf{l}^2] \right| \left| \begin{pmatrix} \mathbf{L} \\ \mathbf{G} \end{pmatrix} \right|$$

$$(3.103) \quad = \left| [\mathbf{l}^1 \ \mathbf{l}^2] \right| \left| \begin{pmatrix} \mathbf{l}_\alpha \\ \mathbf{l}_\beta \\ \mathbf{l}_{1,2} \\ \mathbf{l}_{1,3} \end{pmatrix} \right|$$

$$(3.104) \quad = \left| [\mathbf{l}^1 \ \mathbf{l}^2] \right| \left| \begin{pmatrix} \mathbf{l}_\alpha \\ \mathbf{l}_\beta \\ dQ_{2,9-12}^{4-8} \mathbf{l}_1 - dQ_{1,9-12}^{4-8} \mathbf{l}_2 \\ dQ_{3,9-12}^{4-8} \mathbf{l}_1 - dQ_{1,9-12}^{4-8} \mathbf{l}_2 \end{pmatrix} \right|$$

$$(3.105) \quad = \left| \begin{bmatrix} \mathbf{l}^1 & \mathbf{l}^2 \end{bmatrix} \right| \left| \begin{pmatrix} \mathbf{e}_\alpha \\ \mathbf{e}_\beta \\ dQ_{2,9-12}^{4-8} & -dQ_{1,9-12}^{4-8} & 0 \\ dQ_{3,9-12}^{4-8} & 0 & -dQ_{1,9-12}^{4-8} \end{pmatrix} \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l}_3 \end{pmatrix} \right|$$

$$(3.106) \quad = 0$$

since the product of a 4×3 with a 3×4 matrix is a 4×4 rank deficient matrix.

Step Two. Brute force suffices to find $\mathbf{c}$

$$\begin{aligned} & -[\mathbf{d}^1 \ \mathbf{d}^2] [\mathbf{f}^1 \ \mathbf{f}^2]^{-1} [\mathbf{f}^3 \ \mathbf{f}^4] + [\mathbf{d}^3 \ \mathbf{d}^4] \\ & \equiv -\frac{1}{|\mathbf{f}^1 \mathbf{f}^2|} [\mathbf{d}^1 \ \mathbf{d}^2] \begin{bmatrix} -|\mathbf{f}^2 \mathbf{f}^3| & -|\mathbf{f}^2 \mathbf{f}^4| \\ |\mathbf{f}^1 \mathbf{f}^3| & |\mathbf{f}^1 \mathbf{f}^4| \end{bmatrix} + [\mathbf{d}^3 \ \mathbf{d}^4] \\ & = -\frac{1}{|\mathbf{f}^1 \mathbf{f}^2|} \begin{bmatrix} -|\mathbf{f}^2 \mathbf{f}^3| \mathbf{d}^1 + |\mathbf{f}^1 \mathbf{f}^3| \mathbf{d}^2 & : & -|\mathbf{f}^2 \mathbf{f}^4| \mathbf{d}^1 + |\mathbf{f}^1 \mathbf{f}^4| \mathbf{d}^2 \end{bmatrix} + [\mathbf{d}^3 \ : \ \mathbf{d}^4] \\ & = \frac{1}{|\mathbf{f}^1 \mathbf{f}^2|} \begin{bmatrix} (|\mathbf{f}^2 \mathbf{f}^3| \mathbf{d}^1 - |\mathbf{f}^1 \mathbf{f}^3| \mathbf{d}^2 + |\mathbf{f}^1 \mathbf{f}^2| \mathbf{d}^3) & : & (|\mathbf{f}^2 \mathbf{f}^4| \mathbf{d}^1 - |\mathbf{f}^1 \mathbf{f}^4| \mathbf{d}^2 + |\mathbf{f}^1 \mathbf{f}^2| \mathbf{d}^4) \end{bmatrix} \\ & = \frac{1}{|\mathbf{f}^1 \mathbf{f}^2|} \begin{bmatrix} |\mathbf{f}^2 \mathbf{f}^3| \mathbf{d}^1 - |\mathbf{f}^1 \mathbf{f}^3| \mathbf{d}^2 + |\mathbf{f}^1 \mathbf{f}^2| \mathbf{d}^3 \end{bmatrix} \begin{bmatrix} 1 & \frac{(|\mathbf{f}^2 \mathbf{f}^4| \mathbf{d}^1 - |\mathbf{f}^1 \mathbf{f}^4| \mathbf{d}^2 + |\mathbf{f}^1 \mathbf{f}^2| \mathbf{d}^4)}{(|\mathbf{f}^2 \mathbf{f}^4| \mathbf{d}^1 - |\mathbf{f}^1 \mathbf{f}^4| \mathbf{d}^2 + |\mathbf{f}^1 \mathbf{f}^2| \mathbf{d}^4)} \end{bmatrix} \\ & = \mathbf{c} \mathbf{r}^T \end{aligned} \quad (3.107)$$

for $\mathbf{c} \equiv \begin{bmatrix} |\mathbf{f}^2 \mathbf{f}^3| \mathbf{d}^1 - |\mathbf{f}^1 \mathbf{f}^3| \mathbf{d}^2 + |\mathbf{f}^1 \mathbf{f}^2| \mathbf{d}^3 \end{bmatrix}$. A similar calculation holds to show $\mathbf{L} \begin{pmatrix} -[\mathbf{g}^1 \ \mathbf{g}^2]^{-1} [\mathbf{g}^3 \ \mathbf{g}^4] \\ I \end{pmatrix} = \bar{\mathbf{c}} \bar{\mathbf{r}}^T$ for $\bar{\mathbf{c}} = \begin{bmatrix} |\mathbf{g}^2 \ \mathbf{g}^3| \mathbf{l}^1 - |\mathbf{g}^1 \ \mathbf{g}^3| \mathbf{l}^2 + |\mathbf{g}^1 \ \mathbf{g}^2| \mathbf{l}^3 \end{bmatrix}$.

Step Three. In order to show that 3.77, hold trivially a few more identities are needed. Because Q_{9-16}^{1-8} is generically of rank four, $Q_{9-12}^{5-8} = Q_{9-12}^{1-4} (Q_{13-16}^{1-4})^{-1} Q_{13-16}^{1-4}$ and $dQ_{\alpha,9-12}^{4-8}$ can be expressed as a product of minors

$$\begin{aligned} dQ_{\alpha,9-12}^{4-8} &= \begin{vmatrix} Q_\alpha^4 & : & Q_\alpha^{5-8} \\ Q_{9-12}^4 & : & Q_{9-12}^{5-8} \end{vmatrix} \\ &= \begin{vmatrix} Q_\alpha^4 & : & Q_\alpha^{5-8} \\ Q_{5-8}^4 & : & Q_{9-12}^{1-4} (Q_{13-16}^{1-4})^{-1} Q_{13-16}^{9-12} \end{vmatrix} \\ &= \begin{vmatrix} 1 & & Q_\alpha^4 & : & Q_\alpha^{5-8} \\ & Q_{9-12}^{1-4} & & & \mathbf{e}_4 & : & (Q_{13-16}^{1-4})^{-1} Q_{13-16}^{5-8} \end{vmatrix} \end{aligned}$$

$$\begin{aligned}
&= \left| \begin{array}{c} 1 \\ Q_{9-12}^{1-4} \end{array} \right| \left| \begin{array}{c} 1 \\ (Q_{13-16}^{1-4})^{-1} \end{array} \right| \left| \begin{array}{c} Q_{\alpha}^4 \quad \vdots \quad Q_{\alpha}^{5-8} \\ Q_{13-16}^4 \quad \vdots \quad Q_{13-16}^{5-8} \end{array} \right| \\
(3.108) \quad &= dQ_{\alpha,13-16}^{4-8} \frac{dQ_{9-12}^{1-4}}{dQ_{13-16}^{1-4}}
\end{aligned}$$

The Graßmann relation

$$(3.109) \quad \sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} dQ_{1-3,9-12}^{\sigma_1, \sigma_2, 4-8} dQ_{\alpha,9-12}^{\sigma_3, 5-8} = dQ_{1-3,9-12}^{\sigma_1, \sigma_2, \sigma_3, 5-8} dQ_{\alpha,9-12}^{4-8}$$

as well as Graßmann relations for the minors

$$(3.110) \quad |\mathbf{f}^j \mathbf{f}^k| = dQ_{14,5-8}^{9-13} dQ_{14-6,5-8}^{j,k,9-13}$$

$$(3.111) \quad |\mathbf{g}^j \mathbf{g}^k| = dQ_{1,9-12}^{4-8} dQ_{1-3,9-12}^{j+12, k+12, 4-8}$$

can be used to write $\bar{\mathbf{c}}$ fairly simply

$$\begin{aligned}
\bar{\mathbf{c}} &= \sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} |\mathbf{g}^{\sigma_1} \mathbf{g}^{\sigma_2}| \mathbf{l}^{\sigma_3} \\
&= dQ_{1,9-12}^{4-8} \sum_{\sigma \in S_3(13,14,15)} \frac{\text{sgn} \sigma}{2} dQ_{1-3,9-12}^{\sigma_1, \sigma_2, 4-8} \begin{pmatrix} dQ_{\alpha,9-12}^{\sigma_3, 5-8} \\ dQ_{\beta,9-12}^{\sigma_3, 5-8} \end{pmatrix} \\
&= dQ_{1,9-12}^{4-8} dQ_{1-3,9-12}^{13,14,15,5-8} \begin{pmatrix} dQ_{\alpha,9-12}^{4-8} \\ dQ_{\beta,9-12}^{4-8} \end{pmatrix} \\
(3.112) \quad &= \bar{\kappa} \begin{pmatrix} dQ_{\alpha,13-16}^{4-8} \\ dQ_{\beta,13-16}^{4-8} \end{pmatrix}
\end{aligned}$$

The last line follows from equation 3.108 for some $\bar{\kappa} \in \mathbb{R}$. Computing $\mathbf{c}$ is more complicated. First note that equations 3.54 and 3.55 can be used to rewrite

$$(3.113) \quad \mathbf{d}^i \equiv \begin{pmatrix} \frac{dQ_{\alpha,5-8}^{i,13-16}}{dQ_{5-8}^{13-16}} - \frac{dQ_{\alpha,13-16}^{i,5-8}}{dQ_{13-16}^{5-8}} \\ \frac{dQ_{\beta,5-8}^{i,13-16}}{dQ_{5-8}^{13-16}} - \frac{dQ_{\beta,13-16}^{i,5-8}}{dQ_{13-16}^{5-8}} \end{pmatrix} = \begin{pmatrix} \frac{dQ_{\alpha,5-8}^{i,9-12}}{dQ_{5-8}^{9-12}} - \frac{dQ_{\alpha,13-16}^{i,9-12}}{dQ_{13-16}^{9-12}} \\ \frac{dQ_{\beta,5-8}^{i,9-12}}{dQ_{5-8}^{9-12}} - \frac{dQ_{\beta,13-16}^{i,9-12}}{dQ_{13-16}^{9-12}} \end{pmatrix} = \mathbf{d}_a^i - \mathbf{d}_b^i$$

where

$$(3.114) \quad \mathbf{d}_a^i \equiv \frac{1}{dQ_{5-8}^{9-12}} \begin{pmatrix} dQ_{\alpha,5-8}^{i,9-12} \\ dQ_{\beta,5-8}^{i,9-12} \end{pmatrix} \quad \text{and} \quad \mathbf{d}_b^i \equiv \frac{1}{dQ_{13-16}^{9-12}} \begin{pmatrix} dQ_{\alpha,13-16}^{i,9-12} \\ dQ_{\beta,13-16}^{i,9-12} \end{pmatrix}$$

A combination of Graßmann relations and range conditions are required now

$$\begin{aligned}
&\sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} dQ_{14-6,5-8}^{\sigma_1, \sigma_2, 9-13} dQ_{\alpha,5-8}^{\sigma_3, 9-12} = \\
&= \pm dQ_{14-6,5-8}^{\sigma_1, \sigma_2, \sigma_3, 9-12} dQ_{\alpha,5-8}^{9-13} + dQ_{\alpha,14-6,5-8}^{\sigma_1, \sigma_2, \sigma_3, 9-13} dQ_{5-8}^{9-12} \\
(3.115) \quad &= dQ_{\alpha,14-6,5-8}^{\sigma_1, \sigma_2, \sigma_3, 9-13} dQ_{5-8}^{9-12}
\end{aligned}$$

since consistency conditions force $dQ_{\alpha,5-8}^{9-13} \equiv 0$. Another identity is useful:

$$\begin{aligned}
& \sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} dQ_{14-6,5-8}^{\sigma_1, \sigma_2, 9-13} dQ_{\alpha, 13-16}^{\sigma_3, 9-12} = \\
& = dQ_{13-16}^{9-12} dQ_{\alpha, 14-6, 5-8}^{1-3, 9-13} + dQ_{\alpha, 13-16}^{9-13} dQ_{14-6, 5-8}^{1-3, 9-12} - dQ_{\alpha, 14-6}^{9-12} dQ_{13-16, 5-8}^{1-3, 9-13}
\end{aligned}
\tag{3.116}$$

Therefore,

$$\begin{aligned}
& \sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} |\mathbf{f}^{\sigma_1} \mathbf{f}^{\sigma_2}| \mathbf{d}_a^{\sigma_3} = \frac{dQ_{14,5-8}^{9-13}}{dQ_{5-8}^{9-12}} \sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} dQ_{14-6,5-8}^{\sigma_1, \sigma_2, 9-13} \begin{pmatrix} dQ_{\alpha, 5-8}^{\sigma_3, 9-12} \\ dQ_{\beta, 5-8}^{\sigma_3, 9-12} \end{pmatrix} \\
& = dQ_{14,5-8}^{9-13} \begin{pmatrix} dQ_{\alpha, 14-6, 5-8}^{1-3, 9-13} \\ dQ_{\beta, 14-6, 5-8}^{1-3, 9-13} \end{pmatrix}
\end{aligned}
\tag{3.117}$$

and

$$\begin{aligned}
& \sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} |\mathbf{f}^{\sigma_1} \mathbf{f}^{\sigma_2}| \mathbf{d}_b^{\sigma_3} = \\
& = \frac{dQ_{14,5-8}^{9-13}}{dQ_{13-16}^{9-12}} \sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} dQ_{14-6,5-8}^{\sigma_1, \sigma_2, 9-13} \begin{pmatrix} dQ_{\alpha, 13-16}^{\sigma_3, 9-12} \\ dQ_{\beta, 13-16}^{\sigma_3, 9-12} \end{pmatrix} \\
& = dQ_{14,5-8}^{9-13} \begin{pmatrix} dQ_{\alpha, 14-6, 5-8}^{1-3, 9-13} \\ dQ_{\beta, 14-6, 5-8}^{1-3, 9-13} \end{pmatrix} + \\
& \frac{dQ_{14,5-8}^{9-13}}{dQ_{13-16}^{9-12}} \left(dQ_{14-6,5-8}^{1-3, 9-12} \begin{pmatrix} dQ_{\alpha, 13-16}^{9-13} \\ dQ_{\beta, 13-16}^{9-13} \end{pmatrix} - dQ_{13-16,5-8}^{1-3, 9-13} \begin{pmatrix} dQ_{\alpha, 14-6}^{9-12} \\ dQ_{\beta, 14-6}^{9-12} \end{pmatrix} \right)
\end{aligned}
\tag{3.118}$$

Notice that thanks to range conditions

$$\begin{vmatrix} dQ_{\alpha, 13-16}^{9-13} & dQ_{\alpha, 14-6}^{9-12} \\ dQ_{\beta, 13-16}^{9-13} & dQ_{\beta, 14-6}^{9-12} \end{vmatrix} = \pm dQ_{\alpha, \beta, 14-6}^{9-13} dQ_{13-16}^{9-12} = 0
\tag{3.119}$$

which permits 3.118 to be written more simply. For some $\kappa, \hat{\kappa} \in \mathbb{R}$,

$$\begin{aligned}
& \sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} |\mathbf{f}^{\sigma_1} \mathbf{f}^{\sigma_2}| \mathbf{d}_b^{\sigma_3} = dQ_{14,5-8}^{9-13} \begin{pmatrix} dQ_{\alpha, 14-6, 5-8}^{1-3, 9-13} \\ dQ_{\beta, 14-6, 5-8}^{1-3, 9-13} \end{pmatrix} + \hat{\kappa} \begin{pmatrix} dQ_{\alpha, 13-16}^{9-13} \\ dQ_{\beta, 13-16}^{9-13} \end{pmatrix} \\
& = dQ_{14,5-8}^{9-13} \begin{pmatrix} dQ_{\alpha, 14-6, 5-8}^{1-3, 9-13} \\ dQ_{\beta, 14-6, 5-8}^{1-3, 9-13} \end{pmatrix} + \kappa \begin{pmatrix} dQ_{\alpha, 13-16}^{5-8, 13} \\ dQ_{\beta, 13-16}^{5-8, 13} \end{pmatrix}
\end{aligned}
\tag{3.120}$$

The last line follows from 3.55. Equations 3.117 and 3.120 imply

$$\mathbf{c} = \sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} |\mathbf{f}^{\sigma_1} \mathbf{f}^{\sigma_2}| \mathbf{d}^{\sigma_3} = \sum_{\sigma \in S_3} \frac{\text{sgn} \sigma}{2} |\mathbf{f}^{\sigma_1} \mathbf{f}^{\sigma_2}| (\mathbf{d}_a^{\sigma_3} - \mathbf{d}_b^{\sigma_3}) = \kappa \begin{pmatrix} dQ_{\alpha, 13-16}^{5-8, 13} \\ dQ_{\beta, 13-16}^{5-8, 13} \end{pmatrix}
\tag{3.121}$$

Note the Graßmann relation $\begin{vmatrix} dQ_{\alpha,13-16}^{5-8,13} & dQ_{\alpha,13-16}^{4-8} \\ dQ_{\beta,13-16}^{5-8,13} & dQ_{\beta,13-16}^{4-8} \end{vmatrix} = \pm dQ_{\alpha,\beta,13-16}^{4-8,13} dQ_{13-16}^{5-8}$, which can be used to simplify

$$(3.122) \quad \begin{vmatrix} \mathbf{c} & \bar{\mathbf{c}} \end{vmatrix} = \hat{\kappa} \kappa \begin{vmatrix} dQ_{\alpha,13-16}^{5-8,13} & dQ_{\alpha,9-12}^{4-8} \\ dQ_{\beta,13-16}^{5-8,13} & dQ_{\beta,9-12}^{4-8} \end{vmatrix} = \pm \hat{\kappa} \kappa \left(dQ_{\alpha,\beta,13-16}^{4-8,13} dQ_{13-16}^{5-8} \right) = 0$$

The last line follows from range conditions. Since $\alpha, \beta \in \{1, 2, 3\}$, $\text{rank } Q_{\alpha,\beta,14-6}^{4-8,13} \leq 4$. Therefore, $\text{rank } Q_{\alpha,\beta,13-16}^{4-8,13} \leq 5$, forcing $dQ_{\alpha,\beta,13-16}^{4-8,13} = 0$. Finally, steps one through three imply

$$\begin{aligned} & \begin{vmatrix} Pih_{\alpha}^5 & Pih_{\alpha}^6 & Pih_{\alpha}^{15} \\ Pih_{\beta}^5 & Pih_{\beta}^6 & Pih_{\beta}^{15} \\ Phh_1^5 & Phh_1^6 & Phh_1^{15} \end{vmatrix} = C \left| \mathbf{D} \begin{matrix} A_{1-4}^2 \\ \mathbf{L} \end{matrix} A_{13-16}^{15} \right| \\ &= \left| \mathbf{D} \begin{pmatrix} -[\mathbf{f}^1 & \mathbf{f}^2]^{-1} [\mathbf{f}^3 & \mathbf{f}^4] \\ I \end{pmatrix} \begin{pmatrix} A_3^2 \\ A_4^2 \end{pmatrix} : \mathbf{L} \begin{pmatrix} -[\mathbf{g}^1 & \mathbf{g}^2]^{-1} [\mathbf{g}^3 & \mathbf{g}^4] \\ I \end{pmatrix} \begin{pmatrix} A_{15}^{15} \\ A_{16}^{15} \end{pmatrix} \right| \\ &= \left| \mathbf{c} \mathbf{r}^T \begin{pmatrix} A_3^2 \\ A_4^2 \end{pmatrix} : \bar{\mathbf{c}} \bar{\mathbf{r}}^T \begin{pmatrix} A_{15}^{15} \\ A_{16}^{15} \end{pmatrix} \right| \\ &= \left(\mathbf{r}^T \cdot \begin{pmatrix} A_3^2 \\ A_4^2 \end{pmatrix} \right) * \left(\bar{\mathbf{r}}^T \cdot \begin{pmatrix} A_{15}^{15} \\ A_{16}^{15} \end{pmatrix} \right) * \begin{vmatrix} \mathbf{c} & \bar{\mathbf{c}} \end{vmatrix} \\ (3.123) &= 0 \end{aligned}$$

Similar calculations hold to show that other minors of this ilk are trivially zero and force no additional conditions upon the A_i^j s.

4. Conclusion. That it might prove computationally “too hard” to eliminate parameters from the solution to the modified problem for large systems has been one of the author’s greatest fears for diffuse tomography. The tricks used in section 1.3 will no doubt apply to larger systems in the plane. In 1.3 *all* of the independent range conditions upon the subsystems were utilized to eliminate half of the parameters from the solution to the inverse problem. In order to implement a recursive scheme for solving the inverse problem, all that is required is the solution of the “modified problem” a $n \times n$ system, where $n = 2^k$, $k \in \mathbb{N}$ [7]. Once that is done, other issues should be considered. From a practical point of view, the most important item on the agenda is to implement the recursive recovery algorithm in three dimensions. The algorithm will be analogous to its two dimensional predecessor. Before that can happen, however, range conditions upon three dimensional forward map must be completely understood [6]. Since clinical applications require that these algorithms be robust, a careful stability study is crucial. An extremely general Markovian model of photon transport was considered above. Neither time-of-flight information nor physical information about photon transport through

tissue were taken into account. A priori information about photon transport can and should be incorporated into this model. (The author doubts that clinicians would find a set of Markov transition probabilities helpful diagnostically.)

REFERENCES

- [1] F. A. Grünbaum, J. P. Zubelli, "Diffuse Tomography: computational aspects of the isotropic case", *Inverse Problems*, **8**, 421–433, (1992).
- [2] F. A. Grünbaum, "Diffuse Tomography: the isotropic case", *Inverse Problems*, **8**, 409–419, (1992).
- [3] J. Singer, F. A. Grünbaum, P. Kohn and J. Zubelli, "Image reconstruction of the interior of bodies that diffuse radiation," *Science*, **248**, 990–993, (1990).
- [4] F. A. Grünbaum, "An inverse problem in transport theory: diffuse tomography," *Invariant Imbedding and Inverse Problems*, SIAM 1993, in Honor of R. Kruger, J. Coronas editor, 209–215.
- [5] S. K. Patch, "Consistency Conditions in Diffuse Tomography", *Inverse Problems*, **10**, no. 1, 199–212, (1994).
- [6] S. K. Patch, "A Note on Consistency Conditions in Three Dimensional Diffuse Tomography", *Lectures in Applied Mathematics : Tomography, Impedance Imaging, and Integral Geometry*, bf. 30, eds. E. T. Quinto, M. Cheney, P. Kuchement, Tomography, American Mathematical Society, Providence, RI., 211–230, (1994).
- [7] S. K. Patch, "Recursive Recovery of a family of Markov Transition Probabilities from Boundary Value Data", *Journal of Mathematical Physics*, accepted for publication.
- [8] F. A. Grünbaum, S. K. Patch, "The use of Graßmann Identities for inversion of a general model in diffuse tomography", Proceedings of the Lapland Conference on Inverse Problems, June 14–20, 1992. Saariselkä, Finland.
- [9] S. K. Patch, "Recursive Recovery of Markov Transition Probabilities from Boundary Value Data", Ph. D. thesis, University of California at Berkeley, 1994.
- [10] Hodge, W. V. D., Pedoe, D., *Methods of Algebraic Geometry*, Cambridge University Press, 1968.
- [11] Griffiths, P., Harris, J., *Principles of Algebraic Geometry*, Wiley and Sons, Inc., 1978.
- [12] C. Aitken, *Determinants and Matrices*, 8th ed., Oliver and Boyd Ltd., Edinburgh, 1954.
- [13] Coronas, J., "A Discrete Model of Transport and Invariant Imbedding in Two or More Dimensions", *Lecture Notes in Pure and Applied Math*, Vol. 115, pp. 271–278.
- [14] W. Feller, *An Introduction to Probability Theory and its Applications*, 2nd ed., John Wiley and Sons, Inc., New York, 1962.
- [15] V. Faber, D. L. Seth, G. M. Wing, "Invariant Imbedding in Two Dimensions", *Lecture Notes in Pure and Applied Math*, **115**, 279–300.
- [16] F. A. Grünbaum, *Tomography with Diffusion, Inverse Problems in Action*, ed. P. Sabatier, 16–21, Springer-Verlag, Berlin, 1990.
- [17] R. M. Redheffer, "Transmission Line Theory and Scattering and Transfer", *Journal of Mathematics and Physics*, **41**, 1–41, (1962).
- [18] R. M. Redheffer, "Remarks on the Basis of Network Theory", *Journal of Mathematics and Physics*, **28**, pp. 237–258, (1950).

FIG. 1. Incoming, hidden, and outgoing states are labeled with i 's, h 's, and o 's respectively.

FIG. 2. A 4×4 system. The incoming and outgoing states are labeled; all unlabeled states are hidden states. There are 16 incoming and 16 outgoing states, but 48 hidden states.

FIG. 3. Decomposition of a 4×4 system into four 2×2 subsystems. The thick lines separate the subsystems. The “modified” 4×4 system disregards individual pixels. Only the subsystems are relevant at the first level of this recursive procedure.

#	Author/s	Title
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