**BOUNDARY VALUES OF THE RESOLVENT OF A
SELF-ADJOINT OPERATOR: HIGHER ORDER ESTIMATES**

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BOUNDARY VALUES OF THE RESOLVENT OF A SELF-ADJOINT OPERATOR: HIGHER ORDER ESTIMATES

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Introduction

In this article we intend to describe the regularity properties of the functions $\lambda \mapsto R(\lambda \pm i0)$, where $R(\lambda \pm i0)$ are the boundary values of the resolvent $R(z) = (H - z)^{-1}$ of a self-adjoint operator H . For that purpose we place ourselves in the framework of the conjugate operator theory ¹.

Let $\mathcal{H}$ be a Hilbert space and A a self-adjoint operator in $\mathcal{H}$. We shall identify $\mathcal{H}$ with its adjoint space $\mathcal{H}^*$ by using the Riesz isomorphism and we shall denote by $\{\mathcal{H}_s\}_{s \in \mathbb{R}}$ the Sobolev scale of Hilbert spaces associated to A . So, for $s \geq 0$ $\mathcal{H}_s$ is the domain of $|A|^s$ equipped with the norm $\|f\|_s = \|\langle A \rangle^s f\|$, where $\|\cdot\|$ is the norm of $\mathcal{H}$ and $\langle A \rangle = (1 + A^2)^{1/2}$, while for $s < 0$ $\mathcal{H}_s$ is the completion of $\mathcal{H}$ for the norm $\|\cdot\|_s$ given by the same expression. Then $\mathcal{H}_0 = \mathcal{H}$, $\mathcal{H}_s \subset \mathcal{H}_t$ if and only if $s \geq t$, and $\mathcal{H}_s^* = \mathcal{H}_{-s}$ (canonical identification). For $s \geq 0$ we denote by $\|\cdot\|^{(s)}$ the norm in the Banach space $B(\mathcal{H}_s, \mathcal{H}_s^*)$, more precisely $\|T\|^{(s)} = \|\langle A \rangle^{-s} T \langle A \rangle^{-s}\|$ where the last norm is in $B(\mathcal{H})$. Observe that for $0 < s < t$ we have

$$\mathcal{H}_t \subset \mathcal{H}_s \subset \mathcal{H} \subset \mathcal{H}_s^* \subset \mathcal{H}_t^*,$$

continuous embeddings which decrease the norm; hence

$$B(\mathcal{H}) \subset B(\mathcal{H}_s, \mathcal{H}_s^*) \subset B(\mathcal{H}_t, \mathcal{H}_t^*)$$

and $\|T\|^{(t)} \leq \|T\|^{(s)} \leq \|T\|$ for all $T \in B(\mathcal{H})$.

For technical reasons it is also convenient to introduce the Besov scale $\{\mathcal{H}_{s,p} \mid s \in \mathbb{R}, 1 \leq p \leq \infty\}$ associated to A . One may obtain it from

¹See [18], the paper where E. Mourre introduced the main ideas of this theory; the version that we use is presented in detail in [1].

the Sobolev scale by real interpolation: $(\mathcal{H}_{s_0}, \mathcal{H}_{s_1})_{\theta, p} = \mathcal{H}_{s, p}$ if $s_0 \neq s_1$, $0 < \theta < 1$ and $s = (1 - \theta)s_0 + \theta s_1$. Note that $\mathcal{H}_{s, 2} = \mathcal{H}_s$. One may find a self-contained and elementary description of the spaces $\mathcal{H}_{s, p}$ in Appendix A of [5]. In §1.1 we recall some properties of the usual Besov spaces $\mathcal{H}^{s, p}(\mathbb{R})$ associated to the operator $A = P = -i \frac{d}{dx}$ in $\mathcal{H} = L^2(\mathbb{R})$; all these properties (including the Littlewood-Paley type description of the norm) remain valid for an arbitrary A in an arbitrary $\mathcal{H}$.

The spaces $\mathcal{H}_{s, p}$ with $s > 0$ consist of vectors in $\mathcal{H}$ that have a certain degree of regularity with respect to A . It will be convenient to introduce classes of operators in $B(\mathcal{H})$ which have similar regularity properties with respect to an operator $\mathcal{A}$ acting in $B(\mathcal{H})$ and naturally associated to A . The definition will be better understood with the following observation: for an arbitrary $f \in \mathcal{H}$ the element Af is a well defined vector in $\mathcal{H}_{-1}$ and we have $f \in \mathcal{H}_1 = D(A)$ if and only if $Af \in \mathcal{H}$ (which is a subspace of $\mathcal{H}_{-1}$); on the other hand the property $f \in \mathcal{H}_1$ is equivalent to the fact that the function $\tau \mapsto e^{iA\tau} f \in \mathcal{H}$ is (strongly or weakly) C^1 (Lipschitz would be sufficient). In fact, one may describe all the spaces $\mathcal{H}_{s, p}$ with $s > 0$ in terms of the regularity properties of the functions $\tau \mapsto e^{iA\tau} f \in \mathcal{H}$. The operator A induces a group of automorphisms $\mathcal{W}_\tau$ of $B(\mathcal{H})$, namely $\mathcal{W}_\tau[T] = e^{-iA\tau} T e^{iA\tau}$, and we formally have $\mathcal{W}_\tau = e^{iA\tau}$ with $\mathcal{A}[T] = TA - AT \equiv [T, A]$. For each $T \in B(\mathcal{H})$, $\mathcal{A}[T]$ is a well defined element of $B(\mathcal{H}_1, \mathcal{H}_{-1})$ and the relation $\mathcal{A}[T] \in B(\mathcal{H})$ (which is a subspace of $B(\mathcal{H}_1, \mathcal{H}_{-1})$) is equivalent to the fact that the function $\tau \mapsto T(\tau) \equiv e^{-iA\tau} T e^{iA\tau} \in B(\mathcal{H})$ is of class C^1 in the strong or weak operator topology (and this is also equivalent to the property that this function is Lipschitz in norm). If this is the case we write $T \in C^1(A; \mathcal{H})$. So $C^1(A; \mathcal{H}) \subset B(\mathcal{H})$ is the analogue of $\mathcal{H}_1 \equiv \mathcal{H}_{1, 2} \subset \mathcal{H}$. If $k \geq 2$ is an integer we define $C^k(A; \mathcal{H})$ (analogue of $\mathcal{H}_k \equiv \mathcal{H}_{k, 2}$) by the property that $T(\cdot) : \mathbb{R} \rightarrow B(\mathcal{H})$ be strongly of class C^k .

The classes $C^k(A; \mathcal{H})$ are not really useful in our approach, we rather need classes of regularity that are analogues of $\mathcal{H}_{s, \infty}$; we shall denote them by $\mathcal{C}^s(A; \mathcal{H})$, where s could be an arbitrary strictly positive number. Before giving their definition it is convenient to introduce the space Λ^s consisting of Lipschitz-Zygmund (or Hölder-Zygmund; we use the terminology and the notations of [23]) functions of order s on $\mathbb{R}$. Let E be an arbitrary Banach space and $f : \mathbb{R} \rightarrow E$ a bounded function. We shall distinguish three cases:

- 1) $0 < s < 1$: then f is of class Λ^s if $\|f(x + \varepsilon) - f(x)\| \leq C|\varepsilon|^s$ for some constant $C < \infty$ and all $x, \varepsilon \in \mathbb{R}$;
- 2) $s = 1$: then f is of class Λ^1 if $\|f(x + \varepsilon) + f(x - \varepsilon) - 2f(x)\| \leq C|\varepsilon|$ for a constant $C < \infty$ and all $x, \varepsilon \in \mathbb{R}$;
- 3) $s > 1$: then $s = k + \sigma$ where $k \geq 1$ is an integer and $0 < \sigma \leq 1$ and we say that f is of class Λ^s if f is of class BC^k (i.e. k times continuously differentiable with bounded derivatives) and $f^{(k)}$ is of class Λ^σ .

A better description of the spaces Λ^s will be given in §1.2. If Ω is an open real set, we say that $f : \Omega \rightarrow E$ is locally of class Λ^s on Ω if θf is of class Λ^s for each $\theta \in C_0^\infty(\Omega)$. The space Λ^1 (whose importance in Fourier analysis was stressed by A. Zygmund) is quite different from the space of Lipschitz functions (of order 1; cf. §1.2).

At last, we can define the space $\mathcal{C}^s$: an operator $T \in B(\mathcal{H})$ is of class $\mathcal{C}^s(A; \mathcal{H})$ if the function $\tau \mapsto e^{-i\tau A} T e^{i\tau A} \in B(\mathcal{H})$ is of class Λ^s .

Assume now that a self-adjoint operator H is given in $\mathcal{H}$ and let us denote by $R(z) = (H - z)^{-1}$ the resolvent family of H ; z is a complex number not in the spectrum of H . We say that H is of class $\mathcal{C}^k$ or $\mathcal{C}^s$ with respect to A if $R(z) \in \mathcal{C}^k(A; \mathcal{H})$ or $R(z) \in \mathcal{C}^s(A; \mathcal{H})$ respectively for some z . One may then show that this property will hold for all z and that $\varphi(H) \in \mathcal{C}^k(A; \mathcal{H})$ or $\varphi(H) \in \mathcal{C}^s(A; \mathcal{H})$ respectively for all $\varphi \in C_0^\infty(\mathbb{R})$ (see [3]).

Throughout this paper we shall assume that H is at least of class $\mathcal{C}^1$ with respect to A . If $D(H)$ is the domain of H equipped with the graph topology, then we have dense continuous embeddings $D(H) \subset \mathcal{H} \subset D(H)^*$. One may show (see [6]) that there is a unique continuous symmetric operator $[H, iA] : D(H) \rightarrow D(H)^*$ such that $[R(z), iA] = -R(z)[H, iA]R(z)$ for each complex z outside the spectrum of H . Then for each real $\varphi \in C_0^\infty(\mathbb{R})$ the operator $\varphi(H)[H, iA]\varphi(H)$ is bounded and symmetric in $\mathcal{H}$. We introduce now the set of points $\lambda \in \mathbb{R}$ where a *strict Mourre estimate* holds:

$$\mu(H) = \{\lambda \in \mathbb{R} \mid \exists a > 0 \text{ and } \exists \varphi \in C_0^\infty(\mathbb{R}) \text{ real with} \\ \varphi(\lambda) \neq 0 \text{ and } \varphi(H)[H, iA]\varphi(H) \geq a\varphi(H)^2\}.$$

This is an open real set which is not empty if A is conjugate to H at some point (this assertion may be interpreted as a definition). It was E. Mourre who understood that H has quite remarkable spectral properties in the set $\mu(H)$, see [18].

We shall now quote a particular case of a result obtained in [4]. *Assume that H is of class $\mathcal{C}^{1,1}$ with respect to A , i.e. $R(z_0) \in \mathcal{C}^{1,1}(A; \mathcal{H})$ for some complex z_0 . The class $\mathcal{C}^{1,1}$ will be defined in §2.1; we note here only that $\mathcal{C}^\alpha \subset \mathcal{C}^{1,1} \subset \mathcal{C}^1 \subset \mathcal{C}^1$ for all $\alpha > 1$. We also recall that for each $s > 1/2$ we have continuous embeddings:*

$$\mathcal{H}_s \subset \mathcal{H}_{1/2,1} \subset \mathcal{H} \subset \mathcal{H}_{1/2,1}^* \subset \mathcal{H}_s^*.$$

In particular

$$B(\mathcal{H}) \subset B(\mathcal{H}_{1/2,1}, \mathcal{H}_{1/2,1}^*) \subset B(\mathcal{H}_s, \mathcal{H}_s^*)$$

with continuous embeddings. Hence one may consider the function $z \mapsto R(z) \in B(\mathcal{H}_{1/2,1}, \mathcal{H}_{1/2,1}^*)$ which is well defined and holomorphic in each of

the half-planes $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm \operatorname{Im} z > 0\}$. If H has a spectral gap (i.e. its spectrum is not $\mathbb{R}$) then the preceding function has a weak* continuous extension to $\mathbb{C}_\pm \cup \mu(H)$; in particular, if $\lambda \in \mu(H)$ then the symbols $R(\lambda + i0)$ and $R(\lambda - i0)$ have a well defined meaning as operators $\mathcal{H}_{1/2,1} \rightarrow \mathcal{H}_{1/2,1}^*$. This fact has been proved in [4] in a more general setting, but is also a consequence of the estimates that will be proved later on in this paper (see §4.7). The result is optimal in three respects:

- 1) weak* continuity cannot be replaced by strong continuity (see §4.8);
- 2) the space $\mathcal{H}_{1/2,1}$ cannot be replaced by a space $\mathcal{H}_{s,p}$ with $s < 1/2$ or $s = 1/2$ and $p > 1$;
- 3) the regularity class $\mathcal{C}^{1,1}$ cannot be replaced by a class $\mathcal{C}^{\alpha,p}$ with $\alpha < 1$ or $\alpha = 1$ and $p > 1$.

The function $R(\cdot)$ has better continuity properties when considered as $B(\mathcal{H}_s, \mathcal{H}_s^*)$ -valued if $s > 1/2$ and if H has better regularity properties with respect to A . The following theorem is the main result of this paper.

Theorem *Let $s > 1/2$ be a real number. If H has a spectral gap and is of class $\mathcal{C}^{s+1/2}$ with respect to A , then the functions*

$$\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_s, \mathcal{H}_s^*)$$

are locally of class $\Lambda^{s-1/2}$ on $\mu(H)$.

The result is optimal in the scales $\mathcal{C}^t$, $\mathcal{H}_s$, Λ^α , as it is explained in detail in §1.4 and §1.5. For example, if s is a half-integer $s = k + 1/2$ with $k \geq 1$ integer, then the functions $\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_s, \mathcal{H}_s^*)$ are of class Λ^k but not of class $\mathcal{C}^k$ and, if $k = 1$, they are not locally Lipschitz (even if H is of class $\mathcal{C}^\infty$).

It is easy to deduce from the Theorem the order of regularity of the maps $z \mapsto R(z) \in B(\mathcal{H}_s, \mathcal{H}_s^*)$ defined for $z = \lambda \pm i\mu$, $\lambda \in \mu(H)$ and $\mu \geq 0$. Indeed, the harmonic function $R(z)$ in the half-spaces $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ can be recovered from its boundary values with the help of the Poisson kernel. For simplicity, we state the result only for the case when s is not a half-integer (in order to get Hölder-type conditions that do not involve the second order modulus of continuity). Assume first that $1/2 < s < 3/2$; then for each compact subset K of $\mu(H)$ there is a constant $C < \infty$ such that for all z_1, z_2 with $\operatorname{Re} z_j \in K$ and $\operatorname{Im} z_j \geq 0$:

$$\|R(z_1) - R(z_2)\|^{(s)} \leq C|z_1 - z_2|^{s-1/2}.$$

More generally, assume that $s = k + 1/2 + \sigma$ with $k \geq 0$ integer and $0 < \sigma < 1$. Taking into account that $\frac{d^k}{dz^k} R(z) = k! R(z)^{k+1}$ for $\operatorname{Im} z \neq 0$ we see that for each compact set $K \subset \mu(H)$ there is a constant $C < \infty$ such that for all z_1, z_2 with $\operatorname{Re} z_j \in K$ and $\operatorname{Im} z_j > 0$:

$$\|R(z_1)^{k+1} - R(z_2)^{k+1}\|^{(s)} \leq C|z_1 - z_2|^\sigma.$$

Indeed this is equivalent to the fact that the function $z \mapsto R(z) \in B(\mathcal{H}_s, \mathcal{H}_s^*)$ is locally of class $\Lambda^{k+\sigma}$ on the set of complex numbers z such that $\operatorname{Re} z \in \mu(H)$ and $\operatorname{Im} z \geq 0$. In particular, we obtain that $\lim_{\mu \rightarrow +0} R(\lambda \pm i\mu)^{k+1} := R^{k+1}(\lambda \pm i0)$ exists in the norm topology of $B(\mathcal{H}_s, \mathcal{H}_s^*)$, locally uniformly in $\lambda \in \mu(H)$, the maps $\lambda \mapsto R^{k+1}(\lambda \pm i0) \in B(\mathcal{H}_s, \mathcal{H}_s^*)$ are locally Lipschitz (or Hölder) continuous of order σ on $\mu(H)$, and $\frac{d^k}{d\lambda^k} R(\lambda \pm i0) = k! R^{k+1}(\lambda \pm i0)$ (derivatives in the norm topology of $B(\mathcal{H}_s, \mathcal{H}_s^*)$). Notice that $R^{k+1}(\lambda \pm i0)$ cannot be interpreted as a product of operators $R(\lambda \pm i0)$.

Let us review what was known before our work concerning the regularity of the boundary values $R(\lambda \pm i0)$ in the setting of the conjugate operator method. In [21] it was shown that $\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_s, \mathcal{H}_s^*)$ are Hölder-continuous of order θ if $1/2 < s \leq 1$ and $\theta < 2/3(s - 1/2)$. In [26] this result was improved to $\theta = (s - 1/2)(s + 1/2)^{-1}$. In both papers the regularity conditions imposed to H are stronger than ours (they assume $D(H)$ invariant under the group $e^{iA\tau}$ hence, according to Theorem 6.3.4 from [1], their hypotheses imply that H is of class C^2 with respect to A). In [5] the assumptions on H have been weakened, but the order of Hölder continuity has not been improved (when compared with [26]). The case $s > 1$ has first been treated in [15] (but see [20] and [13], [14] for related results). We are not going to discuss the conditions of regularity imposed on H in these papers (they are stronger than ours, being the natural extension of those of [18] and [21]). On the other hand, their results concerning the regularity of the functions $\lambda \mapsto R(\lambda \pm i0)$ are of the following type. Let $n \geq 1$ be an integer and let $s > n - 1/2$. Then the functions $\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_s, \mathcal{H}_s^*)$ are of class C^{n-1} and their derivatives of order $n - 1$ are locally Hölder continuous of order $(s - n + 1/2)(sn + s - n + 1/2)^{-1}$. If $1/2 < s < 3/4$ this assertion cannot be true (cf. the optimality of our results, established in §1.4).

In the theorem stated above we assumed that H has a spectral gap. This condition is not really necessary (the result remains true in the setting of [4]) but is very convenient for the proof since it allows us to assume that H is a bounded operator. Indeed, let λ_0 be a real number outside the spectrum of H and set $R = (\lambda_0 - H)^{-1}$. Then R is a bounded self-adjoint operator and for $\operatorname{Im} z \neq 0$:

$$(H - z)^{-1} = (\lambda_0 - z)^{-1} [R - (\lambda_0 - z)^{-1}]^{-1} R.$$

By definition, R is of the same class (with respect to A) as H . Moreover, according to Proposition 7.2.5 of [1] (or see [3], [6]), if $\lambda \neq \lambda_0$ is a real number then we have $\lambda \in \mu(H)$ if and only if $(\lambda_0 - \lambda)^{-1} \in \mu(R)$. Hence it is sufficient to prove the Theorem with H replaced by the bounded symmetric operator R . The preceding formula will give us the regularity properties of

the function $z \mapsto (H - z)^{-1}$ in terms of those of $\zeta \mapsto (R - \zeta)^{-1}$ (see the proof of Theorem 7.4.1 in [1]).

The paper is organized as follows. In Section 1, after some preliminary remarks concerning some function spaces on the real line, we show that the assertions of the Theorem are optimal. Section 2 contains several technical results needed for the proof of our main theorem but which, we think, have some independent interest. In Section 3 we prove a special case of the Theorem by elementary means. In the final section we prove the Theorem under the assumption that H is a bounded operator. This implies the general case by the argument sketched above.

1. Preliminary Considerations

1.1. BESOV SPACES

Let $B_q^{s,p}(\mathbb{R})$ be the Besov space of order (s, p) associated to $L^q(\mathbb{R})$, where s is an arbitrary real number and $p, q \in [1, \infty]$ (see [19] and [23]). Special notations will be used in the cases $q = 2$ and $q = \infty$:

$$\mathcal{H} := L^2(\mathbb{R}) \equiv B_2^{0,2}(\mathbb{R}), \quad \mathcal{H}^{s,p} := B_2^{s,p}(\mathbb{R}), \quad \mathcal{H}^s := \mathcal{H}^{s,2}$$

for $s \in \mathbb{R}, 1 \leq p \leq \infty$,

$$\Lambda^{\alpha,p} := B_\infty^{\alpha,p}(\mathbb{R}), \quad \Lambda^\alpha := \Lambda^{\alpha,\infty}$$

for $\alpha > 0$ and $1 \leq p \leq \infty$. In this subsection some properties of the spaces $\mathcal{H}^{s,p}$ will be presented; several remarks concerning $\Lambda^{\alpha,p}$ can be found in §1.2.

All these spaces can be described in terms of the operator $P = -i \frac{d}{dx}$ as follows. We denote $f(P)$ the operator defined by

$$(\mathcal{F}f(P)u)(x) = f(x)(\mathcal{F}u)(x),$$

where $(\mathcal{F}g)(x) \equiv \hat{g}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ixy} g(y) dy$ is the Fourier transform of g . If f is of class C_0^∞ then clearly $f(P)$ is a well defined operator on the space $\mathcal{S}^*(\mathbb{R})$ (of temperate distributions) with values in $C^\infty(\mathbb{R})$. Now let $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(x) \neq 0$ if $|x| < 1$ and $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(x) \neq 0$ if $2^{-1} < |x| < 2$ and $\psi(x) = 0$ otherwise. Then $\mathcal{H}^{s,p}$ is the space of all $u \in \mathcal{S}^*(\mathbb{R})$ such that

$$\|u\|_{s,p} := \|\varphi(P)u\| + \left[\int_1^\infty \|r^s \psi(P/r)u\|^p r^{-1} dr \right]^{1/p} < \infty$$

where $\|\cdot\|$ is the norm of $L^2(\mathbb{R})$; the space $B_q^{s,p}$ is obtained by taking instead the norm in $L^q(\mathbb{R})$. If $p = \infty$ the second term in the r.h.s. should be

thought as being $\sup_{r \geq 1} \|r^s \psi(P/r)u\|$. $\mathcal{H}^{s,p}$ equipped with the norm $\|\cdot\|_{s,p}$ is a Banach space continuously embedded in $\mathcal{S}^*$. If $p = 2$ then $\|\cdot\|_{s,2}$ is equivalent to the norm $\|u\|_s = \|\langle P \rangle^s u\|$ where $\langle P \rangle = (1 + P^2)^{1/2}$; it is convenient to equip $\mathcal{H}^s$ with the Hilbert space structure defined by the norm $\|\cdot\|_s$ ².

If Ω is an open real set, then a function (or distribution) u on Ω is *locally of class* $B_q^{s,p}$ on Ω if $\theta u \in B_q^{s,p}$ for all $\theta \in C_0^\infty(\Omega)$. This defines, in particular, the expressions locally of class $\mathcal{H}^{s,p}$ or locally of class $\Lambda^{\alpha,p}$.

We notice that the scale of Banach spaces $\{\mathcal{H}^{s,p} \mid s \in \mathbb{R}, 1 \leq p \leq \infty\}$ is totally ordered in the following sense: for arbitrary couples (s,p) and (t,q) we have either $\mathcal{H}^{s,p} \subset \mathcal{H}^{t,q}$ or $\mathcal{H}^{t,q} \subset \mathcal{H}^{s,p}$. In order to state this property in more precise terms we shall introduce a total order relation on the set $\mathbb{R} \times [1, \infty]$ by the following rule: $(s,p) > (t,q)$ means $s > t$ (then p, q are arbitrary) or $s = t$ but $p < q$. Then $\mathcal{H}^{s,p} \subset \mathcal{H}^{t,q}$ is equivalent to $(s,p) \geq (t,q)$. One has $\mathcal{S}(\mathbb{R}) \subset \mathcal{H}^{s,p}$ for any s, p but the embedding is dense if and only if $p < \infty$. We denote by $\mathring{\mathcal{H}}^{s,\infty}$ the closure of $\mathcal{S}$ in $\mathcal{H}^{s,\infty}$.

The adjoint space $\mathcal{H}^*$ of $\mathcal{H} = L^2(\mathbb{R})$ is identified with itself as usual. This implies the following duality properties:

$$[\mathcal{H}^{s,p}]^* = \mathcal{H}^{-s,p'} \text{ if } 1 \leq p < \infty, \quad p^{-1} + p'^{-1} = 1, \text{ and } [\mathring{\mathcal{H}}^{s,\infty}]^* = \mathcal{H}^{-s,1}.$$

We discuss now the regularity properties of the elements of $\mathcal{H}^{s,p}$. The largest space in the scale $\{\mathcal{H}^{s,p}\}$ which consists only of continuous functions is $\mathcal{H}^{1/2,1}$. More precisely, $\mathcal{H}^{1/2,1}$ is a subalgebra of $C_\infty(\mathbb{R})$ (which is the space of continuous functions that tend to zero at infinity; one may easily show that the Fourier transform of a function from $\mathcal{H}^{1/2,1}$ is an integrable function and then one applies the Riemann-Lebesgue lemma). This assertion is sharp in the following sense: there is a function f such that $f \in \mathcal{H}^{1/2,p}$ for all $p > 1$ and f is unbounded on each interval. Indeed, let g be the inverse Fourier transform of a function $\hat{g} \in C^\infty(\mathbb{R})$ such that $\hat{g} \geq 0$, $\hat{g}$ is even and $\hat{g}(x) = (x \log x)^{-1}$ if $x \geq 2$. Then g is a real function of class $\mathcal{S}$ outside the origin and $g(x) \sim \log \log |x|^{-1}$ as $x \rightarrow 0$; so g belongs to all the spaces $\mathcal{H}^{1/2,p}$ with $p > 1$, g is bounded from below ($g(x) \geq c > -\infty$) and $\lim_{x \rightarrow 0} g(x) = +\infty$. Now for each rational number r let λ_r be a strictly positive number such that $\sum \lambda_r < \infty$. Then the function $f(x) = \sum \lambda_r g(x - r)$ belongs to $\bigcap_{p>1} \mathcal{H}^{1/2,p}$, $f(x) \geq c \sum \lambda_r > -\infty$ and $\lim_{x \rightarrow r} f(x) = +\infty$ for each rational number r .

²An elementary and self-contained presentation of the properties of the spaces $\mathcal{H}^{s,p}$ may be found in Appendix A of [5], where the context is slightly more general, namely P could be an arbitrary self-adjoint operator in an abstract Hilbert space.

1.2. LIPSCHITZ-ZYGMUND SPACES

The local regularity of the functions from $\mathcal{H}^{s,p}$ with $s > 1/2$ is best described by the embedding $\mathcal{H}^{s,p} \subset \Lambda^{s-1/2,p}$ (cf. Ch.III in [19]). We recall the notation $\Lambda^{\alpha,p} := B_{\infty}^{\alpha,p}(\mathbb{R})$ for $\alpha > 0$ and $1 \leq p \leq \infty$. One has $\Lambda^{\alpha,p} \subset \Lambda^{\beta,q}$ if and only if $(\alpha, p) \geq (\beta, q)$.

The Lipschitz-Zygmund spaces $\Lambda^{\alpha} := \Lambda^{\alpha,\infty}$ may also be defined in terms of the *modulus of continuity of order m* of the function $f \in L^{\infty}(\mathbb{R})$:

$$\omega_m(\delta) = \omega_m(\delta, f) := \sup_{|\varepsilon| \leq \delta} \operatorname{ess-sup}_{x \in \mathbb{R}} \left| \sum_{j=0}^m (-1)^j \binom{m}{j} f(x + j\varepsilon) \right|$$

If $\alpha > 0$ and $f \in L^{\infty}(\mathbb{R})$ then $f \in \Lambda^{\alpha}$ if and only if there is an integer $m > \alpha$ such that $\omega_m(\delta) \leq c\delta^{\alpha}$, where c is a finite constant. So, for $0 < \alpha < 1$, Λ^{α} is exactly the space of bounded Hölder-continuous functions of order α on $\mathbb{R}$ (defined by $|f(x + \varepsilon) - f(x)| \leq c\delta^{\alpha}$ for $x \in \mathbb{R}$ and $|\varepsilon| \leq \delta$). But if $\alpha = 1$ the space Λ^1 is defined by the Zygmund type condition $|f(x + \varepsilon) + f(x - \varepsilon) - 2f(x)| \leq c\delta$ ($x \in \mathbb{R}$, $|\varepsilon| \leq \delta$). Λ^1 contains strictly the space of Lipschitz continuous functions (defined by $\omega_1(\delta) \leq c\delta$; for $f \in \Lambda^1$ we only have $\omega_1(\delta) = O(\delta|\log \delta|)$). Indeed, there are functions in Λ^1 which are nowhere differentiable (see Sections 3 and 4 in Ch.II of [27]), while a Lipschitz continuous function is differentiable almost everywhere (a.e.).

If $\alpha > 1$ and $k \geq 0$ is an integer such that $k < \alpha$, then $f \in \Lambda^{\alpha}$ is equivalent to $f \in BC^k$ (i.e. f and its derivatives of order $\leq k$ are bounded and continuous) and $f^{(k)} \in \Lambda^{\alpha-k}$. Note that if $f \in \Lambda^2$ for example, then f is of class C^1 but its derivative f' is not Lipschitz continuous in general (we have only $f' \in \Lambda^1$).

For $1 \leq p < \infty$ we have $\Lambda^{\alpha,p} \subset \Lambda^{\alpha}$ and the functions from $\Lambda^{\alpha,p}$ have better regularity properties; for example, the functions from $\Lambda^{1,p}$ ($p < \infty$) are differentiable at all points of a dense subset of $\mathbb{R}$, although this set could be of measure zero (see Theorem 3.3 and page 48 in Ch.II of [27]). If k is an integer, then $\Lambda^{k,1} \subset BC^k$ and this is optimal: the elements of $\Lambda^{1,p}$ are not Lipschitz continuous (in general) if $p > 1$. However, if $f \in \Lambda^{1,2}$ then its distributional derivative is a function of class BMO (bounded mean oscillation; §6.18 in [23]).

So, for $s > 1/2$ we have $\mathcal{H}^{s,p} \subset \Lambda^{s-1/2,p} \subset \Lambda^{s-1/2}$ for all $p \in [1, \infty]$ and the embeddings are optimal in an obvious sense (see Example 4 on page 50 of [19]; we mention that $\mathcal{H}^{s,p} \subset B_q^{t,p}(\mathbb{R})$ if $2 \leq q \leq \infty$ and $t = s - 2^{-1} + q^{-1}$). If $1/2 < s < 3/2$, then the functions from $\mathcal{H}^{s,p}$ are Hölder continuous of order $s - 1/2$. If $s = k + 1/2$ with $k \geq 1$ integer then $\mathcal{H}^{s,1} \subset BC^k$. But for example $\mathcal{H}^{3/2} \subset \Lambda^{1,2}$ only and in general a function f from $\mathcal{H}^{3/2}$ is not Lipschitz continuous (f' is of class BMO).

1.3. HILBERT TRANSFORM

If $u : \mathbb{R} \rightarrow \mathbb{C}$ is a function such that $u(x)(1 + |x|)^{-1}$ is integrable, then $\lim_{\varepsilon \rightarrow 0} \pi^{-1} \int_{|x-\lambda|>\varepsilon} (\lambda - x)^{-1} u(x) dx$ exists for almost every real λ (see §5.9 in [25]). We denote $\tilde{u}(\lambda)$ the limit whenever it exists; the (a.e. defined) function $\tilde{u}$ is the Hilbert transform of u . For $1 < p < \infty$ the map $u \mapsto \tilde{u}$ is a continuous linear operator in $L^p(\mathbb{R})$. This operator is isometric in $\mathcal{H} = L^2(\mathbb{R})$ and its square is equal to -1 ; in fact we then have $\tilde{u} = -i(\operatorname{sgn} P)u$ where $\operatorname{sgn} P = P|P|^{-1}$.

Since the unitary operator $\operatorname{sgn} P$ in $\mathcal{H}$ commutes with the operators $\varphi(P)$, $\psi(P/r)$ that appear in the definition of the norm $\|\cdot\|_{s,p}$, it follows that $\operatorname{sgn} P$ induces an isometric bijective operator in $\mathcal{H}^{s,p}$ for all $s \in \mathbb{R}$, $1 \leq p \leq \infty$. So $\tilde{u} \in \mathcal{H}^{s,p}$ if $u \in \mathcal{H}^{s,p}$ for some $s > 0$, $1 \leq p \leq \infty$. In particular, if $u \in \mathcal{H}^{1/2,1}$, then the almost everywhere defined function $\tilde{u}$ is a.e. equal to a continuous function (since $\mathcal{H}^{1/2,1} \subset C_\infty$). But in fact in this case the limit defining $\tilde{u}(\lambda)$ exists for all $\lambda \in \mathbb{R}$, uniformly in λ , and the function $\lambda \mapsto \tilde{u}(\lambda)$ so defined is continuous.

For completeness we give an elementary proof of this assertion in a slightly more general setting. Assume that $u \in L^2(\mathbb{R}) \cap C_\infty(\mathbb{R})$ is a real function and that the function $\tilde{u}$ (which is defined a.e. and belongs to $L^2(\mathbb{R})$; one should take here $\tilde{u} = -i(\operatorname{sgn} x)\hat{u}(x)$ as definition of $\tilde{u}$ is a.e. equal to a function in $C_\infty(\mathbb{R})$). Let $\phi(z) = (\pi i)^{-1} \int_{\mathbb{R}} u(t)(t - z)^{-1} dt$, then ϕ is a holomorphic function in the region $\operatorname{Im} z > 0$. Now it follows easily from the continuity of u that $\lim_{\mu \rightarrow +0} \operatorname{Im} \phi(\lambda + i\mu)$ exists (uniformly in λ) if and only if $\lim_{\varepsilon \rightarrow 0} \pi^{-1} \int_{|t-\lambda|>\varepsilon} (\lambda - t)^{-1} u(t) dt$ exists (uniformly in λ) and in this case they are equal (see the argument in the proof of Theorem 92 in [25]). By using Plancherel theorem and the identity $i(t - \lambda + i\mu)^{-1} = \int_0^\infty \exp[i(t - \lambda + i\mu)s] ds$ we obtain

$$\begin{aligned} \phi(\lambda + i\mu) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{i\lambda s - \mu s} 2\hat{u}(s) ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\lambda s - \mu|s|} (\hat{u}(s) + i\tilde{u}(s)) ds \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{\mu}{(\lambda - t)^2 + \mu^2} (u(t) + i\tilde{u}(t)) dt. \end{aligned}$$

Since $u + i\tilde{u} \in C_\infty(\mathbb{R})$ this clearly implies $\lim_{\mu \rightarrow +0} \phi(\lambda + i\mu) = u(\lambda) + i\tilde{u}(\lambda)$ uniformly in $\lambda \in \mathbb{R}$ (cf. Th.3.1, Ch. I of [10]). Taking the imaginary parts we see that $\lim_{\varepsilon \rightarrow 0} \pi^{-1} \int_{|t-\lambda|>\varepsilon} (\lambda - t)^{-1} u(t) dt$ exists uniformly in $\lambda \in \mathbb{R}$ and is equal to $\tilde{u}(\lambda)$.

If $u \in BC(\mathbb{R})$ then the function $\tilde{u}$ is not more than of class BMO in general (see Th.5.2 in Ch VI of [10]). However, if $\alpha > 0$ and $u \in \Lambda^\alpha$ then $\tilde{u} \in \Lambda^\alpha$ too; on the other hand, if u is of class BC^1 then $\tilde{u} \in \Lambda^1$ but it is not

Lipschitz continuous in general (see §5.15 in [25] for the case $0 < \alpha < 1$ and Th.13.29 in Ch.III of [27] for the general case). In particular, if $u \in \mathcal{H}^{s,p}$ for some $s > 1/2$, then u and $\tilde{u}$ belong to $\Lambda^{s-1/2}$ and this is optimal in the scale of Lipschitz-Zygmund spaces. If $s = k + 1/2$ with $k \geq 1$ integer, then $u, \tilde{u} \in \Lambda^k$ but none of them is in BC^k in general.

1.4. THE OPERATOR $H = Q$

We shall study here the boundary values of the resolvent of the operator Q of multiplication by the independent variable x in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, i.e. $(Qf)(x) = xf(x)$. Then for $f \in \mathcal{H}$ and $z = \lambda + i\mu \in \mathbb{C}$, $\mu \neq 0$, we have

$$\langle f, (Q - z)^{-1}f \rangle = \int_{\mathbb{R}} \frac{|f(x)|^2 dx}{x - z} = \int_{\mathbb{R}} \frac{(x - \lambda)|f(x)|^2 dx}{(x - \lambda)^2 + \mu^2} + i \int_{\mathbb{R}} \frac{\mu|f(x)|^2 dx}{(x - \lambda)^2 + \mu^2}.$$

Set $u = |f|^2 \in L^1(\mathbb{R})$. For the real part of this expression we may use the results cited in §1.3; for the imaginary part we use Th.1, Ch.III of [23] (see also Sections 3 and 5 in Ch.I of [10] or §1.17 of [25] for simpler treatments in the one-dimensional case). It follows that the preceding expression has a limit as $\mu \rightarrow +0$ for almost all $\lambda \in \mathbb{R}$ and the following equality holds almost everywhere:

$$\lim_{\mu \rightarrow +0} \pi^{-1} \langle f, (Q - \lambda - i\mu)^{-1}f \rangle = \tilde{u}(\lambda) + iu(\lambda). \quad (1.1)$$

Let $J \subset \mathbb{R}$ be an open non-void set. Then from $|\operatorname{Im} \langle f, (Q - \lambda - i\mu)^{-1}f \rangle| \leq C$ for $\lambda \in J$ and $\mu > 0$ it follows that $u(\lambda) = |f(\lambda)|^2 \leq C/\pi$ for a.e. $\lambda \in J$. Hence $\mathcal{H}^{1/2,1}$ is the largest space in the scale $\{\mathcal{H}^{s,p}\}$ with the property $|\operatorname{Im} \langle f, (Q - \lambda - i\mu)^{-1}f \rangle| \leq C(f) < \infty$ for $\lambda \in J$, $\mu > 0$ and $f \in \mathcal{H}^{s,p}$ (because $\mathcal{H}^{1/2,p}$ with $p > 1$ contains functions that are unbounded on each interval).

The limit $\lim_{\mu \rightarrow +0} \operatorname{Im} \langle f, (Q - \lambda - i\mu)^{-1}f \rangle$ exists locally uniformly in $\lambda \in J$ if and only if u is a continuous function on J . But in this case the real part of $\langle f, (Q - \lambda - i\mu)^{-1}f \rangle$ could diverge for a dense set of values λ . However, we have shown in §1.3 that the limit in (1.1) exists uniformly in $\lambda \in \mathbb{R}$ for each $f \in \mathcal{H}^{1/2,1}$ (note that $u = |f|^2$ belongs to $\mathcal{H}^{1/2,1}$ too, this space being an algebra).

In conclusion, $\mathcal{H}^{1/2,1}$ is the largest space in the scale $\mathcal{H}^{s,p}$ such that the limit (1.1) exists locally uniformly in λ for each $f \in \mathcal{H}^{s,p}$. We now discuss the continuity properties of the limit function $\varphi(\lambda) := \langle f, (Q - \lambda - i0)^{-1}f \rangle \equiv \pi \tilde{u}(\lambda) + \pi i u(\lambda)$ in terms of the scale $\{\Lambda^{\beta,q}\}$. Note that the spaces $\mathcal{H}^{1/2,1}$ and $\mathcal{H}^{s,p}$ with $s > 1/2$ are algebras, hence $u \in \mathcal{H}^{s,p}$ if $f \in \mathcal{H}^{s,p}$ and $(s,p) \geq (1/2,1)$. Then we have $\tilde{u} \in \mathcal{H}^{s,p}$ too (cf. §1.3). So, if $f \in \mathcal{H}^{s,p}$

with $(s, p) \geq (1/2, 1)$, then $\varphi \in \mathcal{H}^{s,p}$ too (by taking $f \geq 0$ one sees that these assertions are optimal on the scale $\{\mathcal{H}^{t,q}\}$). So for $f \in \mathcal{H}^{1/2,1}$ we have $\varphi \in C_\infty(\mathbb{R})$ (in fact $\varphi \in B_\infty^{0,1}$, which is slightly better locally but not so explicit), while for $f \in \mathcal{H}^{s,p}$ with $s > 1/2$ we have $\varphi \in \Lambda^{s-1/2,p}$, and this is optimal on the scale $\{\Lambda^{\beta,q}\}$.

In terms of the Lipschitz-Zygmund scale $\{\Lambda^\alpha\}$, the best one can have for the boundary value function φ is: $\varphi \in \Lambda^{s-1/2}$ if $f \in \mathcal{H}^{s,p}$ and $s > 1/2$. For example, if $s = 3/2$, then $\varphi \in \Lambda^1$ and (in general) φ is *not* a locally Lipschitz continuous function unless $p = 1$, in which case φ is of class BC^1 (cf. §1.2).

1.5. THE OPERATOR $H = h(Q)$

Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a function of class C^1 such that $h'(x) > 0$ for each real x . We denote by $h(Q)$ the operator of multiplication by the function h in $\mathcal{H} = L^2(\mathbb{R})$. Then for $f \in \mathcal{H}$ and $z = \lambda + i\mu$, $\mu \neq 0$:

$$\langle f, (h(Q) - z)^{-1} f \rangle = \int_{\mathbb{R}} \frac{|f(x)|^2 dx}{h(x) - \lambda - i\mu} = \int_I \frac{|f(g(x))|^2 g'(x) dx}{x - \lambda - i\mu}.$$

Here $I = h(\mathbb{R})$ is an open interval and $g : I \rightarrow \mathbb{R}$ is the inverse diffeomorphism of h . We are in a situation similar to that of §1.4 but this time the function u is defined on I by

$$u(x) = |f(g(x))|^2 g'(x) = |f(h^{-1}(x))|^2 \frac{1}{h'(h^{-1}(x))}.$$

The spectrum of the operator $h(Q)$ is the closure of the interval I , so let us assume $\lambda \in I$. If we replace in (1.1) Q by $h(Q)$ then the limit in (1.1) will exist and the equality will hold for almost every $\lambda \in I$ (because the new function u is still in $L^1(I)$). But now the regularity properties of the function

$$\varphi(\lambda) := \langle f, (h(Q) - \lambda - i0)^{-1} f \rangle \equiv \pi \tilde{u}(\lambda) + \pi i u(\lambda), \quad \lambda \in I$$

are determined not only by f but also by the degree of regularity of h . Assume first $f \in C_0^\infty(I)$, $f \neq 0$. Then u has the same regularity class as g' (on the interior of the support of $f \circ g$). If g' is just continuous, then the imaginary part of φ is well behaved (it is a continuous function) but not its real part (because the Hilbert transform of a bounded continuous function is of class BMO but it could be unbounded on each interval). One may construct a C^1 -diffeomorphism $h : \mathbb{R} \rightarrow I$ with $h \in \Lambda^{1,p}$ for all $p > 1$ and such that $\tilde{u}$ is unbounded on any open non-empty interval $J \subset \text{supp}(f \circ g)$ (see Appendix 7.B in [1]). On the other hand, one can show that if h is

locally of class $\Lambda^{1,1}$ then for any $f \in \mathcal{H}^{1/2,1}$ the functions u and $\tilde{u}$ are continuous and the limit in (1.1) (with Q replaced by $h(Q)$) exists locally uniformly in $\lambda \in I$.

Let us consider now the higher order regularity properties that the function $\varphi(\lambda) = \pi \tilde{u}(\lambda) + \pi i u(\lambda)$ could have; recall that $u = |f \circ g|^2 g'$. Assume that $f \in \mathcal{H}^{s,p} \subset \Lambda^{s-1/2,p}$ for some $s > 1/2, 1 \leq p \leq \infty$. If h is locally of class $\Lambda^{s+1/2,p}$ (which implies $h \in C^1$) then from the hypothesis $h'(x) > 0$ ($\forall x$) we obtain that g is locally of class $\Lambda^{s+1/2,p}$ on I ; and this clearly implies that g' is locally of class $\Lambda^{s-1/2,p}$ on I (see Section 5.2 in [1]). Then $f \circ g$ will be ³ locally of class $\Lambda^{s-1/2,p}$ hence $|f \circ g|^2 g'$ is of the same class ($\Lambda^{\alpha,p}$ being algebras). So u is locally of class $\Lambda^{s-1/2}$ on I , which clearly implies that $\tilde{u}$ has the same property (since $s - 1/2 > 0$).

In conclusion, if h is locally of class $\Lambda^{s+1/2}$ and $f \in \mathcal{H}^{s,p}$ for some $s > 1/2, 1 \leq p \leq \infty$, then the function $\varphi(\lambda) = \langle f, (h(Q) - \lambda - i0)^{-1} f \rangle$ is locally of class $\Lambda^{s-1/2}$ on I . The preceding arguments clearly show that *this assertion is optimal on the scale of Lipschitz-Zygmund spaces, even if $f \in C_0^\infty(\mathbb{R})$* . By taking into account that $\tilde{u}$ is locally of class $\Lambda^{\alpha,p}$ if u has this property (for any $\alpha > 0, 1 \leq p \leq \infty$) one similarly sees that *for h locally of class $\Lambda^{s+1/2,p}$, φ is locally of class $\Lambda^{s-1/2,p}$* . For example, *if h is locally $\Lambda^{3/2,1}$ and $f \in \mathcal{H}^{1,1}$, then φ is of class C^1* .

2. Regularity of an Operator with Respect to a Group of Automorphisms

2.1. CLASSES OF REGULARITY

Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and $W_\tau = e^{iA\tau}$ the unitary group generated by A . We shall denote by $\mathcal{W}_\tau$ the automorphism of the C^* -algebra $B(\mathcal{H})$ of all bounded operators in $\mathcal{H}$ defined by $\mathcal{W}_\tau T \equiv \mathcal{W}_\tau[T] = W_{-\tau} T W_\tau$. Then $\{\mathcal{W}_\tau\}_{\tau \in \mathbb{R}}$ is a group ($\mathcal{W}_{\tau+\sigma} = \mathcal{W}_\tau \mathcal{W}_\sigma$) that has the following continuity property: the map $\tau \mapsto \mathcal{W}_\tau[T]$ is continuous in the strong operator topology for an arbitrary bounded operator T in $\mathcal{H}$.

The preceding setting has a certain similarity with that of §1.2: there the role of the C^* -algebra $B(\mathcal{H})$ is played by the C^* -algebra $L^\infty(\mathbb{R})$ equipped with the group of automorphisms induced by the translation group. We shall now define the analogues of the spaces $\Lambda^{\alpha,p}$. It will be convenient to denote these new spaces by $\mathcal{C}^{\alpha,p}(A; \mathcal{H})$ and to set $\mathcal{C}^\alpha \equiv \mathcal{C}^{\alpha,\infty}$. So $\mathcal{C}^\alpha$ are the

³The only non-trivial case is when $s-1/2 = 1$. But then $g(x \pm \varepsilon) = g(x) \pm g'(x)\varepsilon + O(\varepsilon^\alpha)$ for some $\alpha > 1$ (in fact any $\alpha < 3/2$), locally uniformly in x . Since $\omega_1(\delta, v) = O(\delta \log \delta)$ for $v \in \Lambda^1$, we get $f \circ g(x \pm \varepsilon) = f(g(x) \pm g'(x)\varepsilon) + O(\varepsilon^\alpha \log \varepsilon^\alpha)$ and the last term is $O(\varepsilon^\beta)$ for $1 < \beta < \alpha$, locally uniformly in x . Now it is easy to establish that $f \circ g$ is locally $\Lambda^{1,p}$ if f is locally $\Lambda^{1,p}$.

natural version of the Lipschitz-Zygmund spaces Λ^α in the framework of this section.

If T is a bounded operator in $\mathcal{H}$ and $m \geq 1$ is an integer we define the *modulus of continuity of order m of T with respect to A* by

$$\begin{aligned}\omega_m(\varepsilon) &= \omega_m(\varepsilon; T, A) := \sup_{|\tau| \leq \varepsilon} \|[\mathcal{W}_\tau - 1]^m T\| \\ &= \sup_{|\tau| \leq \varepsilon} \left\| \sum_{j=0}^m (-1)^j \binom{m}{j} e^{-iA\tau j} T e^{iA\tau j} \right\|.\end{aligned}$$

Let α be a strictly positive real number and $p \in [1, \infty]$. We say that T is of class $\mathcal{C}^{\alpha,p}(A; \mathcal{H})$ if there is an integer $m > \alpha$ such that

$$\left[\int_0^1 \left[\frac{\omega_m(\varepsilon)}{\varepsilon^\alpha} \right]^p \varepsilon^{-1} d\varepsilon \right]^{1/p} < \infty. \quad (2.1)$$

If $p = \infty$ this should be interpreted as $\omega_m(\varepsilon) \leq C\varepsilon^\alpha$ for some $C \in \mathbb{R}$ and all $\varepsilon \in (0, 1)$. As we said before, we write $\mathcal{C}^\alpha(A; \mathcal{H})$ for $\mathcal{C}^{\alpha,\infty}(A; \mathcal{H})$.

Much of the classical theory of Lipschitz-Zygmund spaces can be extended to the spaces $\mathcal{C}^{\alpha,p}$; one may find a rather detailed presentation of these developments in Chapter 5 of [1], we shall mention here only the facts that are relevant here. First, if (2.1) is finite for some integer $m > \alpha$, then it is finite for all such integers. Moreover, one may replace $\omega_m(\varepsilon)$ in (2.1) by $\|[\mathcal{W}_\varepsilon - 1]^m T\|$ (cf. the comment after Theorem 3.4.6 in [1]). Each $\mathcal{C}^{\alpha,p}$ has a unique B -space (i.e. complete normable topological vector space) topology such that the embedding $\mathcal{C}^{\alpha,p} \subset B(\mathcal{H})$ is continuous. The family $\{\mathcal{C}^{\alpha,p}\}$ is totally ordered, namely $\mathcal{C}^{\alpha,p} \subset \mathcal{C}^{\beta,q}$ if and only if $(\alpha, p) \geq (\beta, q)$ (in the sense defined in §1.1). Finally, this family is stable under real interpolation: $(\mathcal{C}^{\alpha,p}, \mathcal{C}^{\beta,q})_{\theta,r} = \mathcal{C}^{\gamma,r}$ with $\gamma = (1 - \theta)\alpha + \theta\beta$, $0 < \theta < 1$, $1 \leq r \leq \infty$.

Let us explain the relation between the classes $\mathcal{C}^{\alpha,p}$ and the classes of operators defined by differentiability conditions. If $k \geq 0$ is an integer, then we say that a bounded operator T in $\mathcal{H}$ is of class $C^k(A; \mathcal{H})$ or $C_u^k(A; \mathcal{H})$ if the function $\tau \mapsto \mathcal{W}_\tau[T]$ is of class C^k in the strong operator topology or norm topology, respectively (we mention that $\tau \mapsto \mathcal{W}_\tau[T]$ is strongly C^k if and only if it is weakly C^k ; cf. the proof in §2.2 of the fact that $\mathcal{A}$ is closed in the weak operator topology). The following characterizations of these two classes are also interesting: $T \in C^k(A; \mathcal{H})$ if and only if $\|[\mathcal{W}_\varepsilon - 1]^k T\| \leq c\varepsilon^k$ for a finite constant c and all $\varepsilon \in (0, 1)$, while $T \in C_u^k(A; \mathcal{H})$ if and only if the limit $\lim_{\varepsilon \rightarrow +0} \varepsilon^{-k} [\mathcal{W}_\varepsilon - 1]^k T$ exists in norm in $B(\mathcal{H})$. For example $T \in C^1$ is equivalent with $\tau \mapsto \mathcal{W}_\tau[T]$ being Lipschitz.

For each integer $k \geq 1$ we have $\mathcal{C}^{k,1} \subset C_u^k \subset C^k \subset \mathcal{C}^{k,\infty}$ and all the inclusions are strict (if A is not bounded). If $1 < p < \infty$, then the spaces

$\mathcal{C}^{k,p}$ are not comparable with C^k and C_u^k . In particular, the embedding $C^k(A; \mathcal{H}) \subset \mathcal{C}^k(A; \mathcal{H})$ is strict and optimal in the scale $\{\mathcal{C}^{\alpha,p}\}$.

For an arbitrary operator $T \in B(\mathcal{H})$ and for any integer $k \geq 0$ we shall denote by $\mathcal{A}^k[T]$ the sesquilinear form on $D(A^k)$ defined inductively by $\mathcal{A}^0[T] = T$, $\mathcal{A}[T] = [T, A] = TA - AT$, $\mathcal{A}^{k+1}[T] = \mathcal{A}[\mathcal{A}^k[T]]$. More explicitly we have

$$\mathcal{A}^k[T] = \sum_{i+j=k} \frac{k!}{i!j!} (-1)^i A^i T A^j.$$

Then the following identity holds (in the sense of sesquilinear forms on $D(A^k)$):

$$[\mathcal{W}_\varepsilon - 1]^k T = (i\varepsilon)^k \int_{\mathbb{R}} \mathcal{W}_{\tau\varepsilon} \chi_k(\tau) d\tau \mathcal{A}^k[T].$$

Here χ_k are functions such that

$$0 \leq \chi_k \leq 1, \quad \text{supp } \chi_k = [0, k] \text{ and } \int_{\mathbb{R}} \chi_k(\tau) d\tau = 1$$

(spline functions, see e.g. Proposition 3.3.5 in [1]). As an easy consequence of the preceding identity we obtain a new characterization of the class $C^k(A; \mathcal{H})$: $T \in C^k(A; \mathcal{H})$ if and only if the densely defined sesquilinear form $\mathcal{A}^k[T]$ is a bounded operator in $\mathcal{H}$; and in this case we have $i^k \mathcal{A}^k[T] = \frac{d^k}{d\tau^k} \mathcal{W}_\tau[T]|_{\tau=0}$. In particular, this justifies the notation $\mathcal{W}_\tau = e^{i\mathcal{A}\tau}$ that will be used later on. Thus we have given a meaning to $\mathcal{A}$ as an operator in $B(\mathcal{H})$ with domain $C^1(A; \mathcal{H})$; this operator is closed but not densely defined (if A is not bounded). The closure of the domain of $\mathcal{A}$ is the space $C_u^0(A; \mathcal{H})$ consisting of the operators T such that $\tau \mapsto \mathcal{W}_\tau[T]$ is a norm-continuous function. Then $\{\mathcal{W}_\tau\}$ induces a C_0 -group of isometries (in fact automorphisms) of the C^* -subalgebra $C_u^0(A; \mathcal{H})$ of $B(\mathcal{H})$ and $i\mathcal{A}$ is the infinitesimal generator of the induced group. The domain of the k -th order power of $\mathcal{A}$ is $C^k(A; \mathcal{H})$ and for T of class C^k the k -th power of the operator $\mathcal{A}$ applied to T is just the bounded operator in $\mathcal{H}$ associated to the sesquilinear form $\mathcal{A}^k[T]$. So, the notation $\mathcal{A}^k[T]$ is unambiguous.

There is a description of the class $\mathcal{C}^{\alpha,p}$ in terms of differentiability properties of the function $\tau \mapsto \mathcal{W}_\tau[T]$. Let $\alpha > 0$ be real, $1 \leq p \leq \infty$, and let k be the greatest integer such that $k < \alpha$. So we have $\alpha = k + \sigma$ with $k \geq 0$ integer and $0 < \sigma \leq 1$. Then $T \in \mathcal{C}^{\alpha,p}(A; \mathcal{H})$ is equivalent to $T \in C^k(A; \mathcal{H})$ and $\mathcal{A}^k[T] \in \mathcal{C}^{\sigma,p}(A; \mathcal{H})$ (see Proposition 5.2.2 in [1]). Note that there is a big difference between the case $\alpha = \text{non-integer}$ and $\alpha = \text{integer}$. More explicitly, in the case $p = \infty$ we have $(T^{(k)} \equiv \mathcal{A}^k[T])$:

(i) If α is not an integer, i.e. $0 < \sigma < 1$, then $T \in \mathcal{C}^\alpha(A; \mathcal{H})$ means $T \in C^k(A; \mathcal{H})$ and $\|\mathcal{W}_\varepsilon[T^{(k)}] - T^{(k)}\| \leq c\varepsilon^\sigma$.

(ii) If α is an integer, i.e. $\sigma = 1$ and $k = \alpha - 1$, then $T \in C^\alpha(A; \mathcal{H})$ means $T \in C^k(A; \mathcal{H})$ and $\|\mathcal{W}_\varepsilon[T^{(k)}] + \mathcal{W}_{-\varepsilon}[T^{(k)}] - 2T^{(k)}\| \leq c\varepsilon$; but the function $\varepsilon \mapsto \mathcal{W}_\varepsilon[T^{(k)}]$ is not Lipschitz in general (if it is, then T belongs to $C^\alpha(A; \mathcal{H})$).

An operator $T \in B(\mathcal{H})$ is of class C^∞ with respect to A , written $T \in C^\infty(A; \mathcal{H})$, if the function $\tau \mapsto \mathcal{W}_\tau[T]$ is of class C^∞ . If this function has a holomorphic extension in a complex neighbourhood of $\tau = 0$ then we say that T is analytic with respect to A . More precisely, let $\omega > 0$; then the function $\tau \mapsto \mathcal{W}_\tau[T]$ has a holomorphic extension into the disc $\{\tau \in \mathbb{C} \mid |\tau| < \omega\}$ if and only if $T \in C^\infty(A; \mathcal{H})$ and the radius of convergence of the series $\sum_{k=0}^\infty \frac{\zeta^k}{k!} A^k[T]$ in $B(\mathcal{H})$ ($\zeta \in \mathbb{C}$) is $\geq \omega$. We denote by $C^{(\omega)}(A; \mathcal{H})$ the set of operator T having this property. It is easily shown that $T \in C^{(\omega)}(A; \mathcal{H})$ if and only if there is a function $\zeta \mapsto e^{\zeta A}[T] \in B(\mathcal{H})$ which is holomorphic in the strip $|Re \zeta| < \omega$ and such that $e^{i\tau A}[T] = e^{-iA\tau} T e^{iA\tau}$ if $\tau \in \mathbb{R}$. In this case, and if $0 < a < \omega$, then T leaves invariant the domain of $e^{a|A|}$ and $(e^{\zeta A}[T])f = e^{-\zeta A} T e^{\zeta A} f$ if $|Re \zeta| \leq a$ and $f \in D(e^{a|A|})$.

Finally, let us notice that all the classes C^k ($k \geq 1$), C_u^k , $C^{\alpha,p}$, C^∞ and $C^{(\omega)}$ are dense involutive subalgebras of the C^* -algebra C_u^0 .

2.2. FUNCTIONAL CALCULUS FOR $\mathcal{A}$

The operator $\mathcal{A}$ with domain $D(\mathcal{A}) = C^1(A; \mathcal{H})$ in the Banach space $B(\mathcal{H})$ has been defined by the rule $\mathcal{A}T \equiv \mathcal{A}[T] = \frac{d}{d\tau} \mathcal{W}_\tau[T]|_{\tau=0}$; we have $\mathcal{A}T = [T, A]$ and $\mathcal{W}_\tau = e^{iA\tau}$ in the sense explained above. On $B(\mathcal{H})$ we may consider (at least) three topologies: the norm (or uniform) topology, the strong operator (s.o.) and the weak operator (w.o.) topology. Then $\mathcal{A}$ is a closed operator in the w.o. topology (hence in the s.o. and norm topology too) and it is densely defined in the s.o. topology (hence in the w.o. topology) but not in the norm topology (unless A is bounded). The density will follow easily from the functional calculus that we shall construct below (s-lim $\varphi(\varepsilon \mathcal{A})T = T$ and $\varphi(\varepsilon \mathcal{A})B(\mathcal{H}) \subset C^\infty(A; \mathcal{H})$ if $\varphi \in \mathcal{S}(\mathbb{R})$, $\varphi(0) = 1$). In order to show that $\mathcal{A}$ is closed in the w.o. topology, we note that for $T \in D(\mathcal{A})$

$$\mathcal{W}_\tau T = T + i \int_0^\tau \mathcal{W}_\sigma \mathcal{A} T d\sigma, \quad \tau \in \mathbb{R},$$

(the integral exists in the s.o. topology). Hence, if $T_j \in D(\mathcal{A})$ and $w\text{-}\lim T_j = T$, $w\text{-}\lim \mathcal{A}T_j = S$, then $\mathcal{W}_\tau T = T + i \int_0^\tau \mathcal{W}_\sigma S d\sigma$ which clearly implies $T \in D(\mathcal{A})$.

Now we define a functional calculus for $\mathcal{A}$. Let $\mathcal{M} = \mathcal{M}(\mathbb{R})$ be the Wiener algebra, i.e. the subalgebra of $BC(\mathbb{R})$ consisting of Fourier transforms of bounded measures on $\mathbb{R}$. If $\varphi(t) = \int_{\mathbb{R}} e^{it\tau} \mu(d\tau)$ then we set $\|\varphi\|_{\mathcal{M}} := |\mu|(\mathbb{R})$ (total variation of the measure μ). If we equip $\mathcal{M}$ with the usual

operations of addition and multiplication of functions and with the norm $\|\cdot\|_{\mathcal{M}}$, then $\mathcal{M}$ becomes an abelian Banach algebra with unit. For $\varphi \in \mathcal{M}$ we set $\varphi(\mathcal{A}) := \int_{\mathbb{R}} \mathcal{W}_\tau \mu(d\tau)$ where μ is as above. More precisely, $\varphi(\mathcal{A}) : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ is defined by $\varphi(\mathcal{A})T = \varphi(\mathcal{A})[T] = \int_{\mathbb{R}} e^{-iA\tau} T e^{iA\tau} \mu(d\tau)$ (the integral exists in the strong operator topology). Clearly $\varphi(\mathcal{A}) \in B(B(\mathcal{H}))$, $\|\varphi(\mathcal{A})\|_{B(B(\mathcal{H}))} \leq \|\varphi\|_{\mathcal{M}}$ and $\varphi \mapsto \varphi(\mathcal{A})$ is a unital homomorphism. Moreover, if we define φ^+ by $\varphi^+(x) = \overline{\varphi(-x)}$, then $(\varphi(\mathcal{A})[T])^* = \varphi^+(\mathcal{A})[T^*]$.

The functional calculus $\varphi \mapsto \varphi(\mathcal{A})$ may be extended to larger classes of functions but then it will produce unbounded operators in the Banach space $B(\mathcal{H})$. This is of no interest here, but the following easily proven property will be useful. Let $\varphi \in \mathcal{M}$ and $k \geq 1$ an integer such that the function $\varphi_{(k)}(x) \equiv x^k \varphi(x)$ belongs to $\mathcal{M}$ too (this is equivalent with requiring that the distributional derivative of order k of the measure μ be a bounded measure). Then $\varphi(\mathcal{A})$ improves regularity with respect to A in the following sense: $\varphi(\mathcal{A})C^m(A; \mathcal{H}) \subset C^{m+k}(A; \mathcal{H})$ for all $m \in \mathbb{N}$ (note that $C^0(A; \mathcal{H}) = B(\mathcal{H})$) and for $T \in B(\mathcal{H})$ we have $\mathcal{A}^k[\varphi(\mathcal{A})[T]] = \varphi_{(k)}(\mathcal{A})[T]$. In particular, if $\varphi \in \mathcal{S}(\mathbb{R})$ then $\varphi(\mathcal{A})B(\mathcal{H}) \subset C^\infty(A; \mathcal{H})$. One may also get analytic operators by the following procedure: assume that $\varphi \in \mathcal{M}$, ω is a strictly positive real number and that the functions $e^{\pm \omega x} \varphi(x)$ belong to $\mathcal{M}$; then $\varphi(\mathcal{A})B(\mathcal{H}) \subset C^{(\omega)}(A; \mathcal{H})$ and for $|Re \zeta| < \omega$ we have $e^{\zeta \mathcal{A}}[\varphi(\mathcal{A})[T]] = (e_\zeta \varphi)(\mathcal{A})[T]$ for $T \in B(\mathcal{H})$ (e_ζ is the function $t \mapsto e^{\zeta t}$ and $e_\zeta \varphi \in \mathcal{M}$).

Now let $\varphi \in \mathcal{M}$ and let ε be an arbitrary real number. Then it is clear that $t \mapsto \varphi(\varepsilon t)$ is still a function in $\mathcal{M}$. We shall denote by $\varphi(\varepsilon \mathcal{A})$ the operator on $B(\mathcal{H})$ associated to it by the functional calculus. Since $\varphi(\varepsilon \mathcal{A})[T] = \int_{\mathbb{R}} W_{-\varepsilon \tau} T W_{\varepsilon \tau} \mu(d\tau)$ it is clear that for an arbitrary bounded operator T on $\mathcal{H}$ the map $\varepsilon \mapsto \varphi(\varepsilon \mathcal{A})[T] \in B(\mathcal{H})$ is strongly continuous and $\|\varphi(\varepsilon \mathcal{A})\|_{B(B(\mathcal{H}))} \leq \|\varphi\|_{\mathcal{M}}$. Moreover, $\varphi(0\mathcal{A}) = \varphi(0)$ where $\varphi(0)$ is the operator of multiplication by the constant $\varphi(0)$ on $B(\mathcal{H})$. So $\lim_{\varepsilon \rightarrow 0} \varphi(\varepsilon \mathcal{A})[T] = \varphi(0)T$ for all $T \in B(\mathcal{H})$. Observe that the notation $\varphi(\varepsilon \mathcal{A})$ is consistent with the following interpretation: $\varphi(\varepsilon \mathcal{A})$ is the operator on $B(\mathcal{H})$ associated by the preceding functional calculus to the self-adjoint operator εA on $\mathcal{H}$.

We shall need a version of the ergodic theorem in the present context (see page 560 and Sections VII.7, VIII.7 in [8]; also Thm. 1.1 in [16]). Let $\ker \mathcal{A}$ and $\text{ran } \mathcal{A}$ be the kernel and the range of the operator $\mathcal{A}$ in $B(\mathcal{H})$. We denote $\overline{\text{ran } \mathcal{A}}^s$ the closure of $\text{ran } \mathcal{A}$ in the s.o. topology; note that it coincides with the closure $\overline{\text{ran } \mathcal{A}}^w$ of $\text{ran } \mathcal{A}$ in the w.o. topology (cf. Corollary 1.5 in [24]). We shall prove that

$$B(\mathcal{H}) = \ker \mathcal{A} + \overline{\text{ran } \mathcal{A}}^s. \quad (2.2)$$

Let $T \in B(\mathcal{H})$ and

$$T_\tau = \tau^{-1} \int_0^\tau \mathcal{W}_\sigma[T] d\sigma \equiv (i\tau\mathcal{A})^{-1}(e^{i\tau\mathcal{A}} - 1)[T]$$

for $\tau \neq 0$ (with a slightly formal notation). Since $\|T_\tau\| \leq \|T\|$ and the closed balls in $B(\mathcal{H})$ are w.o. compact there is a sequence $\tau_n \rightarrow \infty$ and $T_\infty \in B(\mathcal{H})$ such that $\text{w-lim}_{n \rightarrow \infty} T_{\tau_n} = T_\infty$. We have $T_\tau \in D(\mathcal{A})$ and $\mathcal{A}T_\tau = (i\tau)^{-1}(\mathcal{W}_\tau - 1)T$, hence $\|\mathcal{A}T_\tau\| \leq \tau^{-1}2\|T\| \rightarrow 0$ as $\tau \rightarrow \infty$. Since $\mathcal{A}$ is closed in the w.o. topology we get that $T_\infty \in D(\mathcal{A})$ and $\mathcal{A}T_\infty = 0$, in other terms $T_\infty \in \ker \mathcal{A}$. Now let $T_0 = T - T_\infty$. Then $\text{w-lim}_{n \rightarrow \infty} \tau_n^{-1} \int_0^{\tau_n} \mathcal{W}_\sigma T_0 d\sigma = 0$. But the formal identity

$$1 = \frac{e^{i\tau\mathcal{A}} - 1}{i\tau\mathcal{A}} - i\tau\mathcal{A} \frac{e^{i\tau\mathcal{A}} - 1 - i\tau\mathcal{A}}{(i\tau\mathcal{A})^2} \quad (\tau \neq 0) \quad (2.3)$$

can easily be made rigorous: if $\varrho(x) = (ix)^{-1}(e^{ix} - 1)$ and $\eta(x) = (ix)^{-2}(e^{ix} - 1 - ix)$, then ϱ and η are functions of class $\mathcal{M}$, $\eta_{(1)}(x) = x\eta(x)$ belongs to $\mathcal{M}$ too and $\varrho(x) - i\eta_{(1)}(x) \equiv 1$. Hence $\eta(\tau\mathcal{A})B(\mathcal{H}) \subset D(\mathcal{A})$ if $\tau \neq 0$ and $\varrho(\tau\mathcal{A}) - i\tau\mathcal{A}\eta(\tau\mathcal{A}) = 1$ as operators on $B(\mathcal{H})$, which is just (2.3). So we have $T_0 = \varrho(\tau\mathcal{A})T_0 - i\tau\mathcal{A}\eta(\tau\mathcal{A})T_0$. We take here $\tau = \tau_n$ and we make $n \rightarrow \infty$. Since $\text{w-lim}_{n \rightarrow \infty} \varrho(\tau_n\mathcal{A})T_0 = 0$ we see that T_0 belongs to the w.o. closure of the range of $\mathcal{A}$. This proves (2.2) because the w.o. closure is equal to the s.o. closure.

We shall make now several remarks in connection with the decomposition (2.2). Observe first that $\ker \mathcal{A} \subset C^\infty(A; \mathcal{H})$ and that the following three assertions are obviously equivalent:

- (i) $T \in \ker \mathcal{A}$;
- (ii) $\mathcal{W}_\tau[T] = T \quad \forall \tau \in \mathbb{R}$;
- (iii) $\varphi(\mathcal{A})T = \varphi(0)T \quad \forall \varphi \in \mathcal{M}$.

It is not difficult to show that the sum (2.2) is not direct in general (the identity operator could belong to $\overline{\text{ran } \mathcal{A}}^s$; we thank G. Skandalis for this remark). However, if we denote by $\overline{\text{ran } \mathcal{A}}^u$ the closure of $\text{ran } \mathcal{A}$ in the norm topology, then

$$B_A(\mathcal{H}) := \ker \mathcal{A} + \overline{\text{ran } \mathcal{A}}^u \quad (2.4)$$

is a norm closed subspace of $B(\mathcal{H})$ and the sum in the r.h.s. of (2.4) is a topological direct sum. Moreover, the projection $\mathcal{P}$ of $B_A(\mathcal{H})$ onto $\ker \mathcal{A}$ along $\overline{\text{ran } \mathcal{A}}^u$ is given by

$$\mathcal{P}[T] = \lim_{\tau \rightarrow \infty} \tau^{-1} \int_0^\tau \mathcal{W}_\sigma[T] d\sigma = \lim_{\tau \rightarrow \infty} \varrho(\tau\mathcal{A})[T] \quad (2.5)$$

where both limits exist in norm and ϱ is an arbitrary function in $\mathcal{M}$ with $\varrho(0) = 1$ and such that the function $\varrho_{(1)}(t) = t\varrho(t)$ belongs to $\mathcal{M}$ too

(note that the middle term in (2.5) corresponds to the choice $\varrho(x) = (e^{ix} - 1)(ix)^{-1}$). In order to prove these assertions observe that $\varrho(\tau\mathcal{A})T = \varrho(0)T = T$ if $T \in \ker \mathcal{A}$, while for $T = \mathcal{A}[S]$ we have $\varrho(\tau\mathcal{A})T = \varrho(\tau\mathcal{A})\mathcal{A}S = \tau^{-1}\varrho_{(1)}(\tau\mathcal{A})S$, so $\|\varrho(\tau\mathcal{A})T\| \leq \tau^{-1}\|\varrho_{(1)}\|_{\mathcal{M}}\|S\| \rightarrow 0$ as $\tau \rightarrow \infty$. Hence $\|\varrho(\tau\mathcal{A})T\| \rightarrow 0$ for all T in the norm closure of $\text{ran } \mathcal{A}$ (because $\|\varrho(\tau\mathcal{A})\| \leq \|\varrho\|_{\mathcal{M}}$). So the last limit in (2.5) exists for all $T \in B_A(\mathcal{H})$ and $\mathcal{P}[T] = T$ if $T \in \ker \mathcal{A}$, $\mathcal{P}[T] = 0$ if $T \in \overline{\text{ran } \mathcal{A}}^u$. Moreover, the norm of the operator $\mathcal{P} : B_A(\mathcal{H}) \rightarrow B_A(\mathcal{H})$ is equal to 1. The norm continuity of $\mathcal{P}$ and the fact that $\ker \mathcal{A}$ and $\overline{\text{ran } \mathcal{A}}^u$ are norm closed imply that $B_A(\mathcal{H})$ is norm closed too. This finishes the proof.

We mention that the limits in (2.5) could exist (in norm, strongly or weakly) for T in spaces larger than $B_A(\mathcal{H})$; however there are, in general, operators T such that the first limit in (2.5) does not exist weakly.

The following observation will be useful later on: if $\theta \in \mathcal{M}$ has the properties $\theta(0) = 0$ and $\theta' \in \mathcal{M}$, then $\theta(\mathcal{A})B(\mathcal{H}) \subset \text{ran } \mathcal{A}$; in particular $\theta(\mathcal{A})[T] \in B_A(\mathcal{H})$ and $\mathcal{P}\theta(\mathcal{A})[T] = 0$ for all $T \in B(\mathcal{H})$. In order to prove this it is sufficient to show that the function φ defined by $\varphi(0) = \theta'(0)$ and $\varphi(x) = \theta(x)/x$ if $x \neq 0$ belongs to $\mathcal{M}$ (because then we may use the identity $\theta(x) = x\varphi(x)$). We clearly have $\varphi(x) = \int_0^1 \theta'(\sigma x) d\sigma$, or $\varphi = \int_0^1 \theta'^\sigma d\sigma$ (with $\theta'^\sigma(x) \equiv \theta'(\sigma x)$). Since $\|\theta'^\sigma\|_{\mathcal{M}} \leq \|\theta'\|_{\mathcal{M}}$ and $\sigma \mapsto \theta'^\sigma \in \mathcal{M}$ is weak* continuous ($\mathcal{M}$ is identified with the dual of the space of continuous functions which converge to zero at infinity) the integral $\int_0^1 \theta'^\sigma d\sigma$ exists in the weak* topology of $\mathcal{M}$, and so $\varphi \in \mathcal{M}$.

2.3. REGULARIZATION WITH RESPECT TO A

We describe here a method of approximating an operator $T \in B(\mathcal{H})$ by operators of class C^∞ (or analytic) with respect to A that is similar to the regularization by convolution in the case of $L^\infty(\mathbb{R})$ (equipped with the translation group). Then we study the rapidity of the approximation for operators having a prescribed order of regularity.

If φ is a function in $C_0^\infty(\mathbb{R})$ and $\varphi(0) = 1$, then for an arbitrary non-zero real number ε the operator $\varphi(\varepsilon\mathcal{A})$ will transform an arbitrary operator $T \in B(\mathcal{H})$ into an analytic operator $\varphi(\varepsilon\mathcal{A})T$ and $\text{s-lim}_{\varepsilon \rightarrow 0} \varphi(\varepsilon\mathcal{A})T = T$. Our purpose now is to determine the rapidity of this convergence under certain regularity assumptions on T . This is equivalent, in our abstract setting, to giving a Littlewood-Paley type description of the spaces $\Lambda^{\alpha,p}$ from §1.4.

Let $\theta \in C_0^\infty(\mathbb{R})$ and $m \geq 1$ an integer such that θ has a zero of order m at zero (so $\theta(0) = \theta'(0) = \dots = \theta^{(m-1)}(0) = 0$). Choose $a > 0$ such that $\text{supp } \theta \subset (-2\pi/a, 2\pi/a)$. Then $e^{ixa} \neq 1$ if $x \neq 0$ and x belongs to $\text{supp } \theta$, hence the function $\eta(x) = \theta(x)[e^{ixa} - 1]^{-m}$ is of class $C_0^\infty(\mathbb{R})$ and $\theta(x) = \eta(x)[e^{ixa} - 1]^m$. So, for any $\varepsilon \in \mathbb{R}$ we have $\theta(\varepsilon\mathcal{A}) = \eta(\varepsilon\mathcal{A})[\mathcal{W}_{\varepsilon a} - 1]^m$

and for an arbitrary $T \in B(\mathcal{H})$

$$\|\theta(\varepsilon\mathcal{A})T\| \leq \|\eta\|_{\mathcal{M}} \|[\mathcal{W}_{\varepsilon a} - 1]^m T\|. \quad (2.6)$$

Hence, if $0 < \alpha < m$ and $1 \leq p \leq \infty$, then

$$\left[\int_0^1 \|\varepsilon^{-\alpha} \theta(\varepsilon\mathcal{A})T\|^p \varepsilon^{-1} d\varepsilon \right]^{1/p} \leq \|\eta\|_{\mathcal{M}} \cdot a^\alpha \left[\int_0^a \left[\frac{\omega_m(\varepsilon; T, A)}{\varepsilon^\alpha} \right]^p \varepsilon^{-1} d\varepsilon \right]^{1/p}. \quad (2.7)$$

In particular, for an arbitrary function $\theta \in C_0^\infty(\mathbb{R})$ having a zero of order $m > \alpha$ at zero we have $[\int_0^1 \|\varepsilon^{-\alpha} \theta(\varepsilon\mathcal{A})T\|^p \varepsilon^{-1} d\varepsilon]^{1/p} < \infty$ if $T \in C^{\alpha,p}(A; \mathcal{H})$.

For $p = \infty$ the preceding assertion is especially simple and important for our later work, so we state it explicitly. *Let $\alpha > 0$ real, $T \in C^\alpha(A; \mathcal{H})$ and $\theta \in C_0^\infty(\mathbb{R})$ such that θ has a zero of order $> \alpha$ at zero. Then we have $\|\theta(\varepsilon\mathcal{A})T\| \leq C\varepsilon^\alpha$ for all $\varepsilon > 0$ and some finite constant C . Reciprocally, if $\theta \neq 0$, then such an estimate characterizes the class $C^\alpha(A; \mathcal{H})$.* In order to prove this converse assertion we find it convenient to establish first an analogue of Calderon's identity (see [9]) in our context.

Let $\xi \in C^\infty(\mathbb{R})$ such that $\xi(0) = 0$ and $\xi(x) = 1$ near infinity. If we set $\eta(x) = x\xi'(x)$ then $\eta \in C_0^\infty(\mathbb{R})$, $\eta(0) = 0$ and we have $\xi(bx) - \xi(ax) = \int_a^b \eta(\tau x) \tau^{-1} d\tau$ for an arbitrary $b > a > 0$ and $x \in \mathbb{R}$. This clearly implies $\xi(b\mathcal{A}) - \xi(a\mathcal{A}) = \int_a^b \eta(\tau\mathcal{A}) \tau^{-1} d\tau$; note that for $T \in B(\mathcal{H})$ the integral $\int_a^b \eta(\tau\mathcal{A})[T] \tau^{-1} d\tau$ exists in $B(\mathcal{H})$ in the norm topology. On the other hand, by using the facts established in §2.2, we get $\text{s-lim}_{a \rightarrow +0} \xi(a\mathcal{A})[T] = \xi(0)T = 0$ for each $T \in B(\mathcal{H})$ and $\text{u-lim}_{b \rightarrow \infty} \xi(b\mathcal{A})[T] = T - \mathcal{P}[T]$ if $T \in B_A(\mathcal{H})$ (use (2.5) with $\varrho = 1 - \xi$). So we have

$$T = \mathcal{P}[T] + \text{s-lim}_{a \rightarrow +0, b \rightarrow \infty} \int_a^b \eta(\tau\mathcal{A})[T] \tau^{-1} d\tau$$

for an arbitrary $T \in B_A(\mathcal{H})$.

Observe that this result can be stated without reference to the function ξ . Indeed, if $\eta \in C_0^\infty(\mathbb{R})$ (and this condition is, in fact, much stronger than needed), $\eta(0) = 0$ and $\int_0^\infty \eta(\pm\tau) \tau^{-1} d\tau = 1$, then we may take $\xi(x) = \int_0^{|x|} \eta(\tau \text{sgn } x) \tau^{-1} d\tau$. In conclusion, we have established the following analogue of Calderon's identity:

Proposition 2.1 *Let η be a function of class $C_0^\infty(\mathbb{R})$ and such that $\eta(0) = 0$, $\int_0^\infty \eta(\pm\tau) \tau^{-1} d\tau = 1$. Then for each operator $T \in B_A(\mathcal{H})$ and each number $\tau > 0$ the operator $\eta(\tau\mathcal{A})[T]$ is analytic with respect to A and the map*

$$(0, +\infty) \ni \tau \mapsto \eta(\tau\mathcal{A})[T] \in B(\mathcal{H})$$

is of class C^∞ (in norm). Moreover, the improper integral

$$\int_{-\infty}^0 \eta(\tau \mathcal{A})[T] \tau^{-1} d\tau$$

exists in the s.o. topology of $B(\mathcal{H})$, in other terms

$$\text{s-lim}_{a \rightarrow +0, b \rightarrow \infty} \int_a^b \eta(\tau \mathcal{A})[T] \tau^{-1} d\tau$$

exists. Finally, if $\mathcal{P}$ is the projection of $B_A(\mathcal{H})$ onto $\ker \mathcal{A}$ determined by the direct sum decomposition $B_A(\mathcal{H}) = \ker \mathcal{A} + \overline{\text{ran } \mathcal{A}^u}$ (cf. (2.4) and (2.5)), then

$$T = \mathcal{P}[T] + \int_{-\infty}^0 \eta(\tau \mathcal{A})[T] \tau^{-1} d\tau. \quad (2.8)$$

Now let $\theta \in \mathcal{M}$ such that $\theta' \in \mathcal{M}$ and $\theta(0) = 0$; then $\mathcal{P}\theta(\mathcal{A})[T] = 0$ (see the end of §2.2) and

$$\theta(\mathcal{A})[T] = \int_{-\infty}^0 (\theta \eta^\tau)(\mathcal{A})[T] \tau^{-1} d\tau \quad (2.9)$$

for each $T \in B(\mathcal{H})$, where $\eta^\tau(x) = \eta(\tau x)$. In fact there are no problems of convergence at infinity of the integral, because $\|\theta \eta^\tau\|_{\mathcal{M}} \leq c \langle \tau \rangle^{-1}$. To prove this for $\tau \geq 1$ we observe that

$$\|\theta \eta^\tau\|_{\mathcal{M}} = \|(\theta^{\tau^{-1}} \eta)^\tau\|_{\mathcal{M}} = \|\theta^{\tau^{-1}} \eta\|_{\mathcal{M}}.$$

Set $t = \tau^{-1}$, then

$$\theta(tx) \eta(x) = t \int_0^1 \theta'(\sigma tx) x \eta(x) d\sigma = t \int_0^1 (\theta'^{\sigma t} \eta_{(1)})(x) d\sigma$$

with $\eta_{(1)}(x) = x \eta(x)$. Hence

$$\|\theta^t \eta\|_{\mathcal{M}} \leq t \int_0^1 \|\theta'^{\sigma t} \eta_{(1)}\|_{\mathcal{M}} d\sigma \leq t \|\theta'\|_{\mathcal{M}} \cdot \|\eta_{(1)}\|_{\mathcal{M}}$$

which is the estimate we were looking for. In conclusion, if θ is a function in $\mathcal{M}$ such that $\theta(0) = 0$ and $\theta' \in \mathcal{M}$ then there is a constant $c < \infty$ such that $\|(\theta \eta^\tau)(\mathcal{A})T\| \leq c \|T\| \langle \tau \rangle^{-1}$ for all $T \in B(\mathcal{H})$ and $\tau > 0$, and we have

$$\theta(\mathcal{A})[T] = \text{s-lim}_{a \rightarrow +0} \int_a^\infty (\theta \eta^\tau)(\mathcal{A})[T] \tau^{-1} d\tau \equiv \int_{-\infty}^0 (\theta \eta^\tau)(\mathcal{A})[T] \tau^{-1} d\tau. \quad (2.10)$$

The integral over $[a, \infty)$ exists in norm in $B(\mathcal{H})$, for each $a > 0$. Moreover, if there is $\alpha > 0$ such that $T \in \mathcal{C}^\alpha(A; \mathcal{H})$ then the integral over $(0, \infty)$ exists

in norm in $B(\mathcal{H})$ (so that the last integral is not improper). Indeed we may assume $0 < \alpha < 1$ and then we have $\|(\theta\eta^\tau)(\mathcal{A})T\| \leq \|\theta\|_{\mathcal{M}}\|\eta(\tau\mathcal{A})T\| < c'\tau^\alpha$ (cf. (2.7)).

One can improve the estimates near infinity if θ has a zero of higher order at the origin. Assume $\theta^{(k)} \in \mathcal{M}$ for $k = 0, 1, \dots, m$ and $\theta(0) = \theta'(0) = \dots = \theta^{(m-1)}(0) = 0$. Then

$$\theta(tx)\eta(x) = \frac{t^m}{m!} \int_0^1 \theta^{(m)}(\sigma tx) x^m \eta(x) m(1-\sigma)^{m-1} d\sigma.$$

Hence, if we set $\eta_{(m)}(x) = x^m \eta(x)$, we obtain for all $t \in \mathbb{R}$:

$$\|\theta^t \eta\|_{\mathcal{M}} \leq \frac{|t|^m}{m!} \|\theta^{(m)}\|_{\mathcal{M}} \cdot \|\eta_{(m)}\|_{\mathcal{M}}. \quad (2.11)$$

We shall use this estimate with a different purpose. Let $\varepsilon > 0$ and consider (2.9) with the function θ replaced by $\theta^\varepsilon(x) = \theta(\varepsilon x)$. Then

$$\|\theta(\varepsilon\mathcal{A})T\| \leq \int_0^\infty \|(\theta^\varepsilon\eta^\tau)(\mathcal{A})T\| \tau^{-1} d\tau.$$

We shall estimate the integrand in two ways, according to the value of τ . If $0 < \tau \leq \varepsilon$, then clearly $\|(\theta^\varepsilon\eta^\tau)(\mathcal{A})T\| \leq \|\theta\|_{\mathcal{M}}\|\eta(\tau\mathcal{A})T\|$. For $\tau > \varepsilon$ we shall use (2.11) but with η replaced by a function $\zeta \in C_0^\infty(\mathbb{R})$ and such that $\zeta\eta = \eta$ (note that (2.11) holds for an arbitrary $\eta \in \mathcal{M}$ such that $\eta_{(m)} \in \mathcal{M}$). Then $\theta^\varepsilon\eta^\tau = \theta^\varepsilon\zeta^\tau\eta^\tau = (\theta^{\varepsilon/\tau}\zeta)^\tau\eta^\tau$ hence:

$$\begin{aligned} \|(\theta^\varepsilon\eta^\tau)(\mathcal{A})T\| &= \|(\theta^{\varepsilon/\tau}\zeta)(\tau\mathcal{A})\eta(\tau\mathcal{A})T\| \leq \|\theta^{\varepsilon/\tau}\zeta\|_{\mathcal{M}}\|\eta(\tau\mathcal{A})T\| \\ &\leq \frac{1}{m!} \left(\frac{\varepsilon}{\tau}\right)^m \|\theta^{(m)}\|_{\mathcal{M}} \cdot \|\zeta_{(m)}\|_{\mathcal{M}}\|\eta(\tau\mathcal{A})T\|. \end{aligned}$$

Now we can prove without difficulty the main estimate of this section.

Theorem 2.2 *Let θ_0, θ be functions on $\mathbb{R}$ with the following properties:*

(1) $\theta_0 \in \mathcal{M}(\mathbb{R})$ and there are $a > 0$ and $b < 0$ such that $\theta_0(a) \neq 0$, $\theta_0(b) \neq 0$.

(2) There is an integer $m \geq 1$ such that $\theta, \theta', \dots, \theta^{(m)} \in \mathcal{M}(\mathbb{R})$ and $\theta(0) = \theta'(0) = \dots = \theta^{(m-1)}(0) = 0$.

Then there is a constant $M < \infty$ such that for all $T \in B(\mathcal{H})$ and all real $\varepsilon > 0$:

$$\|\theta(\varepsilon\mathcal{A})T\| \leq M \int_0^\infty \min(1, (\frac{\varepsilon}{\tau})^m) \|\theta_0(\tau\mathcal{A})T\| \tau^{-1} d\tau. \quad (2.12)$$

Moreover, for $\alpha \in (0, m)$ and $p \in [1, \infty]$ one has

$$\left[\int_0^\infty \|\varepsilon^{-\alpha} \theta(\varepsilon\mathcal{A})T\|^p \varepsilon^{-1} d\varepsilon \right]^{1/p} \leq \frac{Mm}{\alpha(m-\alpha)} \left[\int_0^\infty \|\varepsilon^{-\alpha} \theta_0(\varepsilon\mathcal{A})T\|^p \varepsilon^{-1} d\varepsilon \right]^{1/p}. \quad (2.13)$$

Proof. Taking into account the two estimates obtained before the statement of the theorem, we clearly get (2.12) with θ_0 replaced by η . Now observe that it is sufficient to establish (2.12) under the supplementary hypothesis $\theta_0 \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Indeed, if $\varrho \in \mathcal{M}(\mathbb{R})$ and ϱ is not identically zero on $(0, \infty)$ and on $(-\infty, 0)$ then we can find such a θ_0 with the property that $\text{supp } \theta_0$ is included in the region where $\varrho(x) \neq 0$. Then $\varphi = \theta_0/\varrho$ belongs to $\mathcal{M}(\mathbb{R})$ by Wiener's theorem (see page 78 in [22]) and $\theta_0 = \varphi\varrho$; hence

$$\|\theta_0(\tau\mathcal{A})T\| = \|\varphi(\tau\mathcal{A})\varrho(\tau\mathcal{A})T\| \leq \|\varphi\|_{\mathcal{M}}\|\varrho(\tau\mathcal{A})T\|$$

which gives (2.12) with θ_0 replaced by ϱ .

So we may assume θ_0 of class C^∞ and with support in $\mathbb{R} \setminus \{0\}$. Let $a_\pm = \int_0^\infty |\theta_0(\pm\tau)|^2 \tau^{-1} d\tau$. Then $a_\pm > 0$ and the function η defined on $\mathbb{R}$ by $\eta(\pm\tau) = (a_\pm)^{-1} |\theta_0(\pm\tau)|^2$ for $\tau \geq 0$ has the properties required in Proposition 2.1. So (2.12) holds with θ_0 replaced by η . Since $\eta = \varphi\theta_0$ with $\varphi(\pm\tau) = a_\pm^{-1} \overline{\theta_0(\pm\tau)}$ for $\tau \geq 0$, and since $\varphi \in C_0^\infty(\mathbb{R})$, we obtain now (2.12) by the same argument as above.

Let $\chi_m(\sigma) = \min(1, \sigma^m)$ for $\sigma \geq 0$. The inequality (2.12) may be written as

$$\|\theta(\varepsilon\mathcal{A})T\| \leq M \int_0^\infty \chi_m(\sigma) \left\| \theta_0\left(\frac{\varepsilon}{\sigma}\mathcal{A}\right)T \right\| \frac{d\sigma}{\sigma}.$$

If we set $f(\varepsilon) = \|\theta(\varepsilon\mathcal{A})T\|$, $g(\varepsilon) = \|\theta_0(\varepsilon\mathcal{A})T\|$, so that f, g are positive lower semicontinuous functions on $(0, \infty)$, then we have

$$\varepsilon^{-\alpha} f(\varepsilon) \leq M \int_0^\infty \left(\frac{\varepsilon}{\sigma}\right)^{-\alpha} g\left(\frac{\varepsilon}{\sigma}\right) \sigma^{-\alpha} \chi_m(\sigma) \sigma^{-1} d\sigma$$

for any $\alpha > 0$; in the r.h.s. we recognize the convolution product associated to the multiplication group $(0, \infty)$ (with $\varepsilon^{-1} d\varepsilon$ as Haar measure). If $L_*^p = L^p((0, \infty); \varepsilon^{-1} d\varepsilon)$ with $1 \leq p \leq \infty$ then by the integral Minkowski inequality we get

$$\begin{aligned} \|(\cdot)^{-\alpha} f\|_{L_*^p} &\leq M \int_0^\infty \left\| \left(\frac{\cdot}{\sigma}\right)^{-\alpha} g\left(\frac{\cdot}{\sigma}\right) \right\|_{L_*^p} \chi_m(\sigma) \sigma^{-1-\alpha} d\sigma \\ &= M \|(\cdot)^{-\alpha} g\|_{L_*^p} \int_0^\infty \frac{\chi_m(\sigma)}{\sigma^{\alpha+1}} d\sigma. \end{aligned}$$

The last integral is finite (and equal to $m/[(m-\alpha)\alpha]$) if and only if $0 < \alpha < m$. Note that in this argument L_*^p could be replaced by other Banach spaces of measurable functions on $(0, \infty)$. The proof of (2.13) is finished. $\square$

Corollary 2.3 *Let α be a strictly positive real number, $p \in [1, \infty]$ and $m > \alpha$ an integer. Then for $T \in B(\mathcal{H})$ the following assertions are equivalent:*

- (a) $T \in C^{\alpha,p}(A; \mathcal{H})$;

(b) *there is $\theta \in \mathcal{M}(\mathbb{R})$ that is not identically zero on $(0, \infty)$ and on $(-\infty, 0)$ such that*

$$\left[\int_0^\infty \|\varepsilon^{-\alpha} \theta(\varepsilon \mathcal{A}) T\|^p \varepsilon^{-1} d\varepsilon \right]^{1/p} < \infty. \quad (2.14)$$

(c) *The preceding estimate holds for each $\theta \in \mathcal{M}$ such that $\theta^{(k)} \in \mathcal{M}$ for $0 \leq k \leq m$ and $\theta^{(k)}(0) = 0$ for $0 \leq k \leq m-1$.*

Observe that we may take $\theta(x) = (e^{ix} - 1)^m$. The characterization of $\mathcal{C}^\alpha(A; \mathcal{H})$ stated after (2.7) is a particular case of the preceding corollary. Notice also the following consequence of Corollary 2.3. Let $\varphi \in \mathcal{M}$ with $\varphi(0) = 1$ and such that $\varphi^{(k)} \in \mathcal{M}$ for $0 \leq k \leq m$ and $\varphi'(0) = \dots = \varphi^{(m-1)}(0) = 0$. Then for each $T \in \mathcal{C}^\alpha(A; \mathcal{H})$ (where $0 < \alpha < m$) there is a constant c such that $\|\varphi(\varepsilon \mathcal{A})[T] - T\| \leq c\varepsilon^\alpha$; moreover, if φ is identically equal to 1 neither on $(0, \infty)$ nor on $(-\infty, 0)$, then such an estimate characterizes the property $T \in \mathcal{C}^\alpha$ (for the proof, take $\theta = \varphi - 1$ in the Corollary 2.3). This is an extension to our context of a theorem of Jackson and Zygmund concerning the rapidity of the approximation by trigonometric polynomials of periodic functions of class Λ^α ; cf. (13.14) and (13.20) in Ch.III of [27] (see also [22]).

3. Homogeneous Hamiltonians

We recall that our purpose is to study the boundary values of the resolvent of a self-adjoint operator H (the “hamiltonian”) which has a certain degree of regularity with respect to another self-adjoint operator A (the “conjugate operator”). We think it worthwhile to consider first the particular case of A -homogeneous hamiltonians. In this setting the main ideas of the general method (cf. Section 4) are easier to describe since several important technical difficulties can be completely avoided.

We shall say that a self-adjoint (unbounded) operator H in $\mathcal{H}$ is A -homogeneous if $e^{-iA\tau} H e^{iA\tau} = e^\tau H$ for all real τ . This is formally equivalent to $[H, iA] = H$, i.e. H is an “eigenvector” of the operator $\mathcal{A}$ introduced in §2.1; this is formal because H cannot be a bounded operator without being zero (indeed, if $\lambda > 0$ and $\langle f, Hf \rangle = \lambda$ then $\langle e^{iA\tau} f, H e^{iA\tau} f \rangle = e^\tau \lambda \rightarrow \infty$ as $\tau \rightarrow \infty$). In terms of the resolvent $R(z) = (H - z)^{-1}$ of H the condition of A -homogeneity is clearly equivalent to $\mathcal{W}_\tau[R(z)] = e^{-\tau} R(e^{-\tau} z)$ for all $\tau \in \mathbb{R}$ and some (and hence all) z outside the spectrum of H . An interesting example of couple (H, A) verifying these conditions is $H = \Delta$ in $L^2(\mathbb{R}^n)$ and A a multiple of the generator of the dilation group; see [2] for other non-trivial examples (in the first step of our argument below we follow [2]; cf. also the Introduction of Ch.7 in [1]).

We fix an A -homogeneous operator H and observe that $i\mathcal{A}[R(z)] = -z\frac{d}{dz}R(z) - R(z) \equiv -\frac{d}{dz}(zR(z))$ (take the derivative with respect to τ in $\mathcal{W}_\tau[R(z)] = e^{-\tau}R(e^{-\tau}z)$). It follows that for each $z \in \mathbb{C}$ outside the spectrum of H we have $R(z) \in C^\infty(A; \mathcal{H})$ and for each $k \in \mathbb{N}$:

$$\left(iz\frac{d}{dz}\right)^k R(z) = (\mathcal{A} - i)^k[R(z)]; \quad \left(i\frac{d}{dz}z\right)^k R(z) = \mathcal{A}^k[R(z)]. \quad (3.1)$$

Let $f \in D(A)$ and F the holomorphic function defined outside the spectrum of H by $F(z) = \langle f, R(z)f \rangle$. Then (3.1) with $k = 1$ gives

$$izF'(z) = \langle R(z)^*f, Af \rangle - \langle Af, R(z)f \rangle - iF(z),$$

hence

$$|zF'(z)| \leq 2\|Af\| \cdot \|R(z)f\| + |F(z)| \leq (\|f\| + 2\|Af\|)\|R(z)f\|.$$

Assume, for example, $z = \lambda + i\mu$ with $\mu > 0$. Then $\|R(z)f\|^2 = \mu^{-1} \operatorname{Im} F(z)$ hence, if we denote $\|f\|_A = \|f\| + \|Af\|$, we get:

$$|zF'(z)| \leq 2\|f\|_A \mu^{-1/2} |F(z)|^{1/2}.$$

If $f \neq 0$ then $\operatorname{Im} F(z) > 0$ for $\operatorname{Im} z > 0$, so $F^{1/2}(z) \equiv F(z)^{1/2}$ is a well defined holomorphic function in the upper half-plane and $\frac{d}{dz}F^{1/2}(z) = \frac{1}{2}F(z)^{-1/2}F'(z)$. Hence

$$\left|\frac{d}{dz}F^{1/2}(z)\right| \leq \frac{\|f\|_A}{(|z|\operatorname{Im} z)^{1/2}}, \quad \operatorname{Im} z > 0. \quad (3.2)$$

By a well-known theorem of Hardy and Littlewood (see the Appendix, Corollary A.2) the estimate (3.2) implies that the function $F^{1/2}$ is locally Hölder continuous of order $1/2$ in the set $\{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0 \text{ and } z \neq 0\}$. In particular, for each $f \in D(A)$ the limit $\lim_{\mu \rightarrow +0} \langle f, R(\lambda + i\mu)f \rangle$ exists if $\lambda \neq 0$ and, as a function of $\lambda \in \mathbb{R} \setminus \{0\}$, it is locally of class $\Lambda^{1/2}$.

By using the explicit form (3.2), one may find quite precise estimates on the boundary values of the resolvent. For this we just follow the steps of the proof of the theorem of Hardy and Littlewood. Let $\lambda \neq 0$ and $0 < \mu < \nu$. Then the identity

$$F^{1/2}(\lambda + i\nu) - F^{1/2}(\lambda + i\mu) = i \int_\mu^\nu (F^{1/2})'(\lambda + i\tau) d\tau$$

and the bound (which clearly can be improved to $O(\log \frac{\nu}{|\lambda|})$ for $\frac{\nu}{|\lambda|}$ large)

$$\int_\mu^\nu |(F^{1/2})'(\lambda + i\tau)| d\tau \leq \frac{\|f\|_A}{|\lambda|^{1/2}} \int_0^\nu \tau^{-1/2} d\tau = \frac{2\|f\|_A}{|\lambda|^{1/2}} \nu^{1/2} \quad (3.3)$$

imply that $\lim_{\mu \rightarrow +0} F^{1/2}(\lambda + i\mu) \equiv F^{1/2}(\lambda + i0)$ exists uniformly in λ if $|\lambda| \geq \text{const.} > 0$. Moreover, for each $\nu > 0$ we have

$$|F^{1/2}(\lambda + i0)| \leq |F^{1/2}(\lambda + i\nu)| + \frac{2\|f\|_A}{|\lambda|^{1/2}} \nu^{1/2} \leq \|f\| \nu^{-1/2} + 2\|f\|_A |\lambda|^{-1/2} \nu^{1/2}.$$

By taking the infimum over $\nu > 0$ of the right-hand side we obtain

$$|\langle f, R(\lambda + i0)f \rangle| \leq \frac{8}{|\lambda|^{1/2}} \|f\| \cdot \|f\|_A, \quad f \in D(A). \quad (3.4)$$

Now let $0 < \lambda_1 < \lambda_2$ and $\mu > 0$, then

$$\begin{aligned} F^{1/2}(\lambda_2 + i0) - F^{1/2}(\lambda_1 + i0) &= F^{1/2}(\lambda_2 + i0) - F^{1/2}(\lambda_2 + i\mu) \\ &\quad + F^{1/2}(\lambda_2 + i\mu) - F^{1/2}(\lambda_1 + i\mu) \\ &\quad + F^{1/2}(\lambda_1 + i\mu) - F^{1/2}(\lambda_1 + i0) \\ &= -i \int_0^\nu (F^{1/2})'(\lambda + i\tau) d\tau \\ &\quad + \int_{\lambda_1}^{\lambda_2} (F^{1/2})'(\sigma + i\mu) d\sigma \\ &\quad + i \int_0^\mu F^{1/2}(\lambda_1 + i\tau) d\tau. \end{aligned}$$

By using (3.2) and (3.3) we obtain:

$$|F^{1/2}(\lambda_2 + i0) - F^{1/2}(\lambda_1 + i0)| \leq 2\|f\|_A \left(\frac{1}{\sqrt{\lambda_1}} + \frac{1}{\sqrt{\lambda_2}} \right) \sqrt{\mu} + \frac{\lambda_2 - \lambda_1}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} \frac{\|f\|_A}{2\sqrt{\mu}}.$$

Finally, choosing $\mu = \lambda_2 - \lambda_1$ we get:

$$|F^{1/2}(\lambda_2 + i0) - F^{1/2}(\lambda_1 + i0)| \leq 5\lambda_1^{-1/2}(\lambda_2 - \lambda_1)^{1/2} \|f\|_A. \quad (3.5)$$

Now we use

$$\begin{aligned} &|F(\lambda_2 + i0) - F(\lambda_1 + i0)| \leq \\ &\leq |F^{1/2}(\lambda_2 + i0) - F^{1/2}(\lambda_1 + i0)| \cdot (|F^{1/2}(\lambda_2 + i0)| + |F^{1/2}(\lambda_1 + i0)|) \end{aligned}$$

and the estimates (3.5), (3.4). We obtain

$$|\langle f, R(\lambda_2 + i0)f \rangle - \langle f, R(\lambda_1 + i0)f \rangle| \leq \frac{20}{\lambda_1^{3/4}} \sqrt{\lambda_2 - \lambda_1} [2\|f\| \cdot \|f\|_A^3]^{1/2}. \quad (3.6)$$

This holds for $0 < \lambda_1 < \lambda_2$ and $f \in D(A)$.

It is easily shown that the estimates (3.4), (3.6) may be restated in the following terms. Denote $\mathcal{H}_1 = D(A)$ equipped with the graph norm

$(\|f\|^2 + \|Af\|^2)^{1/2}$ and let $\mathcal{H}_{-1} = \mathcal{H}_1^*$. We identify $\mathcal{H}^* = \mathcal{H}$ and $\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1}$ as usual; in particular $B(\mathcal{H}) \subset B(\mathcal{H}_1, \mathcal{H}_{-1})$ and $\|T\|^{(1)} \leq \|T\|$ where $\|T\|^{(1)}$ denotes the norm in $B(\mathcal{H}_1, \mathcal{H}_{-1})$. Then for each $\lambda \in \mathbb{R}$, $\lambda \neq 0$ the limits $\lim_{\mu \rightarrow \pm 0} R(\lambda + i\mu) \equiv R(\lambda \pm i0)$ exist in norm in $B(\mathcal{H}_1, \mathcal{H}_{-1})$, uniformly in λ if $|\lambda| \geq \text{const.} > 0$, and we have for arbitrary non-zero real $\lambda, \lambda_1, \lambda_2$:

$$\|R(\lambda \pm i0)\|^{(1)} \leq 16|\lambda|^{-1/2} \quad (3.7)$$

$$\|R(\lambda_1 \pm i0) - R(\lambda_2 \pm i0)\|^{(1)} \leq 40|\lambda_1 - \lambda_2|^{1/2} \max(|\lambda_1|^{-3/4}, |\lambda_2|^{-3/4}). \quad (3.8)$$

High energy estimates of the form (3.7) are known in the N -body case, cf. [14] and references therein; see also [17]. The interest of our proof lies only in the fact that we did not use a modified resolvent like in the Mourre approach.

Now let $k \geq 2$ be an integer and let $f \in D(A^k)$. Then (3.1) implies for $0 \leq m \leq k$

$$\left(i \frac{d}{dz}\right)^m F(z) = \langle f, \mathcal{A}^m[R(z)]f \rangle = \sum_{j=0}^m \binom{m}{j} (-1)^{m-j} \langle A^{m-j} f, R(z) A^j f \rangle.$$

The operator $z^k \frac{d^k}{dz^k}$ is a linear combination of the operators $(\frac{d}{dz} z)^m$ with $0 \leq m \leq k$. Hence there is a constant C_k depending only on k such that

$$|z^k F^{(k)}(z)| \leq C_k \sum_{j+m \leq k} |\langle A^j f, R(z) A^m f \rangle|.$$

In each term of the sum at least one of the numbers j, m is $< k$. Assume, for example, that $j < k$. Then $g \equiv A^j f \in D(A)$ hence $(z = \lambda + i\mu$ with $\mu > 0)$:

$$\|R(\bar{z})g\| \leq \frac{1}{\sqrt{\mu}} |Im \langle g, R(z)g \rangle|^{1/2} \leq c(g)\mu^{-1/2}$$

where the constant $C(g)$ is finite and depends only on the norm of g in $D(A)$ and on δ if $|\lambda| \geq \delta > 0$ (one can estimate $c(g)$ by using (3.4) and (3.3) but this is of no interest here). So we get

$$|\langle A^j f, R(z) A^m f \rangle| \leq \|R(\bar{z}) A^j f\| \cdot \|A^m f\| \leq c(g)\mu^{-1/2} \|A^m f\|.$$

In conclusion, for each $f \in D(A^k)$ and each $\delta > 0$ there is a constant $c < \infty$ such that for $|Re z| \geq \delta$ and $Im z > 0$:

$$|F^{(k)}(z)| \leq C(Im z)^{-1/2} = c(Im z)^{-k+(k-1/2)}.$$

By using the theorem of Hardy and Littlewood (Corollary A.2 in the Appendix) we see that the boundary value function $\lambda \mapsto F(\lambda + i0)$ is locally of class $\Lambda^{k-1/2}$ on $\mathbb{R} \setminus \{0\}$ if $f \in D(A^k)$.

Let us state this result in slightly different terms. For $s > 0$ real let $\mathcal{H}_s$ be the domain of $|A|^s$ equipped with the graph norm (see the Introduction). Set $\mathcal{H}_{-s} = \mathcal{H}_s^*$ and identify $\mathcal{H}_s \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{H}_{-s}$. Then $B(\mathcal{H}) \subset B(\mathcal{H}_s, \mathcal{H}_{-s})$ and we denote $\|\cdot\|^{(s)}$ the norm in the Banach space $B(\mathcal{H}_s, \mathcal{H}_{-s})$. By using the preceding results and the uniform boundedness principle it follows that for each integer $s \geq 1$ the functions

$$\mathbb{R} \setminus \{0\} \ni \lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_s, \mathcal{H}_{-s}) \quad (3.9)$$

are locally of class $\Lambda^{s-1/2}$ (in norm).

We shall now prove by complex interpolation that this assertion remains true for all real $s \geq 1$. If s_1, s_2 are real numbers and $0 < \theta < 1$, then $[\mathcal{H}_{s_1}, \mathcal{H}_{s_2}]_\theta = \mathcal{H}_s$ (with equivalent norms) if $s = (1 - \theta)s_1 + \theta s_2$. Hence for $T \in B(\mathcal{H}_{s_1}, \mathcal{H}_{-s_1}) \cap B(\mathcal{H}_{s_2}, \mathcal{H}_{-s_2})$ we have $T \in B(\mathcal{H}_s, \mathcal{H}_{-s})$ and

$$\|T\|^{(s)} \leq c[\|T\|^{(s_1)}]^{1-\theta}[\|T\|^{(s_2)}]^\theta \quad (3.10)$$

with a constant c depending only on the choice of the norms in the space $\mathcal{H}_{s_1}, \mathcal{H}_{s_2}, \mathcal{H}_s$. Now assume that $0 < s_1 < s_2$ are integers and let m be an integer $> s_2 - 1/2$. Our purpose is to estimate the modulus of continuity of order m of the function (3.9) (see §1.2). Assume, for example, that $\lambda > 0, \varepsilon > 0$ and set $T = \sum_{j=0}^m (-1)^j \binom{m}{j} R(\lambda + j\varepsilon + i0)$. By hypothesis, if λ stays in a compact subset of $(0, \infty)$ and $0 < \varepsilon < 1$, there is a constant C such that $\|T\|^{(s_j)} \leq C\varepsilon^{s_j-1/2}$ for $j = 1, 2$. Then (3.10) implies $\|T\|^{(s)} \leq cC\varepsilon^{s-1/2}$. This shows that the function (3.9) is locally of class $\Lambda^{s-1/2}$.

The preceding assertion remains true for $1/2 < s \leq 1$. To prove this we use *real* interpolation as follows. It is known (and easy to prove, cf [2] or the introduction to the Ch.7 of [1]; we shall also prove this assertion in a more general setting in Section 4) that $\|R(\lambda \pm i0)\|^{(1/2,1)} \leq c$ if λ belongs to a compact subset of $\mathbb{R} \setminus \{0\}$. We have denoted by $\|\cdot\|^{(1/2,1)}$ the norm in the space $B(\mathcal{H}_{1/2,1}, \mathcal{H}_{1/2,1}^*)$ where $\mathcal{H}_{1/2,1}$ is the space defined by real interpolation $\mathcal{H}_{1/2,1} = (\mathcal{H}, \mathcal{H}_1)_{1/2,1}$. Then $\mathcal{H}_{1/2,1}^* = \mathcal{H}_{-1/2,\infty} = (\mathcal{H}, \mathcal{H}_{-1})_{1/2,\infty}$ and the estimate (3.10) remains true if $\|\cdot\|^{(s_1)}$ is replaced by $\|\cdot\|^{(1/2,1)}$ and $s_2 > 1/2$.

Summarizing, we have proved that for each real $s > 1/2$ and $\lambda \neq 0$ the operators $R(\lambda \pm i0)$ belong to $B(\mathcal{H}_s, \mathcal{H}_{-s})$ and that the map (3.9) is locally of class $\Lambda^{s-1/2}$. The method of proof clearly extends to other classes of operators having simple commutation relations, e.g. $[Q, iP] = -1$.

4. Proof of the Main Theorem

4.1.

This section is entirely devoted to the proof of the theorem stated in the Introduction. To begin with we assume that H is a bounded self-adjoint

operator in $\mathcal{H}$ of class $C_u^1(A; \mathcal{H})$. We choose a real even function $\theta \in C_0^\infty(\mathbb{R})$, so that $\theta^+(x) \equiv \overline{\theta(-x)} = \theta(x)$. Then for any $\varepsilon \in \mathbb{R} \setminus \{0\}$ the operator $\theta(\varepsilon \mathcal{A})[H]$ is bounded, self-adjoint and analytic with respect to A (in fact entire, i.e. its analyticity strip is equal to $\mathbb{C}$). So for any $\varepsilon \in \mathbb{R}$ we may define $H_\varepsilon := e^{\varepsilon \mathcal{A}} \theta(\varepsilon \mathcal{A})[H] = \xi_0(\varepsilon \mathcal{A})[H]$ where $\xi_0(x) = e^x \theta(x)$. Since $\xi_0^+(x) = e^{-x} \theta(x)$, the bounded operator H_ε is *not self-adjoint* for $\varepsilon \neq 0$, but $H_\varepsilon^* = H_{-\varepsilon}$. We also assume $\theta(0) = 1$ (later on we shall require that $\theta(x) = 1$ in a neighbourhood of $x = 0$); then $\text{s-lim}_{\varepsilon \rightarrow 0} H_\varepsilon = H_0 = H$.

4.2.

We will be interested in the regularity properties of the function $\varepsilon \mapsto H_\varepsilon$. There is no difficulty in showing that this map is of class C^∞ on $\mathbb{R} \setminus \{0\}$ and that for all $k \in \mathbb{N}$ and $\varepsilon \neq 0$:

$$H_\varepsilon^{(k)} \equiv \frac{d^k}{d\varepsilon^k} H_\varepsilon = \mathcal{A}^k \xi_0^{(k)}(\varepsilon \mathcal{A}) H = \varepsilon^{-k} \eta_k(\varepsilon \mathcal{A}) H. \quad (4.1)$$

Here $\eta_k(x) = x^k \xi_0^{(k)}(x)$ is a function in $C_0^\infty(\mathbb{R})$. Moreover, since H is of class $C_u^1(A; \mathcal{H})$, the function $\varepsilon \mapsto H_\varepsilon$ is of class C^1 (in norm) on $\mathbb{R}$ and

$$H'_\varepsilon \equiv \frac{d}{d\varepsilon} H_\varepsilon = \xi'_0(\varepsilon \mathcal{A}) \mathcal{A}[H]. \quad (4.2)$$

From $\xi'_0(0) = 1$ (θ being even) we get $H'_0 = \mathcal{A}[H]$ and

$$\|H_\varepsilon - H\| = \left\| \int_0^\varepsilon H'_\nu d\nu \right\| \leq |\varepsilon| \cdot \|\xi'_0\|_{\mathcal{M}} \|\mathcal{A}[H]\| = c|\varepsilon|. \quad (4.3)$$

If H is only of class $C^1(A; \mathcal{H})$ then the preceding assertions remain true, except the fact that the map $\varepsilon \mapsto H_\varepsilon$ is only strongly C^1 on $\mathbb{R}$. If $H \in C^k(A; \mathcal{H})$ for some integer $k \geq 2$, then $H_\varepsilon^{(k)} = \xi_0^{(k)}(\varepsilon \mathcal{A}) \mathcal{A}^k[H]$ and one has obvious extensions of the results mentioned above. For example, $\varepsilon \mapsto H_\varepsilon$ is strongly C^k (or norm C^k if $H \in C_u^k$) on $\mathbb{R}$ and $H_0^{(k)} = \xi_0^{(k)}(0) \mathcal{A}^k[H] = \mathcal{A}^k[H]$ if $\theta'(0) = \dots = \theta^{(k)}(0) = 0$.

Since $\mathcal{A}[H]$ is of class $C_u^0(A; \mathcal{H})$ and $\mathcal{A}[H_\varepsilon] = \xi_0(\varepsilon \mathcal{A}) \mathcal{A}[H]$, we have

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{A}[H_\varepsilon] - \mathcal{A}[H]\| = 0. \quad (4.4)$$

Set $\varrho(x) = x^{-1} \text{sh } x \cdot \theta(x)$. Then ϱ is a real even function of class $C_0^\infty(\mathbb{R})$, $\varrho(0) = 1$, and $\text{Im } H_\varepsilon = -i\varepsilon \varrho(\varepsilon \mathcal{A}) \mathcal{A}[H]$. Using again the property $\mathcal{A}[H] \in C_u^0(A; \mathcal{H})$ we get

$$\lim_{\varepsilon \rightarrow 0} \|\varepsilon^{-1} \text{Im } H_\varepsilon + i\mathcal{A}[H]\| = 0. \quad (4.5)$$

So, we have $\|Im H_\varepsilon\| < c|\varepsilon|$ (compare with (4.3)) and $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} Im H_\varepsilon = -[H, iA]$ in norm.

4.3.

For each $k \in \mathbb{N}$ we denote by ξ_k the function $\xi_k(x) = e^x x^k \theta^{(k)}(x)$. Using (4.1) we get:

$$H'_\varepsilon = \mathcal{A}\xi'_0(\varepsilon\mathcal{A})H = \mathcal{A}\xi_0(\varepsilon\mathcal{A})H + \mathcal{A}e^{\varepsilon\mathcal{A}}\theta'(\varepsilon\mathcal{A})H = \mathcal{A}H_\varepsilon + \varepsilon^{-1}\xi_1(\varepsilon\mathcal{A})H.$$

For later purposes it is convenient to use the notation $K_\varepsilon = -\varepsilon^{-1}\xi_1(\varepsilon\mathcal{A})H$. So we have

$$H'_\varepsilon = \mathcal{A}H_\varepsilon - K_\varepsilon. \quad (4.6)$$

The map $\varepsilon \mapsto K_\varepsilon$ is of class C^∞ on $(0, \infty)$. We study now the behaviour of its derivatives $K_\varepsilon^{(j)}$ as $\varepsilon \rightarrow 0$ in terms of the degree of regularity of H with respect to A . Until now we assumed only $H \in C_u^1(A; \mathcal{H})$. This allows us to write $K_\varepsilon = -e^{\varepsilon\mathcal{A}}\theta'(\varepsilon\mathcal{A})\mathcal{A}[H]$, which implies $\|K_\varepsilon\| \rightarrow 0$ as $\varepsilon \rightarrow 0$. Indeed, the function $e^x \theta'(x)$ belongs to $C_0^\infty(\mathbb{R})$ and is zero at the origin (θ being even), while $\mathcal{A}[H] \in C_u^0(A; \mathcal{H})$. Now let us apply Corollary 2.3 with $T = H$ and θ replaced by ξ_1 . Since we shall have to assume that ξ_1 , hence θ' , has a zero of a sufficiently high order at the origin, for simplicity we shall assume from now on that $\theta(x) = 1$ in a neighbourhood of $x = 0$. Then ξ_1 belongs to $C_0^\infty(\mathbb{R} \setminus \{0\})$ and is identically zero neither on the positive half-axis nor on the negative half-axis. So we see that $H \in \mathcal{C}^{\alpha,p}(A; \mathcal{H})$ for some $\alpha > 0$ and $1 \leq p \leq \infty$ if and only if $[\int_0^\infty \|\varepsilon^{-\alpha}\xi_1(\varepsilon\mathcal{A})H\|^p \varepsilon^{-1} d\varepsilon]^{1/p} < \infty$; and in fact only the integral over $(0, 1)$ matters. Then $K'_\varepsilon = \varepsilon^{-2}\xi_1(\varepsilon\mathcal{A})H - \varepsilon^{-1}\mathcal{A}\xi'_1(\varepsilon\mathcal{A})H$, so $\varepsilon^2 K'_\varepsilon = \varphi(\varepsilon\mathcal{A})H$ for some $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ and we may apply again Corollary 2.3, etc. Summing-up these results, we obtain:

Proposition 4.1 *The operator H is of class $\mathcal{C}^{\alpha,p}(A; \mathcal{H})$ for some $\alpha > 0$ and $p \in [1, \infty]$ if and only if $[\int_0^1 \|\varepsilon^{-\alpha+1} K_\varepsilon\|^p \varepsilon^{-1} d\varepsilon]^{1/p} < \infty$. If this is the case, then we have for each integer $j \geq 0$:*

$$\left[\int_0^1 \|\varepsilon^{-\alpha+j+1} K_\varepsilon^{(j)}\|^p \varepsilon^{-1} d\varepsilon \right]^{1/p} < \infty. \quad (4.7)$$

We point out explicitly a consequence of the proposition that will be especially important later on: if $\alpha > 0$, then $H \in \mathcal{C}^{\alpha+1}(A; \mathcal{H})$ if and only if for each $j \in \mathbb{N}$ there is a finite constant C_j such that

$$\|K_\varepsilon^{(j)}\| \leq C_j \varepsilon^{\alpha-j}, \quad 0 < \varepsilon < 1. \quad (4.8)$$

We shall also need the following estimate, which is valid for any $H \in C^1(A; \mathcal{H})$: for each integer $j \geq 1$ there is $C_j < \infty$ such that

$$\|H_\varepsilon^{(j)}\| \leq C_j \varepsilon^{1-j}, \quad 0 < \varepsilon < 1. \quad (4.9)$$

This is an immediate consequence of (4.1). Indeed, we have

$$H_\varepsilon^{(j)} = \mathcal{A}^{j-1} \xi_0^{(j)}(\varepsilon \mathcal{A}) \mathcal{A}[H] = \varepsilon^{1-j} (\varepsilon \mathcal{A})^{j-1} \xi_0^{(j)}(\varepsilon \mathcal{A}) \mathcal{A}[H]$$

and the function $x^{j-1} \xi_0^{(j)}(x)$ is of class C^∞ if $j \geq 1$.

4.4.

We denote by E the spectral measure of the operator H , i.e. $H = \int_{\mathbb{R}} \lambda E(d\lambda)$. Our purpose is to study the behaviour of the resolvent $R(z) = (H - z)^{-1}$, $z = \lambda + i\mu$, for λ in a neighbourhood of a point $\lambda_0 \in \mathbb{R}$ where a strict Mourre estimate is satisfied. More precisely, we assume that there are an open interval I and a strictly positive number a_I such that $\lambda_0 \in I$ and

$$a_I E(I) \leq E(I)[H, iA]E(I). \quad (4.10)$$

Lemma 4.2 *For each number a such that $0 < a < a_I$ and for each $\delta > 0$ such that $[\lambda_0 - \delta, \lambda_0 + \delta] \subset I$ there are numbers $\varepsilon_0 > 0$ and $c < \infty$ such that for $|\lambda - \lambda_0| \leq \delta$, $\mu \geq 0$, $0 \leq \varepsilon \leq \varepsilon_0$ and $f \in \mathcal{H}$ the following inequality holds:*

$$(a\varepsilon + \mu) \|f\|^2 \leq \pm \operatorname{Im} \langle [H_{\pm\varepsilon} - (\lambda \pm i\mu)]f, f \rangle + c\varepsilon \| [H_{\pm\varepsilon} - (\lambda \pm i\mu)]f \|^2. \quad (4.11)$$

Proof. The proof is rather standard (cf. Ch.7 in [1] and references therein), we give it for completeness. We treat only the case of $H_\varepsilon - (\lambda + i\mu) \equiv H_\varepsilon - z$. We set $P = E(I)$ and choose a number $a_0 \in (a, a_I)$. (4.10) may be written $a_I P \leq P iA[H]P$. By using (4.5) we see that there is $\varepsilon_1 > 0$ such that $\|\varepsilon^{-1} \operatorname{Im} H_\varepsilon^* - iA[H]\| < a_I - a_0$ if $0 < \varepsilon < \varepsilon_1$. This implies

$$\varepsilon a_0 P \leq \operatorname{Im} P H_\varepsilon^* P, \quad 0 \leq \varepsilon \leq \varepsilon_1. \quad (4.12)$$

Then for $|\lambda - \lambda_0| \leq \delta$ and $f \in \mathcal{H}$:

$$\begin{aligned} \|P^\perp f\|^2 &= \|(H - z)^{-1} P^\perp (H - z)f\|^2 \leq c_1 \|(H - z)f\|^2 \\ &\leq c_2 \|(H_\varepsilon - z)f\|^2 + c_2 \varepsilon^2 \|f\|^2 \end{aligned} \quad (4.13)$$

where (4.3) has been used. On the other hand, using (4.12) we obtain

$$\begin{aligned} \varepsilon a_0 \|f\|^2 &= \varepsilon a_0 \langle f, P f \rangle + \varepsilon a_0 \|P^\perp f\|^2 \\ &\leq \langle f, P(\operatorname{Im} H_\varepsilon^*) P f \rangle + \varepsilon a_0 \|P^\perp f\|^2 \\ &= \langle f, (\operatorname{Im} H_\varepsilon^*) f \rangle - 2 \operatorname{Re} \langle f, (\operatorname{Im} H_\varepsilon^*) P^\perp f \rangle \\ &\quad - \langle P^\perp f, (\operatorname{Im} H_\varepsilon^*) P^\perp f \rangle + \varepsilon a_0 \|P^\perp f\|^2. \end{aligned}$$

Since $\| \operatorname{Im} H_\varepsilon \| \leq c_3 \varepsilon$ (cf. (4.5)) we see that for arbitrary $\nu > 0$ the following inequality holds:

$$\varepsilon a_0 \|f\|^2 \leq \operatorname{Im} \langle H_\varepsilon f, f \rangle + \nu \|f\|^2 + (\nu^{-1} c_3^2 \varepsilon^2 + c_3 \varepsilon + \varepsilon a_0) \|P^\perp f\|^2.$$

Finally, we use (4.13) and get

$$\begin{aligned} [\varepsilon a_0 - \nu - c_2 \varepsilon^3 (c_3^2 \varepsilon \nu^{-1} + c_3 + a_0)] \|f\|^2 &\leq \\ &\leq \operatorname{Im} \langle H_\varepsilon f, f \rangle + \varepsilon (c_3^2 \varepsilon \nu^{-1} + c_3 + a_0) c_2 \| (H_\varepsilon - z) f \|^2. \end{aligned}$$

Now choose $\nu = (a_0 - a - b)\varepsilon$ where $b > 0$ and $a + b < a_0$. Then there is $\varepsilon_0 > 0$ such that (4.11) holds for $0 \leq \varepsilon \leq \varepsilon_0$. $\square$

Proposition 4.3 *Assume that (4.10) holds and let the constants $a, \delta, \varepsilon_0, c$ be as in Lemma 4.2. Let $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ and ε, μ be real numbers having the same sign, one of them being non-zero, and such that $|\varepsilon| \leq \varepsilon_0$; set $z = \lambda + i\mu$. Then $H_\varepsilon - z$ is invertible in $B(\mathcal{H})$ and the operator $G \equiv G_\varepsilon(z) := (H_\varepsilon - z)^{-1}$ has the properties $G_\varepsilon(z)^* = G_{-\varepsilon}(\bar{z})$ and*

$$\|Gf\|^2 \leq \frac{1}{|a\varepsilon + \mu|} |\operatorname{Im} \langle f, Gf \rangle| + \left| \frac{c\varepsilon}{a\varepsilon + \mu} \right| \cdot \|f\|^2, \quad \forall f \in \mathcal{H}. \quad (4.14)$$

In particular, we have

$$\|G\| \leq \frac{1}{|a\varepsilon + \mu|} + \left| \frac{c\varepsilon}{a\varepsilon + \mu} \right|^{1/2}. \quad (4.15)$$

Proof. The equality $(H_\varepsilon - z)^* = H_{-\varepsilon} - \bar{z}$ is obvious. From (4.11) we have

$$|a\varepsilon + \mu| \cdot \|f\| \leq [1 + c|\varepsilon| \cdot \|H_\varepsilon - z\|] \cdot \|(H_\varepsilon - z)f\|.$$

Hence the operator $H_\varepsilon - z$ and its adjoint are injective and with closed range; this implies that $H_\varepsilon - z$ is bijective. (4.14) follows from (4.11) (replace f by Gf). Then (4.14) implies

$$\|G\|^2 \leq \frac{1}{|a\varepsilon + \mu|} \|G\| + \left| \frac{c\varepsilon}{a\varepsilon + \mu} \right|$$

which has (4.15) as a consequence. $\square$

4.5.

From now on we consider only the case $0 \leq \varepsilon \leq \varepsilon_0$, $\mu > 0$ and $|\lambda - \lambda_0| \leq \delta$. G will be thought as a function of (λ, ε) and μ as a parameter that is fixed

but the constants in the estimates have to be independent of it. For example we shall need the following consequence of (4.14):

$$\|Gf\|^2 \leq \frac{1}{a\varepsilon} \operatorname{Im}\langle f, Gf \rangle + \frac{c}{a} \|f\|^2, \quad \forall f \in \mathcal{H}. \quad (4.16)$$

The dependence on ε being especially important we write $G = G_\varepsilon$ and $\frac{d}{d\varepsilon}G = G' = G'_\varepsilon$. It is quite straightforward to show that the function $\varepsilon \mapsto G_\varepsilon \in B(\mathcal{H})$ is of class C^∞ on the open interval $(0, \varepsilon_0)$ and of class C^1 (in norm) on the interval $[0, \varepsilon_0]$ (use (4.15), where $\mu > 0$ is fixed, and the properties of H_ε discussed before). Moreover, for $0 < \varepsilon < \varepsilon_0$ we have $G_\varepsilon \in C^\infty(A; \mathcal{H})$ and $\mathcal{A}[G_\varepsilon] = -G_\varepsilon \mathcal{A}[H_\varepsilon] G_\varepsilon$.

We shall now derive an important identity involving G' and G , namely:

$$\frac{d}{d\varepsilon}G_\varepsilon = \mathcal{A}[G_\varepsilon] + G_\varepsilon K_\varepsilon G_\varepsilon. \quad (4.17)$$

What is remarkable here is that, if H is sufficiently regular with respect to A , then the second term in the r.h.s. is not singular when $\varepsilon \rightarrow 0$ (use $\|G_\varepsilon\| = O(\varepsilon^{-1})$ and (4.8)). In order to prove (4.17) we observe that $G'_\varepsilon = -G_\varepsilon H'_\varepsilon G_\varepsilon$ and then use (4.6):

$$G'_\varepsilon = -G_\varepsilon \mathcal{A}[H_\varepsilon] G_\varepsilon + G_\varepsilon K_\varepsilon G_\varepsilon = \mathcal{A}[G_\varepsilon] + G_\varepsilon K_\varepsilon G_\varepsilon.$$

4.6.

In this subsection we shall deduce from (4.17) one of the main estimates of the theory. We shall denote by $\|\cdot\|^{(1/2,1)}$ the norm in the Banach space of bounded operators from $\mathcal{H}_{1/2,1}$ to $\mathcal{H}_{1/2,1}^* \equiv \mathcal{H}_{-1/2,\infty}$. Similarly, for each $s > 0$ $\|\cdot\|^{(s)}$ will be the norm in the Banach space $B(\mathcal{H}_s, \mathcal{H}_s^*)$, more precisely $\|T\|^{(s)} = \|\langle A \rangle^{-s} T \langle A \rangle^{-s}\|$.

Proposition 4.4 *If H is of class $C^{1,1}(A; \mathcal{H})$ then there is a constant C , independent of $\lambda, \varepsilon, \mu$ such that $\|G_\varepsilon\|^{(1/2,1)} \leq C$. In particular, for each number $s > 1/2$ there is $C_s < \infty$ such that $\|G_\varepsilon\|^{(s)} \leq C_s$.*

Proof. Let $\varepsilon_1 \in [0, \varepsilon_0]$ and $f \in \mathcal{H}$ and let $\{f_\varepsilon\}_{\varepsilon_1 < \varepsilon \leq \varepsilon_0}$ be a family of vectors in $D(A)$ such that the function $\varepsilon \mapsto f_\varepsilon \in \mathcal{H}$ is of class C^∞ and $\lim_{\varepsilon \rightarrow \varepsilon_1} \|f_\varepsilon - f\| = 0$. If we set $F_\varepsilon = \langle f_\varepsilon, G_\varepsilon f_\varepsilon \rangle$, then F_ε is a C^∞ function of $\varepsilon \in (\varepsilon_1, \varepsilon_0)$ and $\lim_{\varepsilon \rightarrow \varepsilon_1} F_\varepsilon = \langle f, G_{\varepsilon_1} f \rangle$. Using (4.17) we get

$$F'_\varepsilon = \langle f'_\varepsilon - A f_\varepsilon, G_\varepsilon f_\varepsilon \rangle + \langle G_\varepsilon^* f_\varepsilon, f'_\varepsilon + A f_\varepsilon \rangle + \langle G_\varepsilon^* f_\varepsilon, K_\varepsilon G_\varepsilon f_\varepsilon \rangle. \quad (4.18)$$

If we set $\ell(\varepsilon) = \|f'_\varepsilon\| + \|A f_\varepsilon\|$ we get

$$|F'_\varepsilon| \leq \ell(\varepsilon)(\|G_\varepsilon f_\varepsilon\| + \|G_\varepsilon^* f_\varepsilon\|) + \|K_\varepsilon\| \cdot \|G_\varepsilon f_\varepsilon\| \cdot \|G_\varepsilon^* f_\varepsilon\|.$$

We estimate $\|G_\varepsilon f_\varepsilon\|$ by using (4.16):

$$\|G_\varepsilon f_\varepsilon\| \leq \frac{1}{\sqrt{a\varepsilon}} |F_\varepsilon|^{1/2} + \left(\frac{c}{a}\right)^{1/2} \|f_\varepsilon\|.$$

Since $G_\varepsilon^* = G_{-\varepsilon}(\bar{z})$, this inequality is valid for $\|G_\varepsilon^* f_\varepsilon\|$ too (cf. (4.14)). Hence there is a constant $b > 0$ (depending only on a and c) such that

$$b|F'_\varepsilon| \leq [\ell(\varepsilon)\|f_\varepsilon\| + \|K_\varepsilon\| \cdot \|f_\varepsilon\|^2] + \frac{\ell(\varepsilon)}{\sqrt{\varepsilon}} |F_\varepsilon|^{1/2} + \frac{\|K_\varepsilon\|}{\varepsilon} |F_\varepsilon|. \quad (4.19)$$

By using a version of Gronwall lemma stated as Proposition 3.1 in [5] one sees that (4.19) implies that for all $\varepsilon \in (\varepsilon_1, \varepsilon_0)$:

$$\begin{aligned} |F_\varepsilon| \leq & 2 \left\{ |F_{\varepsilon_0}| + b^{-1} \int_\varepsilon^{\varepsilon_0} [\ell(\tau)\|f_\tau\| + \|K_\tau\| \cdot \|f_\tau\|^2] d\tau + \right. \\ & \left. + b^{-2} \left(\int_\varepsilon^{\varepsilon_0} \frac{\ell(\tau) d\tau}{\sqrt{\tau}} \right)^2 \right\} \exp \int_\varepsilon^{\varepsilon_0} \|K_\tau\| (b\tau)^{-1} d\tau. \end{aligned}$$

From Proposition 4.1 it follows that the last exponential is finite as $\varepsilon \rightarrow 0$ if and only if $H \in \mathcal{C}^{1,1}(A; \mathcal{H})$. Now let us choose $f_\varepsilon = \theta((\varepsilon - \varepsilon_1)A)f$, where θ is as in §4.3, and let us denote $\theta_1(x) = x\theta'(x)$, $\theta_{(1)}(x) = x\theta(x)$. Then

$$\begin{aligned} \int_\varepsilon^{\varepsilon_0} \frac{\ell(\tau) d\tau}{\sqrt{\tau}} &= \int_{\varepsilon - \varepsilon_1}^{\varepsilon_0 - \varepsilon_1} \frac{\|\theta_1(\sigma A)f\| + \|\theta_{(1)}(\sigma A)f\|}{\sigma(\sigma + \varepsilon_1)^{1/2}} d\sigma \\ &\leq \int_0^{\varepsilon_0 - \varepsilon_1} (\|\theta_1(\sigma A)f\| + \|\theta_{(1)}(\sigma A)f\|) \frac{d\sigma}{\sigma^{3/2}} \end{aligned}$$

and this is bounded by a constant (depending only on θ) times the norm of f in $\mathcal{H}_{1/2,1}$. So there is $C < \infty$, independent of $\varepsilon \in (0, \varepsilon_0)$ and λ, μ as before, such that $|F_\varepsilon| \leq C\|f\|_{\mathcal{H}_{1/2,1}}^2$ for $\varepsilon_1 < \varepsilon < \varepsilon_0$. If we let $\varepsilon \rightarrow \varepsilon_1$ we get $|\langle f, G_{\varepsilon_1} f \rangle| \leq C\|f\|_{\mathcal{H}_{1/2,1}}^2$ and the assertion of the proposition follows from the polarization principle. $\square$

Corollary 4.5 *If H is of class $\mathcal{C}^{1,1}(A; \mathcal{H})$ then there is $C < \infty$, independent of $\lambda, \varepsilon, \mu$, such that*

$$\|G_\varepsilon\|_{B(\mathcal{H}_{1/2,1}, \mathcal{H})} + \|G_\varepsilon\|_{B(\mathcal{H}, \mathcal{H}_{-1/2, \infty})} \leq C\varepsilon^{-1/2}. \quad (4.20)$$

In particular, for each $s > 1/2$ there is a constant $C_s < \infty$ such that

$$\|G_\varepsilon \langle A \rangle^{-s}\| + \|\langle A \rangle^{-s} G_\varepsilon\| \leq C_s \varepsilon^{-1/2}. \quad (4.21)$$

Proof. The assertion concerning the norm of G_ε as operator $\mathcal{H}_{1/2,1} \rightarrow \mathcal{H}$ follows immediately from (4.16) and Proposition 4.4. Then note that $\|G_\varepsilon^*\|_{B(\mathcal{H}_{1/2,1}, \mathcal{H})} \leq C\varepsilon^{-1/2}$ too, because $G_\varepsilon^* = G_{-\varepsilon}(\bar{z})$ and all we have done until now for $\varepsilon \geq 0$, $\mu > 0$ remains valid for $\varepsilon \leq 0$, $\mu < 0$. Finally, G_ε considered as operator from $\mathcal{H}$ to $\mathcal{H}_{1/2,1}^*$ is just the adjoint of G_ε^* considered as operator from $\mathcal{H}_{1/2,1}$ to $\mathcal{H}$. $\square$

We notice that the constants C_s from Proposition 4.4 and Corollary 4.5 diverge as $s \rightarrow 1/2$.

4.7.

We explain now how one may deduce from the proof of Proposition 4.4 the existence of the boundary values $R(\lambda \pm i0) = \lim_{\mu \rightarrow +0} R(\lambda \pm i\mu)$. The limit exists in the weak* topology of the space $B(\mathcal{H}_{1/2,1}, \mathcal{H}_{1/2,1}^*)$ and is uniform in $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$.

We keep the notations of the Proposition 4.4 but assume from the beginning that $f \in \mathcal{H}_{1/2,1}$ and choose $\varepsilon_1 = 0$. Then $\lim_{\varepsilon \rightarrow 0} F_\varepsilon = \langle f, R(z)f \rangle$ and $f_\varepsilon = \theta(\varepsilon A)f$. Clearly:

$$\langle f, R(z)f \rangle = F_{\varepsilon_0}(z) - \int_0^{\varepsilon_0} F'_\varepsilon(z) d\varepsilon. \quad (4.22)$$

We have

$$F_{\varepsilon_0}(z) = \langle \theta(\varepsilon_0 A)f, (H_{\varepsilon_0} - z)^{-1} \theta(\varepsilon_0 A)f \rangle$$

and $\varepsilon_0 > 0$; hence the estimates of Proposition 4.3 imply the existence of the limit $\lim_{\mu \rightarrow +0} F_{\varepsilon_0}(\lambda + i\mu)$ uniformly in $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$. Similarly, taking into account the explicit form (4.18) of F'_ε we see that for each $\varepsilon > 0$ the limit $\lim_{\mu \rightarrow +0} F'_\varepsilon(\lambda + i\mu)$ exists uniformly in λ . Moreover, (4.19) and the bound $|F_\varepsilon| \leq C\|f\|_{\mathcal{H}_{1/2,1}}^2$ gives us an estimate $|F'_\varepsilon| \leq g(\varepsilon)$ with $g : (0, \varepsilon_0) \rightarrow \mathbb{R}$ an integrable function, independent of z . So we may use the dominated convergence theorem in (4.22) and conclude that $\lim_{\mu \rightarrow +0} \langle f, R(\lambda + i\mu)f \rangle$ exists uniformly in $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$.

4.8.

The hypothesis $H \in \mathcal{C}^{1,1}(A; \mathcal{H})$ allowed us to prove the existence of the boundary values $R(\lambda \pm i0)$ of $R(z)$ in the space $B(\mathcal{H}_{1/2,1}, \mathcal{H}_{1/2,1}^*)$ and the weak* continuity of the functions $\lambda \mapsto R(\lambda \pm i0)$ on $[\lambda_0 - \delta, \lambda_0 + \delta]$. This continuity property cannot be improved in general, more precisely, if $f \in \mathcal{H}_{1/2,1}$ then the maps $\lambda \mapsto R(\lambda \pm i0)f \in \mathcal{H}_{-1/2,\infty}$ are not (strongly) continuous in general. This is clear in the context of §1.4, because

$$[(Q - \lambda \mp i0)^{-1}f](x) = PV \frac{1}{x - \lambda} \cdot f(x) \mp \pi i \delta(x - \lambda)f(\lambda)$$

and the function $\lambda \mapsto \delta_\lambda \in \mathcal{H}^{-1/2,\infty}$ is not continuous ($\delta_\lambda(x) = \delta(x - \lambda)$ is the Dirac measure at λ). Indeed $\delta_\lambda = e^{iP\lambda}\delta_0$, we have $\delta_0 \notin \mathring{\mathcal{H}}^{-1/2,\infty}$, and an element g of $\mathcal{H}^{-1/2,\infty}$ belongs to $\mathring{\mathcal{H}}^{-1/2,\infty}$ if and only if $\lambda \mapsto e^{iP\lambda}g \in \mathcal{H}^{-1/2,\infty}$ is strongly continuous.

More generally, note that $R(\lambda + i\mu)f \in \mathring{\mathcal{H}}^{-1/2,\infty}$ if $\mu \neq 0$ but for $\mu = \pm 0$ this holds only under rather restrictive conditions (see Theorem 14.3.6 in [12]).

Our purpose now is to study the continuity properties of the functions $\lambda \mapsto R(\lambda \pm i0)$ when considered as $B(\mathcal{H}_s, \mathcal{H}_s^*)$ -valued for some $s > 1/2$. It is again clear from the considerations of §1.5 that we shall not have a gain of regularity if H is only of class $\mathcal{C}^{1,1}$ with respect to A and that if H is of class $\mathcal{C}^{s+1/2}$ then the best we may expect is that $\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_s, \mathcal{H}_s^*)$ are locally of class $\Lambda^{s-1/2}$. We shall now start the proof of this assertion.

Let us fix some real number $s > 1/2$ and denote $\alpha = s - 1/2$. From now on we assume that $H \in \mathcal{C}^{\alpha+1}(A; \mathcal{H})$. For $0 < \varepsilon < \varepsilon_0$ we set

$$L_\varepsilon \equiv L(\lambda, \varepsilon) = \theta(\varepsilon A)G_\varepsilon\theta(\varepsilon A) = \theta(\varepsilon A)(H_\varepsilon - \lambda - i\mu)^{-1}\theta(\varepsilon A).$$

Here $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ and $\mu > 0$. As in the case of G , L is considered as function of (λ, ε) and μ is thought as a fixed parameter; the constants C and the symbols $O(\varepsilon^\alpha)$ have to be independent of $\lambda, \varepsilon, \mu$. We shall prove that if $m > \alpha$ is a sufficiently large integer, then we have:

$$\sum_{\ell+k=m} \left\| \frac{d^\ell}{d\lambda^\ell} \frac{d^k}{d\varepsilon^k} L(\lambda, \varepsilon) \right\|^{(s)} \leq C\varepsilon^{-m+\alpha}. \quad (4.23)$$

Assuming that this has been done, we apply Proposition A.1 from the Appendix with the following choices: $\Omega = (\lambda_0 - \delta, \lambda_0 + \delta)$, $x \equiv \lambda$, $\tilde{\Omega} = \Omega \times (0, \varepsilon_0)$, $y \equiv \varepsilon$ (the fact that ε_0 is supposed equal to 1 in the Appendix is clearly irrelevant); $F(\lambda, \varepsilon) = \langle f, L(\lambda, \varepsilon)f \rangle$ for some $f \in \mathcal{H}_s$. We obtain that the function $\lambda \mapsto \langle f, L(\lambda, 0)f \rangle = \langle f, R(\lambda + i\mu)f \rangle$ is of class Λ^α on Ω with a bound on its m -th order modulus of continuity independent of μ . Letting $\mu \rightarrow +0$ we see that $\lambda \mapsto \langle f, R(\lambda + i0)f \rangle$ is of class Λ^α on Ω for each $f \in \mathcal{H}_s$. Now we may use the uniform boundedness principle (together with the polarization identity) to get that the operator valued function $\lambda \mapsto R(\lambda + i0) \in B(\mathcal{H}_s, \mathcal{H}_{-s})$ is of class Λ^α on Ω (in fact, (4.23) implies that the bound on the m -th order modulus of continuity of $\lambda \mapsto \langle f, R(\lambda + i\mu)f \rangle$ is of the form $C(\alpha, m)C\|f\|_{\mathcal{H}_s}^2\delta^\alpha$ where C is the same as in (4.23) and $C(\alpha, m)$ depends only on α and m ; so the uniform boundedness principle is not needed). This finishes the proof of the theorem stated in the introduction.

4.9.

We shall first prove (4.23) assuming $1/2 < s \leq 1$ (so $0 < \alpha \leq 1/2$) with $m = 1$. This is not a necessary step for the general case, but it helps to understand one technical point. We shall also see how easily one may get for $s = 1$ Hölder continuity of order $1/2$ (not only $1/3$ like in previous works, cf. [21], [26], [15]).

As before we set $L'_\varepsilon = \frac{d}{d\varepsilon} L_\varepsilon$ and $\theta_1(x) = x\theta'(x)$. Then

$$\begin{aligned} L'_\varepsilon &= A\theta'(\varepsilon A)G_\varepsilon\theta(\varepsilon A) + \theta(\varepsilon A)G_\varepsilon A\theta'(\varepsilon A) + \theta(\varepsilon A)G'_\varepsilon\theta(\varepsilon A) \\ &= \frac{1}{\varepsilon}\theta_1(\varepsilon A)G_\varepsilon\theta(\varepsilon A) + \theta(\varepsilon A)G_\varepsilon \cdot \frac{1}{\varepsilon}\theta_1(\varepsilon A) \\ &\quad + \theta(\varepsilon A)\mathcal{A}[G_\varepsilon]\theta(\varepsilon A) + \theta(\varepsilon A)G_\varepsilon K_\varepsilon G_\varepsilon\theta(\varepsilon A) \end{aligned}$$

where (4.17) has been used. If we set $\theta_{(1)}(x) = x\theta(x)$, we obtain

$$\begin{aligned} \varepsilon L'_\varepsilon &= [\theta_1(\varepsilon A) - \theta_{(1)}(\varepsilon A)]G_\varepsilon\theta(\varepsilon A) \\ &\quad + \theta(\varepsilon A)G_\varepsilon[\theta_1(\varepsilon A) + \theta_{(1)}(\varepsilon A)] + \theta(\varepsilon A)G_\varepsilon \cdot \varepsilon K_\varepsilon \cdot G_\varepsilon\theta(\varepsilon A). \end{aligned} \quad (4.24)$$

We need a bound on $\langle A \rangle^{-s} \varepsilon L'_\varepsilon \langle A \rangle^{-s}$ of the form $O(\varepsilon^\alpha)$. The following observation will be used (here and later on again). Assume that φ is a bounded function with compact support and let $\varphi(x) = O(|x|^p)$ for some $p > 0$ (as $x \rightarrow 0$). Then for $0 < t \leq p$ we have

$$\|\varphi(\varepsilon A)\langle A \rangle^{-t}\| = O(\varepsilon^t) \quad \text{as } \varepsilon \rightarrow 0. \quad (4.25)$$

Indeed, we can write $\varphi(x) = |x|^p \eta(x)$ with η a bounded function with compact support; then

$$\|\varphi(\varepsilon A)\langle A \rangle^{-t}\| \leq \sup_{x \in \mathbb{R}} \left[\frac{|\varepsilon x|^t}{\langle x \rangle^t} |\varepsilon x|^{p-t} |\eta(\varepsilon x)| \right] \leq C\varepsilon^t.$$

Since θ_1 and $\theta_{(1)}$ have a zero of order one at zero and $s \leq 1$, we get $\|[\theta_1(\varepsilon A) \pm \theta_{(1)}(\varepsilon A)]\langle A \rangle^{-s}\| \leq C\varepsilon^s$. So, by using (4.24) and (4.21) we get:

$$\begin{aligned} \|\varepsilon L'_\varepsilon\|^{(s)} &\leq \|[\theta_1(\varepsilon A) - \theta_{(1)}(\varepsilon A)]\langle A \rangle^{-s}\| \cdot \|G_\varepsilon\langle A \rangle^{-s}\| \cdot \|\theta(\varepsilon A)\| + \\ &\quad + \|\theta(\varepsilon A)\| \cdot \|\langle A \rangle^{-s} G_\varepsilon\| \cdot \|[\theta_1(\varepsilon A) + \theta_{(1)}(\varepsilon A)]\langle A \rangle^{-s}\| + \\ &\quad + \|\theta(\varepsilon A)\| \cdot \|\langle A \rangle^{-s} G_\varepsilon\| \cdot \|\varepsilon K_\varepsilon\| \cdot \|G_\varepsilon\langle A \rangle^{-s}\| \cdot \|\theta(\varepsilon A)\| \\ &\leq C\varepsilon^s \cdot \varepsilon^{-1/2} + C\|K_\varepsilon\|. \end{aligned}$$

By using (4.8) we see that $\|K_\varepsilon\| = O(\varepsilon^\alpha)$ and so we get $\|\frac{d}{d\varepsilon} L_\varepsilon\|^{(s)} = O(\varepsilon^{\alpha-1})$.

We need a similar bound for $\|\frac{d}{d\lambda}L_\varepsilon\|^{(s)}$. To get it, we first differentiate with respect to λ in (4.24). Taking into account the equality $\frac{d}{d\lambda}G_\varepsilon = G_\varepsilon^2$, we get

$$\begin{aligned} \varepsilon \frac{d}{d\lambda} L'_\varepsilon &= [\theta_1(\varepsilon A) - \theta_{(1)}(\varepsilon A)] G_\varepsilon^2 \theta(\varepsilon A) + \theta(\varepsilon A) G_\varepsilon^2 [\theta_1(\varepsilon A) + \theta_{(1)}(\varepsilon A)] \\ &\quad + \theta(\varepsilon A) G_\varepsilon^2 \cdot \varepsilon K_\varepsilon \cdot G_\varepsilon \theta(\varepsilon A) + \theta(\varepsilon A) G_\varepsilon \cdot \varepsilon K_\varepsilon \cdot G_\varepsilon^2 \theta(\varepsilon A). \end{aligned}$$

Then we estimate $\langle A \rangle^{-s} \varepsilon \frac{d}{d\lambda} L'_\varepsilon \langle A \rangle^{-s}$ exactly as above but using also (cf. (4.15) and (4.21))

$$\|G_\varepsilon^2 \langle A \rangle^{-s}\| \leq \|G_\varepsilon\| \cdot \|G_\varepsilon \langle A \rangle^{-s}\| = O(\varepsilon^{-3/2})$$

and similarly for $\langle A \rangle^{-s} G_\varepsilon^2$. We obtain

$$\left\| \frac{d}{d\lambda} L'_\varepsilon \right\|^{(s)} = O(\varepsilon^{\alpha-2}).$$

Finally, integrating with respect to ε over on interval of the form $(\varepsilon, \varepsilon_0)$ we get $\|\frac{d}{d\lambda}L_\varepsilon\|^{(s)} = O(\varepsilon^{\alpha-1})$. This finishes the proof of (4.23) with $m = 1$ and $1/2 < s \leq 1$.

4.10.

If $s = 1$ (so H has to be of class $\mathcal{C}^{3/2}(A; \mathcal{H})$) then the preceding argument may be much simplified. Indeed, in this case it is sufficient to repeat the preceding computations but with L_ε replaced by G_ε (i.e. the factors $\theta(\varepsilon A)$ are no more necessary). Then in place of (4.24) we use (4.17) and we get

$$\begin{aligned} \langle A \rangle^{-1} G'_\varepsilon \langle A \rangle^{-1} &= \langle A \rangle^{-1} G_\varepsilon \cdot \frac{A}{\langle A \rangle} - \frac{A}{\langle A \rangle} \cdot G_\varepsilon \langle A \rangle^{-1} \\ &\quad + \langle A \rangle^{-1} G_\varepsilon \cdot K_\varepsilon \cdot G_\varepsilon \langle A \rangle^{-1}. \end{aligned} \quad (4.26)$$

Now we use (4.21) with $s = 1$ and (4.8): $\|K_\varepsilon\| = O(\sqrt{\varepsilon})$. This implies $\|\frac{d}{d\varepsilon}G_\varepsilon\|^{(1)} = O(\varepsilon^{-1/2})$. By differentiating the identity (4.26) with respect to λ we get $\|\frac{d}{d\lambda}G'_\varepsilon\|^{(1)} = O(\varepsilon^{-3/2})$ and so $\|\frac{d}{d\lambda}G_\varepsilon\|^{(1)} = O(\varepsilon^{-1/2})$.

We notice also that if $H \in \mathcal{C}^{3/2}$, once we have shown that $\|R(\lambda_1 + i0) - R(\lambda_2 + i0)\|^{(1)} \leq C|\lambda_1 - \lambda_2|^{1/2}$, then we get $\|R(\lambda_1 + i0) - R(\lambda_2 + i0)\|^{(s)} \leq C_s|\lambda_1 - \lambda_2|^{s-1/2}$ for all $s \in (1/2, 1]$ by interpolation. Indeed, we have $\|R(\lambda_j + i0)\|^{(1/2, 1)} \leq C$ by Proposition 4.4 and we may get $\mathcal{H}_s$ by real interpolation: $\mathcal{H}_s = (\mathcal{H}_{1/2, 1}, \mathcal{H}_1)_{\nu, 2}$ with $\nu = 2s - 1$ (see the end of Section 3).

Similar remarks can be made in the general case. The factors $\theta(\varepsilon A)$ in the definition of L_ε are needed only if s is not an integer. Otherwise one

may take $L_\varepsilon = G_\varepsilon$ which allows one to simplify to a certain extent the computations. We shall use this method for the case of analytic operators in §4.12.

4.11.

Now we start the proof of (4.23) for an arbitrary s . This time it will be convenient to prove it for a (very) large m : we shall assume $m \geq 2s$. For a function F of the variables (λ, ε) we shall use the notations: $F^{(k)} = \frac{d^k}{d\varepsilon^k} F$ and $F^{(\ell, k)} = \frac{d^\ell}{d\lambda^\ell} \frac{d^k}{d\varepsilon^k} F$. We shall also set $\tilde{G} = G_\varepsilon K_\varepsilon G_\varepsilon$ and we sometimes omit the index ε in the notations.

Let $\ell \geq 0$ be an arbitrary integer. If we denote $\theta_i(x) = x^i \theta^{(i)}(x)$ then we have

$$\begin{aligned} L_\varepsilon^{(\ell, m)} &= \frac{d^\ell}{d\lambda^\ell} \sum_{i+j+k=m} \frac{m!}{i!j!k!} \frac{\theta_i(\varepsilon A)}{\varepsilon^i} G_\varepsilon^{(k)} \frac{\theta_j(\varepsilon A)}{\varepsilon^j} \\ &= \sum_{i+j+k=m} \frac{m!}{i!j!k!} \varepsilon^{k-m} \theta_i(\varepsilon A) G_\varepsilon^{(\ell, k)} \theta_j(\varepsilon A). \end{aligned} \quad (4.27)$$

Our main task will be to prove the estimate

$$||L_\varepsilon^{(\ell, m)}||^{(s)} = O(\varepsilon^{\alpha-m-\ell}). \quad (4.28)$$

Once this is done, (4.23) follows easily. Indeed, let $\ell, k \geq 0$ be integers such that $\ell + k = m$. If $\ell = 0$, then (4.28) is just the estimate needed in order to control the term $\ell = 0$ in (4.23). If $\ell \geq 1$, we integrate the function $\tau \mapsto L_\tau^{(\ell, m)}$ over $(\varepsilon, \varepsilon_0)$; taking into account (4.28) and the fact that $\alpha - m < 0$, we get $||L_\varepsilon^{(\ell, m-1)}||^{(s)} = O(\varepsilon^{\alpha-m-\ell+1})$. Repeating this procedure ℓ times, we get $||L_\varepsilon^{(\ell, k)}||^{(s)} = O(\varepsilon^{\alpha-m})$. Thus we get (4.23).

The identity (4.27) shows that (4.28) is a consequence of

$$||\theta_i(\varepsilon A) G_\varepsilon^{(\ell, k)} \theta_j(\varepsilon A)||^{(s)} = O(\varepsilon^{\alpha-k-\ell}). \quad (4.29)$$

Here $i, j, k \in \mathbb{N}$ and $i + j + k = m$. If $k = 0$, then (4.29) follows easily from our preceding estimates. Indeed, we have $i + j = m \geq 2s$ so at least one of the integers i, j is larger than s . Assume for example $i \geq s$. Now observe that

$$G_\varepsilon^{(\ell, 0)} = \frac{d^\ell}{d\lambda^\ell} G_\varepsilon = \ell! G_\varepsilon^{\ell+1} \quad (4.30)$$

and

$$\begin{aligned} ||\langle A \rangle^{-s} \theta_i(\varepsilon A) G_\varepsilon^{\ell+1} \theta_j(\varepsilon A) \langle A \rangle^{-s}|| &\leq ||\langle A \rangle^{-s} \theta_i(\varepsilon A)|| \cdot ||G_\varepsilon||^\ell \\ &\quad \cdot ||G_\varepsilon \langle A \rangle^{-s}|| \cdot ||\theta_j(\varepsilon A)|| \\ &\leq C \varepsilon^s \cdot \varepsilon^{-\ell} \cdot \varepsilon^{-1/2} = C \varepsilon^{\alpha-\ell}. \end{aligned}$$

We have used (4.25), (4.15) and (4.21).

If $k \geq 1$ we shall have first to express $G^{(\ell,k)}$ in a convenient form. Starting with (4.17) we easily get

$$G^{(k)} = \mathcal{A}^k G + \mathcal{A}^{k-1} \tilde{G}' + \mathcal{A}^{k-2} \tilde{G}'' + \dots + \tilde{G}^{(k-1)}.$$

Differentiating this identity ℓ times with respect to λ and using (4.30), we obtain

$$G^{(\ell,k)} = \ell! \mathcal{A}^k [G^{\ell+1}] + \sum_{n=0}^{k-1} \mathcal{A}^{k-n-1} [\tilde{G}^{(\ell,n)}].$$

We see that (4.29) will be a consequence of the next estimate

$$\begin{aligned} & ||\theta_i(\varepsilon A) \mathcal{A}^k [G^{\ell+1}] \theta_j(\varepsilon A)||^{(s)} + \\ & + \sum_{n=0}^{k-1} ||\theta_i(\varepsilon A) \mathcal{A}^{k-n-1} [\tilde{G}^{(\ell,n)}] \theta_j(\varepsilon A)||^{(s)} = O(\varepsilon^{\alpha-k-\ell}). \end{aligned} \quad (4.31)$$

Let us consider the first term above. Using an expression we gave for $\mathcal{A}^k[T]$ in §2.1 and denoting $\varphi_a(x) = x^a \theta_i(x)$, $\psi_b(x) = x^b \theta_j(x)$ for $a, b \in \mathbb{N}$, we get

$$\theta_i(\varepsilon A) \mathcal{A}^k [G^{\ell+1}] \theta_j(\varepsilon A) = \sum_{a+b=k} \frac{k!(-1)^a}{a!b!} \varepsilon^{-k} \varphi_a(\varepsilon A) G^{\ell+1} \psi_b(\varepsilon A).$$

Hence the first term in (4.31) satisfies the required estimate if

$$||\varphi_a(\varepsilon A) G^{\ell+1} \psi_b(\varepsilon A)||^{(s)} = O(\varepsilon^{\alpha-\ell})$$

for $a+b=k$. The functions φ_a, ψ_b have compact support and have zeroes at the origin of the orders $p = a+i$, $q = b+j$. Since $p+q = k+i+j = m \geq 2s$, either p or q is larger than s . If $p \geq s$, then

$$\begin{aligned} ||\varphi_a(\varepsilon A) G^{\ell+1} \psi_b(\varepsilon A)||^{(s)} &\leq ||\langle A \rangle^{-s} \varphi_a(\varepsilon A)|| \cdot ||G||^\ell \cdot ||G \langle A \rangle^{-s}|| \cdot ||\psi_b(\varepsilon A)|| \\ &\leq C \varepsilon^{+s} \cdot \varepsilon^{-\ell} \cdot \varepsilon^{-1/2} = C \varepsilon^{\alpha-\ell} \end{aligned}$$

where (4.25), (4.15) and (4.21) have been used again.

It remains to treat the terms in the sum over n in (4.31). Proceeding exactly as above we see that we have to bound terms of the form $\varepsilon^{-(k-n-1)} \varphi_a(\varepsilon A) G^{(\ell,n)} \psi_b(\varepsilon A)$ with $a+b = k-n-1$ and $0 \leq n \leq k-1$. In fact, it is sufficient to prove that $||\varphi_a(\varepsilon A) \tilde{G}^{(\ell,n)} \psi_b(\varepsilon A)||^{(s)} = O(\varepsilon^{\alpha-\ell-n-1})$. This time the functions φ_a, ψ_b cannot help, since they could be both non-zero at the origin. We shall prove that we have in fact $||\tilde{G}^{(\ell,n)}||^{(s)} = O(\varepsilon^{\alpha-\ell-n-1})$. This will finish the proof of (4.31) and hence of the theorem.

We recall that $\tilde{G} = GKG$ and that K depends only on ε . So by Leibnitz formula the derivative $\tilde{G}^{(\ell,n)}$ is a linear combination of terms of the form $G^{(a,u)}K^{(w)}G^{(b,v)}$ where $a, b, u, v, w \in \mathbb{N}$ and $a + b = \ell$, $u + v + w = n$. By (4.8) we have $\|K^{(w)}\| = O(\varepsilon^{\alpha-w})$. Since

$$\begin{aligned} \|G^{(a,u)}K^{(w)}G^{(b,v)}\|^{(s)} &\leq \| \langle A \rangle^{-s} G^{(a,u)} \| \cdot \|K^{(w)}\| \cdot \|G^{(b,v)} \langle A \rangle^{-s}\| \\ &\leq C\varepsilon^{\alpha-w} \cdot \| \langle A \rangle^{-s} G^{(a,u)} \| \cdot \|G^{(b,v)} \langle A \rangle^{-s}\|, \end{aligned}$$

we conclude that it suffices to show that the last two factors above are of order $O(\varepsilon^{-a-u-1/2})$, $O(\varepsilon^{-b-v-1/2})$ respectively. We show this for $G^{(a,u)} \langle A \rangle^{-s}$ with $a, u \in \mathbb{N}$ arbitrary.

This time it is convenient to use for the derivative with respect to ε of G the expression $G'_\varepsilon = -G_\varepsilon H'_\varepsilon G_\varepsilon$. If $u \geq 0$ is an integer then $G^{(u)}$ will be a linear combination of terms of the form $G_\varepsilon H_\varepsilon^{(u_1)} \cdot G_\varepsilon H_\varepsilon^{(u_2)} \dots G_\varepsilon H_\varepsilon^{(u_n)} \cdot G_\varepsilon$ with $u_j \in \mathbb{N}$ and $u_1 + \dots + u_n = u$. If we take the derivative of order a with respect λ of such a term and use (4.30) we obtain terms of the form

$$G_\varepsilon^{a_1+1} H_\varepsilon^{(u_1)} \cdot G_\varepsilon^{a_2+1} H_\varepsilon^{(u_2)} \dots G_\varepsilon^{a_n+1} H_\varepsilon^{(u_n)} \cdot G_\varepsilon^{b+1}.$$

Here $a_j, b \in \mathbb{N}$ and $a_1 + \dots + a_n + b = a$. We multiply to the right by $\langle A \rangle^{-s}$ and estimate the norm of the operator so obtained by using $\|G_\varepsilon \langle A \rangle^{-s}\| \leq C\varepsilon^{-1/2}$, $\|G_\varepsilon^b\| \leq C\varepsilon^{-b}$ and (cf. (4.9)):

$$\|G_\varepsilon^{a_j+1} H_\varepsilon^{(u_j)}\| \leq \|G_\varepsilon\|^{a_j+1} \|H_\varepsilon^{(u_j)}\| \leq C\varepsilon^{-a_j-1} \cdot \varepsilon^{1-u_j} = C\varepsilon^{-a_j-u_j}.$$

We clearly obtain a bound of the form $C\varepsilon^{-a-u-1/2}$. This finishes the proof of the theorem.

4.12.

It is instructive to see what happens when the operator H is analytic with respect to A . Then it is no more necessary to regularize it, i.e. the factor $\theta(\varepsilon\mathcal{A})$ in the definition of H_ε is useless. So we take $H_\varepsilon = e^{\varepsilon\mathcal{A}}[H] \equiv e^{-\varepsilon\mathcal{A}} H e^{\varepsilon\mathcal{A}}$, which makes sense if $|\varepsilon|$ is small enough. Under the assumption (4.10), Lemma 4.2 and Proposition 4.3 remain true (only (4.3) and (4.5) have been used in their proof; note that $H_\varepsilon^* = H_{-\varepsilon}$). The advantage now is that $K_\varepsilon = 0$, so (4.17) is replaced by the much simpler equation $G'_\varepsilon = \mathcal{A}[G_\varepsilon]$. Then for each integer $k \geq 0$ we have $G^{(k)} = \mathcal{A}^k G$, hence $G^{(\ell,k)} = \ell! \mathcal{A}^k [G^{\ell+1}]$ for $\ell, k \in \mathbb{N}$. So there is a constant C (depending only on m) such that

$$\begin{aligned} \|G^{(\ell,m)}\|^{(m)} &= \ell! \|\mathcal{A}^{m-1} \mathcal{A}[G^{\ell+1}]\|^{(m)} \\ &\leq \ell! \sum_{i+j=m-1} \frac{(m-1)!}{i!j!} \|A^i \mathcal{A}[G^{\ell+1}] A^j\|^{(m)} \end{aligned}$$

$$\begin{aligned}
&\leq C\|\mathcal{A}[G^{\ell+1}]\|^{(1)} \leq C\|G^{\ell+1}A\|^{(1)} + C\|AG^{\ell+1}\|^{(1)} \\
&\leq C\|G^\ell\|(\|\langle A \rangle^{-1}G\| + \|G\langle A \rangle^{-1}\|) \leq C'\varepsilon^{-\ell-1/2}.
\end{aligned}$$

Integrating ℓ times with respect to ε we get for each integer $m \geq 1$:

$$\sum_{\ell+k=m} \|G^{(\ell,k)}\|^{(m)} \leq C_m \varepsilon^{-1/2} = C_m \varepsilon^{-m+(m-1/2)}.$$

This implies that the function $\lambda \mapsto G_0(\lambda) = R(\lambda + i\mu) \in B(\mathcal{H}_m, \mathcal{H}_{-m})$ is of class $\Lambda^{m-1/2}$ on $[\lambda_0 - \delta, \lambda_0 + \delta]$, uniformly in $\mu > 0$ (cf. Proposition A.1). Hence $R(\lambda + i0)$ has the same property. By interpolating between the cases $m = 1$ and $m = \text{large integer}$, we see that the assertion remains true for each real $m \geq 1$. In order to treat the range $1/2 < m < 1$ one has to consider the operator $L_\varepsilon = \theta(\varepsilon A)G_\varepsilon\theta(\varepsilon A)$; the argument is similar but simpler than that in §4.9 ($K_\varepsilon = 0$ in (4.24)).

Appendix

Let $\Omega \subset \mathbb{R}^n$ be an open set and $\tilde{\Omega} = \Omega \times (0, 1) \subset \mathbb{R}^{n+1}$. We denote $z = (x, y)$, with $x \in \Omega$ and $0 < y < 1$, the points of $\tilde{\Omega}$, and we set $\nabla = \nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$, $\partial_2 = \frac{\partial}{\partial y}$. Then $P = -i\nabla$ is the generator of the translation group in $\mathbb{R}^n$, so that $(e^{i\varepsilon P}f)(x) = f(x + \varepsilon)$. For a function F on $\tilde{\Omega}$ and $\alpha = (\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \in \mathbb{N}^{n+1}$ we set

$$F^{(\alpha)} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \frac{\partial^{\alpha_{n+1}}}{\partial y^{\alpha_{n+1}}} F.$$

Our purpose in this Appendix is to prove the following result:

Proposition A.1 *Let $m \geq 1$ be an integer and σ a strictly positive real number. Assume that $F : \tilde{\Omega} \rightarrow \mathbb{C}$ is a function of class C^m and that there is $M < \infty$ such that*

$$\sum_{|\alpha|=m} |F^{(\alpha)}(x, y)| \leq M y^{-m+\sigma}, \quad \forall (x, y) \in \tilde{\Omega}.$$

Then $\lim_{y \rightarrow 0} F(x, y) \equiv f(x)$ exists uniformly in $x \in \Omega$, and there is a constant C depending only on m such that if $x \in \Omega$, $\varepsilon \in \mathbb{R}^n$ and $x + t\varepsilon \in \Omega$ for all $t \in [0, m]$:

$$|(e^{i\varepsilon P} - 1)^m f|(x) \leq CM(1 + \sigma^{-1})|\varepsilon|^\sigma.$$

In particular, if $0 < \sigma < m$ then f is locally of class Λ^σ on Ω ; if $\sigma = m$, then $f \in C^{m-1}(\Omega)$ and $\nabla^\beta f$ are Lipschitz functions on Ω for $|\beta| = m - 1$.

This proposition is a slight extension of some theorems of Hardy and Littlewood concerning the boundary values of a holomorphic function, cf. Theorems 3 and 4 in Ch. IX, §5 of [11], Theorem 5.1 in Ch. VII of [27] and Theorem 2.26 in [7]. In fact, their results are an immediate consequence of Proposition A.1, namely:

Corollary A.2 *Let $\Omega \subset \mathbb{R}$ be an open set and $\tilde{\Omega} = \{z \in \mathbb{C} \mid \operatorname{Re} z \in \Omega, 0 < \operatorname{Im} z < a\}$ for some $a > 0$. Assume that $F : \tilde{\Omega} \rightarrow \mathbb{C}$ is a holomorphic function such that $|F^{(m)}(z)| \leq M(\operatorname{Im} z)^{-m+\sigma}$ for a constant $M < \infty$, an integer $m \geq 1$ and a real $\sigma \in (0, m]$. Then $f(x) \equiv \lim_{y \rightarrow +0} F(x + iy)$ exists uniformly on Ω and f is locally of class Λ^σ on Ω . If $\sigma = m$, then $f \in C^{m-1}(\Omega)$ and $f^{(m-1)}$ is a Lipschitz function on Ω .*

The remainder of the Appendix is devoted to the proof of Proposition A.1. Fix some $x \in \Omega$ and denote $u(y) = F(x, y)$. Then for $0 < a < y < 1$ Taylor's formula gives

$$u(a) = \sum_{k=0}^{m-1} \frac{(-1)^k}{k!} (y-a)^k u^{(k)}(y) + \frac{(-1)^m}{(m-1)!} \int_a^y u^{(m)}(\tau) (\tau-a)^{m-1} d\tau.$$

Then

$$\begin{aligned} \int_a^y u^{(m)}(\tau) (\tau-a)^{m-1} d\tau &= \int_a^y u^{(m)}(\tau) \tau^{m-1} d\tau \\ &\quad + \sum_{j=1}^{m-1} \binom{m-1}{j} (-1)^j \int_a^y u^{(m)}(\tau) \tau^{m-1-j} a^j d\tau. \end{aligned}$$

Using $|u^{(m)}(\tau)| = |\partial_2^m F(x, \tau)| \leq M\tau^{-m+\sigma}$ we see that the first integral in the r.h.s. converges to $\int_a^y u^{(m)}(\tau) \tau^{m-1} d\tau$ as $a \rightarrow 0$, uniformly in $x \in \Omega$, while the other integrals converge to zero. So $\lim_{a \rightarrow 0} F(x, a) \equiv f(x)$ exists uniformly in $x \in \Omega$ and for all $(x, y) \in \tilde{\Omega}$:

$$f(x) = \sum_{k=0}^{m-1} \frac{(-1)^k}{k!} y^k \partial_2^k F(x, y) + \frac{(-1)^m}{m!} \int_0^y \partial_2^m F(x, \tau) d\tau^m.$$

Now let $x \in \Omega$ and $\varepsilon \in \mathbb{R}^n$ be such that $x + t\varepsilon \in \Omega$ if $0 \leq t \leq m$. We have then for all $y \in (0, 1)$:

$$\begin{aligned} [(e^{i\varepsilon P} - 1)^m f](x) &= \sum_{k=0}^{m-1} \frac{(-1)^k}{k!} y^k [(e^{i\varepsilon P} - 1)^m \partial_2^k F](x, y) \\ &\quad + \frac{(-1)^m}{m!} \int_0^y [(e^{i\varepsilon P} - 1)^m \partial_2^m F](x, \tau) d\tau^m. \end{aligned} \tag{A.1}$$

We first estimate the last term in the r.h.s. Since

$$(e^{i\varepsilon P} - 1)^m \partial_2^m F = \sum_{j=0}^m \binom{m}{j} (-1)^{m-j} \partial_2^m e^{ij\varepsilon P} F$$

we have:

$$\begin{aligned} |[(e^{i\varepsilon P} - 1)^m \partial_2^m F](x, \tau)| &\leq \sum_{j=0}^m \binom{m}{j} |\partial_2^m F(x + j\varepsilon, \tau)| \\ &\leq \sum_{j=0}^m \binom{m}{j} M \tau^{-m+\sigma} = 2^m M \tau^{-m+\sigma}. \end{aligned}$$

So the last term in (A.1) is bounded by $2^m [(m-1)!]^{-1} M \sigma^{-1} y^\sigma$.

For the k -th term in the r.h.s. of (A.1), with $0 \leq k \leq m-1$, we write $m = k + \ell$ and use the ℓ -th order mean value theorem:

$$\begin{aligned} |[(e^{i\varepsilon P} - 1)^m \partial_2^k F](x, y)| &= |[(e^{i\varepsilon P} - 1)^\ell \partial_2^k (e^{i\varepsilon P} - 1)^k F](x, y)| \\ &\leq \sup_{0 \leq t \leq \ell} |[(\varepsilon \cdot \nabla)^\ell \partial_2^k (e^{i\varepsilon P} - 1)^k F](x + t\varepsilon, y)| \\ &\leq \sum_{j=0}^k \binom{k}{j} \sup_{0 \leq t \leq \ell} |[(\varepsilon \cdot \nabla)^\ell \partial_2^k F](x + t\varepsilon + j\varepsilon, y)| \\ &\leq \sum_{j=0}^k \sum_{|\beta|=\ell} \binom{k}{j} \frac{\ell!}{\beta!} \sup_{0 \leq s \leq m} |\varepsilon^\beta [\nabla^\beta \partial_2^k F](x + s\varepsilon, y)| \\ &\leq 2^k \ell! |\varepsilon|^\ell \sum_{|\alpha|=m} \sup_{0 \leq s \leq m} |F^{(\alpha)}(x + s\varepsilon, y)| \\ &\leq 2^k \ell! |\varepsilon|^\ell M y^{-m+\sigma}. \end{aligned}$$

If we insert the preceding estimates into (A.1) we obtain:

$$\begin{aligned} |[(e^{i\varepsilon P} - 1)^m f](x)| &\leq \sum_{k=0}^{m-1} \frac{1}{k!} y^k 2^k (m-k)! |\varepsilon|^{m-k} M y^{-m+\sigma} + \frac{2^m}{(m-1)!} \frac{M}{\sigma} y^\sigma \\ &\leq c(m) M \sum_{k=0}^{m-1} y^{k-m+\sigma} |\varepsilon|^{m-k} + \frac{c(m) M}{\sigma} y^\sigma. \end{aligned}$$

By taking $y = |\varepsilon|$ we obtain the estimate stated in Proposition A.1. This finishes the proof.

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