ESTIMATES FOR APPROXIMATE SOLUTIONS TO ACOUSTIC INVERSE SCATTERING PROBLEMS

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Estimates for approximate solutions to acoustic inverse scattering problems

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Introduction

We examine the 'inverse problem' of determining a scatterer $K \subset \mathbb{R}^3$ in terms of information on the scattered waves, particularly solutions to the reduced wave equation $(\Delta + k^2)u = 0$ on $\mathbb{R}^3 \setminus K$, satisfying a Dirichlet boundary condition, $u = f$ on ∂K, and the Sommerfeld radiation condition.

In §1 we state the scattering problem more precisely, and define notation, for solution operators to various scattering problems, the Neumann operator, the scattering amplitude, and other objects, and briefly mention a few common attacks on the inverse problem.

In §2 we study the problem of recovering the near field wave from the scattering amplitude. There is an exact formula, given in terms of a family of functions of the Laplace operator on the sphere S^2, but one obtains seriously unbounded operators, an indication of the 'ill posed' nature of the inverse problem. We discuss a method of regularizing the problem of approximating the near field wave, given an approximate measurement of the scattering amplitude, and of bounding the error, following a general approach of Miller [Mr1]. An estimate on the error on a shell $\Sigma = \{|x| = R\}$ known to contain the (unknown) obstacle well inside is given in (2.40). It is slightly weaker than the $C\varepsilon^r$ estimates one often aims for when regularizing ill posed problems, though not nearly so weak as the $(\log \frac{1}{\varepsilon})^{-r}$ one sometimes fears to obtain. Indeed, this 'bad' estimate does lie in wait if we allow the obstacle to touch the shell Σ; see (2.46). The major result of §2 is then Proposition 2.5, guaranteeing that this disastrous error magnification does not occur if we seek only the features of the wave on Σ on a length scale $\geq C/k$, for an appropriate constant $C > 1$.

In §3 we introduce the linearized inverse problem, which is related to the problem of recovering the Dirichlet data of a wave, on ∂K, given an approximate measurement of the wave on the shell Σ. We want to apply Proposition 2.5 to this problem. To do this, some necessary machinery from microlocal analysis is developed in §4, some global estimates on the solution operator to the direct scattering problem are established in §5, and these results are applied in §6 to a regularization of the linearized inverse problem, given that ∂K is smooth and strictly convex.

Results here provide one step in a program to resolve unknown details of an obstacle. One might envisage a next step: to apply Newton's method, proceeding from an approximation K_0 to the unknown obstacle K, having the property that K_0 is strictly convex and 'smooth', i.e., featureless on length scales $\sim 1/k$, to a finer approximation, stopping the process at an approximation which does have features on a length scale $\sim 1/k$.

This idea is very much consistent with intuition and experience. For example, a well known statement of the limitations of an optical microscope is that, if it has perfect optics, one can use it to examine microscopic detail on a length scale approximately equal to, but not smaller than, the wavelength of visible light. We emphasize that this limitation applies to discerning detail on an obstacle whose diameter is much larger than $1/k$. If one has a single obstacle whose diameter is $\sim 1/k$, then one is said to be dealing with an inverse problem in the 'resonance region,' and, given some a priori hypotheses on the obstacle, one can hope to make out some details of its structure to a higher precision than one wavelength. This sort of problem is discussed in a number of papers on inverse problems, such as [ACK], [AKR], [JM], and [MTW].

One difficulty in applying Newton's method is a problem of phase ambiguity, whose importance was impressed upon the author by W.Symes. In problems in which K is a ball (whose position is unknown) simple explicit calculations show that the 'basin of attraction' of Newton iterations is small: inversely proportional to k. The author has devised a strategy to overcome this difficulty, in cases where scattering data can be obtained at two closely spaced frequencies. We plan to investigate the usefulness of such a strategy in enlarging the basin of atraction for more general problems, in a future paper.

There are modifications of the analysis given here which apply to the linearized inverse problem when K_0 is not necessarily convex, but verifies some hypothesis to the effect that it is 'illuminated.' We will also take this matter up, in a future publication.

1. Direct and inverse problems of acoustical scattering

The basic scattering problem we consider is the following. Let $K \subset \mathbb{R}^3$ be a compact set with smooth boundary, and connected complement Ω. Let $f \in H^s(\partial K)$ be given, and let $k > 0$. We want to solve

$$(\Delta + k^2)v = 0 \text{ on } \Omega, \tag{1.1}$$

$$v = f \text{ on } \partial K. \tag{1.2}$$

In addition, we impose a 'radiation condition,' of the following form:

$$|rv(x)| \leq C, \quad r\Big(\frac{\partial v}{\partial r} - ikv\Big) \longrightarrow 0 \text{ as } r \longrightarrow \infty, \tag{1.3}$$

where $r = |x|$. It is a fundamental result of scattering theory that (1.1)-(1.3) has a unique solution, which we denote

$$v = \mathcal{B}(k)f. \tag{1.4}$$

When we want to emphasize the dependence on K, we denote the operator by $\mathcal{B}_K(k)$. Some methods of producing the solution to (1.1)-(1.3) use integral equations; some results of this sort are recalled in Appendix B.

The solution to (1.1)-(1.3) satisfies the integral identity

$$v(x) = \int_{\partial K} \Big[f(y)\frac{\partial g}{\partial \nu_y}(x,y,k) - g(x,y,k)\frac{\partial v}{\partial \nu}(y)\Big]\, dS(y), \tag{1.5}$$

for $x \in \Omega$, where

$$g(x,y,k) = \big(4\pi|x-y|\big)^{-1}\, e^{ik|x-y|}. \tag{1.6}$$

In light of the appearance of $\partial v/\partial\nu$ in the integrand in (1.5), the operator $\mathcal{N}(k)$, defined by

$$\mathcal{N}(k)f = \frac{\partial}{\partial\nu}\,\mathcal{B}(k)f\big|_{\partial K} \tag{1.7}$$

is of fundamental significance. It is called the Neumann operator.

Let us introduce some convenient notation. If K_1 is contained in the interior $\overset{\circ}{K}_2$ of K_2, then $g = \mathcal{B}_{K_1}(k)f\big|_{\partial K_2}$ defines a bounded operator

$$\mathcal{B}_{K_1K_2}(k) : L^2(\partial K_1) \longrightarrow L^2(\partial K_2), \tag{1.8}$$

whose range is contained in $C^\infty(\partial K_2)$. If either $K_1 = B_r = \{x : |x| \le r\}$ or $K_2 = B_\rho$, we use the notation $\mathcal{B}_{rK_2}(k)$ or $\mathcal{B}_{K_1\rho}(k)$; if both K_1 and K_2 are such balls, we use the notation $\mathcal{B}_{r\rho}(k)$.

Lemma 1.1. *If K_j are compact sets in $\mathbb{R}^3$ (with connected complement) such that $K_1 \subset \overset{\circ}{K}_2$, then for any $k \in \mathbb{R}$ the map $\mathcal{B}_{K_1K_2}(k)$ is injective. If also $\overset{\circ}{K}_2$ is connected, this map has dense range.*

Proof. If $u = \mathcal{B}_{K_1}(k)f$ vanishes on ∂K_2, then u restricted to $\mathbb{R}^3 \setminus K_2$ is an outgoing solution to (1.1), so by uniqueness of solutions to (1.1)-(1.3), we have $u = 0$ on $\mathbb{R}^3 \setminus K_2$. Then unique continuation forces $u = 0$ on $\mathbb{R}^3 \setminus K_1$, so injectivity of (1.8) is established.

As for the second claim, note that, if $y \in \overset{\circ}{K}_1$, then $|x-y|^{-1}e^{ik|x-y|} = g_y(x)$ is clearly in the range of $\mathcal{B}_{K_1}(k)$. Thus if $f \in L^2(\partial K_2)$ is orthogonal to the range of $\mathcal{B}_{K_1K_2}(k)$, we deduce that

$$F(x) = \int_{\partial K_2} f(y) g_x(y)\, dS(y) \tag{1.9}$$

is zero for $x \in \overset{\circ}{K}_1$, hence for $x \in \overset{\circ}{K}_2$ (if $\overset{\circ}{K}_2$ is connected). Also, material in Appendix B implies that F is continuous across ∂K_2, and is an outgoing solution of (1.1) on $\mathbb{R}^3 \setminus K_2$. Uniqueness of solutions to (1.1)-(1.3) forces $F = 0$ on $\mathbb{R}^3 \setminus K_2$. Since, by (B.15), the jump of $\partial_\nu F$ across ∂K_2 is proportional to f, this implies $f = 0$, proving denseness.

Given $K_1 \subset \overset{\circ}{K}_2$ and $\overset{\circ}{K}_2$ connected, denote the right inverse of $\mathcal{B}_{K_1K_2}(k)$ by $\mathcal{C}_{K_2K_1}(k)$:

$$L^2(\partial K_2) \supset \mathcal{D} \xrightarrow{\mathcal{C}_{K_2K_1}(k)} L^2(\partial K_1). \tag{1.10}$$

Here $\mathcal{D} =$ Range $\mathcal{B}_{K_1K_2}(k)$. Thus $\mathcal{C}_{K_2K_1}(k)$ is a closed, densely defined (but unbounded) operator, with domain $\mathcal{D} \subset C^\infty(\partial K_2)$. As above, if $K_1 = B_r$ or $K_2 = B_\rho$, we use the notations

$$\mathcal{C}_{K_2r}(k) = \mathcal{B}_{rK_2}(k)^{-1}, \quad \mathcal{C}_{\rho K_1}(k) = \mathcal{B}_{K_1\rho}(k)^{-1}, \quad \mathcal{C}_{\rho r}(k) = \mathcal{B}_{r\rho}(k)^{-1}. \tag{1.11}$$

A particularly important family of functions defined by a scattering problem is the following. Note that we have

$$(\Delta + |\xi|^2)e^{-ix\cdot\xi} = 0 \text{ on } \mathbb{R}^3, \tag{1.12}$$

for any $\xi \in \mathbb{R}^3$. We define the functions $v(x,\xi)$ on $\Omega \times \mathbb{R}^3$ to satisfy the scattering problem (1.1)-(1.3), with $k^2 = |\xi|^2$ and

$$v(x,\xi) = -e^{-ix\cdot\xi} \text{ on } \partial K. \tag{1.13}$$

These are the <u>scattered waves</u> produced by plane waves striking the obstacle K. It is also common to write $v_+(x,\xi)$ for (1.13).

For any $f \in C^\infty(\partial K)$, by (1.5) we have an asymptotic behavior of the form

$$v(r\theta) = r^{-1}e^{ikr}\alpha(f,\theta,k) + o(r^{-1}), \quad r \to \infty, \tag{1.14}$$

with $\theta \in S^2$, for the solution to the scattering problem (1.1)-(1.3), with a smooth coefficient $\alpha(f,\cdot,\cdot)$. Also

$$(\partial/\partial r)v(r\theta) = (ik/r)e^{ikr}\alpha(f,\theta,k) + o(r^{-1}). \tag{1.15}$$

In particular, the scattered wave $v(x,\xi)$ given by (1.9) has the asymptotic behavior

$$v(r\theta, k\omega) \sim r^{-1}e^{ikr}a(-\omega,\theta,k), \quad r \to \infty, \tag{1.16}$$

for fixed $\theta,\omega \in S^2$, $k \in \mathbb{R}^+$, and its r-derivative has an analogous behavior. The coefficient $a(\omega,\theta,k)$ is called the <u>scattering amplitude</u> and is one of the fundamental objects of scattering theory.

Parallel to the definition of $\mathcal{B}_K(k)$, let $\mathcal{A}_K(k)f(\theta)$ denote the factor $\alpha(f,\theta,k)$ in (1.14), so

$$\mathcal{B}_K(k)f(r\theta) \sim r^{-1}e^{ikr}\,\mathcal{A}_K(k)f(\theta) + o(r^{-1}), \quad r \to \infty. \tag{1.17}$$

Using (1.5) one can easily derive the formula

$$\mathcal{A}_K(k)f(\theta) = \frac{1}{4\pi}\int_{\partial K}\Big[ik\big(\nu(y)\cdot\theta\big)f(y) + \mathcal{N}(k)f(y)\Big]e^{-ik\theta\cdot y}\,dS(y). \tag{1.18}$$

We discuss inverse problems of the following sort. For a fixed $k \in \mathbb{R}$ and one or more directions ω, construct K, given knowledge either of the scattering amplitude $a(-\omega,\theta,k)$, or of the 'near field' scattered wave $v(x,k\omega)$, for x in some set in $\mathbb{R}^3$, typically the sphere $S^2_r = \{x : |x| = r\}$, assuming $K \subset B_r = \{x : |x| \le r\}$.

We give a brief description of a few methods that have been brought to bear on such inverse problems.

Assume you know that $B_1 \subset K \subset B_r$, and that you have a measurement of $v(x, k\omega)$ on $\partial B_r = S_r^2 = \{|x| = r\}$. You want to find (an approximation to) K. One strategy is to minimize

$$\Phi(f, K) = \|\mathcal{B}_{1r}(k)f - v(\cdot, k\omega)\|^2_{L^2(S_r^2)} + \|\mathcal{B}_{1K}(k)f + e^{-ik\omega\cdot x}\|^2_{L^2(\partial K)}, \tag{1.19}$$

with f and K varying over certain compact sets, determined by a priori hypotheses on the scatterer. This is close to some methods of Angell and Kleinman, Kirsch and Kress, as described at the end of [Co2].

Instead of minimizing (1.19) over (f, K), an alternative is first to minimize the first term of (1.19), thus choosing f, within some compact set of functions, and then to pick K to minimize the second term, within some compact set of obstacles.

An attack, similar to one used in [JM], is the following. Given a guess K_μ of K, assumed to be connected, apply the Gramm-Schmidt process in concert with Lemma 1.1 to produce an orthonormal basis $\{w_j : j \in \mathbb{Z}^+\}$ of $L^2(\partial K_\mu)$, such that each w_j is the image under $\mathcal{B}_{1K}(k)$ of a linear combination of the first j eigenfunctions of the Laplace operator on $L^2(S^2)$. In such a case, fix J and pick f in the linear span of $\{w_j : 1 \le j \le J\}$ to minimize the first term of (1.19). Then vary K_μ to minimize the second term.

An attack pursued in [Rog] and [MTW] takes a guess K_μ of K, solves (approximately) a linearized inverse problem, given K_μ, and applies an iteration, provided by Newton's method, to approximate K. See also [Kir].

2. Recovering the near field wave from the scattering amplitude

Depending on circumstances, one might have a measurement of the scattered wave $v_+(x, k\omega)$, for x in a region not far from ∂K, k belonging to some restricted set of frequencies (maybe a single frequency), or one might have only the far-field behavior, i.e., the scattering amplitude $a(\omega, \theta, k)$, which we recall is related to $v_+(x, k\omega)$ by

$$v_+(r\theta, k\omega) \sim \frac{e^{ikr}}{r} a(-\omega, \theta, k), \quad r \to \infty. \tag{2.1}$$

It is useful to have the following explicit relation between the scattered wave $v_+(x, k\omega)$ and the scattering amplitude.

Proposition 2.1. *If $K \subset B_R(0)$, then, for $r \ge R$,*

$$a(-\omega, \theta, k) = -ik^{-1} e^{\frac{1}{2}\pi i(A - \frac{1}{2})} h_{A-\frac{1}{2}}(kr)^{-1} g(\theta) \tag{2.2}$$

where $g = g_{r,\omega,k}$ is given by

$$g(\theta) = v_+(r\theta, k\omega). \tag{2.3}$$

As in Appendix A, $h_{A-\frac{1}{2}}(kr)^{-1}$ is regarded as a family of functions of the self adjoint operator A defined by (A.10) (acting on functions of θ), and $h_m(\lambda)$ is given by (A.17)-(A.18).

Proof. This result follows easily from (A.12), which implies

$$\mathcal{B}(k)f(r\theta) = \frac{h_{A-\frac{1}{2}}(kr)}{h_{A-\frac{1}{2}}(k)}f(\theta), \tag{2.4}$$

for $f \in C^\infty(S^2)$. To prove (2.2)-(2.3), we can suppose without loss of generality that $R = 1$, and apply (2.4) with $f(\theta) = v_+(\theta, k\omega)$, to get

$$v_+(r\theta, k\omega) = \frac{h_{A-\frac{1}{2}}(kr)}{h_{A-\frac{1}{2}}(k)}f(\theta); \quad f(\theta) = v_+(\theta, k\omega).$$

Now compare the asymptotic behavior of both sides as $r \to \infty$. For the left side we have (2.1), while the behavior of the right side is governed by

$$h_m(kr) \sim i^{m-1}\frac{e^{ikr}}{kr}$$

by (A.18)-(A.19), so (2.2) follows. In fact, this argument shows that, more generally, for $f \in L^2(\partial K)$,

$$\mathcal{A}_K(k)f(\theta) = -ik^{-1}e^{\frac{1}{2}\pi i(A-\frac{1}{2})}h_{A-\frac{1}{2}}(kr)^{-1}g(\theta), \quad g(\theta) = \mathcal{B}_K(k)f(r\theta). \tag{2.5}$$

Now we can invert the operator in (2.2), to write

$$v_+(r\theta, k\omega) = ik\ e^{-\frac{1}{2}\pi i(A-\frac{1}{2})}h_{A-\frac{1}{2}}(kr)a(-\omega, \theta, k), \tag{2.6}$$

where the operator acts on functions of θ. The operator $h_{A-\frac{1}{2}}(kr)$ is an unbounded operator on $L^2(S^2)$; indeed, it is not continuous from $C^\infty(S^2)$ to $\mathcal{D}'(S^2)$, which has consequences for the inverse problem as we will see below.

A formula equivalent to (2.2) played a role in Schiffer's original proof of uniqueness of an obstacle with a given scattering amplitude. We recall that argument here, and give a uniqueness statement which is perhaps a little more precise than the typical formulation.

Suppose then that K_1 and K_2 are two compact obstacles in $\mathbb{R}^3$ giving rise to scattered waves which both agree with $v_+(x, k\omega)$ in some open set $\mathcal{O}$ in $\mathbb{R}^3 \setminus (K_1 \cup K_2)$. In other words $v_j(x, k\omega) = v_+(x, k\omega)$ for $x \in \mathcal{O}$, where v_j are solutions to

$$(\Delta + k^2)v_j = 0 \text{ on } \mathbb{R}^3 \setminus K_j, \quad v_j = -e^{-ikx\cdot\omega} \text{ on } \partial K_j, \tag{2.7}$$

satisfying the radiation condition. We suppose the sets K_j have no 'cavities,' i.e., each $\Omega_j = \mathbb{R}^3 \setminus K_j$ has just one connected component. In this case, possibly the complement of $K_1 \cup K_2$ is not connected. We will let $\mathcal{U}$ denote the unbounded connected component of this complement, and consider $\mathbb{R}^3 \setminus \mathcal{U}$, which we denote by $\widetilde{K}_2$, so $K_1 \subset \widetilde{K}_2$. We assume $\mathcal{O} \subset \mathcal{U}$. Let $\mathcal{R}$ be any connected component of the interior of $\widetilde{K}_2 \setminus K_1$. (Switch indices if $K_2 \subset K_1$.)

The functions v_1 and v_2 described above agree on $\mathcal{U}$, since they are real analytic and agree on $\mathcal{O}$. Thus u_1 and u_2 agree on $\mathcal{U}$, where $u_j(x) = v_j(x) + e^{-ikx\cdot\omega}$. Since each u_j vanishes on ∂K_j, it follows that $u = u_1\big|_{\mathcal{R}}$ vanishes on $\partial\mathcal{R}$, so

$$(\Delta + k^2)u = 0 \text{ on } \mathcal{R}, \quad u = 0 \text{ on } \partial\mathcal{R}. \tag{2.8}$$

In fact, one verifies that $u \in H_0^1(\mathcal{R})$. In particular, u provides an <u>eigenfunction</u> of Δ on each connected component $\mathcal{R}$ of the interior of $\widetilde{K}_2 \setminus K_1$, with Dirichlet boundary condition (and with eigenvalue $-k^2$), if u is not identically zero, and if the symmetric difference $K_1 \,\triangle\, K_2$ has nonempty interior. Now there are circumstances where we can obtain bounds on

$$\dim \ker\, (\Delta + k^2)\big|_{H_0^1(\mathcal{R})} = d(k), \tag{2.9}$$

for example if we know the obstacle is contained in a ball B_R. We then have the following uniqueness result.

Proposition 2.2. *Let $k \in (0,\infty)$ be fixed. Suppose $\Sigma = \{\omega_\ell\}$ is a subset of S^2 whose cardinality is known to be greater than $\frac{1}{2}d(k)$. (If ω_ℓ and $-\omega_\ell$ both belong to Σ, do not count them separately.) Then knowledge of $v_+(x, k\omega_\ell)$ for x in an open set $\mathcal{O}$ uniquely determines the obstacle K. Hence knowledge of $a(-\omega_\ell, \theta, k)$ for $\theta \in S^2$ uniquely determines K.*

Proof. If K were not uniquely determined, there would be a nonempty set $\mathcal{R}$ such as described above. The corresponding $u_\ell(x) = v_+(x, k\omega_\ell) + e^{-ikx\cdot\omega_\ell}$, together with their complex conjugates, which are all eigenfunctions on $\mathcal{R}$, must be linearly independent. Indeed, any linear dependence relation valid on $\mathcal{R}$ must continue on all of $\mathbb{R}^3 \setminus (K_1 \cap K_2)$; but near infinity $u_\ell(x) = e^{-ikx\cdot\omega_\ell} + O(|x|^{-1})$ guarantees independence.

We make a few complementary remarks. First, $a(-\omega, \theta, k)$ is analytic in its arguments, so for any given ω, k, it is uniquely determined by its behavior for θ in any open subset of S^2. Next, for k small enough we can say that $d(k) = 0$, so uniqueness holds in that case, for a single $\omega = \omega_\ell$. Note that, even when k^2 is an eigenvalue of $-\Delta$ on $\mathcal{R}$, it would be a real coincidence for a corresponding eigenfunction to happen to continue to $\mathbb{R}^3 \setminus (K_1 \cap K_2)$ with the appropriate behavior at infinity. It is often speculated that knowledge of $a(-\omega, \theta, k)$ for $\theta \in S^2$ (or an open set) and both k and ω fixed, always uniquely determines the obstacle K. This remains an interesting open problem.

Furthermore, suppose $a(-\omega, \theta, k)$ is known on $\theta \in S^2$, for a set $\{\omega_\ell\} \subset S^2$ and a set $\{k_m\} \subset \mathbb{R}^+$. Then one has uniqueness provided card $\{\omega_\ell\} > \min\limits_m \frac{1}{2}d(k_m)$. In particular, if $\{k_m\}$ consists of an <u>interval</u> I (of nonzero length), then $\min\limits_m d(k_m) = 0$, so knowledge of $a(-\omega, \theta, k)$ for $\theta \in S^2$, $k \in I$, and a single ω uniquely determines K.

All of these considerations are subject to the standing assumption made throughout this paper on the smoothness of ∂K. There are interesting cases of non-smooth obstacles, not equal to the closure of their interiors, to which the proof of Proposition 2.2 would not apply. This is discussed further in §12 of [T4].

A consideration of practical importance for inverse problems is how accurately can one recover desired information, given that certain data are observed with good but limited accuracy. We illustrate this in the following simple situation. Suppose you know the obstacle K is contained in a ball B_{R_0}. Fix $k \in (0,\infty)$, $\omega \in S^2$.

Problem A. *Given an approximation $b(-\omega,\theta,k)$ to the scattering amplitude, with*

$$\|a(-\omega,\cdot,k)-b(-\omega,\cdot,k)\|_{L^2(S^2)}\leq\varepsilon, \tag{2.10}$$

how well can you approximate the scattered wave $v_+(x,k\omega)$, for x on the shell $|x|=R_1$, given $R_1>R_0$?

What makes this problem difficult is the failure of the operator $h_{A-\frac{1}{2}}(kr)$ appearing in (2.6) to be bounded, even from $C^\infty(S^2)$ to $\mathcal{D}'(S^2)$. Indeed, for fixed $s\in(0,\infty)$, one has the asymptotic behavior as $\nu\to+\infty$,

$$H_\nu^{(1)}(s)\sim -i\Big(\frac{2}{\pi\nu}\Big)^{\frac{1}{2}}\Big(\frac{2\nu}{es}\Big)^{\nu} \tag{2.11}$$

and hence

$$h_{\nu-\frac{1}{2}}(s)\sim -i(s\nu)^{-\frac{1}{2}}\Big(\frac{2\nu}{es}\Big)^{\nu}. \tag{2.12}$$

Consequently, an attempt to approximate $v_+(x,k\omega)$ for $x=R_1\theta$ by

$$v_0(\theta)=ike^{-\frac{1}{2}\pi i(A-\frac{1}{2})}h_{A-\frac{1}{2}}(kR_1)b(-\omega,\theta,k), \tag{2.13}$$

with the operator acting on functions of θ as in (2.6), could lead to nonsense. We will describe a method below which is well behaved. But first we look further into the question of how well can we possibly hope to approximate $v_+(x,k\omega)$ on the shell $|x|=R_1$ with the data given.

In fact, it is necessary to have some further a priori bound on v_+ to make progress here. We will work under the hypothesis that a bound on $v_+(x,k\omega)$ is known on the sphere $|x|=R_0$:

$$\|v_+(R_0\theta,k\omega)\|_{L^2(S^2_\theta)}\leq E. \tag{2.14}$$

Now, if we are given that (2.10) and (2.14) are both true and we have $b(-\omega,\theta,k)$ in hand (for ω,k fixed, $\theta\in S^2$), then we can consider the set $\mathcal{F}$ of functions $f(\theta)$ such that

$$\|f-b(-\omega,\cdot,k)\|_{L^2(S^2)}\leq\varepsilon, \tag{2.15}$$

and such that

$$\|k\ h_{A-\frac{1}{2}}(kR_0)f\|_{L^2(S^2)}\leq E, \tag{2.16}$$

knowing that $\mathcal{F}$ is nonempty. We know that $a(\theta)=a(-\omega,\theta,k)$ belongs to $\mathcal{F}$, and that is all we know about $a(-\omega,\theta,k)$, in the absence of further data. The greatest accuracy of an approximation $v_1(\theta)$ to $v_+(R_1\theta,k\omega)$ that we can count on, measured in the $L^2(S^2)$ norm, is

$$\|v_1(\theta)-v_+(R_1\theta,k\omega)\|_{L^2(S^2_\theta)}\leq 2\ M(\varepsilon,E), \tag{2.17}$$

where $M(\varepsilon, E)$ is defined as follows. Denote by

$$T_j : \mathcal{F} \longrightarrow L^2(S^2), \quad j = 0, 1, \tag{2.18}$$

the maps

$$T_j f(\theta) = ike^{-\frac{1}{2}\pi i(A-\frac{1}{2})} h_{A-\frac{1}{2}}(kR_j) f(\theta). \tag{2.19}$$

Then we set

$$2M(\varepsilon, E) = \sup \{\|T_1 f - T_1 g\|_{L^2(S^2)} : f, g \in \mathcal{F}\}, \tag{2.20}$$

i.e.,

$$M(\varepsilon, E) = \sup \{\|T_1 f\|_{L^2} : \|f\|_{L^2} \leq \varepsilon \text{ and } \|T_0 f\|_{L^2} \leq E\}. \tag{2.21}$$

One way to obtain as accurate as possible an approximation to v_+ on $|x| = R_1$ would be to pick any $f \in \mathcal{F}$ and evaluate $T_1 f$. However, it might not be straightforward to obtain elements of $\mathcal{F}$. We describe a method, from [Mr2] and [MrV], which is effective in producing a 'nearly best possible' approximation.

We formulate a more general problem. We have a linear equation

$$Sv = a, \tag{2.22}$$

where S is a bounded operator on a Hilbert space H, which is injective, but S^{-1} is unbounded (with domain a proper linear subspace of H) Given an approximate measurement b of a, we want to find an approximation to the solution v. This is a typical ill-posed linear problem. As a priori given information, we assume that

$$\|b - a\|_H \leq \varepsilon, \quad \|T_0 a\|_H \leq E. \tag{2.23}$$

T_0 is an auxiliary operator. In the example above, $H = L^2(S^2_\theta)$, and T_0 and T_1 are given by (2.19). Generalizing (2.20)-(2.21), we have a basic measurement of error:

$$\begin{aligned} M(\varepsilon, E) &= \sup \left\{\|T_1 f\|_H : \|f\|_H \leq \varepsilon \text{ and } \|T_0 f\|_H \leq E\right\} \\ &= \tfrac{1}{2} \sup \left\{\|T_1 f - T_1 g\|_H : f, g \in \mathcal{F}\right\}, \end{aligned} \tag{2.24}$$

where

$$\mathcal{F} = \{f \in H : \|f - b\|_H \leq \varepsilon, \|T_0 f\|_H \leq E\}. \tag{2.25}$$

Now the fact of life is that, if all you know about a in (2.22) is that it belongs to $\mathcal{F}$, then the greatest accuracy of an approximation v_1 to the solution v of (2.22) you can count on is

$$\|v_1 - v\|_H \leq 2M(\varepsilon, E). \tag{2.26}$$

This recaps the estimates (2.14)-(2.21). Now we proceed. An approximation method is called nearly best possible (up to a scalar γ) if it yields a $v_1 \in H$ such that

$$\|v_1 - v\|_H \le 2\gamma M(\varepsilon, E). \tag{2.27}$$

We now describe one nearly best possible method for approximating v, in cases where T_0 is a self adjoint operator, with discrete spectrum accumulating only at $+\infty$. Then, pick an orthonormal basis $\{u_j : 1 \le j < \infty\}$ of H, consisting of eigenvectors, such that

$$T_0 u_j = \alpha_j u_j, \quad \alpha_j \nearrow +\infty. \tag{2.28}$$

When T_0 is given by (2.19) this holds, as a consequence of (2.12). It is essential that the α_j be monotonic, so the eigenvectors need to be ordered correctly. Now let

$$f_\ell = P_\ell b, \quad P_\ell u = \sum_{j=1}^{\ell} (u, u_j) u_j. \tag{2.29}$$

Now let N be the first ℓ such that

$$\|f_\ell - b\|_H \le 2\varepsilon. \tag{2.30}$$

We then claim that

$$\|T_0 f_N\|_H \le 2E. \tag{2.31}$$

This can be deduced from:

Lemma 2.3. *If the set $\mathcal{F}$ defined by (2.25) is nonempty, and if $M+1$ is the first j such that $\alpha_j > E/\varepsilon$, then*

$$\|f_M - b\|_H \le 2\varepsilon \text{ and } \|T_0 f_M\|_H \le 2E. \tag{2.32}$$

Proof of lemma. The key fact about M is that

$$\|P_M g\| \le \varepsilon \Rightarrow \|T_0 P_M g\| \le E, \text{ and } \|T_0(1 - P_M)h\| \le E \Rightarrow \|(1 - P_M)h\| \le \varepsilon. \tag{2.33}$$

We are given that there exists f such that

$$\|f - b\| \le \varepsilon \text{ and } \|T_0 f\| \le E. \tag{2.34}$$

The first part of (2.34) implies $\|P_M f - f_M\| \le \varepsilon$, which via the first part of (2.33) yields the second part of (2.32). The second part of (2.34) implies $\|T_0(1 - P_M)f\| \le E$, which by the second part of (2.33) gives $\|(1 - P_M)f\| \le \varepsilon$. Since $\|f - b\| \le \varepsilon$, this yields the first part of (2.32).

Having the lemma, we see that $N \le M$, so $\|T_0 f_N\| \le \|T_0 f_M\|$, giving (2.31). Then (2.30)-(2.31), together with (2.23), yield

$$\|f_N - a\|_H \le 3\varepsilon \text{ and } \|T_0(f_N - a)\|_H \le 3E. \tag{2.35}$$

We have established:

Proposition 2.4. *Under the hypotheses (2.23), if we set $v_N = T_1 f_N$, where N is the smallest ℓ such that (2.30) holds, we have*

$$\|v_N - v\|_H \le 3\ M(\varepsilon, E). \tag{2.36}$$

Hence this method of approximating v is nearly best possible. Note that the value of the estimate E of (2.14) does not play an explicit role in the method described above for producing the approximation v_N; it plays a role in estimating the error $v_N - v$.

The method described above provides a technique for solving a certain class of 'ill posed' problems. Other related problems involve analytic continuation of functions and solving backwards heat equations. Further discussions of this technique, and other techniques, can be found in papers of K. Miller [Mr1], [Mr2], and references given there.

We now turn to the task of estimating $M(\varepsilon, E)$, for our specific problem, defined by (2.18)-(2.21). Thus, with $R_0 < R_1$, we want to estimate $\|h_{A-\frac{1}{2}}(kR_1)\|_{L^2}$, given that $\|f\|_{L^2} \le \varepsilon$ and $\|H_{A-\frac{1}{2}}(kR_0)f\|_{L^2} \le E$. This is basically equivalent to estimating $\|A^{-\frac{1}{2}}e^{-\beta A}A^A f\|_{L^2}$, given that $\|f\|_{L^2} \le \varepsilon$ and that $\|A^{-\frac{1}{2}}e^{-\alpha A}A^A f\|_{L^2} \le E$, where $e^{-\alpha} = 2/eR_0$ and $e^{-\beta} = 2/eR_1$, so $\alpha < \beta$. We can get a hold of this using the inequality

$$\nu^{-\frac{1}{2}}e^{-\beta\nu}\nu^{\nu} \le e^{-(\beta-\alpha)x}(\nu^{-\frac{1}{2}}e^{-\alpha\nu}\nu^{\nu}) + \sqrt{2}x^x, \tag{2.37}$$

valid for $\nu \ge \frac{1}{2}$, $0 < x < \infty$, to write

$$\|T_1 f\|_{L^2} \le C \inf_{x\in\mathbb{R}^+} \Big(e^{-\gamma x}E + x^x\varepsilon\Big), \tag{2.38}$$

where $\gamma = \beta - \alpha$, given $\|f\| \le \varepsilon$, $\|T_0 f\| \le E$. While picking x to minimize the quantity in brackets is not easy, we can obtain a reasonable estimate by picking

$$x = \frac{\log \frac{E}{\varepsilon}}{\log\log \frac{E}{\varepsilon}}, \tag{2.39}$$

in which case

$$e^{-\gamma x} = \Big(\frac{\varepsilon}{E}\Big)^{\gamma/(\log\log\ E/\varepsilon)}, \quad \varepsilon x^x = E\Big(\frac{\varepsilon}{E}\Big)^{(\log\log\log\ E/\varepsilon)/(\log\log\ E/\varepsilon)}.$$

Consequently,

$$M(\varepsilon, E) \le CE\Big[\Big(\frac{\varepsilon}{E}\Big)^{\gamma/(\log\log\ E/\varepsilon)} + \Big(\frac{\varepsilon}{E}\Big)^{(\log\log\log\ E/\varepsilon)/(\log\log\ E/\varepsilon)}\Big]. \tag{2.40}$$

As for the exponents that appear in (2.40), note the following values (to three digits):

ε/E	$1/(\log\log\ E/\varepsilon)$	$(\log\log\log\ E/\varepsilon)/(\log\log\ E/\varepsilon)$
10^{-2}	.655	.277
10^{-3}	.517	.341
10^{-4}	.450	.359
10^{-5}	.409	.366
10^{-6}	.381	.368
10^{-7}	.360	.368

The close agreement of the last two figures in the right column is due to the fact that $f(y) = (\log\log\log\ y)/(\log\log\ y)$ achieves its maximum value of $1/e$ at $y = e^{e^e} \approx 3.81\times 10^6$, and is very slowly varying in this region. As for the close agreement of the two figures corresponding to $\varepsilon = 10^{-7}$, note that $\log\log\log\ e^{e^e} = 1$. An estimate similar to (2.40) is given in [Isa].

Even though the analysis in (2.13)-(2.39) does not directly deal with the problem of describing ∂K given an approximation $b(\omega,\theta,k)$ to the scattering amplitude $a(\omega,\theta,k)$, to some degree it reduces this problem to that of describing ∂K, given the solution $u = \mathcal{B}(k)f$ to the scattering problem (1.1)-(1.3), with $u(x)$ evaluated near $|x| = R_1$, for a certain class of boundary data, namely $f(x) = e^{-ik\omega\cdot x}\big|_{\partial K}$ (where k and ω belong to a specified subset of $\mathbb{R}$ and S^2, respectively). One assumes it given that $K \subset \{x : |x| < R_0\}$, where $R_0 < R_1$. This reduction is an intermediate step in many studies of inverse problems. Thus Problem A is complemented by:

Problem B. *Approximate $v = \mathcal{B}(k)f$ on $|x| = R_0$, given (an approximation to) v on $|x| = R_1$, and having some a priori estimate of v on $|x| = R_0$, but not on a smaller sphere.*

Rescaling, we can consider the case $R_0 = 1,\ R_1 = R > 1$. By (2.4), we have

$$g = v(\theta) \text{ and } w = v(R\theta) \Longrightarrow g = \frac{h_{A-\frac{1}{2}}(k)}{h_{A-\frac{1}{2}}(kR)} w = \mathcal{C}_R(k)w, \tag{2.41}$$

where the last identity is the definition of the unbounded operator $\mathcal{C}_R(k)$ on $L^2(S^2)$. In view of (2.12), we have, for fixed $k \in (0,\infty),\ R > 1$,

$$\frac{h_{\nu-\frac{1}{2}}(k)}{h_{\nu-\frac{1}{2}}(kR)} \sim R^{\nu+\frac{1}{2}} = C\ e^{\gamma\nu}, \quad \nu \to +\infty, \tag{2.42}$$

where $\gamma = \log\ R > 0$.

Parallel to (2.14)-(2.21), we consider the problem of estimating $\mathcal{C}_R(k)w$ in $L^2(S^2)$, given a small bound on $\|w\|_{L^2(S^2)}$ (estimate on observational error) and an a priori bound on $\mathcal{C}_R(k)w$ in $H^\ell(S^2)$, for some $\ell > 0$. That is, we want to estimate

$$M(\varepsilon, E) = \sup\big\{\|\mathcal{C}_R(k)w\|_{L^2} : \|w\|_{L^2} \le \varepsilon \text{ and } \|\mathcal{C}_R(k)w\|_{H^\ell} \le E\big\}. \tag{2.43}$$

Parallel to (2.37)-(2.38), we can attack this by writing

$$e^{\gamma\nu} \le (\nu/x)^\ell e^{\gamma\nu} + e^{\gamma x}, \tag{2.44}$$

valid for $\nu, x \in (0,\infty)$. Thus, if $\|g\|_{H^\ell} = \|A^\ell g\|_{L^2}$, we have

$$\|\mathcal{C}_R(k)w\|_{L^2} \le C_k \inf_{x>0} \left(x^{-\ell}E + e^{\gamma x}\varepsilon\right). \tag{2.45}$$

We get a decent upper bound by setting $x = \frac{1}{2\gamma}\log(\ell E/\varepsilon\gamma)$. This yields

$$M(\varepsilon, E) \le C_k (2\gamma)^\ell E \left(\log\frac{\ell E}{\gamma\varepsilon}\right)^{-\ell} + C_k\left(\frac{\ell E\varepsilon}{\gamma}\right)^{\frac{1}{2}}. \tag{2.46}$$

This is bad news; ε would have to be terribly tiny for $M(\varepsilon, E)$ to be small. Fortunately, this is not the end of the story.

As a preliminary to deriving a more satisfactory estimate, we produce a variant form of the 'bad' estimate (2.46). Fix $\psi \in C_0^\infty(\mathbb{R})$, supported on $[-1,1]$, such that $\psi(0) = 1$. Instead of (2.43), we estimate

$$M_\delta(\varepsilon, E) = \sup\left\{\|\psi(\delta A)\mathcal{C}_R(k)w\|_{L^2} : \|w\|_{L^2} \le \varepsilon \text{ and } \|\mathcal{C}_R(k)w\|_{L^2} \le E\right\}. \tag{2.47}$$

We proceed via

$$\psi(\delta\nu)e^{\gamma\nu} \le e^{-\gamma x}\left[\psi(\delta\nu)e^{\gamma\nu}\right]e^{\gamma\nu} + e^{\gamma x}\psi(\delta\nu), \tag{2.48}$$

to get

$$\|\psi(\delta A)e^{\gamma A}w\|_{L^2} \le \inf_{x>0}\left[C(\gamma,\delta)e^{-\gamma x}C_k' E + e^{\gamma x}\varepsilon\right], \tag{2.49}$$

where

$$C(\gamma,\delta) = \sup_{\nu>0} \psi(\delta\nu)e^{\gamma\nu} \le e^{\gamma/\delta} = R^{1/\delta}. \tag{2.50}$$

Using (2.42) again, we have the estimate

$$M_\delta(\varepsilon, E) \le C_k\ R^{1/2\delta}\sqrt{E\varepsilon}. \tag{2.51}$$

Now we do want to be able to take δ small, to make $\psi(\delta A)f$ close to f, but $R^{1/2\delta} = e^{\gamma/2\delta}$ blows up very rapidly as $\delta \searrow 0$, so this gives no real improvement over (2.46). Compare (2.51) with the estimate

$$\|\psi(\delta A)\mathcal{C}_R(k)w\|_{L^2} \le C_k R^{1/\delta}\ \varepsilon, \tag{2.52}$$

when $\|w\|_{L^2} \le \varepsilon$, involving no use of the a priori estimate $\|\mathcal{C}_R(k)w\|_{L^2} \le E$.

We now show that a different technique yields a useful bound on $M_\delta(\varepsilon, E)$, when δ lies in the range $\delta > 1/k$.

Proposition 2.5. *Let $R>1$ and $\alpha>1$ be fixed. Then there is an estimate*

$$\|\psi(\delta A)\mathcal{C}_R(k)w\|_{L^2(S^2)} \le C\|w\|_{L^2(S^2)}, \text{ for } \alpha/k \le \delta, \tag{2.53}$$

In particular, C is independent of k.

Proof. Since $\psi(\delta A)$ and $\mathcal{C}_R(k)$ commute, it suffices to show that

$$\|\mathcal{C}_R(k)w\|_{L^2} \le C\|w\|_{L^2}, \text{ for } w \in \text{ Range } \chi(\delta A), \tag{2.54}$$

given $\alpha/k \le \delta$, where $\chi(\lambda)$ is the characteristic function of $[0,1]$. Thus

$$0 \le A \le (k/\alpha)I \text{ on Range } \chi(\delta A). \tag{2.55}$$

Equivalently, we claim that an outgoing solution $u(r,\omega)$ to the reduced wave equation $(\Delta+k^2)u=0$ satisfies

$$\|u(1,\cdot)\|_{L^2(S^2)} \le C\|u(R,\cdot)\|_{L^2(S^2)}, \text{ for } u(R,\cdot) \in \text{ Range } \chi(\delta A), \tag{2.56}$$

given $\alpha/k \le \delta$.

Now u satisfies the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \left(k^2 - r^{-2}L\right)u = 0, \quad L = A^2 - \frac{1}{4} = -\Delta_S. \tag{2.57}$$

We can replace $u(r,\omega)$ by $v(r,\omega)=ru(r,\omega)$, satisfying

$$\frac{\partial^2 v}{\partial r^2} + \left(k^2 - r^{-2}L\right)v = 0, \tag{2.58}$$

and it suffices to establish

$$\|v(1,\cdot)\|_{L^2(S^2)} \le C\|v(R,\cdot)\|_{L^2(S^2)}, \tag{2.59}$$

given $v(R,\cdot) \in$ Range $\chi(\delta A)$, and assuming that $v=ru$, u an outgoing solution to (2.57); let us denote by $\mathcal{V}_{k\delta}$ the vector space of such functions v.

It will be convenient to use a family of norms, depending on r and k, given by

$$N_{kr}(v)^2 = \Big(\big(1-(kr)^{-2}L\big)v, v\Big)_{L^2(S^2)} + k^{-2}\Big\|\frac{\partial v}{\partial r}\Big\|^2_{L^2(S^2)}, \tag{2.60}$$

where $v = v(r,\cdot) \in \mathcal{V}_{k\delta}$. Note that $\partial v/\partial r = [\mathcal{N}_r(k)+r^{-1}]v(r,\cdot)$, where $\mathcal{N}_r(k)$ is the Neumann operator (1.7), for the obstacle $\{|x|=r\}$. By (A.24), extended to treat balls of radius $r\in[1,R]$,

$$\begin{aligned}\|\mathcal{N}_r(k)f\|^2_{L^2(S^2)} &\le C\|f\|^2_{H^1(S^2)} + Ck^2\|f\|^2_{L^2(S^2)} \\ &\le Ck^2\Big[\big(k^{-2}Lf,f\big)_{L^2} + \|f\|^2_{L^2}\Big].\end{aligned} \tag{2.61}$$

Now, by (2.55),

$$0 \le (kr)^{-2} L \le \alpha^{-2} I \text{ on Range } \chi(\delta A), \tag{2.62}$$

given $\alpha/k \le \delta$ and $r \ge 1$. Consequently, if $\alpha > 1$, we have constants $C_j \in (0, \infty)$, independent of k, such that

$$C_0 \|v(r)\|^2_{L^2(S^2)} \le N_{kr}(v(r))^2 \le C_1 \|v(r)\|^2_{L^2(S^2)}, \tag{2.63}$$

for all $v \in \mathcal{V}_{k\delta}$.

We now show that, for $v \in \mathcal{V}_{k\delta}$, $N_{kr}(v(r))^2 = E(r)$ is a monotonically increasing function of $r \in [1, R]$; this will establish the estimate (2.59) and hence complete the proof of Proposition 2.5. To see this, write

$$\frac{dE}{dr} = 2 \text{ Re } \Big(\big(1 - (kr)^{-2} L\big) \frac{\partial v}{\partial r}, v \Big) + \frac{2}{k^2 r^3} (Lv, v) + 2 \text{ Re } \Big(k^{-2} \frac{\partial^2 v}{\partial r^2}, \frac{\partial v}{\partial r} \Big), \tag{2.64}$$

and use (2.58) to replace $k^{-2}\partial^2 v/\partial r^2$ by $-\big(1 - (kr)^{-2} L\big) v$. We obtain

$$\frac{dE}{dr} = \frac{2}{r} \big((kr)^{-2} Lv, v \big) \ge 0, \tag{2.65}$$

and the proof is complete.

We can place the analysis (2.60)-(2.65) in the following more general context. Suppose

$$\frac{\partial^2 v}{\partial r^2} + A(r) v = 0, \tag{2.66}$$

where each $A(r)$ is positive definite, all having the same domain. If we set

$$Q_r(v) = \big(A(r)v, v\big) + \|\partial_r v\|^2, \tag{2.67}$$

then

$$\frac{d}{dr} Q_r(v) = 2 \text{ Re}\big(A(r)v, \partial_r v\big) + 2 \text{ Re } \big(\partial_r^2 v, \partial_r v\big) + \big(A'(r)v, v\big) = \big(A'(r)v, v\big). \tag{2.68}$$

If $A'(r)$ can be bounded by $A(r)$, then we have an estimate

$$\Big| \frac{d}{dr} Q_r(v) \Big| \le C \, Q_r(v). \tag{2.69}$$

Of course, if $A'(r)$ is positive semidefinite, we have monotonicity of $Q_r(v)$, as in (2.65).

3. Linearized inverse problems

We now look at another study that sheds light on the inverse problem, namely the linearized inverse problem. Here, given an obstacle K, denote by $\mathcal{B}_K(k)$ the solution operator (1.4), and by $a_K(\omega, \theta, k)$, the scattering amplitude. We want to compute the 'derivative' with respect to K of these objects, and study their inverses.

More precisely, if K is given, ∂K smooth, we can parametrize nearby smooth obstacles by a neighborhood of 0 in $C^\infty(\partial K)$, via the correspondence that, to $\psi \in C^\infty(\partial K)$ (real valued), we associate the image ∂K_ψ of ∂K under the map

$$F_\psi(x) = x + \psi(x)N(x), \quad x \in \partial K, \tag{3.1}$$

where $N(x)$ is the unit outward pointing normal to ∂K, at x. Then, denote $\mathcal{B}_{K_\psi}(k)$ and $a_{K_\psi}(\omega, \theta, k)$ by $\mathcal{B}_\psi(k)$ and $a_\psi(\omega, \theta, k)$. We want to compute

$$D_\psi \; \mathcal{B}_K(k)f = \frac{d}{ds}\mathcal{B}_{s\psi}(k)f\big|_{s=0}, \tag{3.2}$$

and $D_\psi \; a_K(\omega, \theta, k) = \frac{d}{ds}a_{s\psi}(\omega, \theta, k)\big|_{s=0}$. The following is well-known; proofs can be found in [Kir], [KZ].

Proposition 3.1. *If f is smooth near ∂K and $v_\psi(x) = \frac{d}{ds}\mathcal{B}_{s\psi}(k)f\big|_{s=0}$, for $x \in \mathbb{R}^3 \setminus K$, then $v_\psi(x)$ is uniquely characterized by*

$$\begin{aligned} &(\Delta + k^2)v_\psi = 0 \text{ on } \mathbb{R}^3 \setminus K, \\ &r\Big(\frac{\partial v_\psi}{\partial r} - ikv_\psi\Big) \to 0 \text{ as } r \to \infty, \\ &v_\psi = \psi(x)\Big(\mathcal{N}(k)f - \frac{\partial f}{\partial \nu}\Big) \text{ on } \partial K. \end{aligned} \tag{3.3}$$

Here, $\mathcal{N}(k)$ is the Neumann operator, defined by (1.7). In other words,

$$D_\psi \; \mathcal{B}_K(k)f = \mathcal{B}_K(k)\Big\{\psi(x)\Big(\mathcal{N}(k)f - \frac{\partial f}{\partial \nu}\Big)\Big\}. \tag{3.4}$$

The linearized inverse problem is to find ψ.

Therefore, for a given smooth obstacle K, granted that the operators $\mathcal{B}_K(k)$ and $\mathcal{N}(k)$ have been constructed, e.g., by integral equation methods, or by the 'null field' method, we can to some degree reduce the linearized inverse problem for ψ to the following linear inverse problem:

Problem. *Given (an approximation to) $w = \mathcal{B}(k)g(x)$ on $|x| = R_1$ (and assuming that $K \subset \{x : |x| < R_1\}$), find (an approximation to) g on ∂K.*

Note that this is a generalization of Problem B in §2. In fact, Problem B there is the case where K is the ball B_{R_0}, $R_0 < R_1$.

As for finding $\mathcal{B}_K(k)$ and $\mathcal{N}(k)$ via an integral equation method, we mention that a representation of the form (B.10) is preferable to one of the form (B.8), since it is very inconvenient to deal with the set of values of k for which $I + N(k)$ is not invertible. This point is made in many expositions on the subject, e.g., [Co].

We note that, when we take $f = e^{-ik\omega\cdot x}$, the solution to the linearized inverse problem is unique. Compare Theorem 1 of [KR].

Proposition 3.2. *Given K smooth (nonempty), such that $\mathbb{R}^3 \setminus K$ is connected, define*

$$\mathcal{L}_K(k,\omega) : H^s(\partial K) \to C^\infty(\mathbb{R}^3 \setminus K) \tag{3.5}$$

by

$$\mathcal{L}_K(k,\omega)\psi = D_\psi \mathcal{B}_K(k) f, \quad f(x) = e^{-ik\omega\cdot x}. \tag{3.6}$$

Then $\mathcal{L}_K(k,\omega)$ is always injective.

Proof. By (11.4), our claim is that

$$\mathcal{B}_K(k)\Big\{\psi(x)\Big(\mathcal{N}(k)f - \frac{\partial f}{\partial \nu}\Big)\Big\} = 0 \text{ on } \mathbb{R}^3 \setminus K \Longrightarrow \psi = 0 \text{ on } \partial K. \tag{3.7}$$

Since $\mathcal{B}_K(k)g\big|_{\partial K} = g$, the hypothesis in (3.7) implies $\psi(x)\big(\mathcal{N}(k)f - \partial_\nu f\big) = 0$ on ∂K, so it suffices to show that

$$\mathcal{N}(k)f - \frac{\partial f}{\partial \nu} \text{ vanishes on no open subset } \mathcal{O} \text{ of } \partial K, \tag{3.8}$$

when $f(x) = e^{-ik\omega\cdot x}$. To see this, consider $w = \mathcal{B}(k)f - e^{-ik\omega\cdot x}$, which satisfies

$$(\Delta + k^2)w = 0 \text{ on } \mathbb{R}^3 \setminus K, \quad w = 0 \text{ on } \partial K. \tag{3.9}$$

If $\mathcal{N}(k)f - \partial_\nu f = 0$ on $\mathcal{O}$, then

$$\frac{\partial w}{\partial \nu} = 0 \text{ on } \mathcal{O}. \tag{3.10}$$

But if $\mathcal{O}$ is a nonempty open subset of ∂K, then (3.9)-(3.10) imply that w is identically zero, by uniqueness in the Cauchy problem for $\Delta + k^2$. This is impossible, so the proof is complete.

In light of (1.17), we can deduce from (3.4) that

$$D_\psi\ \mathcal{A}_K(k)f = \mathcal{A}_K(k)\Big\{\psi(x)\Big(\mathcal{N}(k)f - \frac{\partial f}{\partial \nu}\Big)\Big\}. \tag{3.11}$$

Then

$$D_\psi\ a_K(-\omega,\theta,k) = D_\psi\ \mathcal{A}_K(k)f, \quad f(x) = e^{-ik\omega\cdot x}. \tag{3.12}$$

Recall the relation between $\mathcal{A}_K(k)$ and $\mathcal{B}_K(k)$ given by (2.5).

4. Fine and excellent approximate identities

We look at one parameter families of operators $A(\delta), 0 < \delta \le 1$, on functions on a compact n-dimensional manifold M, which have integral kernels $A(\delta, x, y)$ satisfying

$$\delta^{-\ell} A(\delta, x, y) \longrightarrow 0 \text{ in } C^\infty(M \times M \setminus \Delta), \text{ for all } \ell > 0, \tag{4.1}$$

where Δ is the diagonal in $M \times M$, and which, in local coordinates, have the form

$$A(\delta, x, y) \sim \delta^{-n} \sum_{k \ge 0} \delta^k A_k\big(x, (x-y)/\delta\big), \tag{4.2}$$

where each $A_k(x, z)$ is C^∞ in (x, z), and rapidly decreasing as $|z| \to \infty$. An equivalent to (4.2) is the local coordinate representation

$$A(\delta)u(x) = (2\pi)^{-n/2} \int a(\delta, x, \xi) e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi, \tag{4.3}$$

where

$$a(\delta, x, \xi) \sim \sum_{k \ge 0} \delta^k a_k(x, \delta\xi), \quad a_k(x, \xi) \in S^{-\infty}_{1,0}(\mathbb{R}^n). \tag{4.4}$$

Here, the difference between the left side of (4.4) and the sum over $0 \le k < N$ of the right side is $O(\delta^\ell)$ in $S^{-N+\ell}_{1,0}(\mathbb{R}^n)$, $0 \le \ell \le N$. This class was denoted $OP\Sigma^0(M)$ in [ST]. More generally, multiplying (4.2) by $\delta^{-\mu}$ defines $OP\Sigma^\mu(M)$.

The following gives an important class of operators of this type.

Proposition 4.1. *Given $\sigma \in \mathcal{S}(\mathbb{R})$, $\sigma(-\delta^2\Delta) = A(\delta)$ belongs to $OP\Sigma^0$, and in particular has the form (4.3)-(4.4) with*

$$a_0(x, \xi) = \sigma\big(g(x, \xi)\big), \tag{4.5}$$

where $g(x, \xi)$ is the principal symbol of $-\Delta$, i.e., $g(x, \xi) = |\xi|^2$.

This result is a special case of Proposition 1.2 in [ST]. Related results can be found in [T2], [T3], [Hel], [Ro2], and [Sh].

If $\sigma(\lambda) = 1$ for λ in a neighborhood of 0, then $\sigma(-\delta^2\Delta)$ has the following property.

Definition. *A family $A(\delta) \in OP\Sigma^0$ is a fine approximate identity provided it has the form (4.3)-(4.4), and, for some $C_0 > 0$,*

$$a_0(x, \xi) = 1 \text{ for } |\xi| \le C_0. \tag{4.6}$$

The following properties are evident.

Proposition 4.2. *If $A(\delta)$ is a fine approximate identity on M, then*

$$A(\delta) \text{ is bounded in } OPS^0_{1,0}(M), \quad 0 < \delta \le 1, \tag{4.7}$$

and

$$\delta^{-1}\big(I - A(\delta)\big) \text{ is bounded in } OPS^1_{1,0}(M), \quad 0 < \delta \le 1. \tag{4.8}$$

In Proposition 2.2 of [ST] it was shown that $OP\Sigma^0$ is invariant under diffeomorphisms, and that a diffeomorphism χ conjugates $a(\delta, x, D)$, of the form (4.3)-(4.4), to $\tilde{a}(\delta, x, D)$, of a similar form, with

$$\tilde{a}_0\big(\chi(x), \xi\big) = a_0\big(x, D\chi(x)^t \xi\big). \tag{4.9}$$

It follows that conjugation by a diffeomorphism of M preserves the class of fine approximate identities. This is a minor observation. A more incisive use of this conjugation invariance arises in the proof of the following result.

Proposition 4.3. *If $A(\delta) \in OP\Sigma^0(M)$ and if $f \in C^\infty(M)$ is real valued, and if we define $B(\delta)$ for $\delta \in (0,1]$ by*

$$B(\delta)u = e^{-if/\delta}\, A(\delta)\big(e^{if/\delta}u\big), \tag{4.10}$$

then $B(\delta) \in OP\Sigma^0(M)$, and, in the form (3)-(4), with b replacing a, we have

$$b_0(x, \xi) = a_0(x, \xi + df). \tag{4.11}$$

Proof. Via a partition of unity, we can consider $A(\delta)$ of the form (4.3)-(4.4), where $a(\delta, x, \xi)$ is supported on $x \in K$, a compact set in $\mathbb{R}^n$. If f has no critical point on K, one can (using the conjugation invariance (4.9) and perhaps a further partition of unity) reduce the result to the case $f(x) = x_1$. In that case, $B(\delta)$ has the form (4.3) with $a(\delta, x, \xi)$ replaced by $b(\delta, x, \xi) = a(\delta, x, \xi + e_1)$, $e_1 = dx_1$. Thus (4.11) holds in that case. Now if f has a critical point at $x_0 \in M$, we can pick a neighborhood U of x_0 and write $f = f_1 + f_2$ on U, where f_1 and f_2 have <u>no</u> critical points in U. Using

$$e^{-if/\delta} A(\delta) e^{if/\delta} = e^{-if_2/\delta}\Big(e^{-if_1/\delta} A(\delta) e^{if_1/\delta}\Big) e^{if_2/\delta}, \tag{4.12}$$

we have the result (4.10)-(4.11) in general.

Applying this to $\sigma(-\delta^2\Delta)$, we see that

$$e^{-if/\delta}\sigma(-\delta^2\Delta)(e^{if/\delta}u) = A_f(\delta)u, \tag{4.13}$$

where $A_f(\delta)$ has the form (4.3)-(4.4) with

$$a_0(x, \xi) = \sigma\big(g(x, \xi + df)\big). \tag{4.14}$$

If $\sigma(\lambda) = 1$ for $|\lambda| \le C_0^2$, then $a_0(x, \xi) = 1$ for ξ near 0, provided $|df(x)| < C_0$, so $|\xi + df|^2 \le C_0^2$ for $|\xi| \le C_1$, if C_1 is small enough, in fact, provided $C_1 \le C_0 - |df(x)|$. This is a key case of the following result, which is an immediate consequence of Proposition 4.3.

Proposition 4.4. *Suppose $A(\delta)$ is a fine approximate identity on M, such that (4.6) holds. If $f \in C^\infty(M)$ is real valued and $|df(x)| \le B_0 < C_0$ for all $x \in M$, then the conjugated operator $B(\delta)$, given by (4.10), is also a fine approximate identity on M. More generally, if*

$$|df(x)| \le B_0 < C_0 \text{ for } x \in U, \tag{4.15}$$

where $U \subset M$ is open, then $B(\delta)$ is a fine approximate identity on U.

In particular, if (4.15) holds, then, given $\varphi \in C_0^\infty(U)$,

$$\delta^{-1}\Big(I - e^{-if/\delta}\sigma(-\delta^2\Delta)e^{if/\delta}\Big)M_\varphi \text{ is bounded in } OPS^1_{1,0}(M), \tag{4.16}$$

for $\delta \in (0,1]$. Also

$$\delta^{-1}\Big(M_\varphi - M_\varphi e^{-if/\delta}\sigma(-\delta^2\Delta)e^{if/\delta}\Big) \text{ is bounded in } OPS^1_{1,0}(M). \tag{4.17}$$

In addition,

$$\delta^{-1}\psi(-\delta^2\Delta)\Big(M_\varphi - M_\varphi e^{-if/\delta}\sigma(-\delta^2\Delta)e^{if/\delta}\Big) \text{ is bounded in } OPS^0_{1,0}(M), \tag{4.18}$$

provided $\psi \in C_0^\infty(\mathbb{R})$ is supported on $[-a,a]$, with $a \le C_0 - B_0$.

For more refined results, it is convenient to associate to $A(\delta) \in OP\Sigma^0(M)$ an operator $A^\# \in OPS^0(M \times S^1)$, defined by

$$A^\#\big(u(x)e^{ik\theta}\big) = A(k^{-1})u(x)\ e^{ik\theta}, \quad k \in \mathbb{Z}\setminus 0. \tag{4.19}$$

We define the left side to be 0 if $k = 0$. Then $A^\#(x,\theta,D_{x,\theta})$ has symbol

$$A^\#(x,\theta,\xi,\lambda) = a(\lambda^{-1},x,\xi) \sim \sum_{j\ge 0} \lambda^{-j} a_j(x,\xi/\lambda). \tag{4.20}$$

The hypothesis $a_j(x,\xi) \in S^{-\infty}$ implies that the apparent singularity in the symbol at $\lambda = 0$ is removable. $A^\#(x,\theta,\xi,\lambda)$ is C^∞ in all its arguments, away from $(\xi,\lambda) = (0,0)$, and it is a symbol in $S^0(M \times S^1)$. Furthermore, the complete symbol of $A(x,\theta,D_{x,\theta})$ vanishes to infinite order at $\lambda = 0$. Let us denote by $S^m_\varkappa(M \times S^1)$ the set of symbols in $S^m(M \times S^1)$ which vanish to infinite order on $\{\lambda = 0\}$. Denote by $OPS^m_{\varkappa S}(M \times S^1)$ the set of operators with symbols in $S^m_\varkappa(M \times S^1)$ which commute with the S^1 action. As noted in [ST], one easily establishes the fact that $A(\delta) \mapsto A^\#$ gives

$$\mathcal{J} : OP\Sigma^0(M) \longrightarrow OPS^0_{\varkappa S}(M \times S^1), \tag{4.21}$$

which is an <u>isomorphism</u>, modulo $OP\Sigma^{-\infty}$ and $OPS^{-\infty}_{\varkappa S}(M \times S^1)$. Also [ST] notes the following convenient characterization of $OPS^0_\varkappa(M \times S^1)$: Given $P \in OPS^0(M \times S^1)$, P belongs to $OPS^0_\varkappa(M \times S^1)$ if and only if, for each $j \ge 1$, there exists $P_j \in OPS^{-j}(M \times S^1)$ such that

$$P = D^j_\theta P_j \bmod OPS^{-\infty}(M \times S^1). \tag{4.22}$$

Now the correspondence (4.20) of symbols makes the following apparent.

Proposition 4.5. *A family $A(\delta) \in OP\Sigma^0(M)$ is a fine approximate identity if and only if $A^\# = \mathcal{J}A(\delta)$ has principal symbol equal to 1 in some conic neighborhood of $\{\xi = 0\}$ in $T^*(M \times S^1) \setminus 0$.*

We now define an excellent approximate identity as a family $A(\delta) \in OP\Sigma^0(M)$ such that $A^\# = \mathcal{J}A(\delta)$ has total symbol equal to 1 on a conic neighborhood of $\{\xi = 0\}$. Refining the observation that $\sigma(-\delta^2\Delta)$ is a fine approximate identity, we have the following.

Proposition 4.6. *If $\sigma \in \mathcal{S}(\mathbb{R})$ and $\sigma(\lambda) = 1$ for λ near 0, then $\sigma(-\delta^2\Delta)$ is an excellent approximate identity.*

Proof. Given the discussion above, it remains to show that $I - \sigma(-D_\theta^{-2}\Delta)$ has order $-\infty$ on a conic neighborhood of $\{\xi = 0\}$ in $T^*(M \times S^1)\setminus 0$. Note that, via (4.21), the correspondence (4.20) shows that the principal symbol of $\sigma(-D_\theta^{-2}\Delta)$ is 1 on a conic neighborhood of $\{\xi = 0\}$. To go further, we make the following observation. Pick $\tilde{\sigma} \in C_0^\infty(\mathbb{R})$, supported on an interval on which $\sigma(\lambda) = 1$, and such that $\tilde{\sigma}(\lambda) = 1$ on a smaller interval about $\lambda = 0$. Then, by the same reasoning, $\tilde{\sigma}(-D_\theta^{-2}\Delta) \in OPS^0(M \times S^1)$ has principal symbol equal to 1 on a conic neighborhood of $\{\xi = 0\}$, so it is elliptic there. But

$$\left[I - \sigma(-D_\theta^{-2}\Delta)\right]\tilde{\sigma}(-D_\theta^{-2}\Delta) = 0, \tag{4.23}$$

so our conclusion follows from microlocal elliptic regularity.

As a step toward refining Propositions 4.3-4.4, we consider the image under $\mathcal{J}$ of $B(\delta)$, defined by (4.10). Indeed, if $\mathcal{J}A(\delta) = A^\#$, then

$$\mathcal{J}B(\delta) = B^\# = e^{-ifD_\theta}\, A^\#\, e^{ifD_\theta}. \tag{4.24}$$

Now $if(x)D_\theta = f(x)\partial/\partial\theta$ is a real vector field, generating a flow on $M \times S^1$. The fact that the right side of (4.24) belongs to $OPS^0(M \times S^1)$ is hence a consequence of the coordinate invariance of pseudodifferential operators. Clearly all three factors on the right side of (4.24) commute with D_θ. Also, if $A^\#$ possesses the property (4.22), so does $B^\#$. Hence

$$A^\# \in OPS^0_{\varkappa S}(M \times S^1) \Longrightarrow B^\# \in OPS^0_{\varkappa S}(M \times S^1),$$

so, via the isomorphism (4.21), we re-establish Proposition 4.3.

We can say more. If $A_1 = I - A^\#$ has order $-\infty$ on a cone $\Gamma \subset T^*(M \times S^1) \setminus 0$, then the time-one evaluation of the flow generated by $f(x)\partial/\partial\theta$ transforms Γ to a cone $\tilde{\Gamma} \subset T^*(M \times S^1) \setminus 0$, and the conjugated operator $B_1 = e^{-ifD_\theta}A_1e^{ifD_\theta}$ will have order $-\infty$ on $\tilde{\Gamma}$. Under the hypothesis (4.15), $\tilde{\Gamma}$ is still a conic neighborhood of $\{\xi = 0\}$, over $U \times S^1$, so we have the following.

Proposition 4.7. *If $A(\delta)$ is an excellent approximate identity on M, such that (4.6) holds, and if $f \in C^\infty(M)$ is real valued and satisfies (4.15), then $B(\delta) = e^{-if/\delta}A(\delta)e^{if/\delta}$ is an excellent approximate identity on U.*

Finally, we refine Proposition 4.2.

Proposition 4.8. *If $A(\delta)$ is an excellent approximate identity on M, then, for $m > 0$,*

$$\delta^{-m}\big[I - A(\delta)\big] \text{ is bounded in } OPS^m_{1,0}(M), \quad 0 < \delta \le 1. \tag{4.25}$$

Proof. It suffices to show this for $m = 2j, j \in \mathbb{Z}^+$. In such a case, for $k \in \mathbb{Z} \setminus 0$,

$$\begin{aligned} k^{2j}\big[I - A(k^{-1})\big]u(x) &= k^{2j}e^{-ik\theta}(I - A^{\#})\big(u(x)e^{ik\theta}\big) \\ &= e^{-ik\theta}D_\theta^{2j}(I - A^{\#})\big(u(x)e^{ik\theta}\big). \end{aligned} \tag{4.26}$$

Since $I - A^{\#}$ has order $-\infty$ on a conic neighborhood of $\{\xi = 0\}$, we can write

$$I - A^{\#} = P_{2j}\Delta^j + R_{2j}, \quad P_{2j} \in OPS_S^{-2j}(M \times S^1), \quad R_{2j} \in OPS_S^{-\infty}(M \times S^1), \tag{4.27}$$

where as before the subscript S indicates commutativity with D_θ. This yields

$$\delta^{-2j}\big[I - A(\delta)\big] = B_{2j}(\delta)\Delta^j, \tag{4.28}$$

modulo a negligeable error, where $B_{2j}(\delta)$ satisfies

$$B_{2j}(k^{-1})u(x) = e^{-ik\theta}Q_{2j}\big(u(x)e^{ik\theta}\big), \quad Q_{2j} = D_\theta^{2j}P_{2j} \in OPS^0_S(M \times S^1). \tag{4.29}$$

As shown in §3 of [ST], this defines a bounded family in $OPS^0_{1,0}(M)$, of the form (4.4), but with $a_k(x,\xi) \in S^{-k}$ rather than in $S^{-\infty}$. The proposition is proved.

If $A(\delta) \in OP\Sigma^0(M)$ is an excellent approximate identity on $U \subset M$, we have

$$\delta^{-m}M_\varphi\big[I - A(\delta)\big] \text{ and } \delta^{-m}\big[I - A(\delta)\big]M_\varphi \text{ bounded in } OPS^m_{1,0}(M), \tag{4.30}$$

when $\varphi \in C_0^\infty(U)$.

We note the fact that $\delta^{-m}\big[I - \sigma(-\delta^2\Delta)\big]$ is bounded in $OPS^m_{1,0}(M)$ is relatively straightforward, for $\sigma(\lambda)$ described above. However, the fact that $e^{-if/\delta}\sigma(-\delta^2\Delta)e^{if/\delta}$ has the property (4.30), under the hypothesis (4.15), is less apparent.

It will be convenient for us also to associate to $A(\delta) \in OP\Sigma^0(M)$ an operator on functions on the non-compact space $M \times \mathbb{R}$. Let $B(k) = A(\delta)$ be defined for all $k = \delta^{-1} \in \mathbb{R} \setminus 0$, and assume that $B(0)$ can be defined so that $B \in C^\infty\big(\mathbb{R}, \mathcal{L}(L^2(M))\big)$. In that case we say $A(\delta) \in OP\widetilde{\Sigma}^0(M)$. For example, if $M = pt.$, then $A(\delta) = \langle\delta\rangle^{-1}$ (with $B(k) = k/\langle k\rangle$) has this property, but $A(\delta) = \delta/\langle\delta\rangle$ (with $B(k) = \langle k\rangle^{-1}\,\mathrm{sgn}(k)$) does not. Given $A(\delta) \in OP\widetilde{\Sigma}^0(M)$, we set

$$A^{\#\#}\Big(\int u(k,x)e^{ikt}\,dk\Big) = \int B(k)u(k,x)\,e^{ikt}\,dk, \quad B(k) = A(k^{-1}). \tag{4.31}$$

Then $A^{\#\#}(x,t,D_{x,t})$ has a symbol expansion of the same form as (4.20):

$$A^{\#\#}(x,t,\xi,\tau) = a(\tau^{-1},x,\xi) \sim \sum_{j\ge 0}\tau^{-j}a_j(x,\xi/\tau). \tag{4.32}$$

Clearly $A^{\#\#}$ belongs to $OPS^0(M \times \mathbb{R})$, and this operator commutes with t-translation. Thus we can write

$$A^{\#\#}u(t,x) = \int_{-\infty}^{\infty} \int_M F_{A(\delta)}(t-s,x,y)u(s,y)\ dV(y)\ ds, \tag{4.33}$$

where, near $t=s, x=y$, $F_{A(\delta)}(t-s,x,y)$ has the form of a singular integral operator of order zero. Furthermore, $F_{A(\delta)}(t,x,y)$ is smooth outside $\{(t,x,y) : t=0, x=y\}$, and

$$F_{A(\delta)}(t,x,y) = o(|t|^{-N}) \ \text{ as } \ |t| \to \infty, \tag{4.34}$$

for all $N \in \mathbb{Z}^+$, together with all derivatives. An operator $P \in OPS^m(M \times \mathbb{R})$ which commutes with D_t and has the last property will be said to belong to $OPS^m_R(M \times \mathbb{R})$. We say $P \in OPS^m_{\varkappa R}(M \times \mathbb{R})$ if also the complete symbol of P vanishes to infinite order at $\tau = 0$. Parallel to (4.21), we have $A(\delta) \mapsto A^{\#\#}$ defining

$$\widetilde{\mathcal{J}} : OP\widetilde{\Sigma}^0(M) \longrightarrow OPS^0_{\varkappa R}(M \times \mathbb{R}), \tag{4.35}$$

an isomorphism, modulo $OP\widetilde{\Sigma}^{-\infty}(M)$ and $OPS^{-\infty}_{\varkappa R}(M \times \mathbb{R})$. A characterization of the operator class $OPS^0_{\varkappa R}(M \times \mathbb{R})$ parallel to (4.22) also works.

The Poisson summation method defines a map

$$\sigma : OPS^0_{\varkappa R}(M \times \mathbb{R}) \longrightarrow OPS^0_{\varkappa S}(M \times S^1), \quad F_A(t,x,y) \mapsto \sum_{j=-\infty}^{\infty} F_A(t+2\pi j, x, y), \tag{4.36}$$

and we have $\mathcal{J} = \sigma \circ \widetilde{\mathcal{J}}$ on $OP\widetilde{\Sigma}^0(M)$. The following is then a simple reformulation of the definition.

Proposition 4.9. *A family $A(\delta) \in OP\widetilde{\Sigma}^0(M)$ is an excellent approximate identity on $U \subset M$ if and only if $A^{\#\#} = \widetilde{\mathcal{J}} A(\delta)$ has total symbol equal to* 1 *on some conic neighborhood of $\{\tau = 0\}$ in $T^*(U \times \mathbb{R}) \setminus 0$.*

5. Estimates on $\mathcal{B}(k)$

The integral equation methods described in §B give no effective way to estimate the norm of $\mathcal{B}(k)$. In this section we show that decent estimates on $\mathcal{B}(k)$ hold in the special case that there is local exponential decay for the wave equation on $\mathbb{R} \times \Omega$. As is well known [Mel], [MRS], this holds whenever there are no trapped rays.

The relation between the solution to (1.1)-(1.2) and the wave equation is the following. Given $f \in H^s(\partial K)$, let $u(t,x)$ solve

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \text{ for } (t,x) \in \mathbb{R} \times \Omega, \tag{5.1}$$

and

$$u\big|_{\mathbb{R}\times\partial K} = f(x)\delta(t), \quad u = 0 \text{ for } t < 0. \tag{5.2}$$

Then

$$\mathcal{B}(k)f(x) = \hat{u}(k,x), \tag{5.3}$$

where $\hat{u}(k,x)$ is the Fourier transform of $u(t,x)$ with respect to t.

It is convenient to smooth out the t-dependence. Pick $\psi \in C_0^\infty(\mathbb{R})$ such that supp $\psi \subset [0,1]$ and $\hat{\psi}(0) = 1$. Thus $\hat{\psi} \in \mathcal{S}(\mathbb{R})$. Let $\psi_\ell(t) = e^{i\ell t}\psi(t)$, so $\hat{\psi}_\ell(k) = \hat{\psi}(k-\ell)$. Now define $w_\ell(t,x)$ by

$$\partial_t^2 w_\ell - \Delta w_\ell = 0 \text{ on } \mathbb{R}\times\Omega, \quad w_\ell\big|_{\mathbb{R}\times\partial K} = \psi_\ell(t)f(x), \quad w_\ell = 0 \text{ for } t<0. \tag{5.4}$$

We have

$$\hat{w}_\ell(k,x) = \hat{\psi}(k-\ell)\mathcal{B}(k)f(x). \tag{5.5}$$

Since the Dirichlet problem for the wave equation is strongly well-posed, there is an estimate

$$\|w_\ell\|_{L^2([0,T]\times\Omega)} \le C_T\|f\|_{L^2(\partial K)}, \tag{5.6}$$

for any $T<\infty$. Furthermore, for $t>1$, we have

$$\|w_\ell(t,\cdot)\|_{L^2(\Omega)} \le C_t\|f\|_{L^2(\partial K)}. \tag{5.7}$$

Under the hypothesis of local exponential decay, if $\mathcal{O}$ is a fixed bounded open subset of Ω, say $\mathcal{O} = \{x\in\Omega : |x|<R\}$, we have

$$\|w_\ell(t,\cdot)\|_{L^2(\mathcal{O})} \le Ce^{-Lt}\|f\|_{L^2(\partial K)}, \tag{5.8}$$

for some $L>0$. Estimating the Fourier transform of $w_t(t,x)$ with respect to t, we obtain

$$\|\hat{w}_\ell(k,\cdot)\|_{L^2(\mathcal{O})} \le C_1\|f\|_{L^2(\partial K)}. \tag{5.9}$$

Here C_1 is independent of ℓ. Taking $\ell = k$, we then get from (5.5) our first estimate:

$$\|\mathcal{B}(k)f\|_{L^2(\mathcal{O})} \le C_1\|f\|_{L^2(\partial K)}. \tag{5.10}$$

In case K is the unit ball, this follows from (A.13), together with the known monotonic decrease of $|r^{\frac{1}{2}}H_\nu^{(1)}(r)|$, as a function of $r>0$, for fixed $\nu>0$. In fact, if we let

$$\mathcal{B}_{1\rho}(k) : L^2(S^2) \longrightarrow L^2(S^2_\rho) \tag{5.11}$$

be defined by $\mathcal{B}_{1\rho}f(\rho\omega) = \mathcal{B}_K(k)f(\rho\omega)$, when K is the unit ball B_1, we see that

$$\|\mathcal{B}_{1\rho}(k)\|_{\mathcal{L}(L^2)} \le 1, \tag{5.12}$$

keeping in mind that S^2_ρ has area $4\pi\rho^2$. From this it follows easily that, if K is contained in B_1 (which can be arranged by scaling) and if S^2_r is the sphere $|x| = r$, then, as long as K satisfies the local decay hypothesis, we have (say for $r \ge 2$)

$$\mathcal{B}_{Kr}(k) : L^2(\partial K) \longrightarrow L^2(S^2_r), \tag{5.13}$$

given by $\mathcal{B}_{Kr}f = \mathcal{B}_K f\big|_{S^2_r}$, and

$$\|\mathcal{B}_{Kr}(k)f\|_{L^2(S^2_r)} \le C_2\|f\|_{L^2(\partial K)}. \tag{5.14}$$

We emphasize that the right sides of (5.10) and (5.14) are independent of $k \in \mathbb{R}$.

We can estimate the $H^2(\mathcal{O})$ norm of $\mathcal{B}_K(k)f$ by applying regularity estimates to $\Delta v = -k^2 v$, $v\big|_{\partial K} = f$ and using (5.10). We obtain

$$\|\mathcal{B}_K(k)f\|_{H^2(\mathcal{O})} \le C\|f\|_{H^{\frac{3}{2}}(\partial K)} + Ck^2\|f\|_{L^2(\partial K)}. \tag{5.15}$$

6. Resolution of details of an obstacle

Assume you know that $B_1 \subset K \subset B_r$, and that you have a measurement of $v(x, k\omega)$ on $|x| = r$. You want to find (an approximation to) K. As mentioned at the end of §1, one strategy is to minimize

$$\Phi(f, K) = \|\mathcal{B}_{B_1}(k)f - v(\cdot, k\omega)\|^2_{L^2(S^2_r)} + \|\mathcal{B}_{B_1}(k)f + e^{-ik\omega\cdot x}\|^2_{L^2(\partial K)}, \tag{6.1}$$

with f and K varying over certain compact sets, determined by a priori hypotheses on the scatterer.

More generally, one might have measurements of $v(x, k\omega_j)$ on $|x| = r$, for a sequence of incident directions ω_j. Then one might take

$$\Phi = \sum_{j=1}^N \big\|\mathcal{B}_{B_1}(k)f_j - v(\cdot, k\omega_j)\big\|^2_{L^2(S^2_r)} + \sum_{j=1}^N \big\|\mathcal{B}_{B_1}(k)f_j + e^{-ik\omega_j\cdot x}\big\|^2_{L^2(\partial K)}, \tag{6.2}$$

and minimize over $(f_1, \dots f_N; K)$. One might also consider weighted sums. Note that the possibility of making any one of the terms in (6.2) arbitrarily small (if K is given) follows from Lemma 1.1, if K is connected.

In light of Proposition 2.5, we are motivated to take

$$f_j \in \ \text{Range}\ \chi\Big(\frac{\alpha}{k}A\Big) = V_{k/\alpha}, \tag{6.3}$$

for some fixed $\alpha > 1$, i.e., f_j in the sum of the eigenspaces of A with eigenvalue $\le k/\alpha$. In this case, a bound on $\|\mathcal{B}_{B_1}(k)f_j - v(\cdot, k\omega_j)\|_{L^2(S_r^2)}$ implies an L^2 bound on f_j, within the finite dimensional space $V_{k/\alpha}$. Thus this puts f_j in a compact set.

Instead of minimizing (6.2) over $(f_1, \dots, f_N; K)$, an alternative is first to minimize the first term of (6.2), thus choosing f_j, and then to pick K to minimize the second term, within some compact set of obstacles. In this case, f_j is simply given by

$$f_j = \mathcal{C}_r(k)\chi\Big(\frac{\alpha}{k}A\Big)v_j, \quad v_j(\theta) = v(r\theta, k\omega_j). \tag{6.4}$$

In light of the results of §4, we propose instead to take

$$f_j = \mathcal{C}_r(k)\psi\Big(\frac{\alpha}{k}A\Big)v_j, \tag{6.5}$$

where $\psi \in C_0^\infty(\mathbb{R})$ is supported on $[-1,1]$ and equal to 1 on $[-\beta,\beta]$, for some $\beta \in (0,1)$.

It is consequently of interest to investigate the following question. Give an obstacle K, such that $B_1 \subset K \subset B_r$, and given other hypotheses on K, how small can we say

$$e^{-ik\omega_j\cdot x} + \mathcal{B}_{B_1}(k)\mathcal{C}_r(k)\psi\Big(\frac{\alpha}{k}A\Big)v_j\Big|_{\partial K} \tag{6.6}$$

is, given that $v_j(\theta)$ is (an approximation to) $v(r\theta, k\omega_j)$? Recall from §1 the notation $\mathcal{C}_{rK}(k)$ for $\rho\big|_{\partial K}\mathcal{B}_{B_1}(k)\mathcal{C}_r(k)$, so (6.6) is the same as

$$e^{-ik\omega_j\cdot x} + \mathcal{C}_{rK}(k)\psi\Big(\frac{\alpha}{k}A\Big)v_j. \tag{6.7}$$

There is a similar set-up for the linearized problem. Suppose K is given, and $B_1 \subset K \subset B_r$. Suppose you have a measurement of

$$\Gamma_j(x) = D_\varphi \mathcal{B}_K(k)w_j(x) = \mathcal{B}_K(k)\Big\{\varphi(x)\Big(\mathcal{N}(k)w_j - \frac{\partial w_j}{\partial \nu}\Big)\Big\}, \quad x \in S_r^2, \tag{6.8}$$

where $w_j(x) = e^{-ik\omega_j\cdot x}$, $1 \le j \le N$. You want to find an approximation to $\varphi(x)$ on ∂K. A strategy parallel to (6.2) is to minimize

$$\Psi = \sum_{j=1}^N \Big\|\mathcal{B}_{B_1}(k)f_j - \Gamma_j\Big\|^2_{L^2(S_r^2)} + \sum_{j=1}^N \Big\|\mathcal{B}_{B_1}(k)f_j - \varphi\Big(\mathcal{N}(k)w_j - \frac{\partial w_j}{\partial \nu}\Big)\Big\|^2_{L^2(\partial K)}, \tag{6.9}$$

with $\{f_1, \dots, f_N\} \subset L^2(S^2)$ and $\varphi \in L^2(\partial K)$ varying over compact sets, determined by a priori information on the scatterer K and its variation $\delta\varphi$. As before, an alternative is first to minimize the first term in (6.9), thus choosing f_j, and then pick φ to minimize the second term, within some compact set of functions on ∂K. As a further modification, parallel to (6.5), we can set

$$f_j = \mathcal{C}_r(k)\psi\Big(\frac{\alpha}{k}A_r\Big)\Gamma_j, \quad A_r = \Big(-r^2\Delta_{S_r^2} + \frac{1}{4}\Big)^{\frac{1}{2}}. \tag{6.10}$$

where $\psi \in C_0^\infty(\mathbb{R})$ is supported on $[-1,1]$ and equal to 1 on $[-\beta,\beta]$, for some $\beta \in (0,1)$. Note that, if $f \in L^2(S_r^2)$ and $g \in L^2(S^2)$ are related by $f(r\theta) = g(\theta)$, then $A_r f(r\theta) = Ag(\theta)$.

Consequently we are interested in the following two questions. Given an obstacle K such that $B_1 \subset K \subset B_r$, and given other hypotheses on K and on φ, how small can we say

$$(6.11) \qquad \mathcal{C}_{rK}(k)\psi\Big(\frac{\alpha}{k}A_r\Big)\Gamma_j - \varphi\Big(\mathcal{N}(k)w_j - \frac{\partial w_j}{\partial\nu}\Big)$$

is on ∂K, given that Γ_j is (an approximation to) $D_\varphi \mathcal{B}_K(k)w_j\big|_{S_r^2}$? Furthermore, how small can we say

$$(6.12) \qquad \Big(\mathcal{N}(k)w_j - \frac{\partial w_j}{\partial\nu}\Big)^{-1}\mathcal{C}_{rK}(k)\psi\Big(\frac{\alpha}{k}A_r\Big)\Gamma_j - \varphi$$

is on ∂K?

Note that estimates on (6.11) and on (6.6), given K, are both related to the question of producing estimates on

$$(6.13) \qquad g - \mathcal{C}_{rK}(k)\psi\Big(\frac{\alpha}{k}A_r\Big)\mathcal{B}_{Kr}(k)g,$$

given g on ∂K. In fact, suppose $\mathcal{B}_{Kr}(k)g = h$ is measured as $h_1 = h + h_0$, where h_0 is the error. Then

$$(6.14) \qquad g - \mathcal{C}_{rK}(k)\psi\Big(\frac{\alpha}{k}A_r\Big)h_1 = (6.13) - \mathcal{C}_{rK}(k)\psi\Big(\frac{\alpha}{k}A_r\Big)h_0,$$

and the uniform bound on $\mathcal{C}_{rK}(k)\psi(\frac{\alpha}{k}A_r)$ given by Proposition 2.5 controls the last term.

It is easiest to tackle the question of estimating (6.13) in the case that K is the unit ball B_1, since

$$(6.15) \qquad K = B_1 \Longrightarrow \mathcal{C}_{rK}(k)\psi\Big(\frac{\alpha}{k}A_r\Big)\mathcal{B}_{Kr}(k)g = \psi\Big(\frac{\alpha}{k}A\Big)g.$$

In this case, we are estimating $g - \psi(\frac{\alpha}{k}A)g$, and we have a number of strong estimates on this, discussed in §4, following from the fact that

$$(6.16) \qquad I - \psi\Big(\frac{\alpha}{k}A\Big) \text{ is } O(k^{-m}) \text{ in } OPS_{1,0}^m(S^2), \quad m \geq 0.$$

Let us now look at what estimates we can get on (6.12), when $K = B_1$. In (6.12), $\partial_\nu w_j\big|_{S^2} = -ik(\omega_j \cdot x)w_j$, since $x = \nu(x)$ is the normal to S^2 at x. Now, we can analyze $\mathcal{N}(k)w_j$ for large k, away from the shadow boundary

$$\Sigma_j = \{x \in S^2 : \nu(x)\cdot\omega_j = 0\}$$

via Kirchhoff's approximation:

$$\begin{aligned}(6.17)\qquad \mathcal{N}(k)w_j &= +ik\big(\omega_j\cdot\nu(x)\big)w_j + O(1) \qquad \text{in Illum}(\omega_j)\\ &= -ik\big(\omega_j\cdot\nu(x)\big)w_j + O(k^{-\infty}) \text{ in Shad}(\omega_j),\end{aligned}$$

where, for a general convex K,

$$\text{Illum}(\omega) = \{x \in \partial K : \nu(x)\cdot\omega < 0\}, \quad \text{Shad}(\omega) = \{x \in \partial K : \nu(x)\cdot\omega > 0\}$$

are the 'illuminated' and 'shadow' regions of ∂K, for a plane wave travelling in the direction ω. Rigorous justification of (6.17) requires knowledge of the propagation of singularities of solutions to the wave equation. A proof of (6.17), including an estimate of the error near Σ_j, is given in Chapter 10 of [T3]; a detailed analysis of corrections near the shadow boundary is given in [MeT1].

In particular, in the shadow region, $\big(\mathcal{N}(k)w_j - \partial_\nu w_j\big)^{-1}$ is too large, so we have by (6.12) a poor approximation to φ there, if k is large.

On the other hand, within the illuminated region, Proposition 4.4 applies, and we see that we get a good approximation to φ there. More precisely, Proposition 4.4 yields a good estimate on (6.12), for x in the set

$$(6.18)\qquad \mathfrak{I}_{\alpha\beta}(\omega_j) = \big\{x \in \text{ Illum}(\omega_j) : \sqrt{1-(\nu(x)\cdot\omega_j)^2} < (\alpha\beta)^{-1}\big\},$$

provided $\psi \in C_0^\infty(-1,1)$, $\psi(\lambda) = 1$ for $|\lambda| < \beta$, when $\partial K = S^2$.

We now analyze (6.13) for more general K than balls. For now we assume K is strongly convex. We have the following extension of (6.15).

Proposition 6.1. *If $B_1 \subset K \subset B_r$ and K is strongly convex, then, with ψ, α, and A_r as above, we have*

$$(6.19)\qquad \Xi(\delta) = \mathcal{C}_{rK}(k)\psi\Big(\frac{\alpha}{k}A_r\Big)\mathcal{B}_{Kr}(k) \in OP\Sigma^0(\partial K), \quad \delta = 1/k.$$

Furthermore, $\Xi(\delta)$ is an excellent approximate identity.

Proof. We will examine $\widetilde{\mathcal{J}}\Xi(\delta)$, acting on functions on $\partial K \times \mathbb{R}$. We have

$$(6.20)\qquad \widetilde{\mathcal{J}}\Xi(\delta) = \mathbf{C}_{rK}\ \psi\big(\alpha D_t^{-1}A_r\big)\ \mathbf{B}_{Kr},$$

where $\mathbf{B}_{Kr}$, which maps certain functions on $\partial K \times \mathbb{R}$ to functions on $S_r^2 \times \mathbb{R}$, is defined as follows. If $f \in \mathcal{E}'(\partial K \times \mathbb{R})$, $\mathbf{B}_{Kr}f$ is the restriction to $S_r^2 \times \mathbb{R}$ of the unique solution $u = \mathbf{B}_K f$ of

$$(6.21)\qquad \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \text{ on } (\mathbb{R}^3 \setminus K)\times\mathbb{R}, \quad u\big|_{\partial K\times\mathbb{R}} = f, \quad u = 0 \text{ for } t << 0.$$

$\mathbf{B}_K$ and $\mathbf{B}_{Kr}$ extend to more general spaces of functions and distributions, as will be discussed below. Since $\psi \in C_0^\infty(\mathbb{R})$ and $\psi(\lambda) = 1$ for λ near 0, we can set $\psi(\lambda^{-1}) = 1 - \psi_1(\lambda)$, $\psi_1 \in C_0^\infty(\mathbb{R})$, and we have

$$\psi(\alpha D_t^{-1} A_r) = I - \psi_1(\alpha^{-1} A_r^{-1} D_t) \in OPS^0_{\varkappa R}(S_r^2 \times \mathbb{R}). \tag{6.22}$$

The operator $\mathbf{C}_{rK}$ acts on a space of functions on $S_r^2 \times \mathbb{R}$; $\mathbf{C}_{rK} f$ is the restriction to $\partial K \times \mathbb{R}$ of the solution to the wave equation on $(\mathbb{R}^3 \setminus B_1) \times \mathbb{R}$ continuing $\mathbf{B}_{B_r} f$. The domain of $\mathbf{C}_{rK}$ contains the range of $\psi(\alpha D_t^{-1} A_r)\mathbf{B}_{Kr}$.

To say more about the domains and ranges of these three operators, note that, by propagation of singularities and local exponential decay, given $f \in \mathcal{E}'(\partial K \times \mathbb{R})$, $\mathbf{B}_{Kr} f \in \mathcal{D}'(S_r^2 \times \mathbb{R})$ vanishes for $t << 0$, has compact singular support, and is exponentially decreasing, with all derivatives, as $t \nearrow +\infty$. The operator $\mathbf{B}_{Kr}$ is the sum of a Fourier integral operator $\mathbf{B}^\#_{Kr}$ of order 0 and an operator $\mathbf{B}^b_{Kr}$ (whose detailed nature need not concern us here), which smooths out distributions with wave front set disjoint from a small conic neighborhood of glancing. The integral kernel of an operator in $OPS^0_{\varkappa R}(S_r^2 \times \mathbb{R})$ was discussed in §4.

We now look at $\mathbf{C}_{rK}\psi(\alpha D_t^{-1} A_r)$, the restriction to $\partial K \times \mathbb{R}$ of $\mathbf{C}_r \psi(\alpha D_t^{-1} A_r)$, defined by

$$\mathbf{C}_r \psi(\alpha D_t^{-1} A_r) g(s\theta, t) = \left(\frac{r}{s}\right)^{\frac{1}{2}} \frac{H_A^{(1)}(sD_t)}{H_A^{(1)}(rD_t)}\, \psi(\alpha D_t^{-1} A) g(r\theta, t), \quad 1 < s < \infty. \tag{6.23}$$

Parallel to (2.57), this satisfies

$$\frac{\partial^2 w}{\partial s^2} + \frac{2}{s}\frac{\partial w}{\partial s} + \left(D_t^2 + s^{-2}\Delta_S\right) w = 0, \tag{6.24}$$

and, as s decreases from r to 1, if $g(r\theta, t)$ is in the range of $\psi = \psi(\alpha D_t^{-1} A_r)$, then the family of square norms of w :

$$\left((D_t^2 + s^{-2}\Delta_S) w, w\right) + \left\| \frac{\partial w}{\partial s} \right\|^2_{L^2(S^2)} \tag{6.25}$$

is monotone. On the range of ψ, $D_t^2 + s^{-2}\Delta_S$ is comparable to $-(\partial_t^2 + \Delta_S)$, for $1 \le s \le r$, and the estimate (2.67) applies to dominate $\|\partial w/\partial s\|^2_{L^2(S^2)}$. Thus

$$\mathbf{C}_{rK}\ \psi(\alpha D_t^{-1} A_r) : L^2(S_r^2 \times \mathbb{R}) \longrightarrow L^2(\partial K \times \mathbb{R}). \tag{6.26}$$

It is also clear that this is a Fourier integral operator of order 0, which smooths out the elements of the range of $\mathbf{B}^b_{Kr}$ and whose wave front relation inverts that of $\mathbf{B}^\#_{Kr}$. Therefore the operator (6.20) is a pseudodifferential operator of order 0 on $\partial K \times \mathbb{R}$.

Certainly the operator (6.20) commutes with D_t, so it has a Schwartz kernel of the form $L(t-s, x, y)$. We see that the singular support is contained in $\{t = s, x = y\}$, but it is not clear that L is rapidly decreasing as $|t - s| \to \infty$, and we do not claim that

$\Xi(\delta) \in OP\widetilde{\Sigma}^0(\partial K)$. However, we can pass to the quotient of t-space, $\mathbb{R} \to \mathbb{R}/(\ell\mathbb{Z}) = \mathbb{T}_\ell$, if ℓ is large enough, and establish the analogues of (6.23)-(6.26) with $\mathbb{R}$ replaced by $\mathbb{T}_\ell$, together with a similar modification of $\mathbf{B}_{Kr}$, obtaining

$$\mathcal{J}\ \Xi(\delta) \in OPS^0(\partial K \times \mathbb{T}_\ell), \tag{6.27}$$

commuting with D_t. Also, by Egorov's theorem, $\mathcal{J}\ \Xi(\delta)$ is microlocally equal to 0 on a conic neighborhood of $\{\tau = 0\}$ and microlocally equal to I on a conic neighborhood of $\{\xi = 0\}$ in $T^*(\partial K \times \mathbb{T}_\ell) \setminus 0$. This finishes the proof of Proposition 6.1.

A. Some formulas for scattering by a sphere

In this appendix we collect some useful formulas for solutions to problems of scattering by the unit sphere $S^2 \subset \mathbb{R}^3$, particularly the scattering problem

$$(\Delta + k^2)v = 0 \text{ on } \Omega,\ v = f \text{ on } S^2,\ r(\partial v/\partial r - ikv) \to 0 \text{ as } r \to \infty, \tag{A.1}$$

where $\Omega = \{x \in \mathbb{R}^3 : |x| > 1\}$, the complement of the unit ball. As before, we consider real k to start with. This problem can be solved by writing the Laplace operator Δ on $\mathbb{R}^3$ in polar coordinates,

$$\Delta = \partial_r^2 + 2r^{-1}\partial_r + r^{-2}\Delta_S, \tag{A.2}$$

where Δ_S is the Laplace operator on the sphere S^2. Thus v in (A.1) satisfies

$$r^2\partial_r^2 v + 2r\partial_r v + (k^2r^2 + \Delta_S)v = 0 \tag{A.3}$$

for $r > 1$. In particular, if $\{\varphi_j\}$ is an orthonormal basis of $L^2(S^2)$ consisting of eigenfunctions of Δ_S, with eigenvalue $-\lambda_j^2$, and we write

$$v(r\omega) = \sum_j v_j(r)\varphi_j(\omega), \quad r \geq 1, \tag{A.4}$$

then the functions $v_j(r)$ satisfy the ODEs

$$r^2 v_j''(r) + 2rv_j'(r) + (k^2r^2 - \lambda_j^2)v_j(r) = 0, \quad r > 1. \tag{A.5}$$

This is a modified Bessel equation, and the solution satisfying the radiation condition $r(v_j'(r) - ikv_j(r)) \to 0$ as $r \to \infty$ is of the form

$$v_j(r) = a_j r^{-\frac{1}{2}} H_{\nu_j}^{(1)}(kr), \tag{A.6}$$

where $H_\nu^{(1)}(\lambda)$ is the Hankel function. We recall from [Wat] the integral formula

$$H_\nu^{(1)}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \frac{e^{i(z - \frac{\pi\nu}{2} - \frac{\pi}{4})}}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-s} s^{\nu - \frac{1}{2}} \Big(1 - \frac{s}{2iz}\Big)^{\nu - \frac{1}{2}} ds. \tag{A.7}$$

This is valid for Re $\nu > -\frac{1}{2}$ and $-\frac{1}{2}\pi < \arg z < \pi$. Also, in (A.6), ν_j is given by

$$\nu_j = \left(\lambda_j^2 + \frac{1}{4}\right)^{\frac{1}{2}}. \tag{A.8}$$

The coefficients a_j in (A.6) are determined by the boundary condition $v_j(1) = (f, \varphi_j)$, so

$$a_j = (f, \varphi_j)/H^{(1)}_{\nu_j}(k). \tag{A.9}$$

Using these calculations, we can write the solution operator $\mathcal{B}(k)$ to (A.1), $v = \mathcal{B}(k)f$, as follows. Introduce the self adjoint operator

$$A = \left(-\Delta_S + \frac{1}{4}\right)^{\frac{1}{2}}, \tag{A.10}$$

so

$$A\varphi_j = \nu_j \varphi_j. \tag{A.11}$$

Then

$$\mathcal{B}(k)f(r\omega) = r^{-\frac{1}{2}}\varkappa(A, k, kr)f(\omega), \tag{A.12}$$

where $\varkappa(\nu, k, kr) = H^{(1)}_\nu(kr)/H^{(1)}_\nu(k)$, and, for each k, r, $\varkappa(A, k, kr)$ is regarded as a function of the self adjoint operator A. For convenience, we use the notation

$$\mathcal{B}(k)f(r\omega) = r^{-\frac{1}{2}}\,\frac{H^{(1)}_A(kr)}{H^{(1)}_A(k)}f(\omega). \tag{A.13}$$

Similar families of functions of the operator A will arise below.

Taking the r-derivative of (A.13), we have the following formula for the Neumann operator:

$$\mathcal{N}(k)f(\omega) = \left[k\frac{H^{(1)\prime}_A(k)}{H^{(1)}_A(k)} - \frac{1}{2}\right]f(\omega). \tag{A.14}$$

We also denote the operator on the right by $kQ(A, k) - \frac{1}{2}$, with

$$Q(\nu, k) = H^{(1)\prime}_\nu(k)/H^{(1)}_\nu(k). \tag{A.15}$$

We now derive some properties of the operators (A.13) and (A.14) which follow from the special nature of the operator defined by (A.10). The standard analysis of the spectrum of the Laplace operator on S^2 gives

$$\text{spec } A = \{m + \tfrac{1}{2} : m = 0, 1, 2, \dots\}. \tag{A.16}$$

Now , as shown in [Wat], $H^{(1)}_{m+\frac{1}{2}}(\lambda)$ and the other Bessel functions of order $m+\frac{1}{2}$ are all elementary functions of λ. We have

$$H^{(1)}_{m+\frac{1}{2}}(\lambda) = \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{2}} h_m(\lambda) \tag{A.17}$$

where

$$\begin{aligned} h_m(\lambda) &= -i(-1)^m \left(\frac{1}{\lambda}\frac{d}{d\lambda}\right)^m \left(\frac{e^{i\lambda}}{\lambda}\right) \\ &= \lambda^{-m-1} p_m(\lambda) e^{i\lambda} \end{aligned} \tag{A.18}$$

and $p_m(\lambda)$ is a polynomial of order m in λ, given by

$$\begin{aligned} p_m(\lambda) &= i^{-m-1} \sum_{k=0}^{m} \left(\frac{i}{2}\right)^k \frac{(m+k)!}{k!(m-k)!}\, \lambda^{m-k} \\ &= i^{m-1}\lambda^m + \cdots + \frac{1}{2^m i}\frac{(2m)!}{m!}. \end{aligned} \tag{A.19}$$

Consequently,

$$r^{-\frac{1}{2}} \varkappa(m+\tfrac{1}{2}, k, kr) = \frac{h_m(kr)}{h_m(k)} = r^{-m-1} e^{ik(r-1)} \frac{p_m(kr)}{p_m(k)} \tag{A.20}$$

and

$$kQ\big(m+\tfrac{1}{2}, k\big) = ik - \big(m+\tfrac{1}{2}\big) + k\,\frac{p_m'(k)}{p_m(k)}. \tag{A.21}$$

Each polynomial $p_m(\lambda)$ has m complex zeros $\{\zeta_{m1}, \dots, \zeta_{mm}\}$, by the fundamental theorem of algebra, and the collection of all these ζ_{mj} is clearly the set of scattering poles for S^2. Note that (A.21) can be written

$$kQ\big(m+\tfrac{1}{2}, k\big) = ik - \big(m+\tfrac{1}{2}\big) + k \sum_{j=1}^{m} (k-\zeta_{mj})^{-1}. \tag{A.22}$$

It is known (see [Wat]) that, for $\nu \in [\frac{1}{2}, \infty)$, the zeros $\zeta_j(\nu)$ of $H^{(1)}_\nu(z)$ in $\{\mathrm{Im}\, z < 0\}$ are bounded away from the real axis. Consequently, using (A.22), we obtain the estimate

$$\big|kQ(m+\tfrac{1}{2}, k)\big| \le C(|k| + m + 1), \tag{A.23}$$

for $k \in \mathbb{R}, m \ge 0$. Applying this to the formula (A.14) for the Neumann operator, we deduce that, for $s \in \mathbb{R}, k \in \mathbb{R}$,

$$\|\mathcal{N}(k) f\|_{H^s(S^2)} \le C_s \|f\|_{H^{s+1}(S^2)} + C_s |k| \cdot \|f\|_{H^s(S^2)}. \tag{A.24}$$

B. Integral equations for direct scattering problems

Here we collect some well known integral equations which lead to the solution of the scattering problem (1.1)-(1.3). These involve the single layer and double layer potentials,

$$\mathcal{S}\ell(k)f(x) = \int_{\partial K} f(y)\ g(x,y,k)\ dS(y), \tag{B.1}$$

and

$$\mathcal{D}\ell(k)f(x) = \int_{\partial K} f(y)\ \frac{\partial g}{\partial \nu_y}(x,y,k)\ dS(y), \tag{B.2}$$

where

$$g(x,y,k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}. \tag{B.3}$$

Key facts about these are the limiting behaviors

$$\begin{aligned} \mathcal{S}\ell(k)f_+(x) &= \mathcal{S}\ell(k)f_-(x) = G(k)f(x), \\ \mathcal{D}\ell(k)f_\pm(x) &= \pm\tfrac{1}{2}f(x) + \tfrac{1}{2}N(k)f(x), \end{aligned} \tag{B.4}$$

where $u_+(x)$ is the limit of $u(z)$ as $z \to x \in \partial K$ from within $\mathbb{R}^3 \setminus K$, and $u_-(x)$ is the limit as $z \to x$ from within the interior of K. Here, the singular integral operators $G(k)$ and $N(k)$ are given by

$$G(k)f(x) = \int_{\partial K} f(y)\ g(x,y,k)\ dS(y), \tag{B.5}$$

and

$$N(k)f(x) = 2\int_{\partial K} f(y)\ \frac{\partial g}{\partial \nu_y}(x,y,k)\ dS(y). \tag{B.6}$$

The operators $G(k)$ and $N(k)$ are both compact. In fact, an analysis yields

$$G(k), N(k) \in OPS^{-1}(\partial K), \quad G(k) \text{ elliptic}. \tag{B.7}$$

The solution to (1.1)-(1.3) can be written in the form

$$\mathcal{B}(k)f = \mathcal{D}\ell(k)g, \quad g = 2\big(I + N(k)\big)^{-1}f, \tag{B.8}$$

or in the form

$$\mathcal{B}(k)f = \mathcal{S}\ell(k)g, \quad g = G(k)^{-1}f, \tag{B.9}$$

with some exceptions. As is well known, $I + N(k)$ is invertible on $H^s(\partial K)$ for all real k <u>except</u> for $k = \lambda_j$, where $-\lambda_j^2$ is an eigenvalue for Δ on the interior of K, with the <u>Neumann</u> condition, while $G(k) : H^s(\partial K) \to H^{s+1}(\partial K)$ is invertible for every real k <u>except</u> $k = \mu_j$, where $-\mu_j^2$ is an eigenvalue of Δ on the interior of K, with the <u>Dirichlet</u> boundary condition. Since it is undesirable to have to deal with all these exceptions, one also looks at another useful representation of the solution to (1.1)-(1.3), which arises by picking a real $\eta \neq 0$ and using the form

$$\mathcal{B}(k)f = \mathcal{D}\ell(k)g + i\eta\mathcal{S}\ell(k)g, \quad g = 2\big(I + N(k) + 2i\eta G(k)\big)^{-1}f. \tag{B.10}$$

It is known that $I + N(k) + 2i\eta G(k)$ is invertible for all k in the first closed quadrant in $\mathbb{C}$, when $\eta > 0$, and for all k in the second closed quadrant of $\mathbb{C}$ when $\eta < 0$.

We also note that (1.4) gives the representation

$$\mathcal{B}(k)f = \mathcal{D}\ell(k)f - \mathcal{S}\ell(k)\mathcal{N}(k)f, \tag{B.11}$$

where $\mathcal{N}(k)$ is the Neumann operator (1.7). Note that evaluation of this on ∂K, using (B.4), gives

$$\mathcal{N}(k) = \tfrac{1}{2}G(k)^{-1}\big(N(k) - I\big). \tag{B.12}$$

The poles of $G(k)^{-1}$ on the real axis are removable singularities for $\mathcal{N}(k)$, being cancelled by the action of $N(k) - I$.

Another representation of the Neumann operator is obtained by looking at the normal derivative of $\mathcal{S}\ell(k)g$. In rough parallel with (B.4)-(B.7), we have

$$\frac{\partial}{\partial\nu_\pm}\,\mathcal{S}\ell(k)g = \tfrac{1}{2}\big(N^\#(k)g \mp g\big), \tag{B.13}$$

where $\partial/\partial\nu_\pm$ is the normal derivative from outside/inside K, and

$$N^\#(k)f(x) = 2\int_{\partial K} f(y)\,\frac{\partial g}{\partial\nu_x}(x,y,k)\,dS(y); \quad N^\#(k) \in OPS^{-1}(\partial K). \tag{B.14}$$

In particular,

$$\frac{\partial}{\partial\nu_+}\mathcal{S}\ell(k)g - \frac{\partial}{\partial\nu_-}\mathcal{S}\ell(k)g = -g \text{ on } \partial K. \tag{B.15}$$

Now, computing $\partial/\partial\nu_+$ on $\mathcal{S}\ell(k)g = \mathcal{B}(k)f$, we obtain the identity

$$\mathcal{N}(k) = \tfrac{1}{2}\big(N^\#(k) - I\big)G(k)^{-1}. \tag{B.16}$$

Note that comparing (B.16) with (B.12) yields the intertwining relation $N(k)G(k) = G(k)N^{\#}(k)$.

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