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PASSAGE-TIME MOMENTS FOR CONTINUOUS NON-NEGATIVE STOCHASTIC PROCESSES AND APPLICATIONS

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Abstract

We give general criteria for the finiteness or not of the passage-time moments for continuous non-negative stochastic processes in terms of sub/supermartingale inequalities for powers of these processes. We apply these results to one-dimensional diffusions and also to reflected Brownian motion in a wedge. The discrete-time analogue of this problem was studied previously by Lamperti and more recently by Aspandiiarov, Iasnogorodski and Menshikov [2]. Our results are continuous analogues of those in [2].

Key words. Passage-time moments, sub/supermartingle inequalities, diffusions, reflected Brownian motion in a wedge.

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1. Introduction

In this paper we consider the problem of finding general criteria for the finiteness or not of the passage-time moments for continuous non-negative stochastic processes which might not be Markov processes. (Here passage-time means time to hit a compact set about the origin.) The discrete-time analogue of this problem was studied previously by Lamperti [3] and recently Aspandiiarov, Iasnogorodski and Menshikov [2] have given more general criteria than those of Lamperti [3]. In this paper we give continuous analogues of the results in [2]. Specifically, in Section 2 we give conditions for the finiteness or not of the p^{th} moments ($p > 0$, real) of the passage-times of a continuous non-negative process in terms of sub/super-martingale inequalities for powers of the process. (Here we abuse terminology and refer to the mean of the p^{th} power of a passage-time as the p^{th} moment

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of that passage-time, thus extending the usual terminology from integer $p > 0$ to all real $p > 0$.)

In Section 3, to provide some intuition for the reader, we give a simple application of our results to one-dimensional diffusion processes. This case may be viewed as a prototype for the more general results presented in Section 2. The conditions given in Section 3 are analogues of results obtained by Lamperti [3] and Aspandiiarov et al. [2] for one-dimensional discrete-time Markov processes.

A more substantial application is given in Section 4. In [2], Aspandiiarov et al. give explicit criteria for the finiteness or not of passage-time moments for driftless reflected random walks in a quadrant. In [1], using an approximation by these reflected random walks, Aspandiiarov and Iasnogorodski obtain analogous criteria for reflected Brownian motion in a wedge (henceforth abbreviated as RBM). Their results are in terms of a parameter α which depends in an explicit way on the geometric data of the RBM. In Section 4, we give a direct proof of the results of [1] for RBM (in all except the critical case $p = \alpha/2$) using the criteria of Section 2. In addition, we are able to prove finiteness of the p^{th} passage-time moments in the case $0 < p < \alpha/2 \leq 1$ which was not covered in [1]. Throughout Section 4 (as well as in [1, 2]), a martingale function of the reflecting Brownian motion plays a crucial role. This function appeared first in the work of Varadhan and Williams [4] where in particular it was used to obtain conditions for the finiteness or not of the *first* passage-time moment for reflected Brownian motion in a wedge (see Corollary 2.3 of [4]).

We feel the criteria in Section 2 are natural and quite general, and might be usefully applied beyond the applications of Sections 3 and 4. Furthermore, comparison of the results here and in [1, 2] indicates that there is an intimate connection between the discrete and continuous results. These points will be further illustrated in a separate work where we apply the results of [2] and Section 2 to obtain explicit criteria for the finiteness or not of p^{th} passage-time moments for reflected random walks and reflected Brownian motions in wedges with an additional skew reflection on an interior radial line. It would be interesting to find further applications of the results of Section 2 and [2].

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2. Passage-time moments for continuous non-negative stochastic processes

In this section we give analogues of the discrete-time theorems developed in Part I of [2]. Some proofs in [2] carry over to the continuous situation without much change whilst others (most notably the $p \geq 1$ finite moments case and the proof of Lemma 2.2 for the infinite moments case) need some non-trivial modification. An analogue of Corollary 2.4 is not in [2]. For the convenience of the reader, we give complete proofs below indicating which proofs are substantially different from those in [2].

Throughout this section, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ will be a filtered probability space on which is defined a continuous adapted process $\{X_t, t \geq 0\}$ which takes values in the non-negative real numbers. For each $r \geq 0$, let

$$\sigma_r = \inf\{t \geq 0 : X_t \leq r\},$$

and

$$\tilde{X}_t = X_{t \wedge \sigma_r}.$$

Here the term adapted will mean adapted to the filtration $\{\mathcal{F}_t\}$ and all (local) (sub/super) martingales referenced here will be defined relative to this filtration, unless stated otherwise. Here a continuous local sub/super martingale is defined to be the sum of a continuous local martingale and a non-decreasing/non-increasing continuous adapted process that starts from 0.

2.1 Finite Moments

Theorem 2.1 Let $p > 0$, $r > 0$ and $X_0 = x > r$. Suppose that there is $\epsilon > 0$ such that $\{\tilde{X}_t^{2p} + \epsilon \int_0^{t \wedge \sigma_r} \tilde{X}_s^{2p-2}\, ds, t \geq 0\}$ is a supermartingale. Then for any $0 < q < p$ and also for $q = p$ if $p \geq 1$, we have

$$E[\sigma_r^q] \leq c_1 x^{2p} + c_2,$$

where c_1, c_2 are positive constants that depend on ϵ, p and q.

Remark. The proof of this result for $p < 1$ is very similar to that in [2]. For $p \geq 1$, we start with the same idea, namely, to show that the quantity $\left(\tilde{X}_t^2 + \frac{\epsilon(t \wedge \sigma_r)}{p}\right)^p$ is a supermartingale. However, because we are working with continuous time we need to develop and use Taylor expansions obtained via Itô's formula and so our proof is different from that in [2].

Proof. We first consider the case $0 < q < p < 1$. Let

$$Y_t = \tilde{X}_t^{2p} + (t \wedge \sigma_r)^q \text{ for all } t \geq 0.$$

Observe that since $\epsilon > 0$, $\tilde{X}^{2p}$ is a supermartingale and hence $E[Y_t] \leq x^{2p} + t^q < \infty$ for each $t \geq 0$. Furthermore, using the assumption of the theorem, for $0 \leq s < t < \infty$ we have,

$$\begin{aligned} E\left[Y_t - Y_s\right] &\leq -\epsilon\, E\left[\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} \tilde{X}_u^{2p-2} du\right] + E\left[\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} qu^{q-1} du\right] \\ &= (I) + (II) \end{aligned}$$

where for β a fixed constant such that $\frac{q}{2p} < \beta < \frac{1-q}{2(1-p)}$,

$$(I) = -\epsilon E\left[\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} \left(\tilde{X}_u^{2p-2} - \frac{qu^{q-1}}{\epsilon}\right) 1_{[r,u^\beta]}(\tilde{X}_u)\, du\right]$$

and

$$(II) = qE\left[\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} u^{q-1} 1_{(u^\beta,\infty)}(\tilde{X}_u)\, du\right].$$

Now, if $\tilde{X}_u \leq u^\beta$, then because $0 < p < 1$ we have

$$\tilde{X}_u^{2p-2} \geq u^{2(p-1)\beta}$$

and since $2(p-1)\beta > q-1$, there is an $s_0 > 0$ such that the right member above is greater than or equal to $\frac{q}{\epsilon}u^{q-1}$ for all $u \geq s_0$. Thus,

$$(I) \leq 0 \quad \text{for all } s \geq s_0.$$

Now for (II), note that

$$(II) \leq q \int_s^t u^{q-1} P\left(\tilde{X}_u > u^\beta\right) du,$$

where by Chebychev's inequality and since $\tilde{X}^{2p}$ is a supermartingale we have

$$P\left(\tilde{X}_u > u^\beta\right) \leq \frac{E\left[\tilde{X}_u^{2p}\right]}{u^{2p\beta}} \leq \frac{x^{2p}}{u^{2p\beta}}.$$

Combining the above, we have for $t > s \geq s_0$,

$$E[Y_t - Y_s] \leq qx^{2p} \int_s^t u^{q-1-2p\beta}\, du,$$

where $q - 2p\beta < 0$ by the choice of β. Hence, since $c \equiv q \int_{s_0}^\infty u^{q-1-2p\beta}\, du < \infty$ and

$$\sup_{0 \leq s \leq s_0} E[Y_s] \leq x^{2p} + s_0^q,$$

it follows that

$$\sup_{t \geq 0} E[Y_t] \leq x^{2p}(1 + c) + s_0^q.$$

Then by monotone convergence we have

$$E[\sigma_r^q] = \sup_t E[(t \wedge \sigma_r)^q] \leq \sup_t E[Y_t] \leq c_1 x^{2p} + c_2$$

where $c_1 = (1 + c)$ and $c_2 = s_0^q$.

Now we turn to the case $p \geq 1$. For $0 < q \leq p$, $E[\sigma_r^q] \leq E[\sigma_r^p] + 1$ and so it suffices to prove the desired result for $q = p$. Since $\{\tilde{X}_t^{2p}, t \geq 0\}$ is a supermartingale and $p \geq 1$, it follows from Hölder's inequality for conditional expectations that so too are $\{\tilde{X}_t,\, t \geq 0\}$ and $\{\tilde{X}_t^2,\, t \geq 0\}$. (Recall that $\tilde{X}$ is a non-negative process.) Let

$$\tilde{X}_t = M_t - A_t \tag{2.1}$$

be the Doob-Meyer decomposition of $\tilde{X}$ into a continuous local martingale M and a continuous non-increasing process $-A$ with $A_0 = 0$. Then by Itô's formula,

$$\tilde{X}_t^2 = \tilde{X}_0^2 + 2\int_0^t \tilde{X}_s\, dM_s - 2\int_0^t \tilde{X}_s\, dA_s + [M]_t, \tag{2.2}$$

where $[M]$ denotes the quadratic variation process for M. We note here in passing that it follows from the above that the quadratic variation process $[\tilde{X}^2]$ for $\tilde{X}^2$ is given by

$$[\tilde{X}^2]_t = 4\int_0^t \tilde{X}_s^2\, d[M]_s. \tag{2.3}$$

Since $\tilde{X}^2$ is a supermartingale, it follows from (2.2) that

$$U_t \equiv 2\int_0^t \tilde{X}_s\, dA_s - [M]_t$$

is a non-decreasing process. Similarly,

$$\tilde{X}_t^{2p} = \tilde{X}_0^{2p} + 2p\int_0^t \tilde{X}_s^{2p-1}\,dM_s - 2p\int_0^t \tilde{X}_s^{2p-1}\,dA_s + p(2p-1)\int_0^t \tilde{X}_s^{2p-2}d[M]_s$$

and since $\{\tilde{X}_t^{2p} + \epsilon\int_0^{t\wedge\sigma_r} \tilde{X}_s^{2p-2}\,ds, t\geq 0\}$ is a supermartingale, it follows that

$$V_t \equiv 2p\int_0^t \tilde{X}_s^{2p-1}\,dA_s - p(2p-1)\int_0^t \tilde{X}_s^{2p-2}d[M]_s - \epsilon\int_0^{t\wedge\sigma_r} \tilde{X}_s^{2p-2}\,ds \tag{2.4}$$

is a non-decreasing process. Now we shall prove that $(\tilde{X}_t^2 + \frac{\epsilon(t\wedge\sigma_r)}{p})^p$ is a supermartingale. For this we simply compute using Itô's formula. Here we let $\tilde{t} = t\wedge\sigma_r$ and $\tilde{s} = s\wedge\sigma_r$.

$$\begin{aligned}\left(\tilde{X}_t^2 + \frac{\epsilon\tilde{t}}{p}\right)^p &= X_0^{2p} + p\int_0^t \left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)^{p-1} d\left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)\\ &\quad + \frac{p(p-1)}{2}\int_0^t \left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)^{p-2} d[\tilde{X}^2]_s\\ &= X_0^{2p} + 2p\int_0^t \left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)^{p-1} \tilde{X}_s\,dM_s\\ &\quad + p\int_0^t \left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)^{p-1} \left(d[M]_s - 2\tilde{X}_s dA_s + \frac{\epsilon}{p}d\tilde{s}\right)\\ &\quad + 2p(p-1)\int_0^t \left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)^{p-2} \tilde{X}_s^2\,d[M]_s,\end{aligned} \tag{2.5}$$

where we have used (2.2) and (2.3). The integral with respect to dM in (2.5) defines a continuous local martingale N. The second last integral in (2.5) may be rewritten using the definition (2.4) of V as

$$\begin{aligned}&p\int_0^t \left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)^{p-1} \left(d[M]_s - (2p-1)d[M]_s - \frac{\epsilon}{p}d\tilde{s} - \frac{1}{p}\tilde{X}_s^{2-2p}dV_s + \frac{\epsilon}{p}d\tilde{s}\right)\\ &= p\int_0^t \left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)^{p-2} \left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)\left(-2(p-1)d[M]_s - \frac{1}{p}\tilde{X}_s^{2-2p}dV_s\right).\end{aligned}$$

Inserting this in (2.5) we obtain

$$\begin{aligned}\left(\tilde{X}_t^2 + \frac{\epsilon\tilde{t}}{p}\right)^p &= X_0^{2p} + N_t - 2p(p-1)\int_0^t \left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)^{p-2} \frac{\epsilon\tilde{s}}{p}d[M]_s\\ &\quad - \int_0^t \left(\tilde{X}_s^2 + \frac{\epsilon\tilde{s}}{p}\right)^{p-1} \tilde{X}_s^{2-2p}dV_s.\end{aligned}$$

Since V_t and $[M]_t$ are non-decreasing processes and $p \geq 1$ it follows that $\left(\tilde{X}_t^2 + \frac{\epsilon \tilde{t}}{p}\right)^p$ is a local supermartingale. But since it is non-negative, it is in fact a supermartingale. Hence, for any $t \geq 0$,

$$E[(t \wedge \sigma_r)^p] \leq \left(\frac{p}{\epsilon}\right)^p E\left[\left(\tilde{X}_t^2 + \frac{\epsilon \tilde{t}}{p}\right)^p\right] \leq \left(\frac{p}{\epsilon}\right)^p x^{2p}.$$

Letting $t \to \infty$ yields

$$E[\sigma_r^{2p}] \leq \left(\frac{p}{\epsilon}\right)^p x^{2p}.$$

∎

2.2 Infinite Moments

Lemma 2.2 Let $r > 0$ and $X_0 = x > r$. Suppose that there are positive constants A and B, and $\gamma > 1$, such that for any $0 \leq s < t$,

$$E\left[\tilde{X}_t^2 | \mathcal{F}_s\right] \geq \tilde{X}_s^2 - AE\left[t \wedge \sigma_r - s \wedge \sigma_r | \mathcal{F}_s\right] \tag{2.6}$$

and

$$E\left[\tilde{X}_t^{2\gamma} | \mathcal{F}_s\right] \leq \tilde{X}_s^{2\gamma} + BE\left[\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} X_u^{2\gamma-2}\, du | \mathcal{F}_s\right], \tag{2.7}$$

where it is implicit here that all conditional expectations are well defined and finite a.s. (In particular, the random variables inside the conditional expectations are integrable.) Then for any $\nu \in (0,1)$, there are positive constants $\epsilon = \epsilon(\nu, \gamma, A, B)$ and $\delta = \delta(\nu, \gamma, A, B)$ such that for any $s \geq 0$,

$$P[\sigma_r > s + \epsilon \tilde{X}_s^2 | \mathcal{F}_s] \geq \nu \quad \text{on } \{\tilde{X}_s > r(1+\delta)\}. \tag{2.8}$$

Remark. This lemma is a continuous analogue of Lemma 2 of [2] and our proof is a modification of the argument given there. The use of a Gronwall-type inequality to obtain a useful bound on $E\left[\tilde{X}_t^{2\gamma}\right]$ is an innovation here.

Proof. Fix $s \geq 0$. To simplify notation, we set

$$\tau = (\sigma_r - s)^+, \quad Y_t = \tilde{X}_{t+s}^2, \quad \mathcal{G}_t = \mathcal{F}_{t+s} \text{ for all } t \geq 0.$$

It suffices to show that for any $\nu \in (0,1)$, there are $\epsilon, \delta > 0$ such that on $\{\tau > 0\} \cap \{Y_0 > r^2(1+\delta)^2\}$, the following holds

$$P(\tau > \epsilon Y_0 | \mathcal{G}_0) \geq \nu.$$

For this note that $\tilde{X}$ is X stopped at σ_r and so

$$P(\tau > \epsilon Y_0 | \mathcal{G}_0) = P(Y_{\epsilon Y_0} > r^2 | \mathcal{G}_0). \tag{2.9}$$

First we show that $E[Y^\gamma_{\epsilon Y_0}] < \infty$. It follows from (2.7) that $\{Y_t^\gamma - B\int_0^{t\wedge\tau} Y_u^{\gamma-1}\,du,\ \mathcal{G}_t,\ t \geq 0\}$ is a supermartingale. For each positive integer n, let $\eta_n = \inf\{t \geq 0 : |Y_t| \geq n\}$. Fix n and let $\hat{Y}_t = Y_{t\wedge\eta_n}$. Then by Doob's stopping theorem, $\{\hat{Y}_t^\gamma - B\int_0^{t\wedge\eta_n\wedge\tau} \hat{Y}_u^{\gamma-1}\,du, \mathcal{G}_t, t \geq 0\}$ is a supermartingale. Hence, for each $t \geq 0$,

$$\begin{aligned} E[\hat{Y}_t^\gamma | \mathcal{G}_0] &\leq Y_0^\gamma + BE\left[\int_0^{t\wedge\eta_n\wedge\tau} \hat{Y}_u^{\gamma-1}\,du | \mathcal{G}_0\right] \\ &\leq Y_0^\gamma + B\int_0^t E[\hat{Y}_u^\gamma|\mathcal{G}_0]^{\frac{\gamma-1}{\gamma}}\,du. \end{aligned} \tag{2.10}$$

Setting $\hat{v}_t = E[\hat{Y}_t^\gamma|\mathcal{G}_0]$, the above may be rewritten a.s. as

$$\hat{v}_t \leq \hat{v}_0 + B\int_0^t \hat{v}_u^{1-\frac{1}{\gamma}}\,du, \tag{2.11}$$

where the integral on the right is finite a.s. because $\hat{Y}_u \leq Y_0 \vee n$ for all u. Then a Gronwall-type inequality yields that

$$\hat{v}_t \leq \left(\hat{v}_0^{\frac{1}{\gamma}} + \frac{Bt}{\gamma}\right)^\gamma. \tag{2.12}$$

(To prove this, let $w_t = \int_0^t \hat{v}_u^{1-\frac{1}{\gamma}}\,du$ and use (2.11) to obtain the inequality $\dot{w}_t \leq (\hat{v}_0 + Bw_t)^{1-\frac{1}{\gamma}}$. "Solve" this by separation of variables to obtain the desired result.) Since the right member of (2.12) does not depend on n, we can let $n \to \infty$ and use Fatou's lemma to conclude that

$$E\left[Y_t^\gamma|\mathcal{G}_0\right] \leq \left(Y_0 + \frac{Bt}{\gamma}\right)^\gamma \quad \text{for all } \ t \geq 0. \tag{2.13}$$

Since ϵY_0 is $\mathcal{G}_0$-measurable, it can be deduced from the above that

$$E\left[Y^\gamma_{\epsilon Y_0}|\mathcal{G}_0\right] \leq \left(Y_0 + \frac{B\epsilon Y_0}{\gamma}\right)^\gamma, \tag{2.14}$$

where in particular, $E[Y^\gamma_{\epsilon Y_0}] < \infty$, since $Y_0^\gamma = \tilde{X}_s^{2\gamma}$ is integrable, by assumption. Now, by Hölder's inequality for conditional expectations, since $\gamma > 1$ we have

$$E[Y_{\epsilon Y_0}|\mathcal{G}_0] \leq r^2 + \left(E[Y^\gamma_{\epsilon Y_0}|\mathcal{G}_0]\right)^{\frac{1}{\gamma}}\left(P(Y_{\epsilon Y_0} > r^2|\mathcal{G}_0)\right)^{1-\frac{1}{\gamma}}. \tag{2.15}$$

By (2.6),

$$E[Y_{\epsilon Y_0}|\mathcal{G}_0] - r^2 \geq Y_0 - AE[\epsilon Y_0 \wedge \tau|\mathcal{G}_0] - r^2 \tag{2.16}$$
$$\geq Y_0 - A\epsilon Y_0 - r^2.$$

Then, substituting (2.14)–(2.16) into (2.9) yields:

$$\begin{aligned}
P(\tau > \epsilon Y_0|\mathcal{G}_0) &\geq \left(\frac{E[Y_{\epsilon Y_0}|\mathcal{G}_0] - r^2}{E[Y_{\epsilon Y_0}^{\gamma}|\mathcal{G}_0]^{\frac{1}{\gamma}}}\right)^{\frac{\gamma}{\gamma-1}} \\
&\geq \left(\frac{(Y_0(1-A\epsilon) - r^2) \vee 0}{Y_0\left(1+\frac{B\epsilon}{\gamma}\right)}\right)^{\frac{\gamma}{\gamma-1}} \\
&= \left(\frac{(1 - A\epsilon - \frac{r^2}{Y_0}) \vee 0}{1+\frac{B\epsilon}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}} \\
&\geq \left(\frac{\left(1 - A\epsilon - \frac{1}{(1+\delta)^2}\right) \vee 0}{1+\frac{B\epsilon}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{on } \{Y_0 > r^2(1+\delta)^2\}.
\end{aligned} \tag{2.17}$$

Clearly, given $\nu \in (0,1)$, we can choose ϵ and δ depending only on A, B, γ and ν such that the last member above is $\geq \nu$. ∎

Theorem 2.3 Suppose that $\{X_t, t \geq 0\}$ satisfies the hypotheses of Lemma 2.2 and $p > 0$ such that $\{\tilde{X}_t^{2p}, \mathcal{F}_t, t \geq 0\}$ is a submartingale. Then $E[\sigma_r^q]$ is infinite for all $q > p$.

Remark. This theorem is a slightly simpler analogue of Theorem 2 of [2].

Proof. If $P(\sigma_r = \infty) > 0$, then obviously $E[\sigma_r^q] = \infty$. Hence we may assume that $\sigma_r < \infty$ a.s. For a proof by contradiction, suppose that $q > p$ and $E[\sigma_r^q] < \infty$. Then by applying Lemma 2.2 with $\nu = \frac{1}{2}$ we have for all $s \geq 0$,

$$\begin{aligned}
\infty > E[\sigma_r^q] &\geq E\left[\sigma_r^q;\ \tilde{X}_s > r(1+\delta)\right] \\
&\geq \frac{1}{2}E\left[(s+\epsilon\tilde{X}_s^2)^q;\ \tilde{X}_s > r(1+\delta)\right] \\
&\geq \frac{\epsilon^q}{2}E\left[\tilde{X}_s^{2q};\ \tilde{X}_s > r(1+\delta)\right] \\
&\geq \frac{\epsilon^q}{2}E[\tilde{X}_s^{2q}] - \frac{\epsilon^q}{2}(r(1+\delta))^{2q}.
\end{aligned}$$

In the above we have used the semicolon notation as an abbreviation: $E[Y;A] = E[Y1_A]$. It follows from the above that $\{\tilde{X}_s^{2q}, s \geq 0\}$ is uniformly bounded and hence that $\{\tilde{X}_s^{2p}, s \geq 0\}$ is uniformly integrable and so by the submartingale convergence theorem,

$$E[\tilde{X}_s^{2p}] = E[X_{s\wedge\sigma_r}^{2p}] \to E[X_{\sigma_r}^{2p}] \quad \text{as } s \to \infty.$$

Note that the right member here is less than or equal to r^{2p}. On the other hand, since $\{\tilde{X}_s^{2p}, s \geq 0\}$ is a submartingale we have

$$E[\tilde{X}_s^{2p}] \geq x^{2p} \quad \text{for all } s,$$

which yields the contradiction $r > x$. ∎

Corollary 2.4 Let $p > 0$, $r > 0$ and $X_0 = x > r$. Suppose there are positive constants A and B such that $\{\tilde{X}_t^2 + A(t \wedge \sigma_r), t \geq 0\}$ and $\{\tilde{X}_t^{2p}, t \geq 0\}$ are local submartingales and for some $\gamma > (1 \vee p)$, $\{\tilde{X}_t^{2\gamma} - B\int_0^{t\wedge\sigma_r} \tilde{X}_s^{2\gamma-2} ds, t \geq 0\}$ is a local supermartingale. Then, $E[\sigma_r^q]$ is infinite for all $q > p$.

Proof. It suffices to verify that the hypotheses of Lemma 2.2 and Theorem 2.3 are satisfied. Consider the Doob-Meyer decompositions of $\tilde{X}^2$, $\tilde{X}^{2\gamma}$ and $\tilde{X}^{2p}$:

$$\tilde{X}_t^2 = x^2 + M_t + U_t - A(t \wedge \sigma_r),$$

$$\tilde{X}_t^{2\gamma} = x^{2\gamma} + N_t - V_t + B\int_0^{t\wedge\sigma_r} \tilde{X}_s^{2\gamma-2} ds,$$

$$\tilde{X}_t^{2p} = x^{2p} + R_t + W_t,$$

where M, N, R are continuous local martingales and U, V, W are continuous adapted non-decreasing processes that start from zero.

Let $\{\eta_n, n \geq 0\}$ be an increasing sequence of stopping times that converges a.s. to $+\infty$ such that for each n, $M_{\cdot\wedge\eta_n}$, $N_{\cdot\wedge\eta_n}$, $R_{\cdot\wedge\eta_n}$ are martingales and $\tilde{X}^2_{\cdot\wedge\eta_n}$, $\tilde{X}^{2\gamma}_{\cdot\wedge\eta_n}$, $\tilde{X}^{2p}_{\cdot\wedge\eta_n}$, $U_{\cdot\wedge\eta_n}$, $V_{\cdot\wedge\eta_n}$, $W_{\cdot\wedge\eta_n}$ are bounded. Then, for $0 \leq s < t$,

$$E\left[\tilde{X}^{2\gamma}_{t\wedge\eta_n} \middle| \mathcal{F}_s\right] \leq \tilde{X}^{2\gamma}_{s\wedge\eta_n} + BE\left[\int_s^t 1_{[0,\sigma_r\wedge\eta_n]}(u)\tilde{X}_u^{2\gamma-2}\, du \middle| \mathcal{F}_s\right]. \tag{2.18}$$

By the same reasoning (cf. (2.11)) as in the proof of Lemma 2.2, we obtain

$$E\left[\tilde{X}^{2\gamma}_{t\wedge\eta_n}\right] \leq \left(x^2 + \frac{Bt}{\gamma}\right)^{\gamma}. \tag{2.19}$$

Note that the right member in the above does not depend on n. In particular, by Fatou's lemma, $E[\tilde{X}_t^{2\gamma}] < \infty$. Furthermore, since $(1 \vee p) < \gamma$, it follows from (2.19) and Hölder's inequality that $\{\tilde{X}^2_{t\wedge\eta_n}, n \geq 0\}$ and $\{\tilde{X}^{2p}_{t\wedge\eta_n}, n \geq 0\}$ are uniformly integrable for each fixed $t \geq 0$. Now for $\tilde{X}^2$ we have for $0 \leq s < t$,

(2.20)

$$\begin{aligned} E\left[\tilde{X}^2_{t\wedge\eta_n} \middle| \mathcal{F}_s\right] &= \tilde{X}^2_{s\wedge\eta_n} + E\left[U_{t\wedge\eta_n} - U_{s\wedge\eta_n} \middle| \mathcal{F}_s\right] - AE\left[(t\wedge\sigma_r\wedge\eta_n) - (s\wedge\sigma_r\wedge\eta_n) \middle| \mathcal{F}_s\right] \\ &\geq \tilde{X}^2_{s\wedge\eta_n} - AE\left[(t\wedge\sigma_r\wedge\eta_n) - (s\wedge\sigma_r\wedge\eta_n) \middle| \mathcal{F}_s\right]. \end{aligned}$$

Using the uniform integrability stated above on the left and pointwise plus monotone convergence on the right, we can let $n \to \infty$ in (2.20) to conclude that (2.6) holds. Note that $E[\tilde{X}_t^2] < \infty$ by the uniform integrability of $\{\tilde{X}_{t\wedge\eta_n}^2, n \geq 0\}$. In (2.18) we can use Fatou's lemma on the left and pointwise plus monotone convergence on the right to conclude that (2.7) holds. Note that the conditional expectations in (2.7) are well defined and finite a.s. (in particular, their arguments are integrable), by virtue of (2.19) and Hölder's inequality. Finally, we can use the uniform integrability of $\{\tilde{X}_{t\wedge\eta_n}^{2p},\ n \geq 0\}$ to conclude that $\{\tilde{X}_t^{2p}, t \geq 0\}$ is not just a local submartingale but it is in fact a submartingale. ∎

3. Passage-time moments for one-dimensional diffusions

In this section we apply the criteria of Section 2 to the simple case of a one-dimensional diffusion. This case may be viewed as a prototype for our general results developed in Section 2. Of course, the results presented here may be known to experts and they can at least be approximately conjectured from the results for discrete-time Markov processes (cf. [2, 3]). However, we were not able to find a reference that gives these results in the simple form presented here. The closest reference seems to be a paper by Yamazato [5] which gives behavior of the tails of passage-time distributions for some diffusions in terms of tail behavior of the speed measure. In any event, our main aim in this section is simply to illustrate the ease of applying our general results in Section 2 to obtain meaningful conditions for the finiteness of passage-time moments for one-dimensional diffusions, without using a lot of special structure. In particular, note that we do not use the Markov property per se.

In this section, $\{X_t, t \geq 0\}$ will be a one-dimensional diffusion satisfying the stochastic differential equation:

$$X_t = x_0 + \int_0^t \mu(X_s)ds + \int_0^t b(X_s)dB_s,$$

where $x_0 \in \mathbb{R}$ and B is a standard one-dimensional Brownian motion. For simplicity we suppose that μ and b are uniformly Lipschitz continuous on $\mathbb{R}$, which guarantees pathwise existence and uniqueness for the above equation. We take $\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t\}$. The process X will be adapted to this filtration because of the Lipschitz assumptions on the coefficients. Let $\nu = b^2$, $r > 0$ and $X_0 = x_0 > r$. We use the same notation σ_r and $\tilde{X}$ as in Section 2. By Itô's formula we have for any $p > 0$,

$$\tilde{X}_t^{2p} = x_0^{2p} + 2p \int_0^{t\wedge\sigma_r} \tilde{X}_s^{2p-1} b(\tilde{X}_s)\, dB_s \tag{3.1}$$
$$+ \int_0^{t\wedge\sigma_r} \left(2p\tilde{X}_s^{2p-1}\mu(\tilde{X}_s) + p(2p-1)\tilde{X}_s^{2p-2}\nu(\tilde{X}_s)\right)\, ds.$$

The following results (Theorem 3.1 and Corollary 3.3) are analogues of the results of Lamperti [3] and Aspandiiarov et al. [2] on the finiteness or not of passage-time moments for discrete-time Markov processes on $\mathbb{R}_+$. We do not require higher than second moment conditions, though such are required in some cases by [2] and [3] to control jumps. Furthermore, in place of the second moment μ_2 that appears in the conditions of [2] and [3] we have the variance ν of the process X. In the case of finite moments, ours is a slightly weaker condition, but for infinite moments it is the same because of the assumption that the drift μ is asymptotically of order at most $1/x$ as $x \to \infty$.

Theorem 3.1 Let $p > 0$. Suppose that $\epsilon > 0$ such that

$$2x\mu(x) + (2p-1)\nu(x) \leq -\epsilon \quad \text{for all} \quad x \geq r. \tag{3.2}$$

Then for any $0 < q < p$ and also for $q = p$ if $p \geq 1$, we have

$$E[\sigma_r^q] \leq c_1 x_0^{2p} + c_2$$

where c_1, c_2 are positive constants that depend on ϵ, p and q.

Proof. By (3.1) we have

$$\tilde{X}_t^{2p} + \epsilon p \int_0^{t\wedge\sigma_r} \tilde{X}_s^{2p-2}\, ds = x_0^{2p} + 2p \int_0^{t\wedge\sigma_r} \tilde{X}_s^{2p-1} b(\tilde{X}_s)\, dB_s$$
$$+ p \int_0^{t\wedge\sigma_r} \tilde{X}_s^{2p-2}\left(2\tilde{X}_s\mu(\tilde{X}_s) + (2p-1)\nu(\tilde{X}_s) + \epsilon\right)\, ds.$$

In the above, the integral with respect to dB defines a continuous local martingale and the last integral defines an adapted non-increasing process by (3.2). Thus,

$$\left\{\tilde{X}_t^{2p} + \epsilon p \int_0^{t\wedge\sigma_r} \tilde{X}_s^{2p-2}\, ds,\ t \geq 0\right\}$$

is a positive local supermartingale, and hence is a supermartingale. The desired result then follows immediately from Theorem 2.1. ∎

Theorem 3.2 Let $p > 0$. Suppose

$$(3.3) \qquad 2x\mu(x) + (2p-1)\nu(x) \geq 0 \quad \text{for all} \quad x \geq r,$$

and assume that $x\mu(x)$ and $\nu(x)$ are bounded on $\{x : x \geq r\}$. Then $E[\sigma_r^q] = +\infty$ for all $q > p$.

Proof. By (3.1) with $p = 1$,

$$\tilde{X}_t^2 = x_0^2 + 2\int_0^{t\wedge\sigma_r} \tilde{X}_s\, b(\tilde{X}_s)\, dB_s + \int_0^{t\wedge\sigma_r} \left(2\tilde{X}_s\mu(\tilde{X}_s) + \nu(\tilde{X}_s)\right) ds.$$

Now by the boundedness assumptions on $x\mu(x)$ and $\nu(x)$, there is a positive constant A such that $2x\mu(x) + \nu(x) \geq -A$ for all $x \geq r$. It then follows from the above that $\{\tilde{X}_t^2 + A(t \wedge \sigma_r),\, t \geq 0\}$ is a local submartingale. Similarly, for $\gamma > (1 \vee p)$, by choosing B to be a fixed positive constant such that $B \geq \gamma|2x\mu(x) + (2\gamma - 1)\nu(x)|$ for all $x \geq r$, we see from (3.1) with γ in place of p that $\{\tilde{X}_t^{2\gamma} - B\int_0^{t\wedge\sigma_r} \tilde{X}_s^{2\gamma-2}\, ds,\, t \geq 0\}$ is a local supermartingale. Condition (3.3) together with (3.1) implies that $\{\tilde{X}_t^{2p}, t \geq 0\}$ is a local submartingale. Thus, the hypotheses of Corollary 2.4 are satisfied and we may conclude that $E[\sigma_r^q] = +\infty$ for all $q > p$. ∎

Corollary 3.3 Let $p > 0$. Suppose there is $\epsilon > 0$ such that

$$(3.4) \qquad 2x\mu(x) + (2p-1)\nu(x) \geq \epsilon \quad \text{for all} \quad x \geq r,$$

and assume that $x\mu(x)$ and $\nu(x)$ are bounded on $\{x : x \geq r\}$. Then $E[\sigma_r^q] = +\infty$ for all $q \geq p$.

Proof. Since ν is bounded, we can find $0 < \tilde{p} < p$ such that (3.4) still holds with $\tilde{p}$ in place of p and a slightly smaller, but still positive, $\tilde{\epsilon}$ in place of ϵ. It then follows immediately from Theorem 3.2 that $E[\sigma_r^q] = +\infty$ for all $q > \tilde{p}$ and hence for all $q \geq p$. ∎

4. Passage-time moments for reflected Brownian motion in a wedge

Let $\xi \in (0, 2\pi)$. In polar coordinates, let

$$S = \{(r,\theta) : r \geq 0, 0 \leq \theta \leq \xi\},$$

$\partial S_1 = \{(r,\theta) : r \geq 0, \theta = 0\}$ and $\partial S_2 = \{(r,\theta) : r \geq 0, \theta = \xi\}$. The origin, which is the corner of the wedge S, will be denoted by 0. For $i = 1, 2$, let v_i be a vector defined on

$\partial S_i\backslash\{0\}$ whose inward normal component is of length one, and let θ_i denote the angle that v_i makes with the inward normal to $\partial S_i\backslash\{0\}$ where θ_i is positive if and only if v_i points towards the origin.

Let Z, defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, be a reflecting Brownian motion in S with absorption at the origin and directions of reflection on the boundary given by v_i on $\partial S_i\backslash\{0\}$, $i = 1, 2$. We assume that $Z_0 = z \in S\backslash\{0\}$. Heuristically this process behaves like Brownian motion in the interior of the wedge and prior to hitting the origin, it is confined to the wedge by instantaneous reflection (or pushing) at the boundary, where the direction of reflection on $\partial S_i\backslash\{0\}$ is given by v_i, $i = 1, 2$. For the precise definition of this process see Varadhan and Williams [4].

Define

$$\alpha = (\theta_1 + \theta_2)/\xi.$$

We assume that $\alpha > 0$ since this ensures that Z hits the origin with probability one (see [4, Theorem 2.2]). We define

$$\phi(r, \theta) = r^\alpha \cos(\alpha\theta - \theta_1),$$

for $(r, \theta) \in S\backslash\{0\}$. It can be verified (see Varadhan-Williams [4]) that $\phi \in C^2(S\backslash\{0\})$ and

$$\Delta\phi = 0 \quad \text{in } S\backslash\{0\}, \tag{4.1}$$

$$v_i \cdot \nabla\phi = 0 \quad \text{on } \partial S_i\backslash\{0\}, \qquad i = 1, 2, \tag{4.2}$$

$$|\nabla\phi|^2 = \alpha^2 r^{2\alpha - 2} = \phi^{2 - \frac{2}{\alpha}} h \quad \text{on } S\backslash\{0\}, \tag{4.3}$$

where

$$h(r, \theta) = \alpha^2 (\cos(\alpha\theta - \theta_1))^{\frac{2}{\alpha} - 2}.$$

There are positive constants a_1, a_2 such that for all $\theta \in [0, \xi]$,

$$0 < a_1 \leq \cos(\alpha\theta - \theta_1) \leq a_2, \tag{4.4}$$

and hence there are positive constants b_1, b_2 such that for all $(r, \theta) \in S\backslash\{0\}$,

$$0 < b_1 \leq h(r, \theta) \leq b_2. \tag{4.5}$$

Let

$$X_t = \phi^{1/\alpha}(Z_t), \quad t \geq 0.$$

For $r > 0$, let

$$\tau_r = \inf\{t \geq 0 : |Z_t| \leq r\},$$

and

$$\sigma_r = \inf\{t \geq 0 : X_t \leq r\}.$$

Then it follows from (4.4) and the definition of ϕ that

$$\sigma_{r_2} \leq \tau_r \leq \sigma_{r_1}, \tag{4.6}$$

where $r_2 = r a_2^{1/\alpha}$, $r_1 = r a_1^{1/\alpha}$. Thus, for $p > 0$,

$$E[\tau_r^p] < \infty \quad \text{for all } r > 0, \tag{4.7}$$

if and only if

$$E[\sigma_r^p] < \infty \quad \text{for all } r > 0. \tag{4.8}$$

Note also that $\tau_0 = \sigma_0$ and that if $E[\sigma_0^p] < \infty$ then $E[\sigma_r^p] < \infty$ for all $r > 0$.

Theorem 4.1 If $p < \alpha/2$, then $E[\tau_r^p]$ is finite for all $r \geq 0$. If $p > \alpha/2$, then $E[\tau_r^p]$ is infinite for all $0 \leq r < |z|\,(a_1/a_2)^{\frac{1}{\alpha}}$.

Remark. For all except the case $0 < p < \alpha/2 \leq 1$, this result can be found in [1]. Furthermore, the authors in [1] obtain that the critical case $p = \alpha/2$ is like the $p > \alpha/2$ case. Our methods are not sharp enough to recover this result, though one might be able to obtain it by proving analogues of the fine tail estimates in [1].

Before proving Theorem 4.1, we shall develop some preliminary properties of Z and powers of $\phi(Z)$.

Fix $r > 0$ and for all $t \geq 0$, let

$$\tilde{Z}_t = Z_{t \wedge \sigma_r}$$

and

$$\tilde{X}_t = X_{t \wedge \sigma_r} = \phi^{1/\alpha}(\tilde{Z}_t).$$

The process $\tilde{Z}$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ has a semimartingale decomposition:

$$\tilde{Z}_t = B_t + v_1 Y_t^{(1)} + v_2 Y_t^{(2)}, \quad t \geq 0,$$

where B is a two-dimensional Brownian motion martingale starting from z and stopped at σ_r, and for $i = 1, 2$, $Y^{(i)}$ is a continuous, adapted, non-decreasing, one-dimensional process such that $Y^{(i)}(0) = 0$ and $Y^{(i)}$ can increase only when Z is on $\partial S_i \backslash \{0\}$. Note that for all $t \geq \sigma_r$, $B_t = B_{\sigma_r}$ and $Y_t^{(i)} = Y_{\sigma_r}^{(i)}$, $i = 1, 2$.

We shall need the semimartingale decompositions of various powers of $\phi(\tilde{Z})$. We develop these using Itô's formula, the semimartingale decomposition of $\tilde{Z}$, and the properties of ϕ described above. For $t \geq 0$, P-a.s.,

$$\phi(\tilde{Z}_t) = \phi(\tilde{Z}_0) + \int_0^t \nabla\phi(Z_s) \cdot dB_s + \sum_{i=1}^{2} \int_0^t v_i \cdot \nabla\phi(Z_s) dY_s^{(i)} + \frac{1}{2} \int_0^{t \wedge \sigma_r} \Delta\phi(Z_s) ds.$$

The first integral in the right member above is a continuous local martingale, the second integral is zero by equation (4.2) and the property that $Y^{(i)}$ can increase only when Z is on $\partial S_i \backslash \{0\}$, and the third integral is zero by equation (4.1). Thus,

$$M_t \equiv \phi(\tilde{Z}_t) \quad \text{for all } t \geq 0, \tag{4.9}$$

is a continuous local martingale with quadratic variation process

$$[M]_t = [\phi(\tilde{Z})]_t = \int_0^{t \wedge \sigma_r} |\nabla\phi|^2(Z_s) ds = \int_0^{t \wedge \sigma_r} (\phi^{2-\frac{2}{\alpha}} h)(Z_s) ds = \int_0^{t \wedge \sigma_r} M_s^{2-\frac{2}{\alpha}} h(Z_s)\, ds, \tag{4.10}$$

where we have used (4.3). Then by Itô's formula for $\tilde{X}_t^2 = \phi^{\frac{2}{\alpha}}(\tilde{Z}_t) = M_t^{\frac{2}{\alpha}}$ we have P-a.s. for all $t \geq 0$,

$$\begin{aligned} \tilde{X}_t^2 &= \tilde{X}_0^2 + \frac{2}{\alpha} \int_0^t M_s^{\frac{2}{\alpha}-1} dM_s + \frac{1}{2} \int_0^t \frac{2}{\alpha} \left(\frac{2}{\alpha} - 1 \right) M_s^{\frac{2}{\alpha}-2} d[M]_s \\ &= \tilde{X}_0^2 + \frac{2}{\alpha} \int_0^t M_s^{\frac{2}{\alpha}-1} dM_s + \int_0^{t \wedge \sigma_r} \frac{1}{\alpha} \left(\frac{2}{\alpha} - 1 \right) h(Z_s) ds, \end{aligned} \tag{4.11}$$

where we have used (4.10). The quadratic variation process of the semimartingale $\tilde{X}^2$ is given by:

$$[\tilde{X}^2]_t = \frac{4}{\alpha^2} \int_0^{t \wedge \sigma_r} M_s^{2(\frac{2}{\alpha}-1)} M_s^{2-\frac{2}{\alpha}} h(Z_s) ds = \frac{4}{\alpha^2} \int_0^{t \wedge \sigma_r} \tilde{X}_s^2 h(Z_s) ds.$$

Similarly to (4.11), by Itô's formula for $\tilde{X}_t^{2p} = M_t^{\frac{2p}{\alpha}}$ and $p > 0$, we have P-a.s. for $0 \le s < t$:

$$(4.12) \quad \tilde{X}_t^{2p} = \tilde{X}_s^{2p} + \frac{2p}{\alpha} \int_s^t M_u^{\frac{2p}{\alpha}-1} dM_u + \frac{1}{2} \int_s^t \frac{2p}{\alpha} \left(\frac{2p}{\alpha} - 1 \right) 1_{[0,\sigma_r]}(u) M_u^{\frac{2p-2}{\alpha}} h(Z_u) du.$$

The first integral in the right member above is a continuous local martingale. Let $\{\eta_n, n \ge 0\}$ be a localizing sequence for it (i.e., $\{\eta_n, n \ge 0\}$ is an increasing sequence of stopping times which converges a.s. to infinity and is such that when the local martingale is stopped with any one of these times, it becomes a martingale). Taking conditional expectations in (4.12) stopped at η_n, we obtain

$$(4.13) \quad E[\tilde{X}_{t\wedge\eta_n}^{2p} | \mathcal{F}_s] = \tilde{X}_{s\wedge\eta_n}^{2p} + \frac{p}{\alpha} \left(\frac{2p}{\alpha} - 1 \right) E \left[\int_s^t 1_{[0,\sigma_r\wedge\eta_n]}(u) \tilde{X}_u^{2p-2} h(Z_u) du \big| \mathcal{F}_s \right],$$

where we have replaced $M^{\frac{1}{\alpha}}$ by $\tilde{X}$ in the last member above. On letting $n \to \infty$, using Fatou's lemma on the left and monotone convergence for the last term on the right, we obtain:

$$(4.14) \quad E[\tilde{X}_t^{2p} | \mathcal{F}_s] \le \tilde{X}_s^{2p} + \frac{p}{\alpha} \left(\frac{2p}{\alpha} - 1 \right) E \left[\int_s^t 1_{[0,\sigma_r]}(u) \tilde{X}_u^{2p-2} h(Z_u) du \Big| \mathcal{F}_s \right].$$

Proof of Theorem 4.1. We consider conditions for the finiteness of the p^{th} moments first. Recalling the inequality (4.5) for h and (4.14), we see that for $0 < p < \alpha/2$, there is $\epsilon > 0$ (not depending on r) such that

$$E[\tilde{X}_t^{2p} | \mathcal{F}_s] \le \tilde{X}_s^{2p} - \epsilon E \left[\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} \tilde{X}_u^{2p-2} du \big| \mathcal{F}_s \right]$$

and hence the hypotheses of Theorem 2.1 are satisfied for $0 < r < x \equiv \phi^{\frac{1}{\alpha}}(z)$. Now for $0 < q < \alpha/2$ we can always find a p such that $q < p < \alpha/2$. Then by Theorem 2.1,

$$E[\sigma_r^q] \le c_1 x^{2p} + c_2,$$

where c_1, c_2 do not depend on r because ϵ does not depend on it. Then, since $\sigma_0 = \lim_{r\downarrow 0} \sigma_r$, we can let $r \to 0$ to conclude that $E[\sigma_0^q] < \infty$. This, together with the comment just before Theorem 4.1, concludes the treatment of the finite moments case.

We now turn to proving infiniteness of the p^{th} passage-time moments for $p > \alpha/2$ and r sufficiently small. Observe that $X_0 = x \equiv \phi^{\frac{1}{\alpha}}(z)$. We suppose that $r > 0$ is such that

$r < |z|a_1^{\frac{1}{\alpha}}$ and so $r < x$ (cf. (4.4)). Let $p > \alpha/2$ and $\gamma > 1 \vee p$. By (4.12) with p replaced successively by $1, \gamma, p$, we see that the hypotheses of Corollary 2.4 are satisfied with $A = 1$ if $\alpha \leq 2$, $A = -\frac{a_2}{\alpha}\left(\frac{2}{\alpha} - 1\right)$ if $\alpha > 2$, and $B = \frac{\gamma}{\alpha}(\frac{2\gamma}{\alpha} - 1)b_2$. Hence it follows by applying the result with $q > p > \alpha/2$ that $E[\sigma_r^q] = +\infty$ for all $q > \alpha/2$ and $r < |z|a_1^{\frac{1}{\alpha}}$. Then by (4.6), $E[\tau_r^q] \geq E[\sigma_{\tilde{r}}^q] = +\infty$ for all $q > \alpha/2$ and $r < |z|(a_1/a_2)^{\frac{1}{\alpha}}$ where $\tilde{r} = ra_2^{1/\alpha}$. ∎

References

[1] ASPANDIIAROV, S. and IASNOGORODSKI, R. (1994). Tails of passage-times and an application to stochastic processes with boundary reflection in wedges. Preprint.

[2] ASPANDIIAROV, S., IASNOGORODSKI, R. and MENSHIKOV, M. (1994). Passage-time moments for non-negative stochastic processes and an application to reflected random walks in a quadrant. Preprint.

[3] LAMPERTI, J. (1963). Criteria for stochastic processes II: passage-time moments. *J. Math. Anal. Applic.* **7** 127–145.

[4] VARADHAN, S. R. S., and WILLIAMS, R. J. (1985). Reflected Brownian motion in a wedge. *Comm. Pure Appl. Math.* **38** 405–443.

[5] YAMAZATO, M. (1990) Hitting time distribution of single points for 1-dimensional generalized diffusion processes. *Nagoya Math. J.* **119** 143–172.

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