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NEW RESULTS ON THE OPERATOR CARLESON MEASURE CRITERION

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ABSTRACT. We consider control systems of the form $\dot{x} = Ax + Bu$ where A is the generator of a diagonal semigroup $\mathbb{T}$ on l^2 and B is an unbounded operator from a Hilbert space U to l^2 . In a previous paper by Hansen and Weiss, a condition called the operator Carleson measure criterion was shown to be necessary for the admissibility of the control operator B . Furthermore this condition was shown to be sufficient if $\mathbb{T}$ is either analytic or invertible. In this paper we continue the analysis of admissibility as related to the operator Carleson measure criterion. We show that the operator Carleson measure criterion is satisfied if and only if the input-to-state transfer function has a certain decay rate. We also extend the previous sufficiency results of Hansen and Weiss to a more general class of diagonal semigroups. To achieve our aims, we derive some general results (not confined to diagonal semigroups) concerning Lyapunov equations and feedback type perturbations, which are of independent interest.

1. Introduction and statement of the main results.

In the paper Hansen and Weiss [7], a necessary condition for admissibility of control operators for diagonal semigroups on l^2 was proved. This condition, called the operator Carleson measure criterion, was shown to be a sufficient condition as well, if the semigroup is either analytic or invertible. In this paper we continue this analysis of the operator Carleson measure criterion and its relationship to admissible control operators. We prove that a system satisfies this criterion if and only if its input-to-state transfer function possesses a certain decay rate. We also show that the results of [7] concerning analytic or invertible semigroups can be extended to a larger class of diagonal semigroups.

Key words and phrases. Admissibility, Carleson measure, Lyapunov equation.

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Some of our intermediate results relate to arbitrary (nondiagonal) strongly continuous semigroups and are interesting in their own right. We have in mind the connection between admissibility, stability and the Lyapunov equation, as well as the invariance of admissibility under (possibly unbounded) feedback.

All vector spaces considered here are complex. We represent elements of l^2 as infinite column matrices (i.e., matrices consisting of a single column) and elements of the dual l^{2*} as infinite row matrices.

We are concerned with control systems described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1.1)$$

where $x(t) \in l^2$ is the state and $u \in L^2([0, \infty), l^2)$ is the input function. The operators A and B are represented by infinite matrices. A is diagonal and its diagonal elements λ_k (its eigenvalues) are in the open left half of the complex plane:

$$-\operatorname{Re} \lambda_k \in \mathbb{C}_0 \quad \forall k \in \mathbb{N}, \quad (1.2)$$

where

$$\mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}.$$

Therefore A generates a strongly continuous diagonal semigroup $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ on l^2 :

$$(\mathbb{T}_t z)_k = e^{\lambda_k t} z_k, \quad \forall k \in \mathbb{N}, \quad (1.3)$$

where z_k denotes the k -th component of $z \in l^2$. The infinite matrix B may be unbounded in the sense that its range, when applied to l^2 , is not necessarily contained in l^2 . B represents an *admissible control operator* for $\mathbb{T}$ if for any input function $u \in L^2([0, \infty); l^2)$, (1.1) together with the initial condition $x(0) = 0$ has an l^2 -valued strong solution $x(\cdot)$. If this is the case then $x(\cdot)$ is a continuous function from $[0, \infty)$ to l^2 (for more details on this see Section 2). If in addition, the function $x(\cdot)$ is bounded (for every u) then B is called *infinite-time admissible* for $\mathbb{T}$.

The assumption that $\mathbb{T}$ is diagonal is restrictive in that not all semigroups are isomorphic to diagonal ones. However, the stability assumption in (1.2) is imposed mainly to avoid trivialities. All the results in this paper concerning admissibility could be reworded so that (1.2) would not be needed. Our choice of l^2 for the input space also leads to no loss of generality (see Remark 2.7).

To recall the results of [7], we need a notation for certain rectangles in $\mathbb{C}$:

$$R(h, \omega) = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z \leq h, \quad \omega - h \leq \operatorname{Im} z < \omega + h\}, \quad (1.4)$$

for any $h + i\omega \in \mathbb{C}_0$ ($i = \sqrt{-1}$). We denote by b_k the k -th row of B . Then $b_k^* b_k$ is an infinite matrix of rank one.

Definition 1.1. The infinite matrix B , with rows b_k , satisfies the *operator Carleson measure criterion* for the semigroup $\mathbb{T}$ defined by (1.3) if $b_k \in l^{2*}$ for any $k \in \mathbb{N}$ and there is some $M \geq 0$ such that for any $h + i\omega \in \mathbb{C}_0$,

$$\left\| \sum_{-\lambda_k \in R(h, \omega)} b_k^* b_k \right\|_{\mathcal{L}(l^2)} \leq Mh. \quad (1.5)$$

We denote by $\text{OCM}(\mathbb{T})$ the space of infinite matrices which satisfy the operator Carleson measure criterion for $\mathbb{T}$. The following two results concerning the operator Carleson measure criterion were proved in [7].

Theorem 1.2. *Assume $\mathbb{T}$ defined by (1.3) is exponentially stable and the infinite matrix B represents an admissible control operator for $\mathbb{T}$. Then $B \in \text{OCM}(\mathbb{T})$.*

Theorem 1.3. *Assume $\mathbb{T}$ defined by (1.3) is exponentially stable and either analytic or invertible. If $B \in \text{OCM}(\mathbb{T})$ then B represents an admissible control operator for $\mathbb{T}$.*

Remark 1.4. By modifying the definition of the rectangles $R(h, \omega)$ in (1.4) (by left translation), (1.5) can be used to check admissibility in cases where the semigroup is not assumed to be exponentially stable. Furthermore, to use the operator Carleson measure criterion, it is not necessary to verify (1.5) for all $h + i\omega \in \mathbb{C}_0$; see Proposition 6.1. It is also worth noting that the rectangles $R(h, \omega)$ defined in (1.4) could have been replaced by the analogous set of translations and dilations of *any* rectangle, or of a half-disk, or certain other sets, without changing the meaning of Definition 1.1. (Our choice of 2×1 rectangles follows Ho and Russell [8].)

Our main results are in Sections 5 and 6. The following theorem lists some of their consequences, thus indicating the general direction of this paper.

Theorem 1.5. *Assume the semigroup $\mathbb{T}$ is given by (1.2), (1.3) and B is an infinite matrix with rows b_k in l^{2*} .*

- (i) *B is an infinite-time admissible control operator for $\mathbb{T}$ if and only if the infinite matrix $P = (p_{jk})$, whose entries are given by*

$$p_{jk} = -\frac{\langle b_j, b_k \rangle}{\lambda_j + \bar{\lambda}_k}$$

represents a bounded operator on l^2 .

- (ii) *$B \in \text{OCM}(\mathbb{T})$ if and only if there exists a $K \geq 0$ for which*

$$\sup_{\|v\|_{l^2} \leq 1} \sum_{k=1}^{\infty} \left| \frac{b_k v}{s - \lambda_k} \right|^2 \leq \frac{K}{\text{Re } s}, \quad \forall s \in \mathbb{C}_0. \quad (1.6)$$

- (iii) *Suppose that there are numbers $0 < a \leq b$ and $\alpha \geq 0$ such that*

$$a|\text{Im } \lambda_k|^\alpha \leq -\text{Re } \lambda_k \leq b|\text{Im } \lambda_k|^\alpha. \quad (1.7)$$

Then $B \in \text{OCM}(\mathbb{T})$ if and only if B is an infinite-time admissible control operator for $\mathbb{T}$.

Some comments about this theorem are in order.

Concerning (i), the infinite matrix P is the *controllability Gramian* for (1.1). If P is bounded then it is the unique positive solution of a Lyapunov equation, see Section 5. The equivalence between infinite-time admissibility and boundedness of the controllability Gramian holds also for non-diagonal semigroups, see Theorem 3.1.

Concerning (ii), when B satisfies the operator Carleson measure criterion for $\mathbb{T}$, the left-hand side of (1.6) is the norm squared of the input-to-state transfer function, i.e., $\|(sI - A)^{-1}B\|_{\mathcal{L}(l^2)}^2$ (see Sect. 5). The condition $\|(sI - A)^{-1}B\|^2 \leq K/\operatorname{Re} s$ is necessary for the infinite-time admissibility of B also for nondiagonal semigroups, see Section 2. It has been conjectured in [20] that it is sufficient as well. (In [20] it is assumed that $\mathbb{T}$ is exponentially stable, but this difference is not significant.)

A result similar to (iii) was announced in [7, Section 5]. The geometric restriction on the position of the eigenvalues in (1.7) can be substantially relaxed such that it is almost always satisfied in physically motivated systems of the form (1.1)–(1.3); see Propositions 6.7, 6.8. It is unknown whether or not the equivalence in (iii) remains true without any geometric restriction on the eigenvalues (see [7, Conjecture 4.4]).

This paper is organized as follows. In Section 2 we provide background on admissible and infinite-time admissible control operators. In Section 3 we consider general systems of the type (1.1), without the restrictive assumption that A is diagonal. In this setting, we prove that infinite-time admissibility is equivalent to the existence of the controllability Gramian, which in turn is equivalent to solvability of a Lyapunov equation (see Theorem 3.1 for the precise statement). In Section 4 we prove the feedback invariance of admissibility (this is needed for a proof in Section 6). In Section 5 we prove a stronger version of the results in Section 3 for the case where A is diagonal and also prove (i) and (ii) of Theorem 1.5. In Section 6 we extend the results of [7] to a larger class of diagonal semigroups. In particular, we give some conditions less restrictive than (1.7) under which the equivalence in (iii) of Theorem 1.5 holds.

2. Some background.

In this section we give some general facts about admissible and infinite-time admissible control operators, following Hansen and Weiss [7], Ho and Russell [8], Salamon [13], [14] and Weiss [16], [20] (our notation follows [7] and [16]).

We need a notation for some spaces which will be used. Suppose A is the generator of a strongly continuous semigroup $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ on the Hilbert space X . For any $n \in \mathbb{N}$ define the space X_{-n} as the completion of X with respect to the norm

$$\|x\|_{-n} = \|(\beta I - A)^{-n}x\|,$$

where $\beta \in \rho(A)$ (X_{-n} does not depend upon β). We put $X_0 = X$. Then $(\beta I - A)^{-1}$ extends to an isomorphism from X_{-n} to X_{-n+1} . $\mathbb{T}$ extends to a strongly continuous semigroup on X_{-n} , whose generator is an extension of A , with domain X_{-n+1} . The extended semigroup is isomorphic to the initial one. We denote the extensions of $\mathbb{T}$ and A by the same symbols. Let Z_n be the Hilbert space obtained by endowing $D((A^*)^n)$ with the norm

$$\|x\|_n = \|(\beta I - A^*)^n x\|.$$

We identify X with X^* . It follows that $Z_n^* = X_{-n}$ for any $n \in \mathbb{N}$. For $z \in Z_n$ and $x \in X_{-n}$, we denote by $\langle z, x \rangle$ the duality pairing which reduces to the usual scalar product on X if $x \in X$.

Definition 2.1. With the above notation, let U be a Hilbert space and $B \in \mathcal{L}(U, X_{-1})$. Then B is said to be an *admissible control operator* for $\mathbb{T}$, if for some $\tau > 0$ and any $u \in L^2([0, \infty), U)$ we have $\tilde{\Phi}_\tau u \in X$, where $\tilde{\Phi}_\tau u$ is defined by

$$\tilde{\Phi}_\tau u = \int_0^\tau \mathbb{T}_\sigma B u(\sigma) d\sigma.$$

If B is admissible then for any $\tau \geq 0$, $\tilde{\Phi}_\tau$ defined above is a bounded linear operator from $L^2([0, \infty), U)$ to X (this follows from the closed graph theorem). In other words, for each $\tau \geq 0$ there is a $k_\tau \geq 0$ such that

$$\|\tilde{\Phi}_\tau u\|_X \leq k_\tau \|u\|_{L^2} \quad \forall u \in L^2([0, \infty), U). \quad (2.1)$$

The concept of admissibility is important because it is equivalent to the solvability, in a reasonable sense, of the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t). \quad (2.2)$$

More precisely, if B is admissible, then for any $x_0 \in X$ and any $u \in L^2_{loc}([0, \infty), U)$, the X -valued function x defined on $[0, \infty)$ by

$$x(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} B u(s) ds$$

is continuous (in X), and it is a strong solution of (2.2) (in X_{-1}). Any *abstract linear control system* may be represented in the form (2.2), with admissible $B \in \mathcal{L}(U, X_{-1})$, see [15], [16] for the definition and for details.

The space $\mathcal{B}(U, X, \mathbb{T})$ of all admissible control operators for $\mathbb{T}$ with domain U is a subspace of $\mathcal{L}(U, X_{-1})$. This space becomes a Banach space with the norm

$$\|B\|_\tau = \sup_{\|u\|_{L^2} \leq 1} \|\tilde{\Phi}_\tau u\|_X,$$

where the choice of $\tau > 0$ is unimportant for the topology of $\mathcal{B}(U, X, \mathbb{T})$.

Definition 2.2. With the above notation, an operator $B \in \mathcal{L}(U, X_{-1})$ is *infinite-time admissible* for $\mathbb{T}$ if $B \in \mathcal{B}(U, X, \mathbb{T})$ and for any $u \in L^2([0, \infty), U)$, the function $\tau \rightarrow \tilde{\Phi}_\tau u$ (from $[0, \infty)$ to X) is bounded.

If B is infinite-time admissible then the constant k_τ appearing in (2.1) can be chosen to be independent of τ (this follows from the uniform boundedness theorem):

$$\|\tilde{\Phi}_\tau u\|_X \leq k \|u\|_{L^2} \quad \forall u \in L^2([0, \infty), U). \quad (2.3)$$

It is easy to see that an admissible B is infinite-time admissible if and only if, for any $u \in L^2([0, \infty), U)$, the strong solution x of (2.2) with $x(0) = 0$ is bounded (in X).

We denote by $\tilde{\mathcal{B}}(U, X, \mathbb{T})$ the space of all infinite-time admissible control operators for $\mathbb{T}$ with domain U . $\tilde{\mathcal{B}}(U, X, \mathbb{T})$ becomes a Banach space with the norm

$$\|B\|_\infty = \lim_{\tau \rightarrow \infty} \|B\|_\tau. \quad (2.4)$$

(The completeness of this space follows from the completeness of $\mathcal{B}(U, X, \mathbb{T})$.) If the semigroup $\mathbb{T}$ is exponentially stable then $\tilde{\mathcal{B}}(U, X, \mathbb{T}) = \mathcal{B}(U, X, \mathbb{T})$. Otherwise (even if $\mathbb{T}$ is strongly stable) the two notions of admissibility are not equivalent; see Remark 2.8.

A simple but important fact is that $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$ if and only if there exists $\tilde{\Phi} \in \mathcal{L}(L^2([0, \infty), U), X)$ such that for any $u \in L^2([0, \infty), U)$

$$\tilde{\Phi}u = \lim_{\tau \rightarrow \infty} \tilde{\Phi}_\tau u \quad (\text{in } X). \quad (2.5)$$

In this formula, by writing “in X ” we mean that the limit converges in X . In order to prove (2.5), we use (2.3) to show that for $0 \leq \tau \leq t$

$$\|\tilde{\Phi}_t u - \tilde{\Phi}_\tau u\|_X \leq k \|u\|_{L^2([\tau, t], U)}.$$

Clearly $\|\tilde{\Phi}\| \leq k$, k being the constant appearing in (2.3).

Let $\alpha \geq 0$ be such that for any $\beta > \alpha$, $e^{-\beta t} \|\mathbb{T}_t\| \rightarrow 0$ as $t \rightarrow \infty$. Let $\mathbb{C}_\alpha$ denote the set of complex numbers s with $\operatorname{Re} s > \alpha$. By taking in (2.5) $u(t) = v e^{-st}$, where $v \in U$ and $s \in \mathbb{C}_\alpha$, and estimating $\|\tilde{\Phi}u\|$, we obtain that for $K = k^2/2$

$$\|(sI - A)^{-1} B\|_{\mathcal{L}(U, X)}^2 \leq \frac{K}{\operatorname{Re} s}, \quad \forall s \in \mathbb{C}_\alpha.$$

Thus, the above estimate is a necessary condition for the infinite-time admissibility of B . It has been conjectured in Weiss [20] that it is sufficient as well, and various partial results in this direction have been obtained in [7] and [20] (these papers assume that the semigroup is exponentially stable, but this is not a significant restriction). Related material is contained in the recent paper of Grabowski [5].

We give now the dual formulation of the concepts introduced above. For any $B \in \mathcal{L}(U, X_{-1})$ and any $\tau > 0$, the dual of $\tilde{\Phi}_\tau$ is an operator in $\mathcal{L}(Z_1, L^2([0, \infty), U))$ (we make the identification $U = U^*$) which is given by

$$(\tilde{\Phi}_\tau^* x)(t) = \begin{cases} B^* \mathbb{T}_t^* x & t \leq \tau, \\ 0 & t > \tau, \end{cases} \quad \forall x \in Z_1.$$

It follows that $B \in \mathcal{B}(U, X, \mathbb{T})$ if and only if for some (hence for any) $\tau > 0$, $\tilde{\Phi}_\tau^*$ extends continuously to X . In other words, there is a $k_\tau \geq 0$ such that

$$\int_0^\infty \|(\tilde{\Phi}_\tau^* x)(t)\|_U^2 dt \leq k_\tau^2 \|x\|_X^2, \quad \forall x \in Z_1.$$

We have $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$ if and only if B is admissible and the constant k_τ appearing above can be chosen to be independent of τ . Assume $B \in \mathcal{L}(U, X_{-1})$ and define for every $x \in Z_1$ the function Ψx on $[0, \infty)$ by

$$(\Psi x)(t) = B^* \mathbb{T}_t^* x, \quad \forall x \in Z_1.$$

It is now clear that $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$ if and only if there is a $k \geq 0$ (in fact the same as in (2.3)) such that

$$\int_0^\infty \|(\Psi x)(t)\|_U^2 dt \leq k^2 \|x\|_X^2, \quad \forall x \in Z_1. \quad (2.6)$$

Equivalently, $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$ if and only if Ψ has an extension to X which is bounded as an operator from X to $L^2([0, \infty), U)$. This extension, still denoted Ψ , is the adjoint of the operator $\tilde{\Phi}$ defined in (2.5):

$$\Psi = \tilde{\Phi}^*. \quad (2.7)$$

A formula for Ψ which is valid on X will be given in Remark 3.6.

In the remainder of this section we particularize to the case where $\mathbb{T}$ is diagonal, given by (1.2) and (1.3). When $U = \mathbb{C}$, the space $\tilde{\mathcal{B}}(\mathbb{C}, l^2, \mathbb{T})$ can be identified with a subspace of l^2_{-1} , which we denote by $\tilde{\mathfrak{b}}(l^2, \mathbb{T})$. The following is a slight variation of a theorem of Ho and Russell [8] and Weiss [17].

Theorem 2.3. *Assume $\mathbb{T}$ is given by (1.2), (1.3) and $b = (b_k)_{k \in \mathbb{N}}$ is a sequence of complex numbers. Then $b \in \tilde{\mathfrak{b}}(l^2, \mathbb{T})$ if and only if there exists $M > 0$ such that for any $h + i\omega \in \mathbb{C}_0$*

$$\sum_{-\lambda_k \in R(h, \omega)} |b_k|^2 \leq Mh. \quad (2.8)$$

Theorem 2.3 was proved for the case where $\mathbb{T}$ is exponentially stable in [8] (the ‘if’ part) and [17] (the ‘only if’ part). We leave it to the reader to verify that the same proofs remain valid under the weaker hypothesis (1.2).

The elements of $\tilde{\mathfrak{b}}(l^2, \mathbb{T})$ are called *infinite-time admissible input elements* and the inequality (2.8) is called the *Carleson measure criterion* for the infinite-time admissibility of input elements. It is easy to see that the operator Carleson measure criterion (1.5) reduces to (2.8) when the infinite matrix B has only one column.

In Theorems 2.4–2.6, we give some consequences of Theorem 2.3 which parallel Proposition 3.3, Theorem 1.2 and Proposition 5.1, respectively, of [7]. The proofs are exactly the same as those in [7] provided we replace $\mathfrak{b}$ and $\mathcal{B}$ by $\tilde{\mathfrak{b}}$ and $\tilde{\mathcal{B}}$, respectively.

Theorem 2.4. *Assume $\mathbb{T}$ is given by (1.2), (1.3) and B is an infinite matrix. Then $B \in \text{OCM}(\mathbb{T})$ if and only if B represents a bounded operator from l^2 to $\tilde{\mathfrak{b}}(l^2, \mathbb{T})$.*

Theorem 2.5. *With $\mathbb{T}$ and B as above, if $B \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$ then $B \in \text{OCM}(\mathbb{T})$.*

Theorem 2.6. *With $\mathbb{T}$ and B as above, suppose that the rows of B have at most m nonzero entries ($m \in \mathbb{N}$). Then $B \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$ if and only if $B \in \text{OCM}(\mathbb{T})$.*

Remark 2.7. Suppose $\mathbb{T}$ is given by (1.2), (1.3) on the state space $X = l^2$. In what concerns admissibility for $\mathbb{T}$ of control operators defined on an arbitrary infinite dimensional Hilbert space U , we may restrict our attention, without loss of generality, to the case $U = l^2$ (the approach taken in this paper). Indeed, since l^2_{-1} is separable, the orthogonal complement of $\text{Ker } B$ in U is separable as well (it is the range of B^*). Restricting B to this subspace then is equivalent to having B defined on l^2 . Since any operator from l^2

to l_{-1}^2 is represented by an infinite matrix, there is no loss of generality in considering only operators B which are represented by infinite matrices.

Remark 2.8. In the notation of Theorems 2.3 and 2.4, it is *not* true in general that $l^2 \subset \tilde{b}(l^2, \mathbb{T})$. In fact if $\mathbb{T}$ is any semigroup on a Hilbert space X and $X \subset \tilde{b}(X, \mathbb{T})$ then (2.6) implies that

$$\int_0^\infty |\langle \mathbb{T}_t^* x, b \rangle|^2 dt < \infty, \quad \forall x \in X, \quad \forall b \in X,$$

i.e., $\mathbb{T}^*$ is weakly L^2 -stable. This implies (see Weiss [18]) that $\mathbb{T}^*$ (and hence also $\mathbb{T}$) is exponentially stable. On the other hand it is clear from the definition that $X \subset \tilde{b}(X, \mathbb{T})$ regardless of the stability of $\mathbb{T}$.

3. The Lyapunov equation and the controllability Gramian.

In this section we describe the relationship between infinite-time admissibility and the Lyapunov equation. This connection has been investigated by Levan [11] (who assumed that $B \in \mathcal{L}(U, X)$) and by Grabowski [4]. Related results have appeared in Russell and Weiss [12]. The main result of this section is the following theorem, parts of which are contained in [4]. We use the notation of the last section.

Theorem 3.1. *Let $\mathbb{T}$ be a strongly continuous semigroup on the Hilbert space X , with generator A . Let U be a Hilbert space and assume $B \in \mathcal{L}(U, X_{-1})$. Then the following three statements are equivalent:*

- (i) B is an infinite-time admissible control operator for $\mathbb{T}$.
- (ii) There exists an operator $P \in \mathcal{L}(X)$ such that for any $x \in Z_1$,

$$Px = \lim_{\tau \rightarrow \infty} \int_0^\tau \mathbb{T}_t B B^* \mathbb{T}_t^* x dt \quad (\text{in } X). \quad (3.1)$$

- (iii) There exist operators $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$, which satisfy the following equation with terms in $\mathcal{L}(Z_1, X_{-1})$:

$$A \Pi + \Pi A^* = -B B^*. \quad (3.2)$$

Moreover, if B is infinite-time admissible, then the following two statements are true:

- (I) P defined in (3.1) is the smallest positive solution of (3.2). In other words $P \geq 0$, P satisfies (3.2) and if $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$ and (3.2) holds, then $P \leq \Pi$.
- (II) For any $x \in X$,

$$\lim_{t \rightarrow \infty} P^{\frac{1}{2}} \mathbb{T}_t^* x = 0 \quad (\text{in } X).$$

Proof. First we shall prove that (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). As in Section 2, we denote $(\Psi x)(t) = B^* \mathbb{T}_t^* x$, for any $x \in Z_1$ and any $t \geq 0$.

- (i) $\Rightarrow$ (ii): Assume (i) holds. Then $\tilde{\Phi}$ defined in (2.5) is a bounded operator from $L^2([0, \infty), U)$ to X . We define $P = \tilde{\Phi} \tilde{\Phi}^*$, so that $P \in \mathcal{L}(X)$. Then (2.5) and (2.7) show that for any $x \in Z_1$, Px is given by (3.1), so that (ii) holds.

- (ii) $\implies$ (i): Let $P \in \mathcal{L}(X)$ be defined by (3.1) (this formula defines P since Z_1 is dense in X). Since for any $x \in Z_1$, $\|\Psi x\|^2 = \lim_{\tau \rightarrow \infty} \|\Psi x\|_{L^2([0, \tau], U)}^2$, we get

$$\|\Psi x\|^2 = \langle Px, x \rangle, \quad \forall x \in Z_1. \quad (3.3)$$

This shows that (2.6) holds (with $k^2 = \|P\|$), so B is infinite-time admissible.

- (ii) $\implies$ (iii): Let $P \in \mathcal{L}(X)$ be defined by (3.1). We show that (3.2) is satisfied for $\Pi = P$. Let $x, y \in Z_2$ and for $t \geq 0$ define $f(t) = \langle B^* \mathbb{T}_t^* x, B^* \mathbb{T}_t^* y \rangle$. Then f is continuously differentiable and

$$\frac{d}{dt} f(t) = \langle B^* \mathbb{T}_t^* A^* x, B^* \mathbb{T}_t^* y \rangle + \langle B^* \mathbb{T}_t^* x, B^* \mathbb{T}_t^* A^* y \rangle.$$

Integrating both sides on $[0, \tau]$ gives

$$f(\tau) - f(0) = \left\langle \int_0^\tau \mathbb{T}_t B B^* \mathbb{T}_t^* A^* x dt, y \right\rangle + \left\langle \int_0^\tau \mathbb{T}_t B B^* \mathbb{T}_t^* x dt, A^* y \right\rangle. \quad (3.4)$$

Since $A^* x \in Z_1$, by (ii) each of the above integrals converges (in X) as $\tau \rightarrow \infty$. Hence $\lim_{\tau \rightarrow \infty} f(\tau)$ also exists. Since by (ii) the integral $\int_0^\tau f(t) dt$ has a finite limit as $\tau \rightarrow \infty$, we must have $f(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$. We then let $\tau \rightarrow \infty$ in (3.4) to find that

$$\langle P A^* x, y \rangle + \langle A P x, y \rangle = -\langle B B^* x, y \rangle.$$

Since Z_2 is dense in Z_1 , the above equality holds for any $y \in Z_1$. This implies $(AP + PA^*)x = -BB^*x$, for any $x \in Z_2$. Since both $AP + PA^*$ and BB^* are in $\mathcal{L}(Z_1, X_{-1})$, again by a density argument P satisfies (3.2).

- (iii) $\implies$ (i): Assume $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$ and Π satisfies (3.2). For any $x \in X$ and any $t \in [0, \infty)$, we define $E_t(x)$ by $E_t(x) = \langle \Pi \mathbb{T}_t^* x, \mathbb{T}_t^* x \rangle$. Then $E_t(x) \geq 0$ and for any $x \in Z_1$, $E_t(x)$ is a continuously differentiable function of t . By (3.2) we have that for any $x \in Z_1$,

$$\frac{d}{dt} E_t(x) = -\langle B B^* \mathbb{T}_t^* x, \mathbb{T}_t^* x \rangle = -\|B^* \mathbb{T}_t^* x\|^2 \leq 0, \quad (3.5)$$

so that $E_t(x)$ is nonincreasing. Since $E_t(x)$ is a continuous function of x , from the density of Z_1 in X we conclude that for any $x \in X$, $E_t(x)$ is nonincreasing. This can be written in the following form: for $0 \leq \tau \leq t$,

$$\mathbb{T}_t \Pi \mathbb{T}_t^* \leq \mathbb{T}_\tau \Pi \mathbb{T}_\tau^*.$$

It is a well known fact that any decreasing positive operator-valued function has a strong limit. Thus, there exists $\Pi_\infty \in \mathcal{L}(X)$, $\Pi_\infty \geq 0$, such that for all $x \in X$

$$\lim_{t \rightarrow \infty} \mathbb{T}_t \Pi \mathbb{T}_t^* x = \Pi_\infty x \quad (\text{in } X). \quad (3.6)$$

It is clear that $0 \leq \Pi_\infty \leq \Pi$. Integrating (3.5) on $[0, \infty)$ we get that for $x \in Z_1$

$$\langle \Pi x, x \rangle - \langle \Pi_\infty x, x \rangle = \int_0^\infty \|B^* \mathbb{T}_t^* x\|^2 dt = \|\Psi x\|_{L^2([0, \infty), U)}^2. \quad (3.7)$$

From here we see that (2.6) is satisfied, so that (i) holds.

Now assume that B is infinite-time admissible and let us prove statement (I). We see from (3.3) that $P \geq 0$. We have seen earlier that P satisfies (3.2). If $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$ and (3.2) holds, then by (3.3) and (3.7) we have that for all $x \in Z_1$

$$\langle Px, x \rangle = \langle \Pi x, x \rangle - \langle \Pi_\infty x, x \rangle, \quad (3.8)$$

so that $P \leq \Pi$, as claimed in (I). Finally, to prove (II) we take $\Pi = P$ in (3.6) and (3.8) and obtain $\Pi_\infty = 0$. By (3.6) this implies $\lim_{t \rightarrow \infty} \langle \mathbb{T}_t P \mathbb{T}_t^* x, x \rangle = 0$ for any $x \in X$, which is precisely (II). $\square$

If B is infinite-time admissible for $\mathbb{T}$, then P defined in (3.1) is called the *controllability Gramian* of $\mathbb{T}$ and B . Equation (3.2) is called a *Lyapunov equation* (this name is also used for slightly different equations). In several papers, the connection between the solvability of a Lyapunov equation and the stability of $\mathbb{T}$ was investigated; see, e.g., Levan [11] and the references therein. We shall briefly discuss this connection.

Suppose that B is infinite-time admissible for $\mathbb{T}$, so that the Gramian P defined in (3.1) exists. It is clear from (II) of Theorem 3.1 that if P is invertible (i.e., $P \geq \varepsilon I > 0$) then $\mathbb{T}^*$ is *strongly stable*, i.e., $\mathbb{T}_t^* x \rightarrow 0$ as $t \rightarrow \infty$, for any $x \in X$. This is the best possible result under the given assumptions, see Example 3.4 below.

With $\mathbb{T}$, B and P as above, it is clear from (3.6) and (3.8) that if $\mathbb{T}^*$ is strongly stable, then P is the unique positive solution of (3.2). (In [11] it is claimed that $\mathbb{T}$ uniformly bounded implies the uniqueness of P , which is wrong even if X is one dimensional.) If $\mathbb{T}$ is strongly stable but $\mathbb{T}^*$ is not, then (3.2) may have many positive solutions. For example, if $\mathbb{T}$ is the left shift semigroup on $L^2[0, \infty)$ and $B = 0$, then any multiple of the identity I satisfies (3.2).

From the preceding two paragraphs it follows that if B is infinite-time admissible and P is invertible, then P is the unique positive solution of (3.2).

The following proposition is a slight generalization of a result in [11] (where only the case $B \in \mathcal{L}(U, X)$ is considered).

Proposition 3.2. *With the notation of Theorem 3.1, assume that B is an infinite-time admissible control operator for $\mathbb{T}$. If $P > 0$ and $\mathbb{T}$ is uniformly bounded, then $\mathbb{T}$ is weakly stable, i.e., $\langle \mathbb{T}_t x, y \rangle \rightarrow 0$ as $t \rightarrow \infty$, for any $x, y \in X$.*

Weak stability is the strongest possible conclusion under the assumptions of the proposition, as Example 3.5 shows.

Proof. Denote $V = \text{Ran } P^{\frac{1}{2}}$, then V is dense in X (because $\bar{V}$ is the orthogonal complement of $\text{Ker } P^{\frac{1}{2}} = \text{Ker } P = \{0\}$). It follows from (II) of Theorem 3.1 that for any $x \in X$ and any $v \in V$, $\lim_{t \rightarrow \infty} \langle \mathbb{T}_t x, v \rangle = 0$. Let $x, y \in X$ be fixed. We claim that for any $\varepsilon > 0$ we can find $T \geq 0$ such that $\langle \mathbb{T}_t x, y \rangle \leq \varepsilon$ for each $t \geq T$. Indeed, let $v \in V$ be such that $\langle \mathbb{T}_t x, y - v \rangle \leq \frac{\varepsilon}{2}$ for all $t \geq 0$ (this is possible by the uniform boundedness of $\mathbb{T}$). Now if T is such that $\langle \mathbb{T}_t x, v \rangle \leq \frac{\varepsilon}{2}$ for all $t \geq T$, then T is the desired number. The existence of such a T for any $\varepsilon > 0$ means that $\langle \mathbb{T}_t x, y \rangle \rightarrow 0$. $\square$

Remark 3.3. After Proposition 3.2 it is worth mentioning the following facts: Suppose that $\mathbb{T}$ is a weakly stable (hence uniformly bounded) semigroup on the Hilbert space X , and let A denote its generator.

(a) If for some (hence for any) $s \in \rho(A)$, $(sI - A)^{-1}$ is compact, then $\mathbb{T}$ and $\mathbb{T}^*$ are strongly stable. This is well known and easy to prove.

(b) If $\sigma(A) \cap i\mathbb{R}$ is at most countable, then $\mathbb{T}$ and $\mathbb{T}^*$ are strongly stable. This follows from the stability theorem of Arendt and Batty [1].

Example 3.4. Let $\mathbb{T}$ be the right shift semigroup on $X = L^2[0, \infty)$. We take $U = \mathbb{C}$ and $B = \delta_0$, i.e., $B^*x = x(0)$ for each $x \in Z_1 = H^1[0, \infty)$. Then it is not difficult to see that B is infinite-time admissible and $P = I$. Since P is invertible, we have that $\mathbb{T}^*$ is strongly stable, but $\mathbb{T}$ is not strongly stable.

Example 3.5. This is a refinement of the preceding example. Let $\mathbb{T}$ be the right shift semigroup on $X = L^2(\mathbb{R})$. We take $U = l^2$ and decompose $Bv = b_1v_1 + b_2v_2 + b_3v_3\ldots$, for any $v = (v_1, v_2, v_3\ldots) \in l^2$. We define the components of B by $b_k = 2^{-k/2}\delta_{-k}$, i.e., $b_k^*x = 2^{-k/2}x(-k)$ for each $x \in Z_1 = H^1(\mathbb{R})$. Then it is not difficult to see that B is infinite-time admissible and for any $x \in X$, $(Px)(\xi) = \varphi(\xi)x(\xi)$, where $\varphi(\xi) = \sum_{k \geq -\xi} 2^{-k}$. In particular, $\varphi(\xi) = 1$ for $\xi \in [-1, \infty)$ and $\varphi(\xi)$ decreases rapidly as $\xi \rightarrow -\infty$. Since $\varphi(\xi) > 0$ everywhere, we have $P > 0$. By Proposition 3.2 $\mathbb{T}$ is weakly stable, but no stronger stability concepts are true for $\mathbb{T}$, since it is unitary.

Remark 3.6. Assume B is an admissible control operator for $\mathbb{T}$, and let Ψ be as in (2.7). We have seen in Section 2 that for any $x \in Z_1$, $(\Psi x)(t) = B^*\mathbb{T}_t^*x$. In order to obtain a formula valid for any $x \in X$, we may replace B^* by its Λ -extension B_Λ^* . We shall say more about this extension in Section 4. We have that for any $x \in X$ and almost every $t \geq 0$,

$$(\Psi x)(t) = B_\Lambda^*\mathbb{T}_t^*x.$$

Together with $P = \tilde{\Phi}\Psi$ this leads to the following expression for the controllability Gramian P (valid for any $x \in X$):

$$Px = \lim_{\tau \rightarrow \infty} \int_0^\tau \mathbb{T}_t B B_\Lambda^* \mathbb{T}_t^* x \, dt \quad (\text{in } X).$$

4. Feedback invariance of admissibility.

In this section we first briefly recall some facts about regular linear systems and especially about feedback for such systems, following Weiss [21], [22], as needed for the proof of our feedback invariance result which is given at the end of this section. We assume that the reader has some familiarity with the concept of regular linear system, as presented in [21], however we do not assume knowledge of the results on feedback contained in the more recent paper [22].

An *abstract linear system* is a linear time-invariant system such that on any finite time interval, the operator from the initial state and the input function to the final state and the output function is bounded. The input, state and output spaces are Hilbert spaces and the input and output functions are of class L^2_{loc} . For the detailed definition we refer to Salamon [14] or to [21]. The input to output operator of any abstract linear system can be described by a *transfer function*, which is an analytic operator-valued function defined on some right half-plane in $\mathbb{C}$. The transfer function of any abstract

linear system is *well posed*, meaning that it is bounded on some right half-plane. We do not distinguish between two transfer functions defined on two different right half-planes, if one function is a restriction of the other (thus, by a transfer function we mean in fact an equivalence class of analytic functions).

Let Σ be an abstract linear system, with input space U , state space X and output space Y . Let $\mathbb{T}$ be the *semigroup of Σ* , i.e., the strongly continuous semigroup on X which describes the evolution of the state of Σ if the input function is zero. Let A denote the generator of $\mathbb{T}$. The Hilbert space X_{-1} is defined as in Section 2 and we denote the extension of $\mathbb{T}$ to X_{-1} and the extension of A to X by the same symbols. The state of Σ at any moment $\tau \geq 0$ can be expressed by the formula

$$x(\tau) = \mathbb{T}_\tau x(0) + \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) d\sigma$$

(see [15],[16]). Here $u \in L^2_{loc}([0, \infty), U)$ is the input function and $B \in \mathcal{L}(U, X_{-1})$ is the *control operator* of Σ . We have $x(\tau) \in X$ and $x(\tau)$ depends continuously on τ , on $x(0)$ and on the restriction of u to $[0, \tau]$ (which is in $L^2([0, \tau], U)$). Thus B is an admissible control operator for $\mathbb{T}$, as defined in Section 2.

The domain $D(A)$ with the norm $\|x\|_1 = \|(\beta I - A)x\|$ becomes a Hilbert space, which we denote X_1 . Here, the choice of $\beta \in \rho(A)$ is irrelevant for the topology of X_1 . (This space resembles Z_1 defined in Section 2.) If $u = 0$ and $x(0) \in X_1$ then the output function of Σ on $[0, \infty)$ is (see [15],[19])

$$y(t) = C \mathbb{T}_t x(0) .$$

Here $C \in \mathcal{L}(X_1, Y)$ is the *observation operator* of Σ . C is an *admissible observation operator* for $\mathbb{T}$, which means that for some (hence for any) $\tau > 0$, there exists a $k_\tau \geq 0$ such that

$$\int_0^\tau \|C \mathbb{T}_t x\|_Y^2 dt \leq k_\tau^2 \|x\|_X^2, \quad \forall x \in X_1 .$$

It is clear that C is an admissible observation operator for $\mathbb{T}$ if and only if C^* is an admissible control operator for $\mathbb{T}^*$ (see Section 2). The Λ -*extension* of C is defined by

$$C_\Lambda x_0 = \lim_{\lambda \rightarrow +\infty} C \lambda (\lambda I - A)^{-1} x_0 \quad (4.1)$$

(λ is real), for all x_0 in the domain

$$D(C_\Lambda) = \{x_0 \in X \mid \text{the limit in (4.1) exists}\} .$$

(In [21] this operator was denoted $\tilde{C}_\Lambda$.)

Let $\mathbf{G}$ denote the transfer function of Σ . $\mathbf{G}$ is called *regular* if the following limit exists for all $v \in U$:

$$Dv = \lim_{\lambda \rightarrow +\infty} \mathbf{G}(\lambda)v \quad (4.2)$$

(λ is real). Then $D \in \mathcal{L}(U, Y)$ is called the *feedthrough operator* of $\mathbf{G}$. If $\mathbf{G}$ is regular then Σ is called a *regular linear system*, abbreviated RLS. The regularity condition (4.2)

can be formulated in many different ways, of which we mention the following: Σ is regular if and only if the product $C_\Lambda(sI - A)^{-1}B$ makes sense for some (hence for any) $s \in \rho(A)$, i.e.,

$$\text{Ran } (sI - A)^{-1}B \subset D(C_\Lambda)$$

(see [21]). Another equivalent condition is that any step response of Σ has a Lebesgue point at $t = 0$. If Σ is regular then

$$\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D, \quad (4.3)$$

for any $s \in \mathbb{C}$ with $\text{Re } s$ sufficiently large. Moreover, in the time domain, Σ is completely described by the following equations:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (4.4a)$$

$$y(t) = C_\Lambda x(t) + Du(t), \quad (4.4b)$$

which hold for almost every $t \geq 0$ (in particular, $x(t) \in D(C_\Lambda)$ for a.e. $t \geq 0$). For any given $x(0) \in X$ and $u \in L^2_{loc}([0, \infty), U)$, $x(\cdot)$ is the unique strong solution (in X_{-1}) of (4.4a). The output function y belongs to $L^2_{loc}([0, \infty), Y)$.

The operators A , B , C and D appearing in (4.4) are called the *generating operators* of Σ , or its *generators* for short. We mention that in [21] (4.3) and (4.4) were stated (and proved) with C_L in place of C_Λ , where C_L is another extension of C . C_Λ is an extension of C_L and so it is clear that (4.3) and (4.4) are valid as stated here. We prefer to work with C_Λ because its definition is somewhat simpler than that of C_L , and because it is not known if formula (4.11) below is true for C_L as well.

Let U , X and Y be Hilbert spaces and A , B and C linear operators. We say that (A, B, C) is a *regular triple* on U , X and Y if for some (hence for any) $D \in \mathcal{L}(U, Y)$, A , B , C , D are the generators of an RLS with input, state and output spaces U , X and Y . We have the following:

Proposition 4.1. *(A, B, C) is a regular triple on U , X and Y if and only if the following five conditions are satisfied:*

- (1) A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X .
- (2) B is an admissible control operator for $\mathbb{T}$.
- (3) C is an admissible observation operator for $\mathbb{T}$.
- (4) $C_\Lambda(sI - A)^{-1}B$ makes sense for some (hence for any) $s \in \rho(A)$.
- (5) $s \rightarrow \|C_\Lambda(sI - A)^{-1}B\|_{\mathcal{L}(U, Y)}$ is bounded on some right half-plane.

Now we turn to feedback, following [22]. Let U and Y be Hilbert spaces, suppose $\mathbf{G}$ is an $\mathcal{L}(U, Y)$ -valued well posed transfer function and let $K \in \mathcal{L}(Y, U)$. K is an *admissible feedback operator* for $\mathbf{G}$ if $I - K\mathbf{G}$ is invertible on some right half-plane and its inverse is a well posed transfer function (equivalently, if $I - \mathbf{G}K$ has the same property). Then the function $\mathbf{G}^K$ defined by

$$\mathbf{G}^K(s) = \mathbf{G}(s)(I - K\mathbf{G}(s))^{-1} \quad (4.5)$$

is called the *closed loop transfer function* corresponding to $\mathbf{G}$ and K . We have

$$\mathbf{G}^K - \mathbf{G} = \mathbf{G}K\mathbf{G}^K = \mathbf{G}^K K\mathbf{G} .$$

Now suppose $\mathbf{G}$ is regular and let D be its feedthrough operator. If K is an admissible feedback operator for $\mathbf{G}$, then $\mathbf{G}^K$ (given by (4.5)) is regular iff $I - DK$ is invertible (equivalently, iff $I - KD$ is invertible). If $\mathbf{G}^K$ is regular then its feedthrough operator is

$$D^K = (I - DK)^{-1}D = D(I - KD)^{-1} . \quad (4.6)$$

Under certain additional assumptions (e.g., if at least one of the spaces U and Y is finite dimensional), the invertibility of $I - DK$ follows from the admissibility of K .

Fig. 1. The closed loop system Σ^K

Let Σ be an abstract linear system with transfer function $\mathbf{G}$. If K is an admissible feedback operator for $\mathbf{G}$ then there exists a unique *closed loop system* Σ^K corresponding to Σ and K , represented in Fig. 1. Σ^K is an abstract linear system and its transfer function is $\mathbf{G}^K$ from (4.5). For the precise definition of Σ^K we refer to [22].

Now suppose Σ is an RLS with generators A, B, C, D and suppose K and Σ^K are as before. If $I - DK$ is invertible then (as mentioned earlier) Σ^K is regular and we can compute its generators via formulas which are similar to those for finite dimensional systems. Let A^K, B^K, C^K, D^K be the generators of Σ^K . As we already know, D^K is given by (4.6). The formula for A^K is

$$A^K x = (A + BK(I - DK)^{-1}C_\Lambda) x , \quad (4.7)$$

defined for all x in the domain

$$D(A^K) = \{ x \in D(C_\Lambda) \mid (A + BK(I - DK)^{-1}C_\Lambda) x \in X \} . \quad (4.8)$$

We call $D(A^K)$ defined above the *natural domain* of $A + BK(I - DK)^{-1}C_\Lambda$.

The operators C^K and B^K are given by

$$C^K x = (I - DK)^{-1}C_\Lambda x , \quad \forall x \in D(A^K) , \quad (4.9)$$

$$B^K = B(I - KD)^{-1} . \quad (4.10)$$

The formulae for $A^K, D(A^K)$ and C^K given in [22] use C_L instead of C_Λ , but it is easy to see that this makes no difference. As already mentioned, in this paper we use

only C_Λ . To understand (4.10) better, it should be pointed out that $B^K \in \mathcal{L}(U, X_{-1}^K)$, where X_{-1}^K is the analogue of X_{-1} for Σ^K , but in fact both B and B^K are in $\mathcal{L}(U, W)$, where W is a Banach space such that $W \subset X_{-1} \cap X_{-1}^K$. W depends only on A and C , X is dense in W and $C_\Lambda(sI - A)^{-1}$ has a continuous extension to W .

The extensions C_Λ and C_Λ^K (defined as in (4.1)) are related by

$$D(C_\Lambda^K) = D(C_\Lambda), \quad C_\Lambda^K = (I - DK)^{-1}C_\Lambda. \quad (4.11)$$

It follows from (4.6), (4.10) and (4.11), by simple substitution into (4.3), that

$$\mathbf{G}^K(s) = (I - DK)^{-1} [C_\Lambda(sI - A^K)^{-1}B^K + D],$$

and

$$\mathbf{G}^K(s) = [C_\Lambda^K(sI - A^K)^{-1}B + D](I - KD)^{-1}.$$

The resolvents of A and A^K are related by

$$\begin{aligned} (sI - A^K)^{-1} - (sI - A)^{-1} &= (sI - A)^{-1}BK C^K(sI - A^K)^{-1} \\ &= (sI - A^K)^{-1}B^K KC(sI - A)^{-1}, \end{aligned}$$

for any $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large.

Proposition 4.2. *Let Σ be an RLS with generators A, B, C, D and transfer function $\mathbf{G}$. Let K be an admissible feedback operator for $\mathbf{G}$ such that $I - DK$ is invertible. Let $\mathbb{T}^K$ be the semigroup of the closed loop system corresponding to Σ and K . Then B is an admissible control operator for $\mathbb{T}^K$.*

Proof. Let Σ^K denote the closed loop system. Since B^K is the control operator of Σ^K and $\mathbb{T}^K$ is its semigroup, it is clear that B^K is an admissible control operator for $\mathbb{T}^K$. By (4.10) we can write $B = B^K(I - KD)$, so that taking an input function $u \in L^2([0, \infty), U)$ in the integral in Definition 2.1 for B and $\mathbb{T}^K$, is the same as taking the function $(I - KD)u$ in the integral for B^K and $\mathbb{T}^K$. Therefore this integral is in X , so that B is an admissible control operator for $\mathbb{T}^K$. $\square$

The above proof is very simple, but it is based on formula (4.10) whose proof (in [22]) is rather involved. Proposition 4.2 is much more general than what we actually need in Section 6. We shall use the particular case of an RLS Σ with generators A, B, C, D defined as follows: The state space $X = l^2$ and $\mathbb{T}$ is diagonal, given by (1.2) and (1.3). Thus, $A = \operatorname{diag}(\lambda_1, \lambda_2, \dots)$, where $-\lambda_k \in \mathbb{C}_0$. The input space U and the control operator B are decomposed as follows:

$$U = \begin{matrix} l^2 \\ \times \\ l^2 \end{matrix}, \quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

We take $Y = l^2$ and we assume that B_2 and C are represented by infinite diagonal matrices,

$$B_2 = \operatorname{diag}(b_1, b_2, \dots), \quad C = \operatorname{diag}(c_1, c_2, \dots).$$

The sequences (b_k) and (c_k) satisfy the estimates

$$|b_k|^2 \leq \frac{1}{2} |\operatorname{Re} \lambda_k|, \quad |c_k|^2 \leq \frac{1}{2} |\operatorname{Re} \lambda_k|.$$

We assume that the infinite matrix B_1 represents a bounded operator from l^2 to l^2_{-1} and we shall not distinguish between the matrix and this operator.

It is clear that the numbers $\lambda_k + b_k c_k$ are in $\mathbb{C}_0$. We define a new semigroup $\mathbb{T}^+$ on l^2 by

$$(\mathbb{T}^+_t z)_k = e^{(\lambda_k + b_k c_k)t} z_k, \quad \forall k \in \mathbb{N},$$

so that $\mathbb{T}^+$ is generated by $A^+ = \operatorname{diag}(\lambda_1 + b_1 c_1, \lambda_2 + b_2 c_2, \dots)$. Note that the space l^2_{-1} for this semigroup is the same as l^2_{-1} for the original semigroup $\mathbb{T}$.

Proposition 4.3. *With the above notation, if B_1 is such that (A, B_1, C) is a regular triple, then B_1 is an admissible control operator for $\mathbb{T}^+$.*

Proof. First we show that (A, B, C) is a regular triple. This is equivalent to the statement that both (A, B_1, C) and (A, B_2, C) are regular triples. For (A, B_1, C) this has been assumed in the proposition, while for (A, B_2, C) we must check the conditions listed in Proposition 4.1. The first condition is clearly satisfied. Since $B_2 \in \operatorname{OCM}(\mathbb{T})$ and $C^* \in \operatorname{OCM}(\mathbb{T}^*)$, by Theorem 2.6 (with $m = 1$) B_2 is an admissible control operator for $\mathbb{T}$ and C is an admissible observation operator for $\mathbb{T}$. Moreover, it is easy to check that the composition $C_\Lambda(sI - A)^{-1} B_2$ makes sense for any $s \in \mathbb{C}_0$. The function $s \rightarrow \|C_\Lambda(sI - A)^{-1} B_2\|$ is bounded on $\mathbb{C}_0$, since for all $s \in \mathbb{C}_0$

$$\begin{aligned} \|C_\Lambda(sI - A)^{-1} B_2\|_{\mathcal{L}(l^2)} &= \left\| \operatorname{diag} \left(\frac{c_1 b_1}{s - \lambda_1}, \frac{c_2 b_2}{s - \lambda_2}, \dots \right) \right\|_{\mathcal{L}(l^2)} \\ &\leq \sup_{k \in \mathbb{N}} \left| \frac{c_k b_k}{\operatorname{Re} \lambda_k} \right| \leq \frac{1}{2}. \end{aligned}$$

Thus we conclude that (A, B, C) is a regular triple.

We define Σ to be the RLS with generators A, B, C and 0. By (4.3) its transfer function is $\mathbf{G}(s) = [C_\Lambda(sI - A)^{-1} B_1 \quad C_\Lambda(sI - A)^{-1} B_2]$. We take $K = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and claim that K is an admissible feedback operator for $\mathbf{G}$. Indeed, it is not difficult to see that

$$\|\mathbf{G}(s)K\|_{\mathcal{L}(Y)} = \|C_\Lambda(sI - A)^{-1} B_2\|_{\mathcal{L}(l^2)},$$

so that $I - \mathbf{G}(s)K$ is invertible on $\mathbb{C}_0$ and the norm of its inverse is bounded (by the number 2). We conclude that there is a closed loop system Σ^K corresponding to Σ and K and let $\mathbb{T}^K$ denote its semigroup. Then from (4.7) and (4.8) (with $D = 0$) we get that the generator of $\mathbb{T}^K$ is

$$A^K = \operatorname{diag}(\lambda_1 + b_1 c_1, \lambda_2 + b_2 c_2, \dots),$$

with its natural domain. Now we see that $\mathbb{T}^K$ is exactly the semigroup $\mathbb{T}^+$ defined before the proposition. By Proposition 4.2 we have that B is admissible for $\mathbb{T}^K$, in particular B_1 is admissible for $\mathbb{T}^K$. $\square$

5. Results for diagonal semigroups.

In the remainder of this paper we shall restrict our attention to the case in which $\mathbb{T}$ is given by (1.2), (1.3) and B is an infinite matrix with rows $b_k \in l^{2*}$.

The following result contains a restatement of (i) of Theorem 1.5 and is analogous to Theorem 3.1 for the case of diagonal semigroups. Perhaps it is of interest apart from control theory, since it relates the boundedness of an infinite matrix which can be written as a controllability Gramian to infinite-time admissibility. Through (iii) of Theorem 1.5, whether or not B is infinite-time admissible can be tested in many cases by applying the operator Carleson measure criterion. (See also Section 6.)

Proposition 5.1. *Let the semigroup $\mathbb{T}$ be defined by (1.2), (1.3) and assume B is an infinite matrix with rows $b_k \in l^{2*}$. B^* denotes the conjugate transpose matrix of B . Then the following statements are equivalent:*

- (i) B is an infinite-time admissible control operator for $\mathbb{T}$, i.e., $B \in \mathcal{B}(l^2, l^2, \mathbb{T})$.
- (ii) The infinite matrix with entries

$$p_{jk} = -\frac{\langle b_j, b_k \rangle}{\lambda_j + \bar{\lambda}_k} \quad (5.1)$$

represents a bounded operator P on l^2 .

- (iii) The equation (3.2) has a positive solution in $\mathcal{L}(l^2)$.

Moreover, if (ii) holds then P defined there is the unique positive solution of (3.2).

Proof. Note that if we would have assumed that $B \in \mathcal{L}(l^2, l_{-1}^2)$, then this proposition would have followed very easily from Theorem 3.1.

First we prove that (i) $\Leftrightarrow$ (ii). If B is infinite-time admissible, then clearly $B \in \mathcal{L}(l^2, l_{-1}^2)$. By (i) $\Rightarrow$ (ii) of Theorem 3.1 there is a positive operator $P \in \mathcal{L}(l^2)$ satisfying (3.1). An easy calculation shows that the components of the infinite matrix representing P are given by (5.1). Thus, (p_{jk}) defines a bounded operator on l^2 .

Conversely, let $\mathcal{F}$ denote the space of complex sequences with at most finitely many nonzero entries. Then B^* maps $\mathcal{F}$ into l^2 . Let P be the bounded operator on l^2 represented by the matrix (p_{jk}) . For $x \in \mathcal{F}$ it is not hard to check that $\mathbb{T}^*x \in \mathcal{F}$ and

$$\int_0^\infty \|B^* \mathbb{T}_t^* x\|^2 dt = \langle Px, x \rangle \leq \|P\| \cdot \|x\|^2. \quad (5.2)$$

This implies that for any $s \in \mathbb{C}_0$ and any $x \in \mathcal{F}$

$$\|B^*(sI - A^*)^{-1}x\|^2 = \left\| \int_0^\infty e^{-st} B^* \mathbb{T}_t^* x dt \right\|^2 \leq \frac{\|P\|}{2 \operatorname{Re} s} \|x\|^2.$$

Since $\mathcal{F}$ is dense in l^2 , this shows that the operator $B^*(sI - A^*)^{-1}$ from $\mathcal{F}$ to l^2 can be extended to an operator in $\mathcal{L}(l^2)$. Since $(sI - A^*)^{-1}$ is an isomorphism from l^2 to Z_1 (as defined in Section 2), it follows that $B^* \in \mathcal{L}(Z_1, l^2)$, which is equivalent to $B \in \mathcal{L}(l^2, l_{-1}^2)$. This fact, together with the density of $\mathcal{F}$ in Z_1 implies that (5.2) remains

true for any $x \in Z_1$. Thus (2.6) holds, which means that B represents an infinite-time admissible control operator for $\mathbb{T}$.

Let us prove (i) $\Leftrightarrow$ (iii). If B is infinite-time admissible then (iii) follows immediately by Theorem 3.1. Conversely, if (3.2) has a bounded solution then it follows that BB^* represents a bounded operator from Z_1 to l^2_{-1} . This easily implies that $B \in \mathcal{L}(l^2, l^2_{-1})$, and now (i) follows by Theorem 3.1.

Finally, if we assume (i) then P is a positive solution of (3.2), according to (I) of Theorem 3.1. Since $\mathbb{T}^*$ is strongly stable, P is the unique positive solution of (3.2), as explained after the proof of Theorem 3.1. $\square$

Remark 5.2. Consider the following particular case of Proposition 5.1: $\lambda_k = -k$, so that $\mathbb{T}$ is analytic and positive. B has only one nonzero column and it consists only of ones. By Theorem 2.3, B is an infinite-time admissible control operator for $\mathbb{T}$. We have $\langle b_j, b_k \rangle = 1$ for all j, k , so that P is represented by the famous Hilbert matrix $p_{jk} = \frac{1}{j+k}$. As is well known, P is bounded on l^2 , and we have found a new proof of this fact.

The following is a reformulation of (ii) of Theorem 1.5.

Proposition 5.3. *Let the semigroup $\mathbb{T}$ be defined by (1.2), (1.3), let A be its generator and assume B is an infinite matrix. Then $B \in \text{OCM}(\mathbb{T})$ if and only if $B \in \mathcal{L}(l^2, l^2_{-1})$ and there exists a $K \geq 0$ such that*

$$\|(sI - A)^{-1}B\|_{\mathcal{L}(l^2)}^2 \leq \frac{K}{\text{Re } s}, \quad \forall s \in \mathbb{C}_0. \quad (5.3)$$

Proof. Assume first that $B \in \mathcal{L}(l^2, l^2_{-1})$ and (5.3) holds. Then the rows b_k of B must be in l^{2*} . Let $s = h - i\omega \in \mathbb{C}_0$. We observe that

$$\begin{aligned} \left\| \sum_{-\lambda_k \in R(h, \omega)} b_k^* b_k \right\|_{\mathcal{L}(l^2)} &= \sup_{\|v\| \leq 1} \left\langle \sum_{-\lambda_k \in R(h, \omega)} b_k^* b_k v, v \right\rangle \\ &= \sup_{\|v\| \leq 1} \sum_{-\lambda_k \in R(h, \omega)} \langle b_k^* b_k v, v \rangle \\ &= \sup_{\|v\| \leq 1} \sum_{-\lambda_k \in R(h, \omega)} |b_k v|^2. \end{aligned}$$

Since for $-\lambda_k \in R(h, \omega)$ we have $|s - \lambda_k|^2 \leq 5h^2$, we can write

$$\begin{aligned} \sum_{-\lambda_k \in R(h, \omega)} |b_k v|^2 &\leq \sum_{-\lambda_k \in R(h, \omega)} \frac{|b_k v|^2}{|s - \lambda_k|^2} \cdot 5h^2 \\ &\leq 5h^2 \sum_{k=1}^{\infty} \left| \frac{b_k v}{s - \lambda_k} \right|^2 \\ &= 5h^2 \|(sI - A)^{-1}Bv\|_{l^2}^2. \end{aligned}$$

Now (5.3) implies that (1.5) holds with $M = 5K$.

Conversely, assume that $B \in \text{OCM}(\mathbb{T})$. By Theorem 2.4, $B \in \mathcal{L}(l^2, \tilde{\mathfrak{b}}(l^2, \mathbb{T}))$ (in particular, $B \in \mathcal{L}(l^2, l^2_{-1})$). For any $\mathbf{v} \in l^2$ and any $f \in L^2[0, \infty)$ with $\|f\| \leq 1$, we have

$$\left\| \int_0^\infty \mathbb{T}_t B \mathbf{v} f(t) dt \right\|_{l^2} \leq \|B \mathbf{v}\|_\infty \leq m \|\mathbf{v}\|_{l^2}, \quad (5.4)$$

where m is independent of $\mathbf{v}$. Now let $s \in \mathbb{C}_0$ and use (5.4) with $f(t) = e^{-st} \sqrt{2\operatorname{Re} s}$ to obtain

$$\sqrt{2\operatorname{Re} s} \|(sI - A)^{-1} B \mathbf{v}\|_{l^2} \leq m \|\mathbf{v}\|_{l^2},$$

which implies (5.3) (with $K = m^2/2$). $\square$

6. Extensions of Theorem 1.3.

As we have mentioned, for *arbitrary* systems of form (1.1)–(1.3) we do not know if infinite time admissibility is completely determined by the operator Carleson measure criterion, i.e., if $\text{OCM}(\mathbb{T}) = \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$. However, as Theorems 1.2 and 1.3 indicate, for many semigroups $\mathbb{T}$ we know this to be the case. In this final section we show that $\text{OCM}(\mathbb{T}) = \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$ for a much larger class of semigroups than those in Theorems 1.2 and 1.3. Throughout this section we assume $\mathbb{T}$ and B to be as in Theorem 1.5.

We first introduce some notation which we shall use to reformulate the operator Carleson measure criterion. For any $h > 0$ and $\omega \in \mathbb{R}$ let

$$Q(h, \omega) = \left\{ z \in \mathbb{C} \mid \frac{h}{2} < \operatorname{Re} z \leq h, \omega - h \leq \operatorname{Im} z < \omega + h \right\}, \quad (6.1)$$

and let

$$\mathcal{T} = \{(2^m, 2^m(1 + 2n)) \mid m, n \in \mathbb{Z}\}. \quad (6.2)$$

The rectangles $\{Q(h, \omega) \mid (h, \omega) \in \mathcal{T}\}$ form a tiling of $\mathbb{C}_0$, as indicated in Figure 2. Define a partial order $\prec$ on $\mathcal{T}$ by

$$(g, \sigma) \prec (h, \omega) \iff R(g, \sigma) \subset R(h, \omega).$$

For any bounded set $S \subset \mathbb{C}_0$ we define the infinite matrix $\mu(S)$ by

$$\mu(S) = \sum_{-\lambda_k \in S} b_k^* b_k.$$

Proposition 6.1. *Suppose $\mathbb{T}$ is given by (1.2), (1.3) and B is an infinite matrix with rows $b_k \in l^{2*}$. Then $B \in \text{OCM}(\mathbb{T})$ if and only if there exists $M_0 \geq 0$ such that for any $(h, \omega) \in \mathcal{T}$,*

$$\left\| \sum_{(g, \sigma) \prec (h, \omega)} \mu(Q(g, \sigma)) \right\|_{\mathcal{L}(l^2)} \leq M_0 h. \quad (6.4)$$

Proof. The “only if” part is obvious, since

$$\sum_{(g, \sigma) \prec (h, \omega)} \mu(Q(g, \sigma)) = \mu(R(h, \omega)).$$

Conversely, assume (6.4) holds for any $(h, \omega) \in \mathcal{T}$. We wish to show that (1.5) holds for any $h + i\omega \in \mathbb{C}_0$. Thus let $h + i\omega \in \mathbb{C}_0$ and define

$$\mathcal{Z}(h, \omega) = \{(g, \sigma) \in \mathcal{T} \mid Q(g, \sigma) \cap R(h, \omega) \neq \emptyset\},$$

and let $\mathcal{Z}_m(h, \omega)$ denote the set of maximal elements of $\mathcal{Z}(h, \omega)$. There can be at most two distinct pairs, say $(g_1, \sigma_1), (g_2, \sigma_2)$ in $\mathcal{Z}_m(h, \omega)$ since the existence of a third pair would contradict maximality. Furthermore it is clear that $g_1 \leq 2h$ and $g_2 \leq 2h$. We have

$$\begin{aligned} \|\mu(R(h, \omega))\|_{\mathcal{L}(l^2)} &\leq \sum_{(g, \sigma) \in \mathcal{Z}_m(h, \omega)} \|\mu(R(g, \sigma))\|_{\mathcal{L}(l^2)} \\ &\leq 2M_0 h + 2M_0 h \end{aligned}$$

and consequently (1.5) holds with $M = 4M_0$. $\square$

By Theorem 2.2, the estimate (6.4) is necessary for the infinite-time admissibility of B . We do not know if it is sufficient (see [7, Conjecture 4.4]) however by strengthening (6.4) we obtain the following sufficient condition.

Proposition 6.2. *Let $\mathbb{T}$, B and μ be as in Proposition 6.1. If there exists $M_1 \geq 0$ such that for any $(h, \omega) \in \mathcal{T}$,*

$$\sum_{(g, \sigma) \prec (h, \omega)} \|\mu(Q(g, \sigma))\|_{\mathcal{L}(l^2)} \leq M_1 h, \quad (6.5)$$

then $B \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$.

Proof. Let $X = X^* = l^2$ denote the state space and $U = U^* = l^2$ denote the input space. We wish to show that $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$.

Let $\tau_1, \tau_2, \tau_3 \dots$ denote an ordering of those members τ_n of $\mathcal{T}$ for which $Q(\tau_n) \cap \{-\lambda_j \mid j \in \mathbb{N}\} \neq \emptyset$. (The case of only finitely many such τ_n is covered under Theorems 1.2 and 1.3.) By repeated applications of Proposition 4.3 we may assume without loss of generality that each $-\lambda_j$ coincides with some τ_n (for some $n \in \mathbb{N}$). Let $(e_k)_{k \in \mathbb{N}}$ be the standard basis for l^2 and for $n \in \mathbb{N}$ let X^n denote the subspace of X defined by

$$X^n = \text{closed span } \{e_k \mid \lambda_k \in Q(\tau_n)\}.$$

When X^n is infinite dimensional it is isomorphic to l^2 . X may be written as the orthogonal direct sum of the spaces X^k :

$$X = X^1 \oplus X^2 \oplus X^3 \oplus \dots$$

For $n \in \mathbb{N}$ let B_n denote the infinite matrix whose k th row is b_k if $\lambda_k = \tau_n$ and otherwise is the 0 row vector. By hypothesis, the infinite matrix B_n defines a bounded operator from U to X (with range in X^n). Likewise, the conjugate-transpose B_n^* of

B_n defines a bounded operator from X to U (with $B_n^* X^j = 0$ for all $j \neq n$). To avoid trivialities we will assume that for all $n \in \mathbb{N}$, $B_n \neq 0$.

Let P denote the infinite matrix with entries (p_{jk}) defined by

$$p_{jk} = -\frac{\langle b_j, b_k \rangle}{\lambda_j + \bar{\lambda}_k}.$$

Our goal is to show that P defines a bounded operator on l^2 and conclude from Proposition 5.1 that $B \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$. P may be rewritten as

$$P = \sum_{m,n=1}^{\infty} \frac{B_m B_n^*}{\tau_m + \bar{\tau}_n}.$$

Let I denote the identity operator on U and let $\mathbf{K} = P^0 \otimes I$ (the infinite Kronecker product), where $P^0 = (p_{mn}^0)$ is defined by

$$p_{mn}^0 = \frac{\|B_m\| \cdot \|B_n\|}{\tau_m + \bar{\tau}_n}.$$

Thus $\mathbf{K}$ can be viewed in a natural way as a partitioned matrix in which each block is a multiple of the identity matrix.

P may be factored as follows:

$$P = \left(\frac{B_1}{\|B_1\|}, \frac{B_2}{\|B_2\|}, \dots \right) \mathbf{K} \left(\frac{B_1}{\|B_1\|}, \frac{B_2}{\|B_2\|}, \dots \right)^*.$$

Denoting $\mathbf{B} = \left(\frac{B_1}{\|B_1\|}, \frac{B_2}{\|B_2\|}, \dots \right)$, we can write this more compactly: $P = \mathbf{B} \mathbf{K} \mathbf{B}^*$. The matrices $\mathbf{B}$ and $\mathbf{B}^*$ are also partitioned into infinite blocks and multiplication in the last identity is defined blockwise.

Let $l^2(l^2)$ denote the Hilbert space of l^2 -valued sequences with the l^2 norm (so $l^2(l^2)$ is isomorphic to l^2). For each $n \in \mathbb{N}$ the operator $B_n/\|B_n\|$ has norm 1 as an operator from U to X and has range in X^n . It follows from the orthogonality of these subspaces that $\mathbf{B}$ has norm 1 as an operator from $l^2(l^2)$ to X . Likewise $\mathbf{B}^*$ has norm 1 as an operator from X to $l^2(l^2)$. Thus we obtain that $\|P\| \leq \|\mathbf{K}\|$.

It remains to show that $\mathbf{K}$ is a bounded operator on $l^2(l^2)$. To prove this, let $\mathbf{S} = (\mathbf{S}_t)_{t \geq 0}$ be the diagonal semigroup with k -th diagonal element defined by $(\mathbf{S}_t)_k = e^{\tau_k t}$, and let b denote the sequence $(\|B_k\|)_{k \in \mathbb{N}}$. The inequality (6.5) implies that b satisfies the Carleson measure criterion (2.8) for $\mathbf{S}$. (Actually (2.8) is satisfied for values of (h, ω) which are in $\mathcal{T}$, but by Proposition 6.1, this is enough.) Thus by Theorem 2.1, b is infinite-time admissible for $\mathbf{S}$. Now Proposition 5.1 shows that P^0 is bounded on l^2 . Since P^0 and $\mathbf{K}$ are Hermitian, their norms are determined by their spectral radii. Since the spectral radius of the Kronecker product of two Hermitian matrices is the product of the spectral radii of each matrix (see e.g., Lancaster and Tismenetsky [9, pp. 411, 412]) it follows that $\|\mathbf{K}\|_{\mathcal{L}(l^2(l^2))} = \|P^0\|_{\mathcal{L}(l^2)}$, and hence $\mathbf{K}$ is bounded. $\square$

Definition 6.3. A set $S \in \mathbb{C}_0$ is called σ -rectangular if there is an at most countable set $\mathcal{I} \subset (0, \infty) \times \mathbb{R}$ such that (with the notation in (6.1))

$$S = \bigcup_{(h, \omega) \in \mathcal{I}} Q(h, \omega),$$

and for any distinct $(g, \sigma), (h, \omega) \in \mathcal{I}$

$$[\sigma - g, \sigma + g) \cap [\omega - h, \omega + h) = \emptyset.$$

For example, any set $\{x \in \mathbb{C}_0 \mid a|\operatorname{Im} z|^\alpha \leq \operatorname{Re} z \leq b|\operatorname{Im} z|^\alpha\}$, with $0 \leq \alpha < 1$ and $0 < a \leq b < 2a$, is contained in the union of a bounded set and a σ -rectangular set (see Proposition 6.8). Figure 3 shows an intuitive picture of a σ -rectangular set.

Fig. 2. The tiling $\mathcal{T}$ of $\mathbb{C}_0$.

Fig. 3. A σ -rectangular set.

Proposition 6.4. Let $\mathbb{T}$ be as in (1.2), (1.3) and such that $\{-\lambda_k \mid k \in \mathbb{N}\} \subset S$, where S is a σ -rectangular set. Then $\mathcal{B}(l^2, l^2, \mathbb{T}) = \operatorname{OCM}(\mathbb{T})$.

Proof. First note that Proposition 6.1 and Proposition 6.2 remain true (for $\mathbb{T}$ as above) when $\mathcal{T}$ is replaced by $\mathcal{I}$. The proofs are exactly the same. Note that $\succ$ is well-defined on $\mathcal{I}$ and every element of $\mathcal{I}$ is maximal. Hence (6.4) and (6.5) are the same inequality. It thus follows that $\operatorname{OCM}(\mathbb{T}) = \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$. $\square$

Remark 6.5. With $\mathbb{T}$ and S as in Proposition 6.4 and B an infinite matrix with rows in l^{2*} , to check that $B \in \operatorname{OCM}(\mathbb{T})$ it is enough to verify (1.5) for the pairs (h, ω) used in the definition of S . The proof is similar to that of Proposition 6.1.

The following result is a generalization of the part of Theorem 1.3 relating to analytic semigroups since there, the semigroup was assumed to be *exponentially* stable.

Proposition 6.6. *Let $\mathbb{T}$ be as in (1.2), (1.3) and assume that for some $\beta \in \mathbb{R}$ and $\theta \in (0, \pi/2)$, $\{-\lambda_k \mid k \in \mathbb{N}\} \subset S$, where $S = \{s \in \mathbb{C}_0 : |\arg(s - i\beta)| < \theta\}$. Then $\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) = \text{OCM}(\mathbb{T})$.*

Proof. It is enough to consider only the case with $\beta = 0$. Assume for the moment that $\theta \leq \arctg 2$. As in Proposition 6.4, it will suffice to show that (6.4) implies (6.5). Let $(h, \omega) \in \mathcal{T}$. If $Q(h, \omega) \cap S = \emptyset$ then it is easy to see that (6.5) holds since the left-hand side is zero. Thus we may assume (since $\theta \leq \arctg 2$) that $\omega = \pm h$. For definiteness, let us assume $\omega = h$. Likewise if $(g, \sigma) \in \mathcal{T}$ with $(g, \sigma) \prec (h, h)$, then either $Q(g, \sigma) \cap S = \emptyset$, in which case $\mu(Q(g, \sigma)) = 0$, or $Q(g, \sigma) \cap S \neq \emptyset$, in which case $\sigma = g$. By (6.4), $\|\mu(Q(g, \sigma))\|_{\mathcal{L}(l^2)} \leq M_0 g$. It follows that

$$\sum_{(g, \sigma) \prec (h, \omega)} \|\mu(Q(g, \sigma))\|_{\mathcal{L}(l^2)} \leq \sum_{(g, g) \prec (h, h)} M_0 g \leq \sum_{k=0}^{\infty} M_0 h 2^{-k} = 2M_0 h.$$

Hence (6.5) holds with $M_1 = 2M_0$.

A similar argument applies if θ is larger than $\arctg 2$, however it is simplest to note that by redefining the rectangles in (1.4) from 2×1 rectangles to $n \times 1$ rectangles (see Remark 1.4), the above argument (with $\arctg n$ in place of $\arctg 2$) remains valid. $\square$

From Propositions 6.4 and 6.6 we know that for semigroups given by (1.2), (1.3) we have $\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) = \text{OCM}(\mathbb{T})$ in several situations not covered under Theorems 1.2 and 1.3. These results can be summarized as follows:

Proposition 6.7. *Let $\mathbb{T}$ be as in (1.2), (1.3) with $\{-\lambda_k \mid k \in \mathbb{N}\} \subset S \subset \mathbb{C}_0$. Then $\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) = \text{OCM}(\mathbb{T})$ if any of the following conditions hold:*

- (C1) $S \subset \{z \in \mathbb{C}_0 \mid a < \operatorname{Re} z < b\}$, with $0 < a < b$;
- (C2) $S \subset \{z \in \mathbb{C}_0 \mid |\arg z| < \psi\}$, with $\psi < \frac{\pi}{2}$;
- (C3) S is contained in a σ -rectangular set;
- (C4) S is a finite union of sets which satisfy (C1), (C2) or (C3).

Combinations of (C1) and (C2) occur for example in some thermoelastic systems (see Hansen [6] and Leis [10, Chapter 13]). Spectral arrangements which form σ -rectangular sets can occur in certain models for internally damped beams and plates (see S. Chen and Triggiani [3], and G. Chen and Russell [2]).

We conclude this paper with an application of Proposition 6.7. In particular, the following result implies the equivalence in (iii) of Theorem 1.5.

Proposition 6.8. *Let $\mathbb{T}$ be as in (1.2), (1.3) and assume that $\{-\lambda_k \mid k \in \mathbb{N}\} \subset S$, where*

$$S = \{h + i\omega \in \mathbb{C}_0 \mid f(|\omega|) \leq h \leq cf(|\omega|)\}, \quad (6.6)$$

where $c \geq 1$ and $f : (0, \infty) \rightarrow (0, \infty)$ is continuously differentiable on $(0, \infty)$ and f' has limiting values in $[-\infty, \infty]$ at 0 and ∞ . Then $\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) = \text{OCM}(\mathbb{T})$.

Proof. Several simplifying assumptions can be made immediately, with no loss of generality. We may assume that $f(0) = 0$ and $f'(0) = 0$, since otherwise S is the union of

a set of this type (for appropriately chosen f) and a set of type (C2) of Proposition 6.7. For the same reason we may assume that $f'(\infty) = 0$. Also we may assume that $c < 2$ in (6.6) since S is contained in a finite union of sets of this form (with f replaced by a scalar multiples of itself in (6.6)).

We wish to construct a σ -rectangular set Q for which $S - Q$ is contained in a set of the type in (C1) of Proposition 6.7. We let

$$Q = \bigcup_{n \in \mathbb{Z}} (Q(h_n, \omega_n) \cup Q(h_n, -\omega_n)) ,$$

where the rectangles $Q(h_n, \omega_n)$ are defined inductively by the following properties:

(i) $\omega_0 = 1$, (ii) $h_n = (1 + c/2)f(\omega_n)$, (iii) $\omega_n + h_n = \omega_{n+1} - h_{n+1}$, (iv) if the previous steps lead to multiple solutions $\{h_n^\alpha, \omega_n^\alpha\}$ (where α runs through some index set), define h_n and ω_n so that $h_n = \min_\alpha h_n^\alpha$. (That a minimal solution h_n exists follows from the continuity of f and the fact that f is positive.) We can easily show that if $s = h + i\omega \in S$ with $|\omega|$ either sufficiently small or sufficiently large then $s \in Q$. Indeed, let $s \in S$, with $\omega_n - h_n < |\omega| < \omega_n + h_n$. Then

$$\begin{aligned} h &< cf(\omega_n + \tau_n) && \text{(for some } \tau_n \in [-h_n, h_n]) \\ &= cf(\omega_n) + cf'(\xi_n)(\tau_n) && \text{(for some } \xi_n \in (\omega_n - h_n, \omega_n + h_n)) \\ &= (c/(1 + c/2))h_n(1 + (1 + c/2)(\tau_n/h_n)f'(\xi_n)) \\ &< h_n && \text{(for } |n| \text{ sufficiently large).} \end{aligned}$$

Similarly inequalities show that $h > h_n/2$ if $|n|$ is sufficiently large. Thus for $|n|$ sufficiently large $s \in Q_n \subset Q$ and consequently $S - Q$ is a set of the type discussed in (C1) of Proposition 6.7. $\square$

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