**SYMMETRIES AND CONSERVATION LAWS OF
TWO-DIMENSIONAL HYDRODYNAMICS**

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Symmetries and conservation laws of two-dimensional hydrodynamics.

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Abstract

We shall briefly review the Hamiltonian and Lie algebraic structure of 2-D hydrodynamics and its known conservation laws: energy, enstrophy, the moments of vorticity. Then we utilize the eigenmode expansion of the Euler equation (to represent it as an infinite ODE system) and the detailed analysis of the structure coefficients to show that there are no other (nontrivial) conserved integrals among all linear and quadratic functionals of vorticity. This result provides a partial justification of the Kraichnan energy-enstrophy theory of 2-D turbulence.

Two-dimensional inviscid hydrodynamics described by the Euler equations is well known to have the Hamiltonian structure (Poisson bracket) and a large number of conserved integrals, including energy, enstrophy and all higher moments of vorticity. Conserved integrals play an important role both in the deterministic systems (stability of stationary flows: [1], [5], [7]) and the statistical theory (stationary Gibbs ensembles, turbulence: [4], [2], [3]). In particular turbulent states are described by the fluxes (cascades) of the conserved integrals (cf. [4], [9]), while stationary equilibria (in the absence of force and dissipation) involve either the Liouville invariant measure on the joint level sets of conserved integrals, or the Gibbs ensemble $\frac{1}{Z} \exp(-\beta E + \sum \alpha_m Q_m)$, whose exponential combines energy E and/or other conserved integrals Q_m .

A peculiar feature of the hydrodynamics is highly degenerate Poisson (Hamiltonian) structure¹ to be explained below. From the standpoint of the degenerate Hamiltonian structure the enstrophy and the higher vorticity moments become trivial (not to mean insignificant). The unsolved problem of hydrodynamics is whether there exist other nontrivial integrals. The present paper makes a step in this direction. We show that the 2-D Euler equations have no nontrivial linear or quadratic integrals. Thus our result provides a partial justification to the Kraichnan energy-enstrophy theory [4] i.e. the equilibrium (Gibbs) ensembles based on the known (energy-enstrophy) integrals. Our method exploits both the Hamiltonian and the geometric Lie structure of the hydrodynamics. We write the Euler equation as a coupled ODE-system via the eigenmode expansion of the Laplacian, and utilize the explicit Lie structure constants to compute the coefficients of the ODS. At present our results are somewhat limited (they strictly speaking apply to special regions, like boxes with periodic boundary conditions, spheres etc., where the eigenvalues of Δ and the Lie structure constants can be calculated in the closed form), but we hope to overcome these limitations in the future.

Aside of the conservation laws our objective was to emphasize the utility and potential of the eigenmode expansion in general. Indeed, in special cases it reveals some intriguing and peculiar features of the structure coefficients. One can't help feeling that further analysis could yield a great deal more on the interacting eigenmodes, particularly, the energy-enstrophy transfer between different modes and scales. Such study should also find applications in the statistical theory (e.g. the proper "Gibbs ensembles"). This work is currently in progress and will be discussed elsewhere.

1. The hydrodynamic equations: Poisson structure and conservation laws

In this section we shall review some known basic equations and properties of the 2-D fluid dynamics. In the standard Eulerian formulation the states of an incompressible fluid are described by one of the following parameters

- velocity field $\mathbf{u} = (u; v)$

¹By Poisson structure we mean a skew-symmetric form (operator) J_x on the phase-space $\mathcal{P}$, equipped with an inner product $(x|y)$, so that the bracket $\{f; g\} = (J[\partial f]|\partial g)$ for any pair of observables $f; g$ on $\mathcal{P}$.

- stream-function $\psi = \psi(x; t) : x = (x; y); \mathbf{u} = (-\psi_y; \psi_x)$
- vorticity $\omega = \nabla \times \mathbf{u} = v_x - u_y = \Delta\psi$

Incompressibility condition means $\nabla \cdot \mathbf{u} = 0$, while the momentum conservation law reads

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{\nabla p}{\rho} + F$$

the r.h.s. made of the pressure gradient ∇p over (constant) density ρ plus any relevant external force (e.g. gravity). The momentum equation can be recast in a more convenient form

$$\mathbf{u}_t + \nabla \left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} \right) - \mathbf{u} \times (\nabla \times \mathbf{u}) = 0 \quad (1.1)$$

Applying operations *div* to (1.1) one can express the pressure as a quadratic functional of $\mathbf{u}$

$$\frac{p}{\rho} = \Delta^{-1} [\omega^2 + \mathbf{u} \cdot \Delta \mathbf{u}] - \frac{\mathbf{u}^2}{2}$$

Here $\Delta^{-1} = G(x; \xi)$ denotes the Green's function of the Laplacian in a given region (basin) D subject to proper boundary conditions (periodic, slippery etc.). On the other hand operation *curl* takes (1.1) to the standard vorticity form

$$\omega_t - \nabla \times (\mathbf{u} \times \omega) = \omega_t + (\mathbf{u} \cdot \nabla) \omega - (\omega \cdot \nabla) \mathbf{u} = 0 \quad (1.2)$$

All the above considerations apply to the 2-D and 3-D fluids alike. But in the 2-D case system (1.2) simplifies to a single scalar equation for $\omega \sim (0; 0; \omega)$. Furthermore, inserting the velocity field $\mathbf{u} = (-\psi_y; \psi_x)$ one can rewrite the vorticity equation as

$$\omega_t + J(\psi; \omega) = \dots \quad (1.3)$$

where

$$J(f; g) = \partial f \wedge \partial g = f_x g_y - f_y g_x \quad (1.4)$$

denotes the Jacobian of two functions or, better to say, Jacobian of the map:

$$(x; y) \rightarrow (f(x; y); g(x; y))$$

Later on we shall interpret $J(f; g)$ as the Poisson-Lie bracket $[f; g]$ of two formal hamiltonians f and g on the phase-plane $\mathbf{R}^2 = \{(x, y)\}$.

Parameters ψ and ω are related by a linear map: $\omega = \Delta\psi$; $\psi = \Delta^{-1}\omega$, so (1.3) can be written in terms of any one of them

$$\begin{aligned}\omega_t + J(\Delta^{-1}\omega; \omega) &= 0 \text{ or} \\ \Delta\psi_t + J(\psi; \Delta\psi) &= 0\end{aligned}\tag{1.5}$$

Among several variations of (1.5) we shall mention the *quasigeostrophic flow* for rotating fluids over topography

$$\partial_t(\Delta\psi) + J(\psi; \Delta\psi + h) = \dots\tag{1.6}$$

Here the additional topographic factor h depends on the depth variation and the latitudinal change of the local Coriolis parameter. Introducing the so called *potential vorticity* $\omega = \Delta\psi + h$ equation (1.6) acquires the same form as (1.3).

1.1. Conservation laws

Equations (1.1) or (1.3) are easily shown to conserve the vorticity along the flow lines of $\mathbf{u}$

$$D_t\omega = \partial_t\omega + \nabla \cdot (\mathbf{u}\omega) = 0$$

Hence all moments (and more general "functions") of vorticity

$$\left\{ Q_n = \int \omega^n d^2\mathbf{r}, n = 1, 2, \dots \right\} Q_F = \int F(\omega) d^2\mathbf{r} \text{ (any } F) \text{ or}\tag{1.7}$$

are also conserved $\frac{d}{dt}Q_F = 0$. The first two moments of ω represent the total vorticity: $Q_1 = \int \omega dx$ and enstrophy: $Q_2 = \int \omega^2 dx$. Furthermore, the kinetic energy

$$E = \frac{1}{2} \int \mathbf{u}^2 = \frac{1}{2} (\Delta\psi | \psi) = \frac{1}{2} (\Delta^{-1}\omega | \omega)\tag{1.8}$$

is conserved by the flow². Here $(f|g) = \int_D f(\mathbf{r}) g(\mathbf{r}) d^2\mathbf{r}$ denotes the standard L^2 -inner product of functions on region D .

²The energy conservation follow easily from the relation

$$\partial_t \left(\frac{\mathbf{u}^2}{2} \right) = \mathbf{u} \cdot \mathbf{u}_t = -\mathbf{u} \cdot \nabla \left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} \right) + \mathbf{u} \cdot (\mathbf{u} \times \omega) = -\nabla \cdot \left(\left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} \right) \mathbf{u} \right)$$

1.2. Hamiltonian structure.

A general Hamiltonian (Poisson) structure on a linear phase-space $\mathcal{P} = \{\omega\}$ equipped with an inner product $(\cdot|\cdot)$, is given by skew-symmetric operators $J = J_\omega$ at each point ω , called the *symplectic form*. The Poisson bracket of two observables on $\mathcal{P}$ is then defined as

$$\{F; G\}(\omega) = (J_\omega [\delta F] | \delta G) \quad (1.9)$$

Operator-function J_ω is subjected to certain constraints to ensure the Jacobi identity for the bracket $\{F; G\}$. Given any observable H on space $\mathcal{P}$ one can write the Hamiltonian flow of H as

$$\begin{aligned} \dot{\omega} &= J_\omega \left(\frac{\delta H}{\delta \omega} \right) \text{ - evolution of states, or} \\ \dot{F} &= \{F; H\} \text{ - evolution of observables} \end{aligned} \quad (1.10)$$

The standard example is the canonical phase-space $\mathbf{R}^{2n} = \{(x; p)\}$ (n degrees of freedom) with the conjugate position-variables $(x; p)$, the symplectic form:

$$J = \begin{bmatrix} & I \\ -I & \end{bmatrix} \text{ i.e. Poisson bracket } \{f; g\} = \partial_x f \cdot \partial_p g - \partial_p f \cdot \partial_x g \quad (1.11)$$

and the Hamiltonian flow

$$\begin{cases} \dot{x} = H_p \\ \dot{p} = -H_x \end{cases}$$

The hydrodynamic phase-space $\mathcal{P} = \mathcal{P}(D) = \{\omega(\mathbf{r}) : \mathbf{r} \in D\}$ is made of all vorticity configurations. It possesses an analogue of the symplectic form J (1.11) defined by a skew-symmetric operator³

$$J_\omega = \partial \omega \wedge \partial = \omega_x \partial_y - \omega_y \partial_x \quad (1.12)$$

so $J_\omega[f] = \partial \omega \wedge \partial f = J(\omega; f)$ - the Jacobian of two functions. So the Poisson bracket for a pair of functionals on $\mathcal{P}$ becomes

$$\{F; G\}(\omega) = \int J_\omega \left[\frac{\delta F}{\delta \omega} \right] \left(\frac{\delta G}{\delta \omega} \right) d^2 \mathbf{r} = \int \left(\partial \omega \wedge \partial \left[\frac{\delta F}{\delta \omega} \right] \right) \left(\frac{\delta G}{\delta \omega} \right) d^2 \mathbf{r} \quad (1.13)$$

³Skew-symmetry is verified directly

$$(J_\omega f | g) = \int (\partial \omega \wedge \partial f) g d^2 \mathbf{r} = - \int (\partial \omega \wedge \partial g) f d^2 \mathbf{r}$$

for any pair of vectors $f; g$ in $\mathcal{P}$

Hence follows the commutation law for the natural variables

$$\{\omega(\mathbf{r}); \omega(\mathbf{r}')\} = J_\omega[\delta(\mathbf{r} - \mathbf{r}')] = \partial\omega(\mathbf{r}) \wedge \partial\delta(\mathbf{r} - \mathbf{r}') \text{ for all } \mathbf{r}; \mathbf{r}' \quad (1.14)$$

The hydrodynamic energy E (1.8) is a quadratic functional in ω , its gradient $\delta E = \Delta^{-1}\omega$ and the Euler equations (1.3) are easily verified to coincide with the Hamilton equations of E

$$\dot{\omega} = \{\omega; E\} = -J_\omega[\delta E] \quad (1.15)$$

Let us make a few comments on the Hamiltonian structure of 2-DH, comparing it to the canonical $\mathbf{R}^{2n}$ -model

- Unlike the standard symplectic form (1.11) the hydrodynamic J_ω is ω -dependent (varies over $\mathcal{P}$) and highly degenerate. Each operator J_ω ($\omega \in \mathcal{P}$) has a large (infinite) null-space

$$\mathcal{N}_\omega = \{\phi : J_\omega[\phi] = J(\omega; \phi) = \partial\omega \wedge \partial\phi = 0\} \quad (1.16)$$

made of all "functions of ω ", $\mathcal{N}_\omega = \{f(\omega) : \text{all } f\}$.

- The vorticity moments $Q = Q_F$ (1.7) have gradients $\delta Q = F'(\omega)$ lying in the null-space of J_ω at each point ω in $\mathcal{P}$. That makes the conserved integrals $\{Q_F\}$ trivial from the Hamiltonian standpoint. In other words the Poisson relation $\{Q; E\} = 0$ does not imply any nontrivial conservation law for the energy functional E (1.8), but holds automatically for any Hamiltonian H on $\mathcal{P}$, all gradients being annihilated by operators: $J_\omega[\delta Q] = \omega \wedge F'(\omega) = 0$.

Of course, the energy (Hamiltonian) E is automatically conserved for any system (1.10).

An immediate consequence of (1.15), (1.16) are the stationary solutions of the Euler equation. Indeed, $J_\omega[\Delta^{-1}\omega] = 0$ implies

$$\Delta^{-1}\omega = f(\omega) \text{ or } \Delta\psi = f^{-1}(\psi) \quad (1.17)$$

for some function⁴ f .

⁴Let us remark that each stationary solution (1.17) is a critical point of the functional

$$Q = E + Q_F = \int \int \Delta^{-1}\omega(\mathbf{r}) \omega(\mathbf{r}') + \int F(\omega)$$

where $f = F'$.

It the next section we shall review the structure of 2-DH from the stand-point of certain infinite Lie algebras and groups. Arnold [1] was the first to interpret 2-DH in the Lie algebraic language. He also drew an analogy between the Euler equations of rigid body, based on the orthogonal group $\mathbf{SO}(3)$ and the Eulerian equations of an ideal fluid, where the role of $\mathbf{SO}(3)$ is played by the infinite group $\mathcal{G}_D$ of all area preserving deformations of region D . He went on to examine the structure of the corresponding Lie algebra (2-D Hamiltonian vector fields over D) and found a formal resemblance of $\mathcal{G}_D$ and the classical negatively curved (hyperbolic) Lie groups, like Lorentz $\mathbf{SO}(1, n)$. Based on such observation Arnold made certain qualitative predictions (and even crude estimates) regarding the inherent instability of fluid flows typical of the "hyperbolic" geometry (symmetry group).

2. Lie structure of the Euler equations

We shall precede our discussion of the hydrodynamics with a simple finite-D prototype, namely the phase-space $\mathcal{P} = \mathbf{R}^3 = \{\mathbf{x}\}$, equipped with the Poisson bracket

$$\{f; g\}(\mathbf{x}) = \mathbf{x} \cdot \partial f \times \partial g \text{ i.e. symplectic form } J_{\mathbf{x}}(\mathbf{y}) = \mathbf{x} \times \mathbf{y} \quad (2.1)$$

The Hamiltonian evolution of any observable f on $\mathbf{R}^3$ is given by

$$\dot{\mathbf{x}} = \mathbf{x} \times \partial f(\mathbf{x}) \quad (2.2)$$

Clearly, symplectic form $J_{\mathbf{x}}$ (2.1) depends on $\mathbf{x}$ and has a nontrivial null-space $\mathcal{N}_{\mathbf{x}} = \{\lambda \mathbf{x}\}$ for any $\mathbf{x}$. Notice that system (2.2) also possesses a trivially conserved integral $Q = |\mathbf{x}|^2$.

The reader can recognize the underlying Lie structure of (2.1), if space $\mathbf{R}^3$ is identified with the Lie algebra $\mathcal{A} = \mathfrak{so}(3)$ of all 3×3 real antisymmetric matrices

$$\mathbf{x} = (a, b, c) \rightarrow X = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

The $\mathbf{R}^3$ -cross-product then turns into the usual Lie bracket on $\mathcal{A}$

$$\mathbf{x} \times \mathbf{y} \rightarrow [X; Y] = XY - YX$$

The symplectic operators $J_X(Y) = [X; Y] = ad_X(Y)$ (called adjoint) are anti-symmetric relative to the natural inner product $(X|Y) = tr(XY)$, and (2.1) can be restated as

$$\{f; g\}(X) = (X|[\partial f; \partial g]) = -(\partial f| ad_X [\partial g]) \quad (2.3)$$

There are several advantages of using the Lie algebraic language. On the one hand it allows to extend the $\mathbf{R}^3$ -model to a broader class of Poisson spaces based on arbitrary Lie algebras⁵ $\mathcal{A}$. On the other hand it brings in the underlying (adjoint) Lie group action on $\mathcal{A}$ and allows to reduce the large yet degenerate space $\mathcal{A}$ to a family of non-degenerate spaces (symplectic leaves) - orbits of the Lie group. Such reduction could provide some useful insight into the dynamical system (cf. [1], [5]). Let us illustrate the group reduction for the $\mathbf{R}^3$ -model. Notice that the entire phase-space $\mathbf{R}^3$ is foliated into the level sets of the conserved integral:

$$\mathcal{O}_R = \{Q(\mathbf{x}) = R^2; 0 < R < \infty\} \text{-sphere of radius } R$$

Each $\mathcal{O}_R$ is an orbit of group $G = \mathbf{SO}(3)$ acting on $\mathbf{R}^3$ by rotations, or on Lie algebra $\mathcal{A} \cong \mathbf{R}^3$ by conjugations (adjoint action)

$$U : X \rightarrow U^{-1} X U; U \in G$$

All basic structures ($J_{\mathbf{x}}$, Poisson bracket, etc.) remain invariant under the group action. In particular, the Hamiltonian flow of any observable H on $\mathcal{A}$

$$\dot{\mathbf{x}} = [\partial H(\mathbf{x}); \mathbf{x}]$$

is constrained to a single orbit $\mathcal{O}_R$. In fact any initial state $\mathbf{x}_0$ is evolved by a one-parameter family of group transformations: $\mathbf{x}(t) = U_t(\mathbf{x}_0)$, depending on $\mathbf{x}_0$. Let us also observe that each orbit $\mathcal{O}_R$ is itself a non-degenerate (symplectic) phase-space with the J -form (respectively, Poisson bracket) (2.1) restricted to gradient vectors $\partial f; \partial g$ - the tangent plane⁶ to orbit $\mathcal{O}$ at $\mathbf{x}_0$. Geometrically one has the entire phase-space (algebra) $\mathcal{A}$ foliated into a family of group orbits labeled

⁵Clearly, bracket (2.3) can be introduced on an arbitrary Lie algebra equipped with an inner product.

⁶The restricted bracket on the 2-sphere $\mathcal{O} \subset \mathbf{R}^3$ written in the standard spherical angles $\phi; \theta$ yields

$$\{f; g\} = \sin \phi (f_{\phi} g_{\theta} - f_{\theta} g_{\phi})$$

So $\phi; \theta$ obey the "almost canonical" commutation relations on $\mathcal{O}$.

by the values of the conserved integral. Each orbit represents a non-degenerate phase-space, invariant under all (conceivable) Hamiltonian flows on $\mathcal{A}$.

Now we shall turn to the hydrodynamics and describe a similar Lie algebraic structure.

- Phase-space $\mathcal{P} = \{\omega\}$ coincides with the Lie algebra of all 2-D hamiltonians $\{\omega\}$ in region D with the natural Lie-Poisson bracket

$$[\omega; \omega'] = \partial\omega \wedge \partial\omega' = J(\omega; \omega')$$

and the standard L^2 -inner product

$$(\omega | \phi) = \int_D \omega \phi d^2\mathbf{r}$$

- The symplectic (adjoint) map $J_\omega = ad_\omega = \partial\omega \wedge \partial$ is highly degenerate, the null-space coinciding with the isotropic subalgebra (commutator) of ω

$$\begin{aligned} \mathcal{N}_\omega &= \{f : [f; \omega] = 0\} = \{J_\omega[F] = 0\} \Rightarrow \\ &\mathcal{N}_\omega = \{f = F(\omega) : \text{all } F\} \end{aligned}$$

- The Poisson bracket of two observables $F; G$ on $\mathcal{P}$ takes the form

$$\{F; G\}(\omega) = (\omega | [\delta F; \delta G])$$

As we mentioned all vorticity moments $\{Q_F\}$ are trivial integrals as their gradients $\delta Q = F'(\omega)$ belong to the null-space $\mathcal{N}_\omega$ at each ω .

- The Lie group of algebra $\mathcal{P}$ consists of all area preserving coordinate changes (diffeomorphisms) of D :

$$\mathcal{G}_D = \left\{ \phi : D \rightarrow D \text{ s.t. } \det \left(\frac{\partial \phi}{\partial \mathbf{r}} \right) = 1 \right\}$$

Its orbits $\mathcal{O} \subset \mathcal{P}$ are somewhat difficult to describe in geometric terms. The group action $\phi : f \rightarrow f \circ \phi = f(\phi(\mathbf{r}))$ clearly preserves all functionals $\{Q_F\}$ or areas of the level sets $\Sigma_\lambda = \{f \leq \lambda\}$. However, the orbit is not fixed by the vorticity moments alone, i.e. not any pair of "equi-distributed" functions $\omega; \omega'$ could be transformed one to the other by an area-preserving ϕ . Besides the identical level-areas the latter would require also an identical structure of all critical points (minima, maxima, saddle) and the connecting separatrices for ω and ω' . In other words the complete classification of orbits would require along with the integrals $\{Q_F\}$ certain (discrete) set of topological invariants.

A simple corollary of the above Lie structure is that any 2-D Euler flow

$$\dot{\omega} + \left[\omega; \frac{\delta E}{\delta \omega} \right] = 0$$

(in fact, any other Hamiltonian flow $\dot{\omega} + \omega \wedge \frac{\delta H}{\delta \omega} = 0$ on $\mathcal{P}$) could be implemented by a one-parameter family of area preserving maps: $\omega(t) = \omega_0 \circ \phi_t$, depending on the initial state ω_0 .

Another application concerns the invariant Liouville volume on $\mathcal{P}$,

$$\mathcal{D}\omega = \prod_{\mathbf{r} \in D} d\omega(\mathbf{r})$$

important in the statistical theory (cf. [4], [6]). Kraichnan has observed that the Liouville volume is preserved by the Euler flow. The same holds for any Hamiltonian flow, in fact the entire group $\mathcal{G}_D$, acting on $\mathcal{P}$. Indeed, any change of variable $\phi : \mathbf{r} \rightarrow \mathbf{r}'$ on D permutes the labels $\{\mathbf{r}\}$ and multiplies the $\mathbf{r}$ -th coordinate by the Jacobian: $d\omega(\mathbf{r}') = \det\left(\frac{\partial \phi}{\partial \mathbf{r}}\right) d\omega(\mathbf{r})$. So the density factor

$$\frac{\mathcal{D}(\omega \circ \phi)}{\mathcal{D}\omega} = \prod_{\mathbf{r}} \det\left(\frac{\partial \phi}{\partial \mathbf{r}}\right)$$

is clearly 1 for any area-preserving map $\phi \in \mathcal{G}$.

Let us make a few comments on the role of the vorticity moments $\{Q_F\}$ in the hydrodynamics. As we mentioned they define the stationary deterministic solutions of the Euler equations

$$\begin{aligned} J(\Delta^{-1}\omega; \omega) &= (\Delta^{-1}\omega) \wedge \omega = 0 \Rightarrow \\ \Delta^{-1}\omega &= f(\omega) \end{aligned}$$

for some $f(\omega)$. Furthermore, Arnold [1] developed the nonlinear stability theory of deterministic equilibria $\{\omega_0\}$ based on spectral properties of the operator

$$\Delta^{-1} + f'(\omega_0) \geq 0 \text{ - condition for stability}$$

The Kraichnan's energy-entropy theory is based on the first 2 quadratic invariants and the canonical Gibbs ensemble: $d\mu_{\beta, \alpha}(\omega) = e^{-(\beta E + \alpha Q_2)} \mathcal{D}\omega$ on $\mathcal{P}$. Some recent works ([2], [8]) make the case for the higher vorticity integrals and the related (non-Gaussian) Gibbs ensembles

$$d\mu_{\lambda_0 \lambda_1 \lambda_2 \dots}(\psi) = e^{-(\beta_0 E + \beta_1 Q_1 + \beta_2 Q_2 + \dots)} d\psi, \quad (2.4)$$

In the Gaussian case the exponent in (2.4) involves only the first 3 conserved integrals, a combination of the quadratic and linear terms

$$\beta_0 E + \beta_1 Q_1 + \beta_2 Q_2 = \frac{1}{2} (A\omega|\omega) + (b|\omega) \quad (2.5)$$

Here A denotes the operator $\beta_0 \Delta^{-1} + \beta_2$, $b = 1$ (for the total vorticity $Q_1 = \int \omega$). So the mean Gaussian field becomes

$$\langle \omega \rangle = \int \psi \exp \left\{ -\frac{1}{2} (A\omega|\omega) + (b|\omega) \right\} \mathcal{D}\omega = \frac{\text{Const}}{\sqrt{\det A}} \exp \left(\frac{1}{2} A^{-1} b \cdot b \right) A^{-1} b$$

It clearly satisfies the linear equation $A\langle \omega \rangle = \text{Const } b$. Let us remark that the same linear relation appears for the critical point of the quadratic functional $\frac{1}{2} (A\omega|\omega) - C (b|\omega)$ (2.5). Hence the mean Gaussian fields $\langle \omega \rangle$ represent stationary Euler solutions (1.17) with linear functions $f = c_1 \omega + c_2$ [2].

3. Direct eigenmode expansion

The eigendata of the Laplacian $\Delta|_D$ (eigenvalues $\{\lambda_k\}$ and eigenfunctions $\{\phi_k\}$) can be used to expand the vorticity field $\omega = \sum \omega_k \phi_k$ and rewrite the dynamic equations (1.3) as an infinite ODE system

$$\dot{\omega}_n + \sum_{km} B_{km}^n \omega_k \omega_m = 0 \quad (3.1)$$

The coefficients $\{B_{km}^n\}$ depend on the structure constants $\{\gamma_{km}^n\}$ of Lie algebra $\mathcal{P}(D)$ relative to the basis $\{\phi_k\}$, namely

$$[\phi_k; \phi_m] = \sum_{n=1}^n \gamma_{km}^n \phi_n$$

Then one easily computes

$$B_{km}^n = \gamma_{km}^n \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_m} \right)$$

Our basic example will be a 2-D box D with periodic boundary conditions, whose eigendata is made of the standard Fourier modes

$$\left\{ \begin{array}{l} \phi_k = e^{i\mathbf{r} \cdot \mathbf{k}} : \mathbf{k} = (k_1; k_2) \in \mathbf{Z}^2 \text{ - lattice points (wave vectors)} \\ \lambda_k = |\mathbf{k}|^2 \text{ - eigenvalues} \end{array} \right\}$$

Here the Lie structure constants are computed to be

$$\gamma_{km}^n = (k \wedge m) \delta(n - k - m) \quad (3.2)$$

hence the B -coefficients

$$B_{km}^n = (k \wedge m) \left(\frac{1}{k^2} - \frac{1}{m^2} \right) \delta(n - k - m) \quad (3.3)$$

So the dynamical system takes the form

$$\dot{\omega}_n + \sum_{k+m=n} B_{km}^n \omega_k \omega_m = 0 \quad (3.4)$$

The r.h.s. of (3.4) could be viewed as some kind of a "B-deformed convolution" on the lattice $\mathbf{Z}^2$, the standard one corresponding to $B_{km}^n = \delta(n - k - m)$.

The conserved integrals expanded in the eigenmode basis become

$$\begin{aligned} E &= \sum \frac{|\omega_n|^2}{\lambda_n} = \sum \frac{\omega_n \omega_{-n}}{\lambda_n} \text{ - energy} \\ \Omega &= \sum |\omega_n|^2 = \sum \omega_n \omega_{-n} \text{ - enstrophy} \\ Q_F &= \sum B_{\dots}^{\dots} (\omega_{k_1}^{m_1} \dots \omega_{k_p}^{m_p}) \text{ - higher moments} \end{aligned}$$

We remark that the pairing ω_{-n} to ω_n in the energy, enstrophy forms: $E = (\Delta^{-1} \omega | \omega)$ and $\Omega = (\omega | \omega)$ reflects the conjugation property of Fourier coefficients: $\omega_{-n} = \bar{\omega}_n$ to guarantee the real function $\omega(\mathbf{r})$.

As a simple excersize we shall derive the basic conservation laws in terms of the structure coefficients (3.3)

$$\begin{aligned} Q_1 &= \int \omega : \dot{\omega}_0 + \sum k \wedge (-k) \left(\frac{1}{k^2} - \frac{1}{k^2} \right) \omega_k \omega_{-k} = 0 \\ Q_2 &= \int \omega^2 : \sum \dot{\omega}_n \omega_{-n} = - \sum \left(B_{km}^{-n} + B_{mn}^{-k} + B_{nk}^{-m} \right) \omega_k \omega_m \omega_n \\ E &= \int \int \Delta^{-1} \omega \omega : \sum \frac{1}{\lambda_n} \dot{\omega}_n \omega_{-n} = - \sum \left(\frac{1}{\lambda_n} B_{km}^{-n} + \frac{1}{\lambda_k} B_{mn}^{-k} + \frac{1}{\lambda_m} B_{nk}^{-m} \right) \omega_k \omega_m \omega_n \end{aligned}$$

The summation in the second and third lines extends over all baricentric triples $\{k + m + n = 0\}$. Then the conservation laws would read

$$\begin{aligned} B_{km}^{-n} + B_{mn}^{-k} + B_{nk}^{-m} &= 0 \text{ - enstrophy} \\ \frac{1}{\lambda_n} B_{km}^{-n} + \frac{1}{\lambda_k} B_{mn}^{-k} + \frac{1}{\lambda_m} B_{nk}^{-m} &= 0 \text{ - energy} \end{aligned}$$

Both follow readily from (3.3). Indeed, all three coefficients $B_{km}^{-n}; B_{mn}^{-k}; B_{nk}^{-m}$ have the common factor: " $k \wedge m = \text{area of the triangle}$ " times " $\delta(k + m + n)$ ". Pulling it out one ends up with the identities

$$\begin{aligned} \left(\frac{1}{k^2} - \frac{1}{m^2} \right) + \left(\frac{1}{m^2} - \frac{1}{n^2} \right) + \left(\frac{1}{n^2} - \frac{1}{k^2} \right) &= 0 \text{ - enstrophy} \\ \frac{1}{n^2} \left(\frac{1}{k^2} - \frac{1}{m^2} \right) + \frac{1}{k^2} \left(\frac{1}{m^2} - \frac{1}{n^2} \right) + \frac{1}{m^2} \left(\frac{1}{n^2} - \frac{1}{k^2} \right) &= 0 \text{ - energy} \end{aligned}$$

Another simple illustration involves the known deterministic and certain statistical solutions of systems (3.1) and (3.4).

3.1. Deterministic solutions.

The equilibrium condition for (3.1) and (3.4)

$$\sum_{k+m=n} (k \wedge m) \left(\frac{1}{k^2} - \frac{1}{m^2} \right) \omega_k \omega_m = 0$$

Passing to the continuous limit we can rewrite it as

$$\int_{x+y=z} \dots \omega \omega d^2 x = \left(\frac{x}{|x|^2} \omega \right) \wedge^* (y \omega) - (x \omega) \wedge^* \left(\frac{y}{|y|^2} \omega \right)$$

where $\wedge^*$ denotes the wedged-convolution of vector-functions

$$f \wedge^* g = f_1 * g_2 - f_2 * g_1$$

Direct analysis via $\mathcal{F}$ -transform on $\mathbf{R}^2$ yields:

$$\Delta^{-1} (\partial \hat{\omega}) \wedge \partial \hat{\omega} = 0 \Rightarrow \boxed{\Delta^{-1} \hat{\omega} = F(\hat{\omega})}$$

i.e. the stationary Euler flow (1.17).

3.2. Statistical equilibria

Statistical ensembles of random vorticity fields are described by a sequence of moments, namely

$$\begin{aligned} \Gamma(k, m) &= \langle \omega_k \omega_m \rangle \text{ - two-point corellators} \\ \Gamma(k, m, l) &= \langle \omega_k \omega_m \omega_l \rangle \text{ - three-point corellators, etc} \end{aligned}$$

that obey the following Hopf equations

$$\begin{aligned} \sum_{km} B_{km}^n \langle \omega_k \omega_m \rangle &= 0 \text{ all } n \\ \sum_{km} B_{km}^n \langle \omega_k \omega_m \omega_l \rangle + B_l^{km} \langle \omega_k \omega_m \omega_n \rangle &= 0 \text{ all } n, l \end{aligned}$$

We shall mention a few easy examples.

- The Gaussian energy-entropy ensembles give trivial $\Gamma = \delta(k + m)$ as both quadratic functionals E and Ω are diagonalized in the eigenbasis. Clearly, the nontrivial mode correlators result from the higher integrals $\{Q_F\}$ and the related Gibbs ensembles.
- Constant Γ (white noise in the $\mathbf{r}$ -space) and more general (short-range)

$$\begin{aligned}\langle \omega_k \omega_m \rangle &= \Gamma(k + m) \iff \\ \langle \omega(\mathbf{r}) \omega(\mathbf{r}') \rangle &= \hat{\Gamma}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}')\end{aligned}$$

- Any stationary radially-symmetric: $\Gamma(|k - m|)$
- Any rotationally-invariant correlation: $\Gamma(|k + m|; |k - m|; \cos \theta)$.

4. Conservation laws

Our main goal in the last section is to show the energy, enstrophy and the total vorticity are the only conserved integrals of the ideal 2-D fluid among all linear and quadratic functionals of ω .

4.1. Linear functionals

We take a linear form $L[\omega] = (b|\omega)$ on $\mathcal{P}$ and expand both vectors b and ω in the eigen-basis $\{\phi_k\}$

$$b = \delta L = \sum b_i \phi_i; \quad \omega = \sum \omega_i \phi_i$$

so $L[\omega] = \sum b_i \omega_i$. Our is to show that the only surviving component of L is $\omega_0 = \int \omega d^2 \mathbf{r}$. The quadratic energy form is written as

$$E = (A\omega|\omega) = \sum \alpha_i \omega_i \omega_{-i}$$

with the diagonal operator $A = \text{diag} \{\alpha_i = i^{-2}\}$. We compute the Poisson bracket

$$\{L; E\} = \sum_{ij}^n \gamma_{ij}^n b_i \alpha_j \omega_j \omega_{-n} = \sum B_{jn}^{-i} b_i \omega_j \omega_n \quad (4.1)$$

making change $n \rightarrow -n$ in the sum as above. Remembering the explicit Lie structure constants (3.2) the summation in the r.h.s. of (4.1) is easily seen to extend over all balanced triples $\{(i; j; n) : i + j + n = 0\}$. Therefore

$$\sum_{jn} B_{jn}^{-i} b_i \omega_j \omega_n = 0$$

for all mode-configurations $\{\omega_j\}$. Hence all coefficients $B_{jn}^{-i} b_i = 0$ as long $i = -j - n$. But any nonzero lattice point i can be represented as $-j - n$ in infinitely many different ways, so one could easily pick a (jn) -pair that has $|j| \neq |n|$ and $j \wedge n \neq 0$ i.e. $B_{jn}^{-i} \neq 0$. Consequently, $b_i = 0$ for any nonzero i , the only surviving eigenmode b_0 giving the total vorticity, $L = b_0 \omega_0$, Q.E.D.

4.2. Quadratic integrals.

Next we shall demonstrate that any quadratic form $(B\omega|\omega)$ on the phase-space $\mathcal{P}$ that Poisson-commutes with the energy-form $E = (A\omega|\omega)$ is a combination of the known energy and enstrophy integrals:

$$B = c_0 E + c_1 \Omega \quad (4.2)$$

Both quadratic forms E and Ω are represented by the diagonal operators in the eigen-basis

$$E = \sum \alpha_i \omega_i \omega_{-i}; \quad \Omega = \sum \omega_i \omega_{-i}$$

the energy eigenvalues being $\alpha_i = |i|^{-2}$. In what follows we shall only use the fact that eigenvalues α are distinct

$$\{\alpha_i = f(|i|) \neq \alpha_j \text{ for all } |i| \neq |j|\}$$

So our result (4.2) holds for any energy-like Hamiltonian $H = \sum \alpha_i \omega_i \omega_{-i}$ with distinct α 's.

The argument involves 3 steps: first the commutation relation $\{B; E\} = 0$ is reduced to a "triplet" linear system for the undetermined coefficients $\{\beta_{mk}\}$ of the B -form. Next solutions of the triplet system are analyzed to show that all off-diagonal coefficients drop: $\beta_{mk} = 0$ for $m + k \neq 0$. Finally, the diagonal coefficients are shown to be functions of α 's, $\beta_k = f(\alpha_k)$, and f must be linear $f(\alpha) = c_0 \alpha + c_1$

4.3. Commutation relation.

The first step is appropriate for any Lie algebra $\mathcal{P}$ with basis $\{\phi_k\}$ and structure constants γ_{ij}^n . By definition

$$\{F; G\}(\omega) = ([\delta F; \delta G]|\omega) = -([\delta F; \omega]|\delta G)$$

The gradients of quadratic forms are $\delta A = A\omega$; $\delta B = B\omega$. Expanding all 3 vectors ω ; $A\omega$; $B\omega$ in the basis $\{\phi_k\}$ and using the Hermitian symmetry of the product $(\phi_k|\phi_m) = \delta(k+m)$ we get

$$[\omega; A\omega] = \sum_{i < j; n} (\alpha_i - \alpha_j) \gamma_{ij}^n \omega_i \omega_j \phi_n$$

hence

$$([\omega; A\omega]| B\omega) = \sum_{i < j; n; k} (\alpha_i - \alpha_j) \gamma_{ij}^n \beta_{-n; k} \omega_i \omega_j \omega_k \quad (4.3)$$

Introducing symmetric tensor

$$B_{ij}^n = (\alpha_i - \alpha_j) \gamma_{ij}^n \quad (4.4)$$

and taking all symmetric permutations of the triplet $\{ijk\}$ in (4.3) we rewrite it as

$$\sum_{\{ijk\}} \sum_n \left(B_{ij}^n \beta_{-n; k} + B_{jk}^n \beta_{-n; i} + B_{ki}^n \beta_{-n; j} \right) \omega_i \omega_j \omega_k = 0$$

The latter of course means that all coefficients are 0. Thus we get a system of linear "triplet" relations for coefficients $\{\beta\}$

$$\sum_n \left(B_{ij}^n \beta_{-n; k} + B_{jk}^n \beta_{-n; i} + B_{ki}^n \beta_{-n; j} \right) = 0 \text{ for all } ijk$$

In the torus case the system is simplified to

$$B_{ij} \beta_{-(i+j); k} + B_{jk} \beta_{-(j+k); i} + B_{ki} \beta_{-(k+i); j} = 0 \quad (4.5)$$

with coefficients

$$B_{ij} = (\alpha_i - \alpha_j) (i \wedge j)$$

We shall demonstrate that the only solution of (4.5) is $B = c_0 E + c_1 \Omega$.

4.4. Eliminating off-diagonal entries

We observe that coefficients $B_{ij} \neq 0$ for all pairs $\{(ij)\}$, excluding the line $\{j = \lambda i\}$ and the circle $\{|j| = |i|\}$. The elimination of the off-diagonal entries proceeds in three steps.

1° Take $i = j$; $k \neq \lambda i$. Here coefficient $B_{ij} = 0$ and the remaining pair of terms in (4.5) is reduced to

$$2B_{ik} \beta_{n; i} = 0$$

Figure 4.1: A pair of indices $i = j$ and $k \neq \lambda i$ with $n = -(i + k)$

Figure 4.2: The admissible set of indices n s.t. $\beta_{n;i} \neq 0$ is reduced after step 1 to the i -line and the shifted i -circle

where $n = -(i + k)$ (see fig. 4.1). Clearly, $\beta_{n;i} = 0$, provided $B_{ik} \neq 0$, i.e.

$$|k| = |n + i| \neq |i| \text{ and } k = -(n + i) \neq \lambda i$$

Thus for each fixed i the only possible set $\{n\}$ of nonvanishing $\{\beta_{i;n}\}$ is the i -line and the shifted i -circle (fig. 4.2).

2°. Next we shall eliminate all circular points $\{n : |n + i| = |i|\}$ outside the line $\{\lambda i\}$. To that end we let j vary over the i -line, while k will vary over any parallel line. The triplet relation is now reduced to

$$B_{jk}\beta_{-(j+k);i} + B_{ki}\beta_{-(k+i);j} = 0 \quad (4.6)$$

We observe that both coefficients B_{jk} and B_{ik} are nonzero. Furthermore, having fixed points i and j we note that only two special values of parameter k (marked k^* on fig. 4.3) would allow either one of two points $\{-(i + k); -(j + k)\}$ to lie

Figure 4.3: Two exceptional directions $\pm k^*$ (thick lines) allow both coefficients β in (4.6) to be nonzero. All other (thing lines) yield vanishing β 's.

on the shifted circle of the other (cf. fig. 4.2), i.e. only two exceptional k would obey both relations

$$\begin{cases} |i - (j + k)| = |i| \\ |j - (i + k)| = |j| \end{cases} \quad (4.7)$$

If k is chosen to satisfy the first equation (4.7) but is different from the exceptional value k^* (fig.4.3), so that the second equation (4.7) fails, we get $\beta_{-(k+i);j} = 0$. Hence $\beta_{n;i} = 0$ for all non-exceptional $n = -(j + k)$. If we let now the second parameter $j = \lambda i$ vary (to make any point $n \neq \lambda i$ on the shifted i -circle non-exceptional (fig. 4.3)), all matrix entries $\beta_{n;i} = 0$ with non-colinear indices $n \neq \lambda i$ are eliminated.

3°. The remaining colinear pairs $n = \lambda i$ will be shown to reduce a single diagonal element, i.e. $\lambda = -1$ and $n = -i$. To that end we pick a balanced triplet $i + j + n = 0$ (i.e. $n = -(i + j)$) with $B_{ij} \neq 0$ and dilate k by a nontrivial factor $\lambda \neq 1$ in the direction n (see fig.4.4). So the pair $\{(i + j); k\}$ is colinear, whereas

Figure 4.4: Elimination of colinear non-diagonal coefficients β

two remaining pairs $\{(k+j); i\}$ and $\{(k+i); j\}$ are clearly not. By the previous argument: $\beta_{-(k+j);i} = \beta_{-(k+i);j} = 0$, so the triplet relation (4.5) implies

$$\beta_{-(i+j);k} = \beta_{n;\lambda n} = 0 \text{ as long as } \lambda \neq 1$$

4.5. Diagonal reduction

We have shown that matrix B is diagonalized along with the energy and enstrophy forms $B = \sum \beta_i \omega_i \omega_{-i}$ in the basis of eigenmodes. To proceed let us first observe that coefficients $\{\beta_i\}$ depend only on the norm $|i|$ as α_i did. Indeed, by (4.5) any pair $|i| = |j|$ on the circle (fig. 4.1) and a complimentary $k = -(i+j)$ obey

$$(\alpha_j - \alpha_k) \beta_i + (\alpha_k - \alpha_i) \beta_j = 0$$

whence $\beta_i = \beta_j = f(|i|)$.

Figure 4.5: Plot of a hypothetical function $\beta = f(\alpha)$

Hence we can write $\beta_i = f(\alpha_i)$ with yet undetermined function f . The basic relation (4.5) is now transformed into a functional equation

$$(\alpha_i - \alpha_j) f(\alpha_k) + (\alpha_j - \alpha_k) f(\alpha_i) + (\alpha_k - \alpha_i) f(\alpha_j) = 0 \quad (4.8)$$

for any balanced triplet: $i + j + k = 0$.

At this point it is convenient to allow parameters $\{\alpha_i\}$ vary, in other words to look at a family of "energy-like" Hamiltonians⁷ $H_\alpha = \sum \alpha_i |\omega_i|^2$. Differentiating (4.8) in α_i we can rewrite it as

$$f'(\alpha_i) = \frac{f(\alpha_j) - f(\alpha_k)}{\alpha_j - \alpha_k} \text{ for all } i; j; k \quad (4.9)$$

Choosing a particular ordering of $\{i; j; k\}$ so that $\alpha_i < \alpha_j < \alpha_k$, the direct examination (fig. 4.5) shows that function f is neither convex $f'' > 0$ nor concave $f'' < 0$. So the only choice is flat (linear)

$$f = c_0 \alpha + c_1$$

Hence we get the final result

$$B = c_0 E + c_1 \Omega, \text{ Q. E. D.}$$

The analysis of higher order commuting integrals (cubic etc.) is more involved and will be carried out elsewhere.

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⁷Any continuous change of coefficients $\{\alpha_i\}$ from the "energy" values $\alpha_i = |i|^{-2}$ to another set would allow to deform the β -coefficients in such a way that preserves the zero bracket of $H = \sum \alpha_i |\omega_i|^2$ and $B = \sum \beta_i |\omega_i|^2$, $\{H; B\} = 0$

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