RANDOMISED APPROXIMATION SCHEMES FOR TUTTE-GRÖTHENDIECK INVARIANTS

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IMA Preprint Series # 1226

April 1994

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1 Introduction

Consider the following very simple counting problems associated with a graph G.

(i) What is the number of connected subgraphs of G?

(ii) How many subgraphs of G are forests?

(iii) How many acyclic orientations has G?

Each of these is a special case of the general problem of evaluating the Tutte polynomial of a graph (or matroid) at a particular point of the (x, y)-plane — in other words is a Tutte-Gröthendieck invariant. Other invariants include:

(iv) the chromatic and flow polynomials of a graph;

(v) the partition function of a Q-state Potts model;

(vi) the Jones polynomial of an alternating link;

(vii) the weight enumerator of a linear code over $GF(q)$.

It has been shown that apart from a few special points and 2 special hyperbolae, the exact evaluation of any such invariant is $\#P$-hard even for the very restricted class of planar bipartite graphs. However the question of which points have a fully polynomial randomised approximation scheme is wide open. I shall discuss this problem and give a survey of what is currently known.

The graph terminology used is standard. The complexity theory and notation follows Garey and Johnson (1979). The matroid terminology follows Oxley (1992). Further details of most of the concepts treated here can be found in Welsh (1993).

2 Tutte-Gröthendieck invariants

First consider the following recursive definition of the function $T(G;x,y)$ of a graph G, and two independent variables x, y.

If G has no edges then $T(G;x,y) = 1$, otherwise for any $e \in E(G)$;

(2.1) $T(G;x,y) = T(G'_e;x,y) + T(G''_e;x,y)$, where G'_e denotes the deletion of the edge e from G and G''_e denotes the contraction of e in G,

(2.2) $T(G;x,y) = xT(G'_e;x,y)$ e an isthmus,

(2.3) $T(G;x,y) = yT(G''_e;x,y)$ e a loop.

From this, it is easy to show by induction that T is a 2-variable polynomial in x, y, which we call the *Tutte polynomial* of G.

In other words, T may be calculated recursively by choosing the edges in *any* order and repeatedly using (2.1-3) to evaluate T. The remarkable fact is that T is well defined in the sense that the resulting polynomial is independent of the order in which the edges are chosen.

Alternatively, and this is often the easiest way to prove properties of T, we can show that T has the following expansion.

First recall that if $A \subseteq E(G)$, the *rank* of A, $r(A)$ is defined by

$$r(A) = |V(G)| - k(A), \tag{2.4}$$

where $k(A)$ is the number of connected components of the graph $G : A$ having vertex set $V = V(G)$ and edge set A.

It is now straightforward to prove:

(2.5) The Tutte polynomial $T(G;x,y)$ can be expressed in the form

$$T(G;x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

This relates T to the *Whitney rank generating function* $R(G;u,v)$ which is a 2-variable polynomial in the variables u, v, and is defined by

$$R(G;u,v) = \sum_{A \subseteq E} u^{r(E)-r(A)} v^{|A|-r(A)}. \tag{2.6}$$

It is easy and useful to extend these ideas to matroids.

A *matroid* M is just a pair (E, r) where E is a finite set and r is a submodular *rank function* mapping $2^E \to Z$ and satisfying the conditions

$$0 \leq r(A) \leq |A| \quad A \subseteq E, \tag{2.7}$$

$$A \subseteq B \Rightarrow r(A) \leq r(B), \tag{2.8}$$

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B) \quad A, B \subseteq E. \tag{2.9}$$

The edge set of any graph G with its associated rank function as defined by (2.4) is a matroid, but this is just a very small subclass of matroids:- known as graphic matroids.

Given $M = (E, r)$ the *dual matroid* $M^* = (E, r^*)$ where r^* is defined by

$$r^*(E \backslash A) = |E| - r(E) - |A| + r(A). \tag{2.10}$$

We now just extend the definition of the Tutte polynomial from graphs to matroids by,

$$T(M; x, y) = \sum_{A \subseteq E(M)} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}. \tag{2.11}$$

Much of the theory developed for graphs goes through in this more general setting and there are many other applications as we shall see. For example, routine checking shows that

$$T(M; x, y) = T(M^*; y, x). \tag{2.12}$$

In particular, when G is a planar graph and G^* is any plane dual of G, (2.12) becomes

$$T(G; x, y) = T(G^*; y, x). \tag{2.13}$$

A set X is *independent* if $r(X) = |X|$, it is a *base* if it is a maximal independent subset of E. An easy way to work with the dual matroid M^* is not via the rank function but by the following definition.

(2.15) M^* has as its bases all sets of the form $E \backslash B$, where B is a base of M.

We close this section with what I call the "recipe theorem" from Oxley and Welsh (1979). Its crude interpretation is that whenever a function f on some class of matroids can be shown to satisfy an equation of the form $f(M) = af(M_e') + b(M_e'')$ for some $e \in E(M)$, then f is essentially an evaluation of the Tutte polynomial. More precisely it says:

(2.16) Theorem. Let $\mathcal{C}$ be a class of matroids which is closed under direct sums and the taking of minors and suppose that f is well defined on $\mathcal{C}$ and satisfies

$$f(M) = af(M_e') + bf(M_e'') \quad e \in E(M) \tag{2.17}$$
$$f(M_1 \oplus M_2) = f(M_1)f(M_2) \tag{2.18}$$

then f is given by

$$f(M) = a^{|E|-r(E)} b^{r(E)} T(M; \frac{x_0}{b}, \frac{y_0}{a})$$

where x_0 and y_0 are the values f takes on coloops and loops respectively.

Any invariant f which satisfies (2.17)-(2.18) is called, a *Tutte-Gröthendieck (TG)-invariant.*

Thus, what we are saying is that any TG-invariant has an interpretation as an evaluation of the Tutte polynomial.

3 Reliability and flows

We illustrate the above with two applications.

Reliability theory deals with the probability of points of a network being connected when individual links or edges are unreliable. Early work in the area was by Moore and Shannon (1956) and now it has a huge literature, see for example Colbourne (1987).

Let G be a connected graph in which each edge is independently *open* with probability p and *closed* with probability $q = 1 - p$. The (*all terminal*) *reliability* $R(G;p)$ denotes the probability that in this random model there is a path between each pair of vertices of G. Thus

$$R(G;p) = \sum_A p^{|A|}(1-p)^{|E\backslash A|} \tag{3.1}$$

where the sum is over all subsets A of edges which contain a spanning tree of G, and $E = E(G)$.

It is immediate from this that R is a polynomial in p and a simple conditioning argument shows the following connection with the Tutte polynomial.

(3.2) If G is a connected graph and e is not a loop or coloop then

$$R(G;p) = qR(G'_e;p) + pR(G''_e;p).$$

where $q = 1 - p$.

Using this with the recipe Theorem 2.16 it is straightforward to check the following statement.

(3.3) Provided G is a connected graph,

$$R(G;p) = q^{|E|-|V|+1}p^{|V|-1}T(G;1,q^{-1}).$$

We now turn to flows. Take any graph G and orient its edges arbitrarily. Take any finite Abelian group H and call a mapping $\phi : E(G) \to H\backslash\{0\}$ a *flow* (or an *H-flow*) if Kirchhoff's laws are obeyed at each vertex of G, the algebra of course being that of the group H.

Note: Standard usage is to describe what we call an H-flow a *nowhere zero* H-flow.

The following statement is somewhat surprising.

(3.4) The number of H-flows on G depends only on the order of H and not on its structure.

This is an immediate consequence of the fact that the number of flows is a TG-invariant. To see this, let $F(G;H)$ denote the number of H-flows on G. Then a straightforward counting argument shows that the following is true.

(3.5) Provided the edge e is not an isthmus or a loop of G then

$$F(G;H) = F(G''_e;H) - F(G'_e;H).$$

Now it is easy to see that if C, L represents respectively a coloop (= isthmus) and loop then

$$F(C;H) = 0 \quad F(L;H) = o(H) - 1 \tag{3.6}$$

where $o(H)$ is the order of H.

Accordingly we can apply the recipe theorem and obtain:

(3.7) For any graph G and any finite abelian group H,

$$F(G;H) = (-1)^{|E|-|V|+k(G)}T(G;0,1-o(H)).$$

The observation (3.4) is an obvious corollary.

A consequence of this is that we can now speak of G *having a k-flow* to mean that G has a flow over *any* or equivalently *some* Abelian group of order k.

Moreover it follows that there exists a polynomial $F(G;\lambda)$ such that if H is Abelian of order k, then $F(G;H) = F(G;k)$. We call F the *flow polynomial* of G.

The duality relationship (2.12) gives:

(3.8) If G is planar then the flow polynomial of G is essentially the chromatic polynomial of G^*, in the sense that

$$\lambda^{k(G)}F(G;\lambda) = P(G^*;\lambda).$$

A consequence of this and the Four Colour Theorem is that:

(3.9) Every planar graph having no isthmus has a 4-flow.

What is much more surprising is that the following statement is believed to be true:

(3.10) Tutte's 5-Flow Conjecture: Any graph having no isthmus has a 5-flow.

It is far from obvious that there is any universal constant k such that graphs without isthmuses have a k-flow. However Seymour (1981) showed:

(3.11) Theorem: Every graph having no isthmus has a 6-flow.

For more on this and a host of related graph theoretic problems we refer to Jaeger (1988a).

4 A catalogue of invariants

We now collect together some of the naturally occurring interpretations of the Tutte polynomial. Throughout G is a graph, M is a matroid and E will denote $E(G), E(M)$ respectively.

(4.1) At (1,1) T counts the number of bases of M (spanning trees in a connected graph).

(4.2) At (2,1) T counts the number of independent sets of M, (forests in a graph).

(4.3) At (1,2) T counts the number of spanning sets of M, that is sets which contain a base.

(4.4) At (2,0), T counts the number of acyclic orientations of G. Stanley (1973) also gives interpretations of T at $(m, 0)$ for general positive integer m, in terms of acyclic orientations.

(4.5) Another interpretation at (2,0), and this for a different class of matroids, was discovered by Zaslavsky (1975). This is in terms of counting the number of different arrangements of sets of hyperplanes in n-dimensional Euclidean space.

(4.6) $T(G; -1, -1) = (-1)^{|E|}(-2)^{d(B)}$ where B is the bicycle space of G, see Read and Rosenstiehl (1978). When G is planar it also has interpretations in terms of the Arf invariant of the associated knot.

(4.7) The chromatic polynomial $P(G; \lambda)$ is given by

$$P(G; \lambda) = (-1)^{r(E)} \lambda^{k(G)} T(G; 1 - \lambda, 0)$$

where $k(G)$ is the number of connected components.

(4.8) The flow polynomial $F(G; \lambda)$ is given by

$$F(G; \lambda) = (-1)^{|E| - r(E)} T(G; 0, 1 - \lambda).$$

(4.9) The (all terminal) reliability $R(G : p)$ is given by

$$R(G; p) = q^{|E| - r(E)} p^{r(E)} T(G; 1, 1/q)$$

where $q = 1 - p$.

In each of the above cases, the interesting quantity (on the left hand side) is given (up to an easily determined term) by an evaluation of the Tutte polynomial. We shall use the phrase "*specialises to*" to indicate this. Thus for example, along $y = 0$, T specialises to the chromatic polynomial.

It turns out that the hyperbolae H_α defined by

$$H_\alpha = \{(x,y) : (x-1)(y-1) = \alpha\}$$

seem to have a special role in the theory. We note several important specialisations below.

(4.10) Along H_1, $T(G;x,y) = x^{|E|}(x-1)^{r(E)-|E|}$.

(4.11) Along H_2; when G is a graph T specialises to the partition function of the Ising model.

(4.12) Along H_q, for general positive integer q, T specialises to the partition function of the Potts model of statistical physics.

(4.13) Along H_q, when q is a prime power, for a matroid M of vectors over $GF(q)$, T specialises to the weight enumerator of the linear code over $GF(q)$, determined by M. Equation (2.12) relating $T(M)$ to $T(M^*)$ gives the MacWilliams identity of coding theory.

(4.14) Along H_q for any positive, not necessarily integer, q, T specialises to the partition function of the random cluster model introduced by Fortuin and Kasteleyn (1971).

(4.15) Along the hyperbola $xy = 1$ when G is planar, T specialises to the Jones polynomial of the alternating link or knot associated with G. This connection was first discovered by Thistlethwaite (1987).

Other more specialised interpretations can be found in the survey of Brylawski and Oxley (1992).

5 The complexity of the Tutte plane

We have seen that along different curves of the x, y plane, the Tutte polynomial evaluates such diverse quantities as reliability probabilities, the weight enumerator of a linear code, the partition function of the Ising and Potts models of statistical physics, the chromatic and flow polynomials of a graph, and the Jones polynomial of an alternating knot. Since it is also the case that for particular curves and at particular points the computational complexity of the evaluation can vary from being polynomial time computable to being $\#P$-hard a more detailed analysis of the complexity of evaluation is needed in order to give a better understanding of what is and is not computationally feasible for these sort of problems. The section is based on the paper of Jaeger, Vertigan and Welsh (1990) which will henceforth be referred to as [JVW].

First consider the problem:

$\pi_1[\mathcal{C}]$: *TUTTE POLYNOMIAL OF CLASS* $\mathcal{C}$

INSTANCE: Graph G belonging to the class $\mathcal{C}$.
OUTPUT: The coefficients of the Tutte polynomial of G.

We note first that for all but the most restricted classes this problem will be $\#P$-hard. This follows from the following observations.

(5.1) Determining the Tutte polynomial of a planar graph is $\#P$-hard.

Proof. Determining the chromatic polynomial of a planar graph is $\#P$- hard and this problem is the evaluation of the Tutte polynomial along the line $y = 0$. □

It follows that:

(5.2) If $\mathcal{C}$ is any class of graphs which contains all planar graphs then $\pi_1[\mathcal{C}]$ is $\#P$-hard.

However it does not follow that it may not be easy to determine the value of $T(G; x, y)$ at particular points or along particular curves of the x, y plane. For example, the evaluation of the Tutte polynomial at (1,1) gives the number of spanning trees of the underlying graph and hence the Kirchhoff determinantal formula shows:

(5.3) Evaluating $T(G; 1, 1)$ for general graphs is in P.

We now consider two further problems.

$\pi_2[\mathcal{C} : L]$ *TUTTE POLYNOMIAL OF CLASS* $\mathcal{C}$ *ALONG CURVE* L
INSTANCE: Graph G belonging to the class $\mathcal{C}$.
OUTPUT: The Tutte polynomial along the curve L as a polynomial with rational coefficients.

$\pi_3[\mathcal{C} : a, b]$ *TUTTE POLYNOMIAL OF CLASS* $\mathcal{C}$ *AT* (a, b)
INSTANCE: Graph G belonging to the class $\mathcal{C}$.
OUTPUT: Evaluation of $T(G; a, b)$.

Note: There are some technical difficulties here, inasmuch as we have to place some restriction on the sort of numbers on which we do our arithmetic operations, and also the possible length of inputs. Thus we restrict our arithmetic to be within a field F which is a finite dimensional algebraic extension of the rationals. We also demand (for reasons which become apparent) that F contains the complex numbers i and $e^{2\pi i/3}$. Similarly we demand that any curve L under discussion will be a rational algebraic curve over such a field F and that L is given in standard parametric form. For more details see [JVW] and for more on this general question see Grötschel, Lovász and Schrivjer (1988).

Now it is obvious that for any class $\mathcal{C}$, if evaluating T at (a, b) is hard and $(a, b) \in L$ then evaluating T along L is hard. Similarly if evaluating T along L is hard then determining T is hard. In other words:

(5.4) For any class $\mathcal{C}$, curve L and point (a,b) with $(a,b) \in L$,

$$\pi_3[\mathcal{C}; a, b] \propto \pi_2[\mathcal{C}; L] \propto \pi_1[\mathcal{C}].$$

Two of the main results of [JVW] are that except when (a,b) is one of a few very special points and L one of a special class of hyperbolae, then the reverse implications hold in (5.4).

Before we can state the two main theorems from [JVW] we need one more definition. Call a class of graphs *closed* if it is closed under the operations of taking minors and series and parallel extensions. That is $\mathcal{C}$ shall remain closed under the four operations of deletion and contraction of an edge together with the insertion of an edge in series or in parallel with an existing edge.

The first result of [JVW] relates evaluations in general with evaluation along a curve.

(5.5) Theorem. If $\mathcal{C}$ is any closed class then the problem $\pi_1[\mathcal{C}]$ of determining the Tutte polynomial of members of $\mathcal{C}$ is polynomial time reducible to the problem $\pi_2[\mathcal{C}; L]$ for any curve L, except when L is one of the hyperbolae defined by

$$H_\alpha \equiv (x-1)(y-1) = \alpha \quad \alpha \neq 0,$$

or the degenerate hyperbolae

$$H_0^x \equiv \{(x,y) : x = 1\}$$

$$H_0^y \equiv \{(x,y) : y = 1\}.$$

We call the hyperbolae H_α *special hyperbolae* and an immediate corollary of the theorem is:

(5.6) Corollary. If L is a curve in the x, y plane which is not one of the special hyperbolae, and $\pi_1(\mathcal{C})$ is $\#P$-hard then $\pi_1(\mathcal{C}; L)$ is $\#P$-hard.

The second main theorem of [JVW] relates the complexity of determining the Tutte polynomial along a curve with determining its value at a particular point on the curve.

(5.7) Theorem. The problem $\pi_3[\mathcal{C}; a, b]$ of evaluating $T(G; a, b)$ for members G of $\mathcal{C}$ (a closed class) is polynomial time reducible to evaluating T along the special hyperbola through (a,b) except when (a,b) is one of the special points (1,1),(0,0),(-1,-1),(0,-1),(- 1,0),$(i,-i)$, $(-i,i)$ and (j,j^2), (j^2,j) where $i^2 = -1$ and $j = e^{2\pi i/3}$.

In other words unless a point is one of these 9 special points evaluating the Tutte polynomial at that point is no easier than evaluating it along the special hyperbola through that point.

As an illustration of the applicability of these results we state without proof the following theorem from [JVW].

(5.8) Theorem. The problem of evaluating the Tutte polynomial of a graph at a point (a,b) is $\#P$-hard except when (a,b) is on the special hyperbola

$$H_1 \equiv (x-1)(y-1) = 1$$

or when (a,b) is one of the special points (1,1),(-1,-1),(0,-1), (-1,0),$(i,-i)$, $(-i,i),(j,j^2)$ and (j^2,j), where $j = e^{2\pi i/3}$. In each of these exceptional cases the evaluation can be done in polynomial time.

As far as planar graphs are concerned, there is a significant difference. The technique developed using the Pfaffian to solve the Ising problem for the plane square lattice by Fisher (1966) and Kasteleyn (1961) can be extended to give a polynomial time algorithm for the evaluation of the Tutte polynomial of any planar graph along the special hyperbola

$$H_2 \equiv (x-1)(y-1) = 2.$$

Thus this hyperbola is also "easy" for planar graphs. However it is easy to see that H_3 cannot be easy for planar graphs since it contains the point (- 2,0) which counts the number of 3-colourings and since deciding whether a planar graph is 3-colourable is NP-hard, this must be at least NP-hard. However it does not seem easy to show that H_4 is hard for planar graphs. The decision problem is after all trivial by the four colour theorem. The fact that it is $\#P$-hard is just part of the following extension of Theorem (5.8) due to Vertigan and Welsh (1992)

(5.9) Theorem. The evaluation of the Tutte polynomial of bipartite planar graphs at a point (a,b) is $\#P$-hard except when

$$(a,b) \in H_1 \cup H_2 \cup \{(1,1),(-1,-1),(j,j^2),(j^2,j)\}$$

when it is computable in polynomial time.

6 Approximating to within a ratio

We know that computing the number of 3-colourings of a graph G is $\#P$- hard. It is natural therefore to ask how well can we approximate it?

For positive numbers a and $r \geq 1$, we say that a third quantity $\hat{a}$ *approximates a within ratio r* or is an *r-approximation* to a, if

$$r^{-1}a \leq \hat{a} \leq ra. \tag{6.1}$$

In other words the ratio $\hat{a}/a$ lies in $[r^{-1}, r]$.

Now consider what it would mean to be able to find a polynomial time algorithm which gave an approximation within r to the number of 3-colourings of a graph. We would clearly have a polynomial time algorithm which would decide whether or not a graph is 3-colourable. But this is NP-hard. Thus no such algorithm can exist unless $NP = P$.

But we have just used 3-colouring as a typical example and the same argument can be applied to any function which counts objects whose existence is NP- hard to decide. In other words:

(6.2) Proposition. If $f : \Sigma^* \to N$ is such that it is NP-hard to decide whether $f(x)$ is non-zero, then for any constant r there cannot exist a polynomial time r-approximation to f unless $NP = P$.

We now turn to consider a randomised approach to counting problems and make the following definition.

An *ϵ-δ-approximation scheme* for a counting problem f is a Monte Carlo algorithm which on every input $\langle x, \epsilon, \delta \rangle$, $\epsilon > 0$, $\delta > 0$, outputs a number $\tilde{Y}$ such that

$$Pr\{(1-\epsilon)f(x) \leq \tilde{Y} \leq (1+\epsilon)f(x)\} \geq 1 - \delta.$$

It is important to emphasize that there is no mention of running time in this definition.

Now let f be a function from input strings to the natural numbers. A *randomised approximation scheme* for f is a probabilistic algorithm that takes as an input a string x and a rational number ϵ, $0 < \epsilon < 1$, and produces as output a random variable Y, such that Y approximates $f(x)$ within ratio $1+\epsilon$ with probability $\geq 3/4$.

In other words,

$$Pr\left\{\frac{1}{1+\epsilon} \leq \frac{Y}{f(x)} \leq 1+\epsilon\right\} \geq \frac{3}{4}. \tag{6.3}$$

A *fully polynomial randomised approximation scheme* (fpras) for a function $f : \Sigma^* \to N$ is a randomised approximation scheme which runs in time which is a polynomial function of n *and* ϵ^{-1}.

Suppose now we have such an approximation scheme and suppose further that it works in polynomial time. Then we can boost the success probability up to $1 - \delta$ for any desired $\delta > 0$, by using the following trick of Jerrum, Valiant and Vazirani (1986). This consists of running the algorithm $O(\log \delta^{-1})$ times and taking the median of the results.

We make this precise as follows:

(6.4) Proposition. If there exists a fpras for computing f then there exists an $\epsilon - \delta$ approximation scheme for f which on input $\langle x, \epsilon, \delta \rangle$ runs in time which is bounded by $O(\log \delta^{-1})\text{poly}(x, \epsilon^{-1})$.

It is worth emphasising here that the existence of a fpras for a counting problem is a very strong result, it is the analogue of an RP algorithm for a decision problem and corresponds to the notion of tractability. However we should also note, that by an analogous argument to that used in proving Proposition (6.2) we have:

(6.5) Proposition: If $f : \Sigma^* \to N$ is such that deciding if f is nonzero is NP-hard then there cannot exist a fpras for f unless NP is equal to random polynomial time RP.

Since this is thought to be most unlikely, it makes sense only to seek out a fpras when counting objects for which the decision problem is not NP-hard.

Hence we have immediately from the NP-hardness of k-colouring that:

(6.6) Unless $NP = RP$ there cannot exist a fpras for evaluating $T(G; -k, 0)$ for any integer $k \geq 2$.

Recall now from (4.12) that along the hyperbola, H_Q, for positive integer Q, T evaluates the partition function of the Q-state Potts model.

In an important paper, Jerrum and Sinclair (1990), have shown that there exists a fpras for the ferromagnetic Ising problem. This corresponds to the $Q = 2$ Potts model and thus, their result can be restated in the terminology of this paper as follows.

(6.7) There exists a fpras for estimating T along the positive branch of the hyperbola H_2.

However it seems to be difficult to extend the argument to prove a similar result for the Q-state Potts model with $Q > 2$ and this remains one of the outstanding open problems in this area.

A second result of Jerrum and Sinclair is the following:

(6.8) There is no fpras for estimating the antiferromagnetic Ising partition function unless $NP = RP$.

Since it is regarded as highly unlikely that $NP = RP$, this can be taken as evidence of the intractability of the antiferromagnetic problem.

Examination of (6.8) in the context of its Tutte plane representation shows that it can be restated as follows.

(6.9) Unless $NP = RP$, there is no fpras for estimating T along the curve

$$\{(x, y) : (x-1)(y-1) = 2, \quad 0 < y < 1\}.$$

The following extension of this result is proved in Welsh (1993b).

(6.10) **Theorem.** On the assumption that $NP \neq RP$, the following statements are true.

(6.11) Even in the planar case, there is no fully polynomial randomised approximation scheme for T along the negative branch of the hyperbola H_3.

(6.12) For $Q = 2, 4, 5, \ldots$, there is no fully polynomial randomised approximation scheme for T along the curves

$$H_Q^- \cap \{x < 0\}.$$

It is worth emphasising that the above statements do not rule out the possibility of there being a fpras at *specific points* along the negative hyperbolae. For example;

(6.13) T can be evaluated exactly at $(-1, 0)$ and $(0, -1)$ which both lie on H_2^-.

(6.14) There is no inherent obstacle to there being a fpras for estimating the number of 4-colourings of a planar graph.

I do not believe such a scheme exists but cannot see how to prove it. It certainly is not ruled out by any of our results. I therefore pose the specific question:

(6.15) **Problem.** Is there a fully polynomial randomised approximation scheme for counting the number of k-colourings of a planar graph for any fixed $k \geq 4$?

I conjecture that the answer to (6.15) is negative.

Similarly, since by Seymour's theorem, every bridgeless graph has a nowhere zero 6-flow, there is no obvious obstacle to the existence of a fpras for estimating the number of k-flows for $k \geq 6$. Thus a natural question, which is in the same spirit is the following.

(6.16) Show that there does not exist a fpras for estimating T at $(0, -5)$. More generally, show that there is no fpras for estimating the number of k-flows for $k \geq 6$.

Again, although because of Theorem 6.10, a large section of the relevant hyperbola has no fpras, there is nothing to stop such a scheme existing at isolated points.

Another point of special interest is $(0, -2)$. Mihail and Winkler (1991) have shown, among other things, that there exists a fpras for counting the number of ice configurations in a 4-regular graph. This is equivalent to the statement:

(6.17) For 4-regular graphs counting nowhere-zero 3-flows has a fpras.

In other words:

(6.18) There is a fpras for computing T at $(0, -2)$ for 4-regular graphs.

The reader will note that all these 'negative results' are about evaluations of T in the region outside the quadrant $x \geq 1$, $y \geq 1$. In Welsh (1993) it is conjectured that the following is true:

(6.19) **Conjecture:** There exists a fpras for evaluating T at all points of the quadrant $x \geq 1$, $y \geq 1$.

Some further recent evidence in support of this is given in the next section.

7 Denseness helps

One mild curiosity about quantities which are hard to count is that the counting problem seems to be easier when the underlying structure is dense. A well known example of this is in approximating the permanent. Less widely known is the result of Edwards (1986) which shows that it is possible to exactly count the number of k-colourings of graphs in which each vertex has at least $(k-2)/(k-1)$ neighbours.

More precisely, if we let $\mathcal{G}_\alpha$ be the collection of graphs $G = (V, E)$ such that each vertex has at least $\alpha |V|$ neighbours then we say G is *dense* if $G \in \mathcal{G}_\alpha$ for some α. Edwards shows:

(7.1) For $G \in \mathcal{G}_\alpha$, evaluation of $T(G; 1-k, 0)$ is in P provided $\alpha > (k-2)/(k-1)$.

As far as approximation is concerned, a major advance is the recent result of Annan (1993) who showed that:

(7.2) There exists a fpras for counting forests in dense graphs.

Now the number of forests is just the evaluation of T at the point (2.1) and a more general version of (7.2) is the following result, also by Annan.

(7.3) For dense G, there is a fpras for evaluating $T(G; x, 1)$ for positive integer x.

The natural question suggested by (7.3) is about the matroidal dual - namely, does there exist a fpras for evaluating T at $(1, x)$? This is the reliability question, and in particular, the point (1,2) enumerates the number of connected subgraphs. It is impossible to combine duality with denseness so Annan's methods don't seem to work.

It turns out that I can obtain the following result:

(7.4) For dense G there exists a fpras for evaluating $T(G; x, y)$ for all (x, y) in the region

$$\{1 \leq x\} \cap \{1 < y\} \cap \{(x-1)(y-1) \leq 1\}.$$

An immediate corollary is:

(7.5) There exists a fpras for estimating the reliability probability in dense graphs.

The proof of (7.4) hinges on a result of Joel Spencer (1993). In its simplest form it can be stated as follows:

(7.6) For $G \in \mathcal{G}_\alpha$, if G_p denotes a random subgraph of G obtained by deleting edges from G with probability $1-p$, independently for each edge, then there exists $d(\alpha, p) < 1$, such that

$$Pr\{G_p \text{ is disconnected}\} \leq d(\alpha, p).$$

At this meeting, I have discovered that Alan Frieze has also obtained a result which is very similar to (7.5), and since the meeting, using some ideas of Noga Alon, together we have been able to eliminate the constraint $(x-1)(y-1) \leq 1$ in (7.4). We aim to write this up in the near future.

I close with three open problems related to (7.4).

(7.7) Does there exist a fpras for counting (a) acyclic orientations, (b) forests of size k, or (c) connected subgraphs of size k, in dense graphs?

Two of these problems are particular instances of the problem of estimating the number of bases in a matroid of some class. For example the question about forests is just counting the number of bases in truncations of graphic matroids.

A very interesting new light on such problems is outlined in the recent paper of Feder and Mihail (1992) who show that a sufficient condition for the natural random walk on the set of bases of M to be rapidly mixing is that M and its minors are *negatively correlated.* That is, they satisfy the constraint, that if B_R denotes a random base of M and e, f is any pair of distinct elements of $E(M)$, then

$$Pr\{e \in B_R \mid f \in B_R\} \leq P_r\{e \in B_R\}.$$

This concept of negative correlation goes back to Seymour and Welsh (1975). This was concerned with the Tutte polynomial and the log-concave nature of various of its evaluations and coefficients. It was shown there that not all matroids have this property of negative correlation; however graphic and cographic ones do, and it is mildly intrigueing that this concept has reappeared now in an entirely new context.

Acknowledgement

The original impetus to look at evaluations of T in the dense case, arose in discussions with Marek Karpinski in a RAND meeting in Oxford in September 1992. Further discussions with Joel Spencer, Alan Frieze, Noga Alon, Nabil Kahale and James Annan have also been very helpful.

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