INFINITE DIMENSIONAL HAMILTON-JACOBI EQUATIONS AND DIRICHLET BOUNDARY CONTROL PROBLEMS OF PARABOLIC TYPE

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Abstract

The paper is concerned with an infinite dimensional Hamilton–Jacobi equation. This equation is related to boundary control problems of Dirichlet type for semilinear parabolic systems.

The viscosity solution approach is adapted to the equation under consideration, using the properties of fractional powers of generators of analytic semigroups. An existence and uniqueness result for such problem is obtained.

Key words: Boundary control, viscosity solutions, Hamilton–Jacobi equation, parabolic equations, Dirichlet boundary conditions.

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1 Introduction

In this paper we study the existence and uniqueness of viscosity solutions to the infinite dimensional Hamilton–Jacobi equation

$$\lambda v(x) + < Ax + \Phi(x), Dv(x) > + H(A^\beta x, Dv(x)) = 0, \quad x \in X, \tag{1.1}$$

where X is a real Hilbert space, $\lambda > 0$ and $H : X \times X \to \mathbb{R}$ is continuous. Moreover, $A : D(A) \subset X \to X$ is a closed linear operator with a compact and dense inclusion $D(A) \subset X$. Also, we assume A to be positive and self–adjoint. We denote by A^β the fractional power of A. Finally $\Phi : D(A^\beta) \to D(A^{-\beta})$ is Lipschitz continuous.

There is an increasing interest and a growing literature on Hamilton–Jacobi equations in infinite dimensions. These equations were first studied by V.Barbu & G.Da Prato (see e.g. [2]), setting the problem in classes of convex functions and using semigroup and perturbation methods.

The viscosity solution approach was then adapted to infinite dimensional equations by M.G.Crandall & P.L.Lions in a sequence of papers [9]. This approach was introduced in [8], (see also [7]), for finite dimensional problems. It allows to obtain uniqueness and comparison results for weak solutions of nonlinear first order PDEs. Additional contributions to the viscosity solution method were obtained by M.Soner [18], H.Ishii [14] and D.Tataru [19], [20]. The last two authors treated equations with a maximal monotone operator A, possibly multivalued. On the other hand, due to the presence of the unbounded term A^β inside the Hamiltonian H, the results proved in these papers do not apply to equation (1.1) except for the case of $\beta = 0$.

In this paper we study the above equation for $\beta \in \left(\frac{3}{4}, 1\right]$. We are interested in this problem because it is related to boundary control of parabolic equations under Dirichlet boundary conditions. We now briefly describe such a problem, more details being given in Section 2.

It is well known that an abstract formulation modelling parabolic systems controlled at the boundary is given by

$$\begin{cases} x'(t) + Ax(t) + F(x(t)) = A^\beta B\gamma(t) \\ x(0) = x_0 \end{cases} \tag{1.2}$$

where $x_0 \in X$ and $\gamma : [0, +\infty) \to U$ is measurable, U being another Hilbert space. Moreover, $B : U \to X$ is a bounded operator, $F : X \to X$ is Lipschitz, and A is maximal accretive. Several kinds of boundary conditions are included in the above formulation: for example, Neumann type boundary conditions allow to take $\beta \in \left(\frac{1}{4}, 1\right]$ in (1.2), whereas Dirichlet data restrict the range of β to $\left(\frac{3}{4}, 1\right]$.

Denoting by $x(\cdot; x_0, \gamma)$ the mild solution of (1.2), one then seeks to minimize a suitable cost functional over all controls γ. In this paper, we consider the functional

$$J(x_0; \gamma) = \int_0^\infty e^{-\lambda t} L(x(t; x_0, \gamma), \gamma(t)) dt, \tag{1.3}$$

where $\lambda > 0$ and $L : X \times U \to \mathbb{R}$ is a given running cost.

Boundary control plays a central role in the theory of distributed parameter systems. There is a vast literature dealing with Linear Quadratic problems, see for instance [1], [11], [15], [3]. In this theory, the main tool for constructing optimal boundary controls is represented by the Riccati equation. The technique used to study this equation for Neumann boundary conditions differs substantially from the one used for Dirichlet conditions. In particular, the way to solve Riccati equations for Neumann data does not apply to Dirichlet data, see for instance [12], [13], [10], [15]. In fact, the latter problem requires a much more careful choice of wheighted norms and function spaces, see e.g. [3].

For boundary control problems that are not Linear Quadratic, the role of the Riccati equation is played by the Dynamic Programming equation

$$\lambda u(x) + < Ax + F(x), Du(x) > + H(x, A^{\beta} Du(x)) = 0, \quad x \in X, \tag{1.4}$$

where $H : X \times X \to \mathbb{R}$ is defined as

$$H(x,p) = \sup_{\gamma \in U} \left[- < B\gamma, p > - L(x,\gamma) \right]. \tag{1.5}$$

The value function of problem (1.3), defined as

$$u(x_0) = \inf \left\{ \int_0^{\infty} e^{-\lambda t} L(x(t; x_0, \gamma), \gamma(t)) dt \mid \gamma : [0, +\infty) \to U \right\}, \tag{1.6}$$

is characterized as the unique solution of equation (1.4).

In [5], the viscosity solution approach has been adapted to equation (1.4) for $\beta \in \left(\frac{1}{4}, \frac{1}{2}\right)$. Therefore the results of [5] yield an existence and uniqueness theorem for the Dynamic Programming equation of boundary control problems of Neumann type. On the other hand, similarly to the Linear Quadratic case, the method of [5] does not apply to (1.4) if $\beta \geq \frac{1}{2}$ and, in particular, to boundary conditions of Dirichlet type.

In this paper we transform the state equation (1.2) by the change of variable $y = A^{-\beta} x$. In this way, we obtain a state equation with continuous trajectories. Accordingly, the Dynamic Programming equation (1.4) is tranformed into equation (1.1) with H defined as in (1.5) and

$$\Phi(x) = A^{-\beta} F(A^{\beta} x).$$

Following the approach of [14], in Section 3 we give a definition of solution to (1.1) which requires the equation to be satisfied in a suitable viscosity sense only on $D(A)$. Using this definition, we obtain a comparison result for Hölder continuous viscosity solutions of (1.1), see Theorem 3.2. In Section 4 we prove a Hölder continuity result for the function $v(x) = u(A^{\beta} x)$. Moreover, we show that v is the unique viscosity solution of (1.1), see Corollary 4.3. In particular, our results characterize the value function u in (1.6) as well.

We conclude this introduction with some comments on possible extensions and applications of our approach. First, we note that the method proposed in this paper applies to both Dirichlet and Neumann boundary control problems. In fact, each of them can be written in the abstract formulation (1.2) with $\beta \in \left(\frac{3}{4}, 1\right]$. Second, the assumption that $A = A^*$ has been made just to simplify the exposition. Using similar ideas one can treat systems governed by operators that are not necessarely self-adjoint. On the other hand, to prove that the function v is a viscosity solution of (1.1), we need to assume that $-A$ generates an analytic semigroup of compact operators. Therefore, the results of this paper concerning existence, typically apply to parabolic boundary control problems in bounded space domains.

Finally, the techniques of this paper can also be used to study boundary control problems of Dirichlet type with finite horizon. In this case, the Dynamic Programming equation is an evolution equation. For the corresponding Cauchy problem one can prove existence and uniqueness results. The analogous equation for Neumann boundary control is treated in [6].

2 Preliminaries

Let X and U be two real Hilbert spaces and let $\tilde{U} \subset U$ be closed and bounded. We set $R = \sup_{\gamma \in \tilde{U}} |\gamma|$.

Let $x_0 \in X$ and consider the problem of minimizing the functional

$$J(x_0;\gamma) = \int_0^\infty e^{-\lambda t} L(x(t;x_0,\gamma),\gamma(t))dt \tag{2.1}$$

over all measurable functions $\gamma : [0,\infty) \to \tilde{U}$ (usually called controls). Here $x(\cdot;x_0,\gamma)$ is the mild solution of

$$\begin{cases} x'(t) + Ax(t) + F(x(t)) = A^\beta B\gamma(t) \\ x(0) = x_0 \end{cases} \tag{2.2}$$

that is the solution of the integral equation

$$x(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A}F(x(s))ds + A^\beta \int_0^t e^{-(t-s)A}B\gamma(s)ds\ . \tag{2.3}$$

In (2.2), A^β denotes the fractional powers of the operator A, see [17]. The discount factor λ is positive and L satisfies the following assumptions

$$\begin{aligned} &(i) \quad L \in C(X \times \tilde{U}),\ \ |L(x,\gamma)| \le C_L\ ,\ \forall (x,\gamma) \in X \times \tilde{U}; \\ &(ii) \quad |L(x,\gamma) - L(y,\gamma)| \le K_L|x-y|,\ \forall \gamma \in \tilde{U},\ x,y \in X\ , \end{aligned} \tag{2.4}$$

for some $C_L > 0$ and $K_L > 0$. Moreover we assume

(2.5)

- (i) $A : D(A) \subset X \to X$ is a closed linear operator such that $A = A^*$ and $< Ax, x > \geq \omega|x|$ for some $\omega > 0$ and all $x \in D(A)$;
- (ii) the inclusion $D(A) \subset X$ is dense and compact ;
- (iii) $F : X \to X$, $|F(x) - F(y)| \le K_F|x-y|$, $|F(x)| \le C_F$ $\forall x, y \in X$;
- (iv) $\beta \in \left(\frac{3}{4}, 1\right)$;
- (v) there exists $\rho > 0$, such that $B \in \mathcal{L}(U, D(A^\rho))$.

for some constants K_F, $C_F > 0$.

We note that (i) and (ii) imply that $-A$ is the infinitesimal generator of an analytic semigroup satisfying $||e^{-tA}|| \le e^{-\omega t}$ for some $\omega > 0$ and all $t \ge 0$. In assumption (v) above, we have denoted by $\mathcal{L}(U, D(A^\rho))$ the Banach space of all bounded linear operator $B : U \to D(A^\rho)$, where $D(A^\rho)$ is equipped with the graph norm.

It is well known that, under the above assumptions, problem (2.3) has a unique solution in $L^2(0,T;X)$ for any $T > 0$. We define the value function of problem (2.1)–(2.2) as

$$u(x_0) = \inf \left\{ \int_0^\infty e^{-\lambda t} L(x(t;x_0,\gamma),\gamma(t))dt \,\Big|\, \gamma : [0,+\infty) \to \tilde{U} \text{ measurable} \right\} \tag{2.6}$$

Control processes as above are very important for applications. In fact, (2.2) describes the evolution of a system which is governed by a parabolic PDE and controlled by Dirichlet type boundary data. We explain this fact below. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Consider the following problem

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = \Delta_\xi x(t,\xi) + f(x(t,\xi)) & \text{in } (0,\infty)\times\Omega \\ x(0,\xi) = x_0(\xi) & \text{on } \Omega \\ x(t,\xi) = \gamma(t,\xi) & \text{on } (0,\infty)\times\partial\Omega \end{cases} \tag{2.7}$$

where $x_0 \in L^2(\Omega)$, $\gamma \in L^2(0,\infty;L^2(\partial\Omega))$, and $f : \mathbb{R} \to \mathbb{R}$.

Problem (2.7) may be rewritten in abstract form as follows. Let $X = L^2(\Omega), U = L^2(\partial\Omega)$ and define an unbounded operator A in X by

$$\begin{aligned} D(A) &= H^2(\Omega) \cap H_0^1(\Omega) \\ Ax &= -\Delta x. \end{aligned}$$

Next, we define the Dirichlet map $\mathbf{D} : U \to X$ as

$$\mathbf{D}\gamma = x \quad \Leftrightarrow \quad \begin{cases} \Delta x = 0 & \text{in } \Omega \\ x = \gamma & \text{on } \partial\Omega \end{cases}$$

Formally, equation (2.7) may be written as

$$\begin{cases} x'(t) + Ax(t) + F(x(t)) = A\mathbf{D}\gamma(t) \\ x(t_0) = x_0 \end{cases} \tag{2.8}$$

where

$$F(x)(\xi) = -f(x(\xi)), \ \forall x \in X.$$

The right–hand side of equation (2.8) is not well defined because the range of $\mathbf{D}$ is not contained in $D(A)$. However, we note that $\mathbf{D}$ has some regularizing effect. Indeed, by classical results (see e.g. [16]), $\mathbf{D} : L^2(\partial\Omega) \to H^{\frac{1}{2}}(\Omega)$, which may be expressed in abstract form using the fractional powers of A. In fact,

$$D(A^\theta) = \begin{cases} H^{2\theta}(\Omega) & \text{if } 0 \le \theta < \frac{1}{4} \\ \left\{ x \in H^{2\theta}(\Omega) \ : \ x = 0 \text{ on } \partial\Omega \right\} & \text{if } \frac{1}{4} < \theta \le 1. \end{cases}$$

Hence $\mathbf{D} : U \to D(A^\alpha)$ for all $\alpha \in \left[0, \frac{1}{4}\right)$. Consequently, having fixed $\beta \in \left(\frac{3}{4}, 1\right]$ equation (2.8) can be written as

$$\begin{cases} x'(t) + Ax(t) + F(x(t)) = A^\beta \mathbf{D}_\beta \gamma(t) \\ x(t_0) = x_0 \end{cases} \tag{2.9}$$

where $\mathbf{D}_\beta = A^{1-\beta}\mathbf{D} \in \mathcal{L}(U, X)$. Moreover $\mathbf{D}_\beta$ satisfies (2.5) (v) for any $\rho < \beta - \frac{3}{4}$.

Using the same technique described above, one can show that Neumann type boundary control problems may be formulated in the same abstract form (2.2). In this case β may be taken in the interval $\left(\frac{1}{4}, 1\right]$.

We now return to the analysis of problem (2.6). We note that equation (2.2) has discontinuous trajectories. Therefore, we transform (2.2) by the change of variable

$$y(t) = A^{-\beta}x(t). \tag{2.10}$$

More precisely, let $y_0 \in X$ and denote by $y(\cdot; y_0, \gamma)$ the solution of

$$\begin{cases} y'(t) + Ay(t) + A^{-\beta}F(A^\beta y(t)) = B\gamma(t) \\ y(0) = y_0 \in X \end{cases} \tag{2.11}$$

Again the above equation has to be understood in mild form

$$y(t) = e^{-tA}y_0 + A^{-\beta}\int_0^t e^{-(t-s)A}F(A^\beta y(s))ds + \int_0^t e^{-(t-s)A}B\gamma(s)ds\,. \tag{2.12}$$

The solution of (2.12), turns out to be continuous, as we show below.

We recall that, since operator $-A$ is the generator of an analytic semigroup in X, for every $\theta \in [0,1]$ there exists a constant $M_\theta > 0$ such that

$$|A^\theta e^{-tA}x| \le \frac{M_\theta}{t^\theta}|x|, \quad \forall t > 0, \forall x \in X. \tag{2.13}$$

Moreover let $\gamma \in (0,1]$ and $\alpha \in (0,\gamma)$. Then, a well known interpolation inequality, see e.g. [17], states that for every $\sigma > 0$ there exists $C_\sigma > 0$ such that

$$|A^\alpha x| \le \sigma|A^\gamma x| + C_\sigma|x|, \quad \forall x \in D(A^\gamma) \tag{2.14}$$

and there exists $C_{\alpha\gamma} > 0$ such that

$$|A^\alpha x| \le C_{\alpha\gamma}|A^\gamma x|^{\frac{\alpha}{\gamma}}|x|^{1-\frac{\alpha}{\gamma}}, \quad \forall x \in D(A^\gamma). \tag{2.15}$$

Proposition 2.1 *Assume that* (2.5) *holds. Let* $\gamma : [0,\infty) \to \tilde{U}$ *and fix* $T > 0$. *Then for any* $y_0 \in X$ *there exists a unique solution*

$$y \in C([0,T]; X) \cap L^1(0,T; D(A^\beta)). \tag{2.16}$$

Moreover, if $y_0 \in D(A^{\frac{1}{2}})$, *then*

$$y \in C([0,T]; D(A^{\frac{1}{2}})) \cap L^2(0,T; D(A)) \cap W^{1,2}(0,T; X). \tag{2.17}$$

Finally, if $y_0 \in D(A)$ *and* $\gamma(\cdot)$ *is constant, then*

$$y \in C([0,T]; D(A)). \tag{2.18}$$

Proof – The argument is well known. We sketch the proof for the reader's convenience. First we show that (2.12) has a unique solution $y \in L^1(0,T;D(A^\beta))$. Fix $y_0 \in X$ and let $T_1 = \frac{1}{2K_F}$. Define the map Φ on $L^1(0,T_1;D(A^\beta))$ by

$$\Phi y(t) = e^{-tA}y_0 + A^{-\beta}\int_0^t e^{-(t-s)A}F(A^\beta y(s))ds + \int_0^t e^{-(t-s)A}B\gamma(s)ds$$

for any $0 \le t \le T_1$. Let us prove that

$$\Phi : L^1(0,T_1;D(A^\beta)) \to L^1(0,T_1;D(A^\beta)) .$$

Indeed, recalling (2.14), we have

$$\int_0^{T_1} |A^\beta \Phi y(t)|dt \le \int_0^{T_1} |A^\beta e^{-tA}y_0|dt$$

$$+ \int_0^{T_1} \left|\int_0^t e^{-(t-s)A}F(A^\beta y(s))ds\right| dt + \int_0^{T_1} \left|A^\beta \int_0^t e^{-(t-s)A}B\gamma(s)ds\right| dt$$

$$\le M_\beta \int_0^{T_1} \frac{|y_0|}{t^\beta}dt + C_F\int_0^{T_1}\int_0^t \left(|A^\beta y(s)|+1\right) dsdt + M_\beta \int_0^{T_1}\int_0^t \frac{|B\gamma(s)|}{(t-s)^\beta}dsdt$$

$$\le M_\beta|y_0|T_1^{1-\beta} + C_f T_1||y||_{L^1(0,T_1;D(A^\beta))} + C_F T_1^2 + M_\beta R||B||T_1^{1-\beta},$$

recalling that $|\gamma(s)| \le R$. Hence $\Phi y \in L^1(0,T_1;D(A^\beta))$.

Next we prove that Φ is a contraction. For any $y, z \in L^1(0,T_1;D(A^\beta))$ we have

$$\int_0^{T_1} \left|A^\beta(\Phi y(s) - \Phi z(s))\right| ds$$

$$\le K_F \int_0^{T_1}\int_0^t \left|A^\beta(y(s)-z(s))\right| dsdt = K_F T_1 ||y(s)-z(s)||_{L^1(0,T_1;D(A^\beta))}.$$

By the Contraction Map Theorem it follows that equation (2.12) has a unique solution $y \in L^1(0,T_1;D(A^\beta))$. Then by classical results, (see e.g. [17]), $y(t) \in C([0,T_1];X)$. Therefore, iterating this procedure, we can cover the interval $[0,T]$ with a finite number of steps.

As for (2.17), the maximal regularity result $y \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;X)$ is well known, see e.g. [3]. The fact that $y \in C([0,T];D(A^{\frac{1}{2}}))$ is also a well known consequence of the maximal regularity result.

Finally, if $y_0 \in D(A)$ and $\gamma(\cdot) = \gamma_0$ is constant, then writing y as

$$y(t) = e^{-tA}y_0 + A^{-\beta}\int_0^t e^{-(t-s)A}F(A^\beta y(s))ds + \int_0^t e^{-(t-s)A}B\gamma_0 ds = y_1(t) + A^{-\beta}y_2(t) + y_3(t)$$

we easily see that, since e^{-tA} is a strongly continuous semigroup, then Ay_1 is continuous. In addition, $Ay_3(t) = (e^{-tA} - I)B\gamma$ is continuous and so is $A^{1-\beta}y_2$ since we know that $y_2 \in C([0,T];D(A^{\frac{1}{2}}))$ and $1-\beta < \frac{1}{2}$. ■

By inserting the change of variable (2.10) in the cost functional (2.1), we obtain a new optimal control problem whose value function v is given by

$$v(y_0) = \inf_{\gamma(t)\in\tilde{U}} \int_0^\infty e^{-\lambda t}L(A^\beta y(t;y_0,\gamma),\gamma(t))dt. \tag{2.19}$$

It is easy to realize that value functions v and u are related by the formula

$$u(x) = v(A^{-\beta}x) , \ \forall x \in X. \tag{2.20}$$

In particular, u is uniquely determined once v has been characterized. Therefore, we will study problem (2.11)–(2.19) instead of (2.2)–(2.6).

We will show that v is the unique solution of the following Hamilton–Jacobi–Bellman equation

$$\lambda v(x) + H(A^\beta x, Dv(x)) + \left\langle Ax + A^{-\beta}F(A^\beta x), Dv(x)\right\rangle = 0 \tag{2.21}$$

where

$$H(x,p) = \sup_{\gamma\in\bar{U}} [- < B\gamma, p > -L(x,\gamma)] . \tag{2.22}$$

Clearly, one needs a suitable notion of weak solution of problem (2.21), since v is not everywhere differentiable and the coefficients of the equation are discontinuous. In the sequel, we use viscosity solutions to overcome these difficulties.

3 Definition of viscosity solution and comparison result

In this Section we study the Hamilton–Jacobi equation

$$\lambda u(x) + H(A^\beta x, Du(x)) + \left\langle Ax + A^{-\beta}F(A^\beta x), Du(x)\right\rangle = 0. \tag{3.1}$$

We assume that (2.5) holds and that $H : X \times X \to \mathbb{R}$ is a function, not necessarily given by (2.22), satisfying

$$|H(A^\beta x,p) - H(A^\beta y,q)| \le K_H \left(|A^\beta(x-y)| + |p-q|\right) \text{ for some } K_H > 0. \tag{3.2}$$

Let $w, \varphi : D(A^{\frac{1}{2}}) \to \mathbb{R}$ be given. For any $\delta > 0$ we define $M_\delta^+(w,\varphi)$ to be the set of all points $x \in D(A^{\frac{1}{2}})$ such that

$$w(x) - \varphi(x) - \frac{\delta}{2}|A^{\frac{1}{2}}x|^2 \ge w(y) - \varphi(y) - \frac{\delta}{2}|A^{\frac{1}{2}}y|^2 \tag{3.3}$$

for all $y \in D(A^{\frac{1}{2}})$. Similarly, we denote by $M_\delta^-(w,\varphi)$ the set of all points $x \in D(A^{\frac{1}{2}})$ such that

$$w(x) - \varphi(x) + \frac{\delta}{2}|A^{\frac{1}{2}}x|^2 \le w(y) - \varphi(y) + \frac{\delta}{2}|A^{\frac{1}{2}}y|^2 \tag{3.4}$$

for all $y \in D(A^{\frac{1}{2}})$.

Definition 3.1 *We say that a bounded continuous function $w : X \to \mathbb{R}$ is a viscosity subsolution of* (3.1) *if w is sequentially weakly upper semicontinuous, and, for every $\varphi \in C^1(D(A^{\frac{1}{2}}))$ and $\delta > 0$,*

$$(i) \quad M_\delta^+(w,\varphi) \subset D(A) ;$$

$$\begin{aligned}(ii) \quad & \lambda w(x) + H(A^\beta x, D\varphi(x) + \delta Ax) + \left\langle Ax + A^{-\beta}F(A^\beta x), D\varphi(x)\right\rangle \\ & +\delta|Ax|^2 + \delta\left\langle Ax, A^{-\beta}F(A^\beta x)\right\rangle \le 0 , \forall x \in M_\delta^+(w,\varphi).\end{aligned} \tag{3.5}$$

We say that w is a viscosity supersolution of (3.1) *if w is sequentially weakly lower semicontinuous, and, for every $\varphi \in C^1(D(A^{\frac{1}{2}}))$ and $\delta > 0$,*

$$(i) \quad M_\delta^-(w,\varphi) \subset D(A)\ ;$$

$$(ii) \quad \lambda w(x) + H(A^\beta x, D\varphi(x) - \delta Ax) + \left\langle Ax + A^{-\beta}F(A^\beta x), D\varphi(x)\right\rangle \tag{3.6}$$

$$-\delta|Ax|^2 - \delta\left\langle Ax, A^{-\beta}F(A^\beta x)\right\rangle \geq 0\ , \forall x \in M_\delta^-(w,\varphi).$$

We say that w is a viscosity solution of (3.1) *if it is both a viscosity subsolution and a supersolution of* (3.1).

Now we give a comparison result between viscosity subsolutions and supersolutions of (3.1).

Theorem 3.2 *Assume that* (2.5) *and* (3.2) *hold true and define $\alpha_\beta \in (0,1)$ as*

$$\alpha_\beta = \frac{4\beta - 3}{2\beta - 1}. \tag{3.7}$$

Let u and v be a viscosity subsolution and supersolution of the Hamilton–Jacobi equation (3.1) *respectively. If u and v are Hölder continuous of exponent $\alpha > \alpha_\beta$, then*

$$u(x) \leq v(x), \quad \forall x \in X. \tag{3.8}$$

Proof – For semplicity we take $\lambda = 1$. For ε and δ positive, we define a function $\phi : D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) \to \mathbb{R}$ as

$$\phi(x,y) = u(x) - v(y) - \frac{1}{2\varepsilon} < A^{\frac{1}{2}}(x-y), x-y > -\frac{\delta}{2}[< Ax, x > + < Ay, y >]. \tag{3.9}$$

Notice that ϕ is weakly upper–semicontinuous. Let $(x_{\varepsilon,\delta}, y_{\varepsilon,\delta}) \in D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}})$ be such that

$$\phi(x_{\varepsilon,\delta}, y_{\varepsilon,\delta}) = \max_{D(A^{\frac{1}{2}})\times D(A^{\frac{1}{2}})} \phi(x,y)\ .$$

First of all we prove that

$$|x_{\varepsilon,\delta} - y_{\varepsilon,\delta}| \leq C_1 \varepsilon^{\frac{1}{2-\alpha}}\ , \tag{3.10}$$

where $C_1 > 0$ and α is the Hölder exponent of u and v. Since

$$\phi(x_{\varepsilon,\delta}, x_{\varepsilon,\delta}) + \phi(y_{\varepsilon,\delta}, y_{\varepsilon,\delta}) \leq 2\phi(x_{\varepsilon,\delta}, y_{\varepsilon,\delta})\ ,$$

from the Hölder continuity of u and v we derive

$$\frac{1}{\varepsilon}|x_{\varepsilon,\delta} - y_{\varepsilon,\delta}|^2 \leq C|x_{\varepsilon,\delta} - y_{\varepsilon,\delta}|^\alpha, \tag{3.11}$$

for some positive constant C. Therefore (3.10) holds.

Now let us consider

$$\varphi(x) = v(y_{\varepsilon,\delta}) + \frac{1}{2\varepsilon} < A^{\frac{1}{2}}(x - y_{\varepsilon,\delta}), x - y_{\varepsilon,\delta} > +\frac{\delta}{2} < Ay_{\varepsilon,\delta}, y_{\varepsilon,\delta} >$$

$$\psi(y) = u(x_{\varepsilon,\delta}) - \frac{1}{2\varepsilon} < A^{\frac{1}{2}}(x_{\varepsilon,\delta} - y), x_{\varepsilon,\delta} - y > -\frac{\delta}{2} < Ax_{\varepsilon,\delta}, x_{\varepsilon,\delta} >$$

Notice that $\varphi, \psi \in C^1(D(A^{\frac{1}{2}}))$. Also, $x_{\varepsilon,\delta} \in M_\delta^+(u,\varphi)$ and $y_{\varepsilon,\delta} \in M_\delta^-(v,\psi)$ by construction. Since u is a viscosity subsolution, using φ as a test function, we have

$$\begin{aligned} &u(x_{\varepsilon,\delta}) + H\left(A^\beta x_{\varepsilon,\delta}, \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})}{\varepsilon} + \delta A x_{\varepsilon,\delta}\right) + \delta |A x_{\varepsilon,\delta}|^2 \\ &+\delta \left\langle A x_{\varepsilon,\delta}, A^{-\beta} F(A^\beta x_{\varepsilon,\delta})\right\rangle + \left\langle A x_{\varepsilon,\delta} + A^{-\beta}F(A^\beta x_{\varepsilon,\delta}), \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})}{\varepsilon}\right\rangle \le 0 \end{aligned} \tag{3.12}$$

Since v is a viscosity supersolution, using ψ as a test function, we have

$$\begin{aligned} &v(y_{\varepsilon,\delta}) + H\left(A^\beta y_{\varepsilon,\delta}, \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})}{\varepsilon} - \delta A y_{\varepsilon,\delta}\right) - \delta |A y_{\varepsilon,\delta}|^2 \\ &-\delta \left\langle A y_{\varepsilon,\delta}, A^{-\beta} F(A^\beta y_{\varepsilon,\delta})\right\rangle + \left\langle A y_{\varepsilon,\delta} + A^{-\beta}F(A^\beta y_{\varepsilon,\delta}), \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})}{\varepsilon}\right\rangle \ge 0 \end{aligned} \tag{3.13}$$

Subtracting (3.13) from (3.12), we obtain

$$\begin{aligned} &u(x_{\varepsilon,\delta}) - v(y_{\varepsilon,\delta}) + \delta\left[|A x_{\varepsilon,\delta}|^2 + |A y_{\varepsilon,\delta}|^2\right] + \frac{1}{\varepsilon}|A^{\frac{3}{4}}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})|^2 \\ &\le H\left(A^\beta y_{\varepsilon,\delta}, \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})}{\varepsilon} - \delta A y_{\varepsilon,\delta}\right) - H\left(A^\beta x_{\varepsilon,\delta}, \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})}{\varepsilon} + \delta A x_{\varepsilon,\delta}\right) \\ &-\delta\left[\left\langle A x_{\varepsilon,\delta}, A^{-\beta}F(A^\beta x_{\varepsilon,\delta})\right\rangle + \left\langle A y_{\varepsilon,\delta}, A^{-\beta}[F(A^\beta y_{\varepsilon,\delta})\right\rangle\right] \\ &+\left\langle A^{-\beta}\left[F(A^\beta y_{\varepsilon,\delta}) - F(A^\beta x_{\varepsilon,\delta})\right], \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})}{\varepsilon}\right\rangle. \end{aligned} \tag{3.14}$$

Recalling assumption (3.2) on H and assumption (2.5) on F, the above inequality yields

$$\begin{aligned} &u(x_{\varepsilon,\delta}) - v(y_{\varepsilon,\delta}) + \delta\left[|A x_{\varepsilon,\delta}|^2 + |A y_{\varepsilon,\delta}|^2\right] + \frac{1}{\varepsilon}|A^{\frac{3}{4}}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})|^2 \\ &\le K_H\delta\left[|A x_{\varepsilon,\delta}| + |A y_{\varepsilon,\delta}|\right] + K_H|A^\beta(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})| \\ &+\delta C_F||A^{-\beta}||[|A x_{\varepsilon,\delta}| + |A y_{\varepsilon,\delta}|] + K_F|A^\beta(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})|\frac{|A^{\frac{1}{2}-\beta}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})|}{\varepsilon}. \end{aligned} \tag{3.15}$$

Now we estimate the right hand side of (3.15). Recalling the interpolation inequality (2.15), we get

$$K_H|A^\beta(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})| \le C_2|A(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})|^{4\beta-3}|A^{\frac{3}{4}}(x_{\varepsilon,\delta}-y_{\varepsilon,\delta})|^{4-4\beta}, \tag{3.16}$$

for some $C_2 > 0$. Moreover recall the following well known inequality

$$ab \le \frac{\sigma^p}{p}a^p + \frac{1}{\sigma^q q}b^q \tag{3.17}$$

for every $a, b \in \mathbb{R}^+$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\sigma > 0$. Choosing $\sigma = \left(\frac{\delta}{8}\right)^{\frac{4\beta-3}{2}}$ and $p = \frac{2}{4\beta - 3}$ in (3.16) and applying it to (3.16) we derive

$$\begin{aligned} & C_2 |A(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^{4\beta-3} |A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^{4-4\beta} \\ & \leq \frac{\delta}{8} |A(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{C_3}{\delta^{\frac{4\beta-3}{5-4\beta}}} \left| A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta}) \right|^{\frac{8-8\beta}{5-4\beta}}, \end{aligned} \tag{3.18}$$

where C_3 is some positive constant. On the other hand, again applying (3.17) with $p = \frac{5 - 4\beta}{4 - 4\beta}$ and $\sigma = \left(\frac{1}{4\varepsilon}\right)^{\frac{4-4\beta}{5-4\beta}}$, we find

$$\frac{C_3}{\delta^{\frac{4\beta-3}{5-4\beta}}} \left| A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta}) \right|^{\frac{8-8\beta}{5-4\beta}} \leq \frac{1}{4\varepsilon} |A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{C_4 \varepsilon^{4-4\beta}}{\delta^{4\beta-3}}, \tag{3.19}$$

for $C_4 > 0$. From estimates (3.18) and (3.19), inequality (3.16) can be rewritten as

$$K_H |A^\beta(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \leq \frac{\delta}{8} \left[|Ax_{\varepsilon,\delta}|^2 + |Ay_{\varepsilon,\delta}|^2 \right] + \frac{1}{4\varepsilon} |A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{C_4 \varepsilon^{4-4\beta}}{\delta^{4\beta-3}}. \tag{3.20}$$

Moreover

$$K_H \delta \left[|Ax_{\varepsilon,\delta}| + |Ay_{\varepsilon,\delta}| \right] \leq \frac{\delta}{8} \left[|Ax_{\varepsilon,\delta}|^2 + |Ay_{\varepsilon,\delta}|^2 \right] + C_5 \delta, \tag{3.21}$$

where C_5 is a positive constant. On the other hand we get

$$\delta C_F ||A^{-\beta}|| [|Ax_{\varepsilon,\delta}| + |Ay_{\varepsilon,\delta}|] \leq \frac{\delta}{8} \left[|Ax_{\varepsilon,\delta}|^2 + |Ay_{\varepsilon,\delta}|^2 \right] + C_6 \delta, \tag{3.22}$$

for $C_6 > 0$. Finally from estimate (3.11), it follows

$$K_F |A^\beta(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \frac{|A^{\frac{1}{2}-\beta}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|}{\varepsilon} \leq \frac{C_7 |A^\beta(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|}{\varepsilon^{\frac{1-\alpha}{2-\alpha}}}, \tag{3.23}$$

where $C_7 > 0$. Applying the interpolation inequality (2.15) and inequality (3.17) to (3.23) as we did in (3.16) we find

$$\frac{C_7 |A^\beta(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|}{\varepsilon^{\frac{1-\alpha}{2-\alpha}}} \leq \frac{\delta}{8} |A(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{C_8 |A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^{\frac{8-8\beta}{5-4\beta}}}{\delta^{\frac{4\beta-3}{5-4\beta}} \varepsilon^{\frac{2-2\alpha}{(2-\alpha)(5-4\beta)}}},$$

for some positive constant C_8. Again, using (3.17) in the last term of the above inequality we rewrite (3.23) as

$$\begin{aligned} & K_F |A^\beta(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \frac{|A^{\frac{1}{2}-\beta}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|}{\varepsilon} \\ & \leq \frac{\delta}{8} |A(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{1}{4\varepsilon} |A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{C_9 \varepsilon^{4-4\beta}}{\delta^{4\beta-3} \varepsilon^{\frac{2-2\alpha}{2-\alpha}}}, \end{aligned} \tag{3.24}$$

with C_9 positive constant. Substituting estimates (3.20), (3.21), (3.22) and (3.24) inequality (3.15) we get

$$u(x_{\varepsilon,\delta}) - v(y_{\varepsilon,\delta}) + \frac{\delta}{2}\left[|Ax_{\varepsilon,\delta}|^2 + |Ay_{\varepsilon,\delta}|^2\right] + \frac{1}{2\varepsilon}|A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 \leq C_{10}\delta + \frac{C_9\varepsilon^\gamma}{\delta^{4\beta-3}}, \tag{3.25}$$

where $C_{10} > 0$ and $\gamma = 4 - 4\beta - \dfrac{2-2\alpha}{2-\alpha}$ is positive as $\alpha > \alpha_\beta$. Therefore, if $x \in D(A^{\frac{1}{2}})$ we have

$$u(x) - v(x) = \phi(x,x) + \delta < Ax, x > \leq \phi(x_{\varepsilon,\delta}, y_{\varepsilon,\delta}) + \delta < Ax, x >$$

$$\leq u(x_{\varepsilon,\delta}) - v(y_{\varepsilon,\delta}) + \delta < Ax, x > \leq C_{10}\delta + \frac{C_9\varepsilon^\gamma}{\delta^{4\beta-3}} + \delta < Ax, x > .$$

Letting $\varepsilon \to 0$ and then $\delta \to 0$ we conclude that

$$u(x) \leq v(x)\ ,\ \forall x \in D(A^{\frac{1}{2}})\ .$$

Since $D(A^{\frac{1}{2}})$ is dense in X, we have $u(x) \leq v(x)$ for every $x \in X$. ■

4 Properties of the value function and existence results

In this Section we prove that the value function v of problem (2.11)–(2.19) is the unique viscosity solution of

$$\lambda v(x) + H(A^\beta x, Dv(x)) + \left\langle Ax + A^{-\beta}F(A^\beta x), Dv(x)\right\rangle = 0 \tag{4.1}$$

where

$$H(x,p) = \sup_{\gamma\in\tilde{U}}\left[- < B\gamma, p > - L(x,\gamma)\right].$$

We first show a Hölder continuity result for v. We will exploit the technique of [5].

Proposition 4.1 *Assume* (2.5), (2.4). *Then, the value function v defined in* (2.19) *is Hölder continuous in X with any exponent $\alpha \in (0,1]$ satisfying $\alpha < \frac{\lambda}{K_F}$. Moreover for any $\theta \in [0, 1-\beta)$ there exists a constant $C_{\alpha\theta} > 0$ such that*

$$|v(x) - v(y)| \leq C_{\alpha\theta}|A^{-\theta}(x-y)|^\alpha \tag{4.2}$$

for all $x, y \in X$.

Proof – Let $x_0, y_0 \in X$ and $\gamma(t) \in \tilde{U}$ be given. Let us set $x(\cdot) = x(\cdot; x_0, \gamma)$ and $y(\cdot) = y(\cdot; y_0, \gamma)$. Then

$$x(t) = e^{-tA}x_0 + A^{-\beta}\int_0^t e^{-(t-s)A}F(A^\beta x(s))ds + \int_0^t e^{-(t-s)A}B\gamma(s)ds$$

and

$$y(t) = e^{-tA}y_0 + A^{-\beta}\int_0^t e^{-(t-s)A}F(A^\beta y(s))ds + \int_0^t e^{-(t-s)A}B\gamma(s)ds$$

for any $t \geq 0$. Now we estimate $|A^\beta(x(t) - y(t))|^\alpha$. From assumption (2.5) and from inequality (2.13) we have

$$|A^\beta(x(t) - y(t))| \leq \frac{M_\theta}{t^{\beta+\theta}}|A^{-\theta}(x_0 - y_0)| + K_F\int_0^t |A^\beta(x(s) - y(s))|ds\ , \tag{4.3}$$

for any $\theta \in [0, 1-\beta)$. Now set $\eta(t) = \int_0^t |A^\beta(x(s) - y(s))| ds$. Integrating the above inequality we get

$$\eta(t) \leq \frac{M_\theta}{1-(\beta+\theta)} |A^{-\theta}(x_0 - y_0)| t^{1-\beta-\theta} + K_F \int_0^t \eta(s) ds$$

By Gronwall's Lemma we obtain an estimate on $\eta(t)$. Applying this estimate to the right hand side of (4.3) we derive

$$|A^\beta(x(t) - y(t))| \leq \left(\frac{C}{t^{\beta+\theta}} + C e^{K_F t} t^{1-\beta-\theta} \right) |A^{-\theta}(x_0 - y_0)| \tag{4.4}$$

Then, for every $\alpha \in (0, \frac{\lambda}{K_F})$, $\alpha \leq 1$, we have

$$|A^\beta(x(t) - y(t))|^\alpha \leq 2^\alpha \left(\frac{C}{t^{\alpha(\beta+\theta)}} + C e^{K_F \alpha t} t^{(1-\beta-\theta)\alpha} \right) |A^{-\theta}(x_0 - y_0)|^\alpha \tag{4.5}$$

Moreover, by (2.4),

$$\begin{aligned} &\left| L(A^\beta x(t), \gamma(t)) - L(A^\beta y(t), \gamma(t)) \right| \\ &\leq (2C_L)^{1-\alpha} \left| L(A^\beta x(t), \gamma(t)) - L(A^\beta y(t), \gamma(t)) \right|^\alpha \leq \tilde{L} |A^\beta (x(t) - y(t))|^\alpha \end{aligned} \tag{4.6}$$

where $\tilde{L} = (2C_L)^{1-\alpha} K_L^\alpha$. Now choose T such that

$$\frac{2e^{-\lambda T} C_L}{\lambda} \leq |A^{-\theta}(x_0 - y_0)|^\alpha$$

From the definition of value function and from the Dynamic Programming Principle it follows that there exists a control $\gamma(\cdot)$ such that

$$v(y_0) > \int_0^T e^{-\lambda t} L(A^\beta y(t), \gamma(t)) dt + e^{-\lambda T} v(y(T)) - |A^{-\theta}(x_0 - y_0)|^\alpha .$$

Here, we may suppose that $|A^{-\theta}(x_0 - y_0)|^\alpha > 0$, as (4.2) is trivial if $|A^{-\theta}(x_0 - y_0)|^\alpha = 0$.

From the Dynamic Programming Principle and from the above estimate it follows

$$\begin{aligned} &v(x_0) - v(y_0) \leq |A^{-\theta}(x_0 - y_0)|^\alpha \\ &+ \int_0^T e^{-\lambda t} \left| L(A^\beta x(t), \gamma(t)) - L(A^\beta y(t), \gamma(t)) \right| dt + e^{-\lambda T} [v(x(T)) - v(y(T))] \\ &\leq 2 |A^{-\theta}(x_0 - y_0)|^\alpha + \tilde{L} \int_0^T e^{-\lambda t} |A^\beta (x(t) - y(t))|^\alpha dt . \end{aligned} \tag{4.7}$$

Substituting (4.5) in (4.7), we get

$$\begin{aligned} &v(x_0) - v(y_0) \leq 2 |A^{-\theta}(x_0 - y_0)|^\alpha \\ &+ \tilde{L} 2^\alpha |A^{-\theta}(x_0 - y_0)|^\alpha \int_0^T \left(\frac{C e^{-\lambda t}}{t^{\alpha(\beta+\theta)}} + C e^{(K_F \alpha - \lambda) t} t^{(1-\beta-\theta)\alpha} \right) dt . \end{aligned}$$

The result follows since $\alpha < \frac{\lambda}{K_F}$ and $\theta \in [0, 1-\beta)$. ∎

We now give an existence result for (4.1).

Theorem 4.2 *Assume that* (2.5) *and* (2.4) *hold true. Then the value function* v *is a viscosity solution of* (2.21) *in the sense of Definition* 3.1.

Proof – Recalling the compactness assumption (2.5) (ii), from (4.2) we conclude that v is sequentially weakly continuous in X. It remains to prove that v satisfies (3.5) and (3.6). We show this fact in the next two steps.

Step I – Proof of (3.5).
Let $\gamma \in \tilde{U}$ be a constant control, $\varphi \in C^1(D(A^{\frac{1}{2}}))$ and $x \in M_\delta^+(v, \varphi)$. Moreover let $y(\cdot) = y(\cdot; x, \gamma)$. Then, recalling Proposition 2.1, we obtain

$$v(x) - \varphi(x) - \frac{\delta}{2} < Ax, x > \geq v(y(t)) - \varphi(y(t)) - \frac{\delta}{2} < Ay(t), y(t) > \tag{4.8}$$

for every $t \geq 0$. From (4.8) and from the Dynamic Programming Principle it follows

$$\begin{aligned} &\frac{\varphi(x) - \varphi(y(t))}{t} + \frac{\delta}{2}\frac{< Ax, x > - < Ay(t), y(t) >}{t} \\ &\leq \frac{v(x) - v(y(t))}{t} \leq \frac{1}{t}\int_0^t e^{-\lambda s} L(A^\beta y(s), \gamma) ds + \frac{e^{-\lambda t} - 1}{t} v(y(t)). \end{aligned} \tag{4.9}$$

Notice that, since by Proposition 2.1 $y \in L^2(0, T; D(A))$, we get

$$\varphi(x) - \varphi(y(t)) = -\int_0^t \left\langle D\varphi(y(s)), -Ay(s) - A^{-\beta} F(A^\beta y(s)) + B\gamma \right\rangle ds \tag{4.10}$$

and

$$\begin{aligned} &\frac{1}{2}(< Ax, x > - < Ay(t), y(t) >) = -\int_0^t \left\langle Ay(s), -Ay(s) - A^{-\beta} F(A^\beta y(s)) + B\gamma \right\rangle ds \\ &= \int_0^t \left[|Ay(s)|^2 + \left\langle Ay(s), A^{-\beta} F(A^\beta y(s)) - B\gamma \right\rangle \right] ds. \end{aligned} \tag{4.11}$$

Therefore, exploiting (4.10) and (4.11), (4.9) can be rewritten as

$$\begin{aligned} &\frac{1}{t}\int_0^t \left\langle D\varphi(y(s)), Ay(s) + A^{-\beta} F(A^\beta y(s)) - B\gamma \right\rangle ds \\ &+ \frac{\delta}{t}\int_0^t \left[|Ay(s)|^2 + \left\langle Ay(s), A^{-\beta} F(A^\beta y(s)) - B\gamma \right\rangle \right] ds \\ &\leq \frac{1}{t}\int_0^t e^{-\lambda s} L(A^\beta y(s), \gamma) ds + \frac{e^{-\lambda t} - 1}{t} v(y(t)). \end{aligned} \tag{4.12}$$

By Proposition 2.1 $y \in L^2(0, T; D(A))$ and so

$$\frac{1}{t}\int_0^t \left\langle D\varphi(y(s)), Ay(s) + A^{-\beta} F(A^\beta y(s)) - B\gamma \right\rangle ds \leq \frac{\delta}{4t}\int_0^t |Ay(s)|^2 ds + C_\delta \tag{4.13}$$

and

$$\begin{aligned} &\frac{\delta}{t}\int_0^t \left\langle Ay(s), A^{-\beta} F(A^\beta y(s)) - B\gamma \right\rangle ds \\ &\leq \frac{\delta}{t}\int_0^t ||A^{-\beta}|| C_F |Ay(s)| ds + \frac{\delta}{t}\int_0^t |Ay(s)| ||B|| R ds \leq \frac{\delta}{4t}\int_0^t |Ay(s)|^2 ds + C_\delta \end{aligned} \tag{4.14}$$

for some positive C_δ.

From (4.12) , (4.13) and (4.14), since v and L are bounded it follows

$$\frac{1}{t}\int_0^t |Ay(s)|^2 ds \leq C_\delta$$

for C_δ positive. Hence, there exists a sequence $\{t_n\}$, $t_n \downarrow 0$ such that

$$|Ay(t_n)| \leq C_\delta.$$

Taking a subsequence we have that $Ay(t_n) \rightharpoonup z$ and $y(t_n) \to x$. Therefore we get $y(t_n) = A^{-1}Ay(t_n) \rightharpoonup A^{-1}z = x$ and so $x \in D(A)$. This proves (3.5) (i).

In order to show that (ii) holds, we recall that if $x \in D(A)$ and $\gamma(\cdot) = \gamma$, then $y \in C([0,T]; D(A))$, see Proposition 2.1. Then, passing to the limit as $t \downarrow 0$ in (4.12), we derive

$$\left\langle D\varphi(x), Ax + A^{-\beta}F(A^\beta x)\right\rangle + \left[- < D\varphi(x) + \delta Ax, B\gamma > -L(A^\beta x, \gamma)\right]$$

$$+\delta|Ax|^2 + \delta\left\langle Ax, A^{-\beta}F(A^\beta x)\right\rangle + \lambda v(x) \leq 0.$$

Taking the supremum over $\gamma \in \tilde{U}$ we obtain (3.5) (ii).

Step II – Proof of (3.6).

Let $\varphi \in C^1(D(A^{\frac{1}{2}}))$ and $x \in M_\delta^-(v,\varphi)$. For every $n \in \mathbb{N}$ there exists a control $\gamma_n(\cdot)$ such that

$$v(x) + \frac{1}{n^2} \geq \int_0^{\frac{1}{n}} e^{-\lambda s} L(A^\beta y_n(s), \gamma_n(s))ds + e^{-\frac{\lambda}{n}} v\left(y_n\left(\frac{1}{n}\right)\right), \tag{4.15}$$

where $y_n(\cdot) = y_n(\cdot; x, \gamma_n)$. Moreover we get

$$v(x) - \varphi(x) + \frac{\delta}{2} < Ax, x > \leq v\left(y_n\left(\frac{1}{n}\right)\right) - \varphi\left(y_n\left(\frac{1}{n}\right)\right) + \frac{\delta}{2}\left\langle Ay_n\left(\frac{1}{n}\right), y_n\left(\frac{1}{n}\right)\right\rangle$$

From (4.15) and from the above inequality we obtain

$$\begin{aligned} &n\left[\varphi(x) - \varphi\left(y_n\left(\frac{1}{n}\right)\right)\right] + \frac{\delta n}{2}\left[\left\langle Ay_n\left(\frac{1}{n}\right), y_n\left(\frac{1}{n}\right)\right\rangle - < Ax, x >\right] \\ &\geq n\int_0^{\frac{1}{n}} e^{-\lambda s} L(A^\beta y_n(s), \gamma_n(s))ds + n(e^{-\frac{\lambda}{n}} - 1)v\left(y_n\left(\frac{1}{n}\right)\right) + \omega\left(\frac{1}{n}\right). \end{aligned} \tag{4.16}$$

Here and in the sequel of the proof we denote by $\omega(t)$ a function such that $\omega(t) \downarrow 0$ as $t \downarrow 0$. Similarly to Step I we have

$$\varphi(x) - \varphi\left(y_n\left(\frac{1}{n}\right)\right) = \int_0^{\frac{1}{n}} \left\langle D\varphi(y_n(s)), Ay_n(s) + A^{-\beta}F(A^\beta y_n(s)) - B\gamma_n(s)\right\rangle ds \tag{4.17}$$

and

$$\begin{aligned} &\frac{1}{2}\left[\left\langle Ay_n\left(\frac{1}{n}\right), y_n\left(\frac{1}{n}\right)\right\rangle - < Ax, x >\right] \\ &= \int_0^{\frac{1}{n}} \left\langle Ay_n(s), -Ay_n(s) - A^{-\beta}F(A^\beta y_n(s)) + B\gamma_n(s)\right\rangle ds \\ &= \int_0^{\frac{1}{n}} \left[-|Ay_n(s)|^2 - \left\langle Ay_n(s), A^{-\beta}F(A^\beta y_n(s))\right\rangle + \langle Ay_n(s), B\gamma_n(s)\rangle\right] ds. \end{aligned} \tag{4.18}$$

Therefore inequality (4.16) can be rewritten as

$$
\begin{aligned}
& n \int_0^{\frac{1}{n}} \left\langle D\varphi(y_n(s)), Ay_n(s) + A^{-\beta}F(A^\beta y_n(s)) - B\gamma_n(s) \right\rangle ds \\
& + n\delta \int_0^{\frac{1}{n}} \left[-|Ay_n(s)|^2 - \left\langle Ay_n(s), A^{-\beta}F(A^\beta y_n(s)) \right\rangle + \langle Ay_n(s), B\gamma_n(s) \rangle \right] ds \qquad (4.19) \\
& \geq n \int_0^{\frac{1}{n}} e^{-\lambda s} L(A^\beta y_n(s), \gamma_n(s)) ds + n(e^{-\frac{\lambda}{n}} - 1) v\left(y_n\left(\frac{1}{n}\right)\right) + \omega\left(\frac{1}{n}\right)
\end{aligned}
$$

Again, reasoning as in Step I, from the above estimate we derive

$$
n \int_0^{\frac{1}{n}} |Ay_n(s)|^2 ds \leq C_\delta \ ,
$$

hence, there exists a sequence $\{s_n\}$, $s_n \downarrow 0$, such that

$$
|Ay_n(s_n)| \leq C, \quad y_n(s_n) \to x \quad \text{and} \quad Ay_n(s_n) \rightharpoonup z \ . \qquad (4.20)
$$

As in Step I we conclude, that $x \in D(A)$. Therefore (3.6) (i) holds.

In order to show that (ii) is verified, we have to estimate the terms contained in (4.19). First we note that, by easy computations exploiting estimate (2.13), as $x \in D(A)$

$$
\lim_{n\to\infty} \sup_{0 \leq t \leq \frac{1}{n}} |A^\alpha(y_n(t) - x)| = 0 \ , \qquad (4.21)
$$

where $\alpha \in [0, 1)$. Since $\varphi \in C^1(D(A^{\frac{1}{2}}))$ and $x \in D(A)$, by (4.21) we obtain

$$
-n \int_0^{\frac{1}{n}} < D\varphi(y_n(s)), B\gamma_n(s) > ds = -n \int_0^{\frac{1}{n}} < D\varphi(x), B\gamma_n(s) > ds + \omega\left(\frac{1}{n}\right). \qquad (4.22)
$$

Moreover, by (4.21)

$$
n \int_0^{\frac{1}{n}} < D\varphi(y_n(s)), A^{-\beta}F(A^\beta y_n(s)) > ds =< D\varphi(x), A^{-\beta}F(A^\beta x) > + \omega\left(\frac{1}{n}\right). \qquad (4.23)
$$

On the other hand we recall that by assumption (2.5) (v), there exists ρ such that $A^\rho B$, is bounded. Hence, (4.21) yields

$$
\begin{aligned}
n \int_0^{\frac{1}{n}} < Ay_n(s), B\gamma_n(s) > ds &= n \int_0^{\frac{1}{n}} \left\langle A^{1-\rho}x, A^\rho B\gamma_n(s) ds \right\rangle + \omega\left(\frac{1}{n}\right) \\
&= n \int_0^{\frac{1}{n}} < Ax, B\gamma_n(s) > ds + \omega\left(\frac{1}{n}\right).
\end{aligned} \qquad (4.24)
$$

In addition, from (4.21) we have

$$
-n \int_0^{\frac{1}{n}} \left\langle Ay_n(s), A^{-\beta}F(A^\beta y_n(s)) \right\rangle ds = -\left\langle Ax, A^{-\beta}F(A^\beta x) \right\rangle + \omega\left(\frac{1}{n}\right). \qquad (4.25)
$$

Finally, again from (4.21)

$$n\int_0^{\frac{1}{n}} e^{-\lambda s}L(A^\beta y_n(s),\gamma_n(s))ds = n\int_0^{\frac{1}{n}} L(A^\beta x,\gamma_n(s))ds + \omega\left(\frac{1}{n}\right). \tag{4.26}$$

Since v is continuous, substituting estimates (4.22), (4.23), (4.24), (4.25) and (4.26) in (4.19), we derive

$$\lambda v(x) + n\int_0^{\frac{1}{n}}\left[-\langle D\varphi(x)-\delta Ax, B\gamma_n(s)\rangle - L(A^\beta x,\gamma_n(s))\right]ds + \left\langle A^{-\beta}F(A^\beta x), D\varphi(x)\right\rangle$$

$$+n\int_0^{\frac{1}{n}}\left[< D\varphi(y_n(s)), Ay_n(s) > -\delta|Ay_n(s)|^2\right]ds - \delta\left\langle Ax, A^{-\beta}F(A^\beta x)\right\rangle \geq \omega\left(\frac{1}{n}\right).$$

On the other hand, recalling the definition of the Hamiltonian (2.22),

$$n\int_0^{\frac{1}{n}}\left[-\langle D\varphi(x)-\delta Ax, B\gamma_n(s)\rangle - L(A^\beta x,\gamma_n(s))\right]ds \leq H(A^\beta x, D\varphi(x)-\delta Ax).$$

Therefore, for some sequence $\{s_n\}$, $0\leq s_n\leq \frac{1}{n}$, as in (4.20) it follows that

$$\lambda v(x) + H(A^\beta x, D\varphi(x)-\delta Ax) + < A^{-\beta}F(A^\beta x), D\varphi(x) > -\delta < Ax, A^{-\beta}F(A^\beta x) >$$

$$\geq < D\varphi(y_n(s_n)), Ay_n(s_n) > +\delta|Ay_n(s_n)|^2 + \omega\left(\frac{1}{n}\right).$$

By (4.20), taking the $\liminf_{n\to\infty}$ of the right hand side of the above inequality, we derive that (3.6) (ii) holds. ■

Combining Theorem 4.2 with Theorem 3.2 we obtain the following existence and uniqueness result for the Hamilton–Jacobi equation (4.1).

Corollary 4.3 *Assume that* (2.5) *and* (2.4) *hold true and let* $\lambda_F = \min\left\{1, \frac{\lambda}{K_F}\right\}$. *Fix*

$$\beta \in \left(\frac{3}{4}, \frac{3-\lambda_F}{4-2\lambda_F}\right). \tag{4.27}$$

Then the value function v *defined in* (2.19) *is the unique viscosity solution of the Hamilton-Jacobi equation* (4.1) *satisfying a Hölder condition with exponent* $\alpha\in(\alpha_\beta,1)$, *where* α_β *is defined in* (3.7).

Proof – Applying Theorem 4.2, we obtain that v is a viscosity solution of equation (4.1). From Proposition 4.1 it follows that v is Hölder continuous of exponent α for any $0<\alpha<\lambda_F$. From (4.27) it is easily seen that $\lambda_F>\alpha_\beta$. The proof of existence is thus complete. As for uniqueness we note that assumption (2.4) implies that the Hamiltonian (2.22) satisfies hypothesis (3.2). Therefore uniqueness follows from Theorem 3.2. ■

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