**SUPERCONVERGENCE RESULTS FOR GALERKIN METHODS
FOR WAVE PROPAGATION IN VARIOUS POROUS MEDIA**

By

Zhangxin Chen

IMA Preprint Series # 1203

January 1994

SUPERCONVERGENCE RESULTS FOR GALERKIN METHODS FOR WAVE PROPAGATION IN VARIOUS POROUS MEDIA

ZHANGXIN CHEN[†]

ABSTRACT. This paper deals with superconvergence phenomena for Galerkin approximations of solutions of Biot's dynamic equations describing elastic wave propagation in fluid-saturated porous media. An asymptotic expansion to high order of Galerkin solutions is used to derive the superconvergence results. Linear elastic and elastodynamic equations for wave propagation in composite isotropic and anisotropic inhomogeneous media are also treated.

1. Introduction.

An isotropic, elastic porous solid saturated by a compressible viscous fluid can be described by the system of partial differential equations [1], [15],

$$(1.1) \quad \mathcal{A} \frac{\partial^2 u}{\partial t^2} + \mathcal{C} \frac{\partial u}{\partial t} - L(u) = F(x, t), \quad (x, t) \in \Omega \times [0, T],$$

where Ω is a bounded domain in $\mathbb{R}^2$, $u(x, t) = (u_1, u_2)$ is the displacement vector on Ω , $u_1 = (u_{11}, u_{12})$ and $u_2(x, t) = (u_{21}, u_{22})$ are the displacement of the solid and the average fluid displacement, respectively, and $F(x, t)$ is the force applied to the system. The differential operator $L(u)$ is defined by

$$L(u) = (\nabla \cdot \theta_1(u), \nabla \cdot \theta_2(u), \nabla s(u)),$$

where the vectors $\theta_i(u)$, $i = 1, 2$, and the scalar $s(u)$ are

$$\begin{aligned} \theta_i(u) &= (\theta_{i1}, \theta_{i2}), \quad i = 1, 2, \\ s(u) &= Q \nabla \cdot u_1 + R \nabla \cdot u_2, \end{aligned}$$

and

$$\theta_{ij}(u) = \sigma_{ij}(u_1) + Q \delta_{ij} \nabla \cdot u_2, \quad i, j = 1, 2,$$

[†]Department of Mathematics and the Institute for Scientific Computation, Texas A&M University, College Station, TX 77843. Partly supported by the Department of Energy under contract DE-ACOS-840R21400.

with δ_{ij} denoting the Kronecker symbol and the stress tensors σ_{ij} and the strain tensors ϵ_{ij} for Ω being related by

$$\begin{aligned}\epsilon_{ij}(u_1) &= \frac{1}{2} \left(\frac{\partial u_{1i}}{\partial x_j} + \frac{\partial u_{1j}}{\partial x_i} \right), \quad 1 \leq i, j \leq 2, \\ \sigma_{ij}(u_1) &= A \delta_{ij} \sum_{k=1}^2 \epsilon_{kk}(u_1) + 2N \epsilon_{ij}(u_1), \quad 1 \leq i, j \leq 2.\end{aligned}$$

$A = A(x)$, $N = N(x)$, $Q = Q(x)$, and $R = R(x)$ are the elastic coefficients for Ω . They are assumed to satisfy the constraints

$$\begin{aligned}0 &< N_* \leq N(x) \leq N^* < \infty, \quad x \in \overline{\Omega} = \Omega \cup \partial\Omega, \\ 0 &< A_* \leq A(x) \leq A^* < \infty, \quad x \in \overline{\Omega}, \\ 0 &< Q_* \leq Q(x) \leq Q^* < \infty, \quad x \in \overline{\Omega}, \\ 0 &< R_* \leq R(x) \leq R^* < \infty, \quad x \in \overline{\Omega}, \\ R(A + N) - Q^2 &> 0, \quad x \in \overline{\Omega}.\end{aligned}$$

In (1.1), $\mathcal{A} \in \mathbb{R}^{4 \times 4}$ and $\mathcal{C} \in \mathbb{R}^{4 \times 4}$ denote the density matrix and the dissipative matrix given by

$$\mathcal{A} = \begin{bmatrix} \rho_{11} & 0 & \rho_{12} & 0 \\ 0 & \rho_{11} & 0 & \rho_{12} \\ \rho_{12} & 0 & \rho_{22} & 0 \\ 0 & \rho_{12} & 0 & \rho_{22} \end{bmatrix}, \quad \mathcal{C} = b(x) \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

where $\rho_{11} = \rho_1 - \rho_{12}$, $\rho_{22} = \rho_2 - \rho_{12}$, $\rho_1 = \rho_1(x)$ (respectively, $\rho_2 = \rho_2(x)$) is the mass of solid (respectively, fluid) per unit of the aggregate, $\rho_{12} = \rho_{12}(x)$ is a mass coupling parameter between fluid and solid, and $b = b(x)$ is the dissipation coefficient for Ω .

From physical consideration, it is assumed that

$$(1.2) \quad \rho_{11} \rho_{22} - \rho_{12}^2 > 0, \quad x \in \overline{\Omega},$$

$$(1.3) \quad 0 < b_* \leq b(x) \leq b^* < \infty, \quad x \in \overline{\Omega}.$$

Then, it follows that $\mathcal{A}$ is positive definite and $\mathcal{C}$ is nonnegative.

We impose initial conditions

$$(1.4a) \quad u(x, 0) = u^0, \quad x \in \Omega,$$

$$(1.4b) \quad \frac{\partial u}{\partial t}(x, 0) = v^0, \quad x \in \Omega,$$

and the homogeneous boundary conditions

$$(1.5) \quad (\theta_1(u) \cdot n, \theta_2(u) \cdot n, s(u)) = 0, \quad (x, t) \in \partial\Omega \times [0, T],$$

where $n = n(x)$ is the outward unit normal to $\partial\Omega$.

In this paper we analyze superconvergence phenomena for the numerical solution of (1.1). The analysis is based on a regularity assumption on the solution of (1.1) [15], [16] and the general method of an asymptotic expansion, called a quasi-projection, of the approximate solution [10]. We also rely on some earlier results on the subject [15], [16], which analyzed the existence and uniqueness of solution of (1.1) and the Galerkin procedure for the approximate solution of such equations.

The paper is organized as follows. In §2, we give some notation and quote some known results. In §3, we present the Galerkin procedure of (1.1). Then, in §4 we develop the quasi-projection for the procedure. In §5, we give the superconvergence results of the same type as those of Bramble and Schatz [4] for Galerkin methods. In §6, we treat wave propagation in inhomogeneous elastic media. In §7, we consider a composite isotropic model in elastodynamics. Finally, in §8 we describe wave propagation in composite anisotropic media.

2. Notation and preliminaries.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. For m a nonnegative integer, let $H^m(\Omega) = W^{m,2}(\Omega)$ be the usual Sobolev space with norm

$$\|v\|_m^2 = \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha v(x)|^2 dx.$$

For $n \geq 1$, the norm of $v = (v_1, \dots, v_n)$ in $[H^m(\Omega)]^n$ will be given by

$$\|v\|_m^2 = \sum_{i=1}^n \|v_i\|_m^2.$$

The inner product and norm in $[L^2(\Omega)]^2$ will be denoted by

$$(v, w) = \sum_{i=1}^2 \int_{\Omega} v_i w_i dx, \quad \|v\|_0^2 = (v, v).$$

We shall also use the norm on the dual space $H^{-m}(\Omega) = (H^m(\Omega))'$; let

$$\|v\|_{-m} = \sup \left\{ \frac{(v, u)}{\|u\|_m} : u \in H^m(\Omega), \|u\|_m \neq 0 \right\}.$$

Let $H(\text{div}, \Omega) = \{q \in [L^2(\Omega)]^2 : \nabla \cdot q \in L^2(\Omega)\}$ with the norm

$$\|q\|_{H(\text{div}, \Omega)}^2 = \|q\|_0^2 + \|\nabla \cdot q\|_0^2.$$

Set $V = [H^1(\Omega)]^2 \times H(\text{div}, \Omega)$ with the norm

$$\|v\|_V^2 = \|v_1\|_1^2 + \|v_2\|_{H(\text{div}, \Omega)}^2, \quad v = (v_1, v_2) \in V.$$

Finally, if X is a Banach space with norm $\|\cdot\|_X$ and if $v : [0, T] \rightarrow X$, we use

$$\begin{aligned}\|v\|_{L^2(X)}^2 &= \int_0^T \|v(t)\|_X^2 dt, \\ \|v\|_{L^\infty(X)}^2 &= \operatorname{ess\,sup}_{0 \leq t \leq T} \|v(t)\|_X^2.\end{aligned}$$

Let B be the bilinear form associated with L in (1.1)

$$B(v, w) = M(v_1, w_1) + (Q \nabla \cdot v_2, \nabla \cdot w_1) + (Q \nabla \cdot v_1 + R \nabla \cdot v_2, \nabla \cdot w_2),$$

for $v = (v_1, v_2)$ and $w = (w_1, w_2) \in V$, with M being defined by

$$M(v_1, w_1) = \int_{\Omega} [A \nabla \cdot v_1 \nabla \cdot w_1 + 2N \sum_{i,j=1}^2 \epsilon_{ij}(v_1) \epsilon_{ij}(w_1)] dx.$$

Test (1.1) against $v \in V$, integrate by parts over Ω , and apply (1.5) to the $(L(u), v)$ -term. Then,

$$(2.1) \quad (\mathcal{A} \frac{\partial^2 u}{\partial t^2}, v) + (\mathcal{C} \frac{\partial u}{\partial t}, v) + B(u, v) = (F, v), \quad v \in V, \quad t \in [0, T].$$

Let the matrix $E \in \mathbb{R}^{4 \times 4}$ be given by

$$E = \begin{bmatrix} A + 2N & A & 0 & Q \\ A & A + 2N & 0 & Q \\ 0 & 0 & 4N & 0 \\ Q & Q & 0 & R \end{bmatrix}.$$

Then, it follows from the assumptions on the elastic coefficients that E is positive definite. If λ_* denotes the minimum eigenvalue of E , there exists a constant $c > 0$ such that [15]

$$(2.2) \quad B(v, v) \geq c \|v\|_V^2 - \lambda_* \|v\|_0^2, \quad v \in V.$$

It is convenient to introduce the differential operator L^* by

$$(2.3) \quad L^*(u) = -L(u) + \lambda_* u.$$

Then, the bilinear form associated with L^* is given by

$$B_1(v, w) = B(v, w) + \lambda_*(v, w), \quad v, w \in V.$$

Thus the symmetric form B_1 satisfies that

$$(2.4a) \quad |B_1(v, w)| \leq c \|v\|_V \|w\|_V, \quad v, w \in V,$$

$$(2.4b) \quad B_1(v, v) \geq c \|v\|_V^2, \quad v \in V.$$

Let $s \geq 0$ and assume that $\psi \in [H^s(\Omega)]^2$. Let φ be determined as the solution of the boundary problem

$$(2.5a) \quad L^*(\varphi) = \psi, \quad x \in \Omega,$$

$$(2.5b) \quad (\theta_1(\varphi) \cdot n, \theta_2(\varphi) \cdot n, s(\varphi)) = 0, \quad x \in \partial\Omega.$$

To obtain the superconvergence estimate, we shall use an asymptotic expansion to high order of Galerkin solutions for which the following regularity assumption is needed

$$(2.6) \quad \|\varphi_1\|_{s+2} + \|\varphi_2\|_{s+1} + \|\nabla \cdot \varphi_2\|_{s+1} \leq c\|\psi\|_s.$$

3. A Galerkin procedure.

Let $r \geq 1$ be an integer and let $0 < h < 1$. Let $\mathcal{T}_h$ and T_h be quasiregular partitions of Ω into triangles or rectangles of diameter bounded by h . Let $M_h \subset [H^1(\Omega)]^2$ be a standard finite element space associated with $\mathcal{T}_h$ such that

$$(3.1) \quad \inf_{v \in M_h} \{\|\varphi - v\|_0 + h\|\varphi - v\|_1\} \leq c\|v\|_s h^s, \quad 1 \leq s \leq r+1.$$

Also, let W_h be a finite dimensional subspace of $H(\text{div}, \Omega)$ associated with T_h such that

$$(3.2a) \quad \inf_{v \in W_h} \|w - v\|_0 \leq c\|w\|_s h^s, \quad 1 \leq s \leq r+1,$$

$$(3.2b) \quad \inf_{v \in W_h} \|w - v\|_{H(\text{div}, \Omega)} \leq c(\|w\|_s + \|\nabla \cdot w\|_s) h^s, \quad 1 \leq s \leq r+1.$$

Here we can take W_h to be one of the Raviart-Thomas vector spaces [14] or the Brezzi-Douglas-Marini vector spaces [7] of index r associated with T_h .

Set $V_h = M_h \times W_h$. Then, by (3.1) and (3.2), we have

$$(3.3a) \quad \inf_{v \in V_h} \|u - v\|_0 \leq c\|u\|_s h^s, \quad 1 \leq s \leq r+1,$$

$$(3.3b) \quad \inf_{v \in V_h} \|u - v\|_0 \leq c(\|u_1\|_{s+1} + \|u_2\|_s + \|\nabla \cdot u_2\|_s) h^s,$$

$$u = (u_1, u_2), \quad 1 \leq s \leq r.$$

The continuous-time Galerkin approximation to the solution of (1.1) is defined as the twice-differentiable map $u_h : [0, T] \rightarrow V_h$ such that

$$(3.4) \quad (\mathcal{A} \frac{\partial^2 u_h}{\partial t^2}, v) + (\mathcal{C} \frac{\partial u_h}{\partial t}, v) + B(u_h, v) = (F, v), \quad v \in V_h, \quad t \in [0, T].$$

The initial conditions $u_h(0)$ and $\partial u_h(0)/\partial t$ will be specified later.

4. Analysis of quasi-projection.

In this section we shall consider the quasi-projection for the Galerkin approximation (3.4). Set $J = (0, T)$ and let $\tilde{u}_h : J \rightarrow V_h$ be defined by

$$(4.1) \quad B_1(\tilde{u}_h - u, v) = 0, \quad v \in V_h, \quad t \in J.$$

Let $u_0 = \tilde{u}_h$, $z_0 = u_0 - u$, and $\theta_0 = u_0 - u_h$. Then, it follows from (2.1), (3.4), and (4.1) that

$$(4.2) \quad (\mathcal{A} \frac{\partial^2 \theta_0}{\partial t^2}, v) + (\mathcal{C} \frac{\partial \theta_0}{\partial t}, v) + B(\theta_0, v) = (\mathcal{A} \frac{\partial^2 z_0}{\partial t^2}, v) + (\mathcal{C} \frac{\partial z_0}{\partial t}, v) - \lambda_*(z_0, v), \quad v \in V_h, \quad t \in J.$$

Define maps $z_j : J \rightarrow V_h$ recursively by

$$(4.3) \quad B_1(z_j, v) = -(\mathcal{A} \frac{\partial^2 z_{j-1}}{\partial t^2}, v) - (\mathcal{C} \frac{\partial z_{j-1}}{\partial t}, v) + \lambda_*(z_{j-1}, v), \quad v \in V_h, \quad t \in J, \quad j = 1, 2, \dots.$$

Set

$$(4.4) \quad u_j = u_0 + z_1 + \dots + z_j, \quad j \geq 1,$$

and

$$(4.5) \quad \theta_j = u_j - u_h, \quad j \geq 1.$$

Then, it is easy to see from (4.2) and (4.3) that

$$(4.6) \quad (\mathcal{A} \frac{\partial^2 \theta_j}{\partial t^2}, v) + (\mathcal{C} \frac{\partial \theta_j}{\partial t}, v) + B(\theta_j, v) = (\mathcal{A} \frac{\partial^2 z_j}{\partial t^2}, v) + (\mathcal{C} \frac{\partial z_j}{\partial t}, v) - \lambda_*(z_j, v), \quad v \in V_h, \quad t \in J, \quad j = 0, 1, 2, \dots.$$

Let $t \in J$ and let $0 \leq j < r$, $0 \leq s < r - j - 1$, and $k \geq 0$. Since the coefficients of L are independent of t , it follows from (4.3) that

$$(4.7) \quad B_1(\frac{\partial^k z_j}{\partial t^k}, v) = -(\mathcal{A} \frac{\partial^{k+2} z_{j-1}}{\partial t^{k+2}}, v) - (\mathcal{C} \frac{\partial^{k+1} z_{j-1}}{\partial t^{k+1}}, v) + \lambda_*(\frac{\partial^k z_{j-1}}{\partial t^k}, v), \quad v \in V_h.$$

By setting $v = \frac{\partial^k z_j}{\partial t^k}$ in (4.7) and noting the properties of $\mathcal{A}$ and $\mathcal{C}$, we see that

$$(4.8) \quad \|\frac{\partial^k z_j}{\partial t^k}\|_V \leq c \left\{ \|\frac{\partial^{k+2} z_{j-1}}{\partial t^{k+2}}\|_0 + \|\frac{\partial^{k+1} z_{j-1}}{\partial t^{k+1}}\|_0 + \|\frac{\partial^k z_{j-1}}{\partial t^k}\|_0 \right\}.$$

Let $s \geq 0$ and assume that $\psi \in [H^s(\Omega)]^2$; let φ be determined by (2.5). Then,

$$\begin{aligned} \left(\frac{\partial^k z_0}{\partial t^k}, \psi\right) &= \left(\frac{\partial^k z_0}{\partial t^k}, L^*(\varphi)\right) \\ &= B_1\left(\frac{\partial^k z_0}{\partial t^k}, \varphi\right) \\ &= B_1\left(\frac{\partial^k z_0}{\partial t^k}, \varphi - v\right), \quad v \in V_h. \end{aligned}$$

Hence, (3.3) and (2.6) imply that

$$(4.9) \quad \left|\left(\frac{\partial^k z_0}{\partial t^k}, \psi\right)\right| \leq c \left\|\frac{\partial^k z_0}{\partial t^k}\right\|_V \|\psi\|_s h^{s+1}, \quad s \leq r-1.$$

Let $1 \leq q \leq r+1$. Then, it follows by the usual argument that

$$(4.10) \quad \left\|\frac{\partial^k z_0}{\partial t^k}\right\|_V \leq c \left\|\frac{\partial^k u}{\partial t^k}\right\|_q h^{q-1}, \quad \frac{\partial^k u}{\partial t^k} \in H^q(\Omega).$$

Hence, (4.9) and (4.10) imply that

$$(4.11) \quad \left\|\frac{\partial^k z_0}{\partial t^k}\right\|_{-s} \leq c \left\|\frac{\partial^k u}{\partial t^k}\right\|_q h^{q+s}, \quad s \leq r-1, \quad 1 \leq q \leq r+1.$$

Next, it follows from (4.7) that

$$\begin{aligned} (4.12) \quad \left(\frac{\partial^k z_j}{\partial t^k}, \psi\right) &= \left(\frac{\partial^k z_j}{\partial t^k}, L^*(\varphi)\right) \\ &= B_1\left(\frac{\partial^k z_j}{\partial t^k}, \varphi\right) \\ &= B_1\left(\frac{\partial^k z_j}{\partial t^k}, \varphi - v\right) + B_1\left(\frac{\partial^k z_j}{\partial t^k}, v\right) \\ &= B_1\left(\frac{\partial^k z_j}{\partial t^k}, \varphi - v\right) + \left(\mathcal{A} \frac{\partial^{k+2} z_{j-1}}{\partial t^{k+1}}, \varphi - v\right) \\ &\quad - \left(\mathcal{A} \frac{\partial^{k+2} z_{j-1}}{\partial t^{k+2}}, \varphi\right) + \left(\mathcal{C} \frac{\partial^{k+1} z_{j-1}}{\partial t^{k+1}}, \varphi - v\right) \\ &\quad - \left(\mathcal{C} \frac{\partial^{k+1} z_{j-1}}{\partial t^{k+1}}, \varphi\right) - \lambda_* \left(\frac{\partial^k z_{j-1}}{\partial t^k}, \varphi - v\right) \\ &\quad + \lambda_* \left(\frac{\partial^k z_{j-1}}{\partial t^k}, \varphi\right). \end{aligned}$$

Thus, together with (3.3) and (4.8), this implies that

$$\begin{aligned} \left\|\frac{\partial^k z_j}{\partial t^k}\right\|_{-s} &\leq c \left\{ \left(\left\|\frac{\partial^{k+2} z_{j-1}}{\partial t^{k+2}}\right\|_0 + \left\|\frac{\partial^{k+1} z_{j-1}}{\partial t^{k+1}}\right\|_0 + \left\|\frac{\partial^k z_{j-1}}{\partial t^k}\right\|_0\right) h^{s+1} \right. \\ &\quad \left. + \left\|\frac{\partial^{k+2} z_{j-1}}{\partial t^{k+2}}\right\|_{s-1} + \left\|\frac{\partial^{k+1} z_{j-1}}{\partial t^{k+1}}\right\|_{-s-1} + \left\|\frac{\partial^k z_{j-1}}{\partial t^k}\right\|_{-s-1} \right\}. \end{aligned}$$

Then, (4.11) and an induction on j imply that with the reduction on the range of s going from $j - 1$ to j ,

$$(4.13) \quad \left\| \frac{\partial^k z_j}{\partial t^k} \right\|_{-s} \leq c \sum_{i=0}^{2j} \left\| \frac{\partial^{i+k} u}{\partial t^{i+k}} \right\|_q h^{s+q+j}, \quad t \in J.$$

The results above can be summarized in the following lemma.

Lemma 4.1. *Let $1 \leq q \leq r + 1$ and let $0 \leq j < r$, $0 \leq s < r - j - 1$, and $k \geq 0$. If $\partial^{i+k} u / \partial t^{i+k} \in H^q(\Omega)$ for $t \in J$, $i = 1, \dots, 2j$, then (4.13) holds.*

We now turn to the estimation of θ_j , which satisfies (4.6). First, we shall set up initial conditions for (3.4). Set

$$(4.14) \quad u_h(0) = u_k(0), \quad \frac{\partial u_h}{\partial t}(0) = \frac{\partial u_k}{\partial t}(0), \quad k \leq r - 1.$$

It follows immediately from (4.14) that

$$(4.15) \quad \theta_k(0) = 0, \quad \frac{\partial \theta_k}{\partial t}(0) = 0, \quad k \leq r - 1.$$

Take $v = \frac{\partial \theta_k}{\partial t}$ in (4.6). Then,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\left(\mathcal{A} \frac{\partial \theta_k}{\partial t}, \frac{\partial \theta_k}{\partial t} \right) + B(\theta_k, \theta_k) \right] + \left(\mathcal{C} \frac{\partial \theta_k}{\partial t}, \frac{\partial \theta_k}{\partial t} \right) \\ &= \left(\mathcal{A} \frac{\partial^2 z_k}{\partial t^2}, \frac{\partial \theta_k}{\partial t} \right) + \left(\mathcal{C} \frac{\partial z_k}{\partial t}, \frac{\partial \theta_k}{\partial t} \right) - \lambda_*(z_k, \frac{\partial \theta_k}{\partial t}) \\ &\leq c \left\{ \left\| \frac{\partial^2 z_k}{\partial t^2} \right\|_0^2 + \left\| \frac{\partial z_k}{\partial t} \right\|_0^2 + \|z_k\|_0^2 + \left\| \frac{\partial \theta_k(t)}{\partial t} \right\|_0^2 \right\}. \end{aligned}$$

Adding

$$\frac{\lambda_*}{2} \frac{d}{dt} \|\theta_k\|_0^2 \leq \frac{\lambda_*}{2} (\|\theta_k\|_0^2 + \left\| \frac{\partial \theta_k}{\partial t} \right\|_0^2)$$

to the inequality above and noting the nonnegativeness of $\mathcal{C}$, we find that

$$(4.16) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\left(\mathcal{A} \frac{\partial \theta_k}{\partial t}, \frac{\partial \theta_k}{\partial t} \right) + B_1(\theta_k, \theta_k) \right] \\ &\leq c \left\{ \left\| \frac{\partial^2 z_k}{\partial t^2} \right\|_0^2 + \left\| \frac{\partial z_k}{\partial t} \right\|_0^2 + \|z_k\|_0^2 + \|\theta_k\|_V^2 + \left\| \frac{\partial \theta_k}{\partial t} \right\|_0^2 \right\}. \end{aligned}$$

Then, if we integrate (4.16) in time from 0 to t and use (4.5) and the Gronwall lemma, we derive that

$$(4.17) \quad \begin{aligned} & \left\| \frac{\partial \theta_k}{\partial t} \right\|_{L^\infty([L^2(\Omega)]^2)} + \|\theta_k\|_{L^\infty(V)} \\ &\leq c \left\{ \left\| \frac{\partial^2 z_k}{\partial t^2} \right\|_{L^2([L^2(\Omega)]^2)} + \left\| \frac{\partial z_k}{\partial t} \right\|_{L^2([L^2(\Omega)]^2)} + \|z_k\|_{L^2([L^2(\Omega)]^2)} \right\}. \end{aligned}$$

Theorem 4.2. *If $k < r$, $1 \leq q \leq r+1$, and $u_h(0)$ and $\frac{\partial u_h}{\partial t}(0)$ are defined by (4.14), then,*

$$(4.18) \quad \left\| \frac{\partial \theta_k}{\partial t} \right\|_{L^\infty([L^2(\Omega)]^2)} + \|\theta_k\|_{L^2(V)} \leq c \sum_{i=0}^{2k+2} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{L^2([H^q(\Omega)]^2)} h^{q+k}.$$

Proof. The theorem follows from (4.13) and (4.17). $\square$

Corollary 4.3. *If $k \leq r-1$, $1 \leq q \leq r+1$, and $u_h(0)$ and $\frac{\partial u_h}{\partial t}(0)$ are defined by (4.14), then,*

$$(4.19) \quad \|u - u_h\|_{L^\infty([H^{-s}(\Omega)]^2)} \leq c \left\{ \sum_{i=0}^{2k} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{L^\infty([H^q(\Omega)]^2)} + \sum_{i=1}^2 \left\| \frac{\partial^{2k+i} u}{\partial t^{2k+i}} \right\|_{L^2([H^q(\Omega)]^2)} \right\} h^{q+s},$$

for $0 \leq s \leq \min(k, r-k-1)$.

Proof. Note that $u - u_h = (u - u_0) + \theta_k - (z_1 + \dots + z_k)$. Then (4.19) follows from (4.13) and (4.18). $\square$

We now discuss the evaluation of $u_h(0)$ and $\frac{\partial u_h}{\partial t}(0)$. Let $k \leq r-1$. To evaluate $z_k(0)$ using (4.3), we must evaluate $\partial^2 z_{k-1}(0)/\partial t^2$, $\partial z_{k-1}(0)/\partial t$, and $z_{k-1}(0)$ first, which in turn require $\partial^2 z_{k-2}(0)/\partial t^2, \dots, \partial^k z_0(0)/\partial t^k$ by (4.7). Also, it follows from (4.1) that

$$B_1\left(\frac{\partial^s \tilde{u}_h}{\partial t^s}(0) - \frac{\partial^s u}{\partial t^s}(0), v\right) = 0, \quad v \in V_h;$$

hence, $u_h(0)$ and $\frac{\partial u_h}{\partial t}(0)$ can be evaluated from (4.14) and using F , u^0 , v^0 , and time-derivatives of the differential equation.

5. Superconvergence.

In this section we shall consider a simple case which assumes that Ω is the unit square and that the problem (1.1) has a periodic solution of period Ω in place of the previous boundary condition. To use the general argument of Bramble and Schatz [4] to obtain superconvergence by means of postprocessing the computed approximate solution, we shall assume that T_h (and T_h) has a translation invariance. Let $h = (h_1, h_2)$, $h_i > 0$, and let $\alpha = (\alpha_1, \alpha_2)$, α_i an integer. Define the translation operator G_h^α by

$$G_h^\alpha u(x) = u(x_1 + \alpha_1 h_1, x_2 + \alpha_2 h_2).$$

Assume that

$$G_h^\alpha u_h \in V_h, \quad \forall u_h \in V_h, \quad \forall \alpha.$$

Set

$$\partial_j u(x) = [u(x + h_j e_j) - u(x)]/h_j,$$

where e_j is the j th unit vector, and let h_1 and h_2 be comparable in the following sense. Let h be the parameter of M_h . Then, there exists a constant Q , independent of h , such that

$$h_i \in [h, Qh], \quad i = 1, 2.$$

Assume that the coefficients of (1.1) are constant. Then $\partial^\alpha u$ and $\partial^\alpha u_h$ are the solutions of (1.1) and (3.4) with data $\{\partial^\alpha F, \partial^\alpha u^0, \partial^\alpha v^0\}$ and $\{\partial^\alpha F, \partial^\alpha u_h^0, \partial^\alpha v_h^0\}$, respectively. Hence, it follows from (4.9) that

$$(5.1) \quad \begin{aligned} \|\partial^\alpha(u - u_h)\|_{L^\infty([H^{-s}(\Omega)]^2)} &\leq c \left\{ \sum_{i=0}^{2k} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{L^\infty([H^{q+|\alpha|}(\Omega)]^2)} \right. \\ &\quad \left. + \sum_{i=1}^2 \left\| \frac{\partial^{2k+i} u}{\partial t^{2k+i}} \right\|_{L^2([H^{q+|\alpha|}(\Omega)]^2)} \right\} h^{q+s}. \end{aligned}$$

Let K_h be the Bramble-Schatz kernel [4]

$$K_h(x) = K_{h,s}^{2s}(x).$$

Then, we define the post-processed approximation u_h^* to u by convolution with K_h :

$$(5.2) \quad u_h^*(x) = K_h \star u_h.$$

It follows from [4] that

$$(5.3a) \quad \|K_h \star u - u\|_0 \leq c \|u\|_q h^q, \quad q = 0, \dots, 2r,$$

$$(5.3b) \quad \|D^\alpha(K_h \star u)\|_m \leq c \|\partial^\alpha u\|_m, \quad m = 0, \pm 1, \pm 2, \dots,$$

$$(5.3c) \quad \|u\|_0 \leq c \sum_{|\alpha| \leq m} \|D^\alpha u\|_{-m}, \quad m = 1, 2, \dots$$

Thus, by (5.1) and (5.3),

$$\begin{aligned} \|u - u_h^*\|_{L^\infty([L^2(\Omega)]^2)} &\leq \|u - K_h \star u\|_{L^\infty([L^2(\Omega)]^2)} + \|K_h \star (u - u_h)\|_{L^\infty([L^2(\Omega)]^2)} \\ &\leq c \left\{ \|u\|_{L^\infty([H^{q+s}(\Omega)]^2)} h^{q+s} \right. \\ &\quad \left. + \sum_{|\alpha| \leq s} \|D^\alpha(K_h \star (u - u_h))\|_{L^\infty([H^{-s}(\Omega)]^2)} \right\} \\ &\leq c \left\{ \sum_{i=0}^{2k} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{L^\infty([H^{q+s}(\Omega)]^2)} \right. \\ &\quad \left. + \sum_{i=1}^2 \left\| \frac{\partial^{2k+i} u}{\partial t^{2k+i}} \right\|_{L^2([H^{q+s}(\Omega)]^2)} \right\} h^{q+s}. \end{aligned}$$

Theorem 5.1. *Let $k \leq r - 1$, $1 \leq q \leq r + 1$, and u_h^* be defined by (5.2). Then,*

$$\begin{aligned} \|u - u_h^*\|_{L^\infty([L^2(\Omega)]^2)} \leq c \left\{ \sum_{i=0}^{2k} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{L^\infty([H^{q+s}(\Omega)]^2)} \right. \\ \left. + \sum_{i=1}^2 \left\| \frac{\partial^{2k+i} u}{\partial t^{2k+i}} \right\|_{L^2([H^{q+s}(\Omega)]^2)} \right\} h^{q+s}, \end{aligned}$$

for $0 \leq s \leq \min(k, r - k - 1)$.

6. Inhomogeneous elastic media.

In this section we consider the approximate solution of wave propagation in inhomogeneous elastic media.

6.1. The differential model. An inhomogeneous, linearly elastic material in a state of plane strain can be described by the system of partial equations

$$(6.1a) \quad \sigma = 2\mu\epsilon(u) + \lambda \operatorname{tr}(\epsilon(u))I, \quad (x, t) \in \Omega \times J,$$

$$(6.1b) \quad \rho \frac{\partial^2 u}{\partial t^2} - L(u) = F, \quad (x, t) \in \Omega \times J,$$

where $u : \Omega \rightarrow \mathbb{R}^2$ is the displacement vector, $\sigma : \Omega \rightarrow \mathbb{R}^{2 \times 2}$ is the stress tensor, $\epsilon(u)$ is the strain tensor defined as in Introduction, $\mu > 0$ and $\lambda \geq 0$ are the standard Lamé constants specifying the material properties, $\operatorname{tr}(\epsilon) = \epsilon_{11} + \epsilon_{22}$, $I \in \mathbb{R}^{2 \times 2}$ is the identity matrix, and the differential operator L is given by

$$(6.2) \quad L(u) = (\operatorname{div} \sigma_1(u), \operatorname{div} \sigma_2(u)),$$

where $\sigma = (\sigma_1, \sigma_2)$. We impose the initial conditions as in (1.4), and the boundary conditions

$$(6.3a) \quad (\sigma_1(u) \cdot n, \sigma_1(u) \cdot n) = 0, \quad x \in \partial\Omega \setminus \Gamma, \quad t \in J,$$

$$(6.3b) \quad (\sigma_1(u) \cdot n, \sigma_1(u) \cdot n) = -\rho D \frac{\partial u}{\partial t}, \quad x \in \Gamma, \quad t \in J,$$

where $\Gamma \subset \partial\Omega$ is such that Γ has a strictly positive $d\sigma$ -measure and $d\sigma$ is the arc length on $\partial\Omega$. Equation (1.4a) represents the free surface condition, while condition (1.4b) is to minimize the transmitted energy ratio between reflected and incident waves on Γ , so that the amount of energy reflected by Γ is minimized [12], [17]. Finally, D is a positive matrix given by

$$D = \begin{bmatrix} n_1 & n_2 \\ n_2 & -n_1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} n_1 & n_2 \\ n_2 & -n_1 \end{bmatrix},$$

where $n = (n_1, n_2)$ and

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \beta = \sqrt{\frac{\mu}{\rho}}.$$

To formulate (6.1) in a weak form, recall that $\gamma_0 u = u|_{\partial\Omega} \in [H^{m-1/2}(\partial\Omega)]^2$ for any $u \in [H^m(\Omega)]^2$ and that

$$|g|_{m-1/2} = \inf \left\{ \|u\|_m, \gamma_0 u = g, u \in [H^m(\Omega)]^2 \right\}.$$

Let $[H^{-m-1/2}(\partial\Omega)]^2$ be the dual space of $[H^{m+1/2}(\partial\Omega)]^2$ with the norm

$$|g|_{-m-1/2} = \sup \left\{ \frac{\langle g, v \rangle}{\|v\|_{m+1/2}} : v \in [H^{m+1/2}(\partial\Omega)]^2, v \neq 0 \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $[H^{-m-1/2}(\partial\Omega)]^2$ and $[H^{m+1/2}(\partial\Omega)]^2$. Note that, if $g \in [H^{-1/2}(\partial\Omega)]^2$ and $v \in [H^1(\Omega)]^2$, then

$$|\langle g, v \rangle| \leq |g|_{-1/2} \|v\|_1.$$

Set

$$\begin{aligned} (6.4) \quad M(v, w) &= ((\sigma_1(u), \sigma_2(u)), (\nabla w_1, \nabla w_2)) \\ &= \int_{\Omega} \left(\lambda(\operatorname{div} v)(\operatorname{div} w) + 2\mu \sum_{i,j=1}^2 \epsilon_{ij}(v) \epsilon_{ij}(w) \right) dx \\ &= \int_{\Omega} [\mathcal{A} \nabla v, \nabla w]_e dx, \end{aligned}$$

where $[\cdot, \cdot]_e$ denotes the usual scalar product in $\mathbb{R}^4$ and the matrix $\mathcal{A} \in \mathbb{R}^{4 \times 4}$ is given by

$$\mathcal{A} = \begin{bmatrix} \lambda + 2\mu & 0 & 0 & \lambda \\ 0 & \mu & \mu & 0 \\ 0 & \mu & \mu & 0 \\ \lambda & 0 & 0 & \lambda + 2\mu \end{bmatrix}.$$

Testing (6.1b) against $v \in [H^1(\Omega)]^2$, integrating by parts over Ω , and applying (6.3) the $(L(u), v)$ -term, we obtain that

$$(6.5) \quad \left(\rho \frac{\partial^2 u}{\partial t^2}, v \right) + M(u, v) + \left\langle \rho D \frac{\partial u}{\partial t}, v \right\rangle_{\Gamma} = (F, v), \quad t \in J.$$

From the physical consideration, it is assumed that

$$0 < \rho_* \leq \rho(x) \leq \rho^* < \infty, \quad x \in \overline{\Omega}.$$

Note that (6.5) is well-defined for $\mu \in (0, \infty)$ and $\lambda \in [0, \infty)$. It is easy to see that M has the properties

$$\begin{aligned} M(v, w) &= M(w, v), \quad v, w \in [H^1(\Omega)]^2, \\ |M(v, w)| &\leq c \|v\|_1 \|w\|_1, \quad v, w \in [H^1(\Omega)]^2. \end{aligned}$$

Suppose that $\mu \in [\mu_0, \mu_1]$. Then it follows from Korn's second inequality [11] that

$$(6.6) \quad M(v, v) \geq c \|v\|_1^2 - 2\mu_0 \|v\|_0^2, \quad v \in [H^1(\Omega)]^2.$$

Let $\gamma \geq 2\mu_0$ and set

$$M_\gamma(v, w) = M(v, w) + \gamma(v, w).$$

Then it follows from (6.6) that

$$M_\gamma(v, v) \geq c \|v\|_1^2, \quad v \in [H^1(\Omega)]^2.$$

6.2. A Galerkin method. The continuous-time Galerkin approximation to the solution of (6.1) is defined as the twice-differentiable map $u_h : J \rightarrow V_h$ such that

$$(6.7) \quad \left(\rho \frac{\partial^2 u_h}{\partial t^2}, v \right) + M(u_h, v) + \left\langle \rho D \frac{\partial u_h}{\partial t}, v \right\rangle_\Gamma = (F, v), \quad v \in M_h, \quad t \in J.$$

The initial conditions $u_h(0)$ and $\partial u_h(0)/\partial t$ can be defined as in §4. It is convenient to define the differential operator L^* by

$$L^*(u) = -L(u) + \gamma u.$$

Let $s \geq 0$ and assume that $\psi \in [H^s(\Omega)]^2$ and $z \in [H^{s+1/2}(\partial\Omega)]^2$. Let φ be determined as the solution of the boundary problem

$$(6.8a) \quad L^*(\varphi) = \psi, \quad x \in \Omega,$$

$$(6.8b) \quad (\sigma_1(\varphi) \cdot n, \sigma_2(\varphi) \cdot n) = z, \quad x \in \partial\Omega.$$

If (6.1a) is substituted into (6.8a), we have the modified Lamé equation

$$-\mu \Delta \varphi - (\lambda + \mu) \nabla \operatorname{div} \varphi + \gamma \varphi = \psi.$$

Together with (6.8b) this is a strongly elliptic Neumann problem. Hence well-known regularity theory applies here. In particular, for $\mu \in [\mu_0, \mu_1]$ and $\lambda \in [0, \infty)$, we have the regularity estimate [20]

$$\|\varphi\|_{s+2} \leq c (\|\psi\|_s + |z|_{s+1/2}).$$

With the above modifications the same quasi-projection analysis as before can be carried out here (also see [8]) and the results in §4 and §5 remain valid.

7. A composite model in elastodynamics.

In this section we consider elastic wave propagation in a composite system Ω consisting of an elastic solid with an imbedded porous medium saturated by a compressible viscous fluid. We assume that the whole system is isotropic. Let $\Omega = \Omega_p \cup \Omega_s \in \mathbb{R}^2$ identify the system where Ω_p and Ω_s are the fluid-saturated porous medium and the elastic solid part of Ω , respectively. Let $\Gamma_0 \subset \partial\Omega$ and $\Gamma_1 \subset \partial\Omega \setminus \Gamma_0$ be the stress-free and artificial boundaries of the model, respectively. Finally, set $\Gamma_2 = \partial\Omega_p$.

7.1. The differential model. Let $u_1 = (u_{11}, u_{12})$ and $u_2(x, t) = (u_{21}, u_{22})$ be the vectors representing the displacement in the solid part of Ω and the averaged fluid displacement in Ω_p , respectively. Then the propagation of elastic waves on Ω_s and Ω_p can be described by the system of partial differential equations [1], [18] for $u = (u_1, u_2)$

$$(7.1a) \quad \rho \frac{\partial^2 u_1}{\partial t^2} - L_1(u_1) = F_1(x, t), \quad (x, t) \in \Omega_s \times J,$$

$$(7.1b) \quad \mathcal{A} \frac{\partial^2 u}{\partial t^2} + \mathcal{C} \frac{\partial u}{\partial t} - L(u) = F(x, t), \quad (x, t) \in \Omega_p \times J,$$

where ρ is the mass density in Ω_s , the differential operator L_1 is given by

$$L_1(u_1) = (\operatorname{div} \sigma_1(u_1), \operatorname{div} \sigma_2(u_1)),$$

and the other quantities are defined as in Introduction. We impose the boundary conditions

$$\begin{aligned} (\sigma_1(u_1) \cdot n_s, \sigma_1(u_1) \cdot n_s) &= 0, \quad (x, t) \in \Gamma_0 \times J, \\ (\sigma_1(u_1) \cdot n_s, \sigma_1(u_1) \cdot n_s) &= -\rho D \frac{\partial u_1}{\partial t}, \quad (x, t) \in \Gamma_1 \times J, \\ (\sigma_1(u) \cdot n_p, \sigma_1(u) \cdot n_p) + s(u) n_p + (\sigma_1(u_1) \cdot n_s, \sigma_1(u_1) \cdot n_s) &= 0, \\ &\quad (x, t) \in \Gamma_2 \times J, \\ (u_2 - u_1) \cdot n_p &= 0, \quad (x, t) \in \Gamma_2 \times J, \end{aligned}$$

where n_s and n_p are the unit outward normals along $\partial\Omega_s$ and $\partial\Omega_p$, respectively, and D is given as in the last section. The initial conditions are imposed as in (1.4).

Set $V = [H^1(\Omega)]^2 \times H(\operatorname{div}, \Omega_p)$ with the norm

$$\|v\|_V^2 = \|v_1\|_1^2 + \|v_2\|_{H(\operatorname{div}, \Omega_p)}^2, \quad v = (v_1, v_2) \in V.$$

For $v = (v_1, v_2)$ and $w = (w_1, w_2)$ in V , we define

$$\begin{aligned} M_l(v_1, w_1) &= \int_{\Omega_l} \left[A \nabla \cdot v_1 \nabla \cdot w_1 + 2N \sum_{i,j=1}^2 \epsilon_{ij}(v_1) \epsilon_{ij}(w_1) \right] dx, \quad l = s, p, \\ B(v, w) &= M_s(v_1, w_1) + M_p(v_1, w_1) + (Q \nabla \cdot v_2, \nabla \cdot w_1)_{\Omega_p} \\ &\quad + (Q \nabla \cdot v_1 + R \nabla \cdot v_2, \nabla \cdot w_2)_{\Omega_p}. \end{aligned}$$

Let $\tilde{V} = \{v = (v_1, v_2) \in V : (v_2 - v_1) \cdot n_p = 0 \text{ on } \Gamma_2\}$. Then $\tilde{V}$ is a closed subspace of V [18].

For simplicity, we assume that Γ_1 is of zero $d\sigma$ -measure. Then if we take $v = (v_1, v_2) \in \tilde{V}$, test (7.1a) against v_1 and (7.1b) against v_2 , and add the resulting equations, we obtain the equation

$$(7.2) \quad (\rho \frac{\partial^2 u_1}{\partial t^2}, v_1)_{\Omega_s} + (\mathcal{A} \frac{\partial^2 u}{\partial t^2}, v)_{\Omega_p} + (\mathcal{C} \frac{\partial u}{\partial t}, v)_{\Omega_p} + B(u, v) = (F_1, v_1) + (F_2, v_2)_{\Omega_p},$$

where $F = (F_1, F_2)$ is the force applied the system.

There are constants $c > 0$ and $\gamma > 0$ [18] such that

$$(7.3) \quad B(v, v) \geq c\|v\|_V^2 - \gamma\|v\|_0^2, \quad v \in V.$$

As in the last section we introduce the differential operators L_1^* on Ω_s and L^* on Ω by

$$\begin{aligned} L_1^*(u_1) &= -L_1(u_1) + \gamma u_1, \\ L^*(u) &= -L(u) + \gamma u. \end{aligned}$$

Let B_γ be the bilinear form associated with L_1^* and L^* :

$$B_\gamma(v, w) = B(v, w) + \gamma(v_1, w_1) + \gamma(v_2, w_2)_{\Omega_p},$$

for $v = (v_1, v_2) \in V$ and $w = (w_1, w_2) \in V$. Then the symmetric form B_γ satisfies

$$\begin{aligned} |B_\gamma(v, w)| &\leq c\|v\|_V\|w\|_V, \quad v, w \in V, \\ B_\gamma(v, v) &\geq c\|v\|_V^2, \quad v \in V. \end{aligned}$$

7.2. A Galerkin method. Let $W_h \subset H(\text{div}, \Omega_p)$ be one of the Raviart-Thomas vector spaces [14] or the Brezzi-Douglas-Marini vector spaces [7] of index r associated with T_h^p , where T_h^p is a quasiregular partition of Ω_p . Set $V_h = M_h \times W_h$ and

$$\tilde{V}_h = \{v = (v_1, v_2) \in V_h : (v_2 - v_1) \cdot n_p = 0 \text{ on } \Gamma_2\}.$$

Then the continuous-time Galerkin approximation to the solution of (7.1) is defined as the twice-differentiable map $u_h : J \rightarrow \tilde{V}_h$ such that

$$\begin{aligned} (\rho \frac{\partial^2 u_{1h}}{\partial t^2}, v_1)_{\Omega_s} + (\mathcal{A} \frac{\partial^2 u_h}{\partial t^2}, v)_{\Omega_p} + (\mathcal{C} \frac{\partial u_h}{\partial t}, v)_{\Omega_p} + B(u_h, v) \\ = (F_1, v_1) + (F_2, v_2)_{\Omega_p}, \quad v = (v_1, v_2) \in \tilde{V}_h, \quad t \in J. \end{aligned}$$

The initial conditions $u_h(0)$ and $\partial u_h(0)/\partial t$ can be defined as in §4.

Let $m \geq 0$ and assume that $\psi = (\psi_1, \psi_2) \in [H^m(\Omega)]^2 \times [H^m(\Omega_p)]^2$, and let $\varphi = (\varphi_1, \varphi_2)$ be determined by the problem

$$(7.4a) \quad L_1^*(\varphi_1) = \psi_1, \quad x \in \Omega_s,$$

$$(7.4b) \quad L^*(\varphi) = \psi, \quad x \in \Omega_p,$$

with the boundary conditions given in §7.1. Regularity assumptions have been used in §4 and §5 to obtain the superconvergence results. Accordingly, we make the following regularity assumption for (7.4):

$$(7.5) \quad \|\varphi_1\|_{m+2} + \|\varphi_2\|_{m+1, \Omega_p} + \|\operatorname{div} \varphi_2\|_{m+1, \Omega_p} \leq c(\|\psi_1\|_m + \|\psi_2\|_{m, \Omega_p}).$$

Under this assumption the same quasi-projection analysis as before can be carried out here and the results in §4 and §5 remain valid. However, we emphasize that (7.5) has not been proven yet.

8. Composite anisotropic media.

In this section we consider elastic wave propagation in a composite anisotropic system $\Omega \subset \mathbb{R}^3$ consisting of an elastic solid Ω_s containing a porous medium Ω_p saturated by a compressible viscous fluid.

8.1. The differential model. In the present case the wave propagation on Ω_s and Ω_p is described by the system of partial differential equations [2], [3]

$$(8.1a) \quad \rho \frac{\partial^2 u_1}{\partial t^2} - L_1(u_1) = F_1(x, t), \quad (x, t) \in \Omega_s \times J,$$

$$(8.1b) \quad \mathcal{A} \frac{\partial^2 u}{\partial t^2} + \mathcal{C} \frac{\partial u}{\partial t} - L(u) = F(x, t), \quad (x, t) \in \Omega_p \times J,$$

where ρ is the total mass density of bulk material in Ω , and $\mathcal{A} \in \mathbb{R}^{6 \times 6}$ and $\mathcal{C} \in \mathbb{R}^{6 \times 6}$ denote the mass matrix and the dissipative matrix given by

$$\mathcal{A} = \begin{bmatrix} \rho I & \rho_f I \\ \rho_f I & G \end{bmatrix}, \quad \mathcal{C} = \mu \begin{bmatrix} 0 & 0 \\ 0 & K^{-1} \end{bmatrix},$$

where ρ_f denotes the mass density of fluid in Ω_p , $G \in \mathbb{R}^{3 \times 3}$ is a symmetric and positive definite matrix, which depends on the coordinates of the pores and the pore geometry, $I \in \mathbb{R}^{3 \times 3}$ is the identity matrix, μ is the fluid viscosity, and $K \in \mathbb{R}^{3 \times 3}$ is the symmetric and positive definite permeability matrix. Finally, the differential operators L_1 and L are given by

$$\begin{aligned} L_1(u_1) &= (\operatorname{div} \sigma_1(u_1), \operatorname{div} \sigma_2(u_1), \operatorname{div} \sigma_3(u_1)), \\ L(u) &= (\operatorname{div} \tau_1(u), \operatorname{div} \tau_2(u), \operatorname{div} \tau_3(u), -\nabla P(u)), \end{aligned}$$

where $\sigma_i(u_1) = (\sigma_{i1}(u_1), \sigma_{i2}(u_1), \sigma_{i3}(u_1))$ and $\tau_i(u) = (\tau_{i1}(u), \tau_{i2}(u), \tau_{i3}(u))$, $i = 1, 2, 3$, are the stress tensor in the solid and the total stress tensor for the bulk material on Ω_p , respectively, and $P(u)$ is the fluid pressure in Ω_p . The stress tensor $\sigma_{ij}(u_1)$ is associated with the strain tensor $\tau_{ij}(u_1)$ by Hooke's law for anisotropic elastic solids:

$$\sigma_{ij}(u_1) = \sum_{k,l=1}^3 A_{ijkl} \epsilon_{ij}(u_1), \quad i, j = 1, 2, 3,$$

where the A_{ijkl} 's are such that

$$A_{ijkl} = A_{klij} = A_{ijlk}.$$

The boundary conditions are imposed as in §7. Namely,

$$\begin{aligned} \sigma(u_1) \cdot n_s &= 0, \quad (x, t) \in \Gamma_0 \times J, \\ \sigma(u_1) \cdot n_s &= -\rho^{1/2} D^{1/2} \frac{\partial u_1}{\partial t}, \quad (x, t) \in \Gamma_1 \times J, \\ \sigma(u) \cdot n_s + \tau(u) \cdot n_p &= 0, \quad (x, t) \in \Gamma_2 \times J, \\ u_2 \cdot n_p &= 0, \quad (x, t) \in \Gamma_2 \times J. \end{aligned}$$

The initial conditions are imposed as in (1.4).

Let l_G be the minimum eigenvalue of G . We assume that

$$\begin{aligned} 0 &< \rho_* \leq \rho(x) \leq \rho^*, \quad x \in \overline{\Omega}, \\ 0 &< \rho_{f*} \leq \rho_f(x) \leq \rho_f^*, \quad x \in \overline{\Omega}_p, \\ \rho_f^2(x) &< l_G(x) \rho(x), \quad x \in \overline{\Omega}_p. \end{aligned}$$

Thus $\mathcal{A}$ is positive definite and $\mathcal{C}$ is nonnegative.

Applying the previous notation, for $v = (v_1, v_2) \in V$ and $w = (w_1, w_2) \in V$, let

$$\begin{aligned} B_s(v_1, w_1) &= \sum_{ij} (\sigma_{ij}(v_1), \epsilon_{ij}(w_1))_{\Omega_s}, \\ B_p(v, w) &= \sum_{ij} (\tau_{ij}(v), \epsilon_{ij}(w))_{\Omega_p} - (P(v), \operatorname{div} w_2)_{\Omega_p}, \\ B(v, w) &= B_s(v_1, w_1) + B_p(v, w). \end{aligned}$$

Also, let

$$\begin{aligned} \tilde{V} &= [H^1(\Omega)]^3 \times H_0(\operatorname{div}, \Omega_p) \\ &= \{v = (v_1, v_2) \in V : v_2 \cdot n_p = 0 \text{ on } \Gamma_2\}. \end{aligned}$$

$\tilde{V}$ is a closed separable subspace of V .

As in the last section, for simplicity, we assume that Γ_1 is of zero $d\sigma$ -measure. Then if we take $v = (v_1, v_2) \in \tilde{V}$, test (8.1a) against v_1 and (8.1b) against v_2 , and add the resulting equations, we obtain the equation

$$(8.2) \quad (\rho \frac{\partial^2 u_1}{\partial t^2}, v_1)_{\Omega_s} + (\mathcal{A} \frac{\partial^2 u}{\partial t^2}, v)_{\Omega_p} + (\mathcal{C} \frac{\partial u}{\partial t}, v)_{\Omega_p} + B(u, v) = (F_1, v_1) + (F_2, v_2)_{\Omega_p}.$$

There are constants $c > 0$ and $\gamma > 0$ [19] such that

$$(8.3) \quad B(v, v) \geq c \|v\|_V^2 - \gamma (\|v_1\|_0^2 + \|v_2\|_{0, \Omega_p}^2), \quad v \in V.$$

We introduce the differential operators L_1^* on Ω_s and L^* on Ω_p by

$$\begin{aligned} L_1^*(u_1) &= -L_1(u_1) + \gamma u_1, \\ L^*(u) &= -L(u) + \gamma u. \end{aligned}$$

Let B_γ be the bilinear form associated with L_1^* and L^* :

$$B_\gamma(v, w) = B(v, w) + \gamma(v_1, w_1) + \gamma(v_2, w_2)_{\Omega_p},$$

for $v = (v_1, v_2) \in V$ and $w = (w_1, w_2) \in V$. Then the symmetric form B_γ satisfies

$$\begin{aligned} |B_\gamma(v, w)| &\leq c \|v\|_V \|w\|_V, \quad v, w \in V, \\ B_\gamma(v, v) &\geq c \|v\|_V^2, \quad v \in V. \end{aligned}$$

8.2. A Galerkin method. Let $W_h \subset H(\text{div}, \Omega_p)$ be one of the Raviart-Thomas-Nedelec vector spaces [14], [13], the Brezzi-Douglas-Fortin-Marini vector spaces [6], Brezzi-Douglas-Durán-Fortin vector spaces [5], or the Chen-Douglas vector spaces [9] of index r associated with T_h^p , where T_h^p is a quasiregular partition of Ω_p . Let $W_h^0 = W_h \cap H_0(\text{div}, \Omega_p)$ and set $\tilde{V}_h = M_h \times W_h^0$. Then the continuous-time Galerkin approximation to the solution of (8.1) is defined as the twice-differentiable map $u_h : J \rightarrow \tilde{V}_h$ such that

$$\begin{aligned} (\rho \frac{\partial^2 u_{1h}}{\partial t^2}, v_1)_{\Omega_s} + (\mathcal{A} \frac{\partial^2 u_h}{\partial t^2}, v)_{\Omega_p} + (\mathcal{C} \frac{\partial u_h}{\partial t}, v)_{\Omega_p} + B(u_h, v) \\ = (F_1, v_1) + (F_2, v_2)_{\Omega_p}, \quad v = (v_1, v_2) \in \tilde{V}_h, \quad t \in J. \end{aligned}$$

The initial conditions $u_h(0)$ and $\partial u_h(0)/\partial t$ can be defined as in §4.

Again, a quasi-projection analysis similar to that given in §4 can be carried out in the present case if the same regularity assumption as in (7.5) is made. As mentioned before, there is no proof for this regularity result.

REFERENCES

1. M. A. Biot, *Theory of propagation of elastic waves in a fluid-saturated porous solid. I. low-frequency range*, Journal of the Acoustical Society of America **28** (1956), 168–178.
2. M. A. Biot, *Mechanics of deformation and acoustic propagation in porous media*, J. Appl. Phys. **33** (1962), 11482–14498.
3. M. A. Biot and D. G. Willis, *The elastic coefficient of the theory of consolidation*, J. Appl. Mech. **24** (1957), 594–601.
4. J. H. Bramble and A. H. Schatz, *Higher order local accuracy by averaging in the finite element*, Math. Compt. **31** (1977), 94–111.
5. F. Brezzi, J. Douglas, Jr., R. Durán, and M. Fortin, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math. **51** (1987), 237–250.
6. F. Brezzi, J. Douglas, Jr., M. Fortin, and L. D. Marini, *Efficient rectangular mixed finite elements in two and three space variables*, R.A.I.R.O. Modél. Math. Anal. Numér. **21** (1987), 581–604.
7. F. Brezzi, J. Douglas Jr., and L.D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math. **47** (1985), 217–235.
8. Z. Chen, *A quasi-projection analysis for elastic wave propagation in fluid saturated porous media*, in the Proceedings of the Bakelman Symposium, S. Fulling, ed. Discourses in Mathematics and its Applications (1993).
9. Z. Chen and J. Douglas, Jr., *Prismatic mixed finite elements for second order elliptic problems*, Calcolo **26** (1989), 135–148.
10. J. Douglas, Jr., T. Dupont, and M. F. Wheeler, *A quasi-projection analysis of Galerkin methods for parabolic and hyperbolic equations*, Math. Comput. **32** (1978), 345–362.
11. G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.
12. J. Lysmer and R. L. Kuhlemeyer, *Finite dynamic model for infinite media*, Eng. Mech. Division, Proc. Amer. Soc. Civil Eng. **95** (1969), 859–877.
13. J. C. Nedelec, *Mixed finite elements in $\mathbf{R}^3$* , Numer. Math. **35** (1980), 315–341.
14. P. A. Raviart and J.M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, *Mathematical Aspects of the Finite Element Method*, Lecture Notes in Math. 606, Springer-Verlag, Berlin, 1977.
15. J. E. Santos, *Elastic wave propagation in fluid-saturated porous media, Part I. The existence and uniqueness theorems*, Math. Model. Numer. Anal. **20** (1986), 113–128.
16. J.E. Santos and E. J. Orena, *Elastic wave propagation in fluid-saturated porous media, Part II. The Galerkin procedures*, Math. Model. Numer. Anal. **20** (1986), 129–139.
17. J.E. Santos, *Finite element methods for the simulation of wave propagation in two dimensional inhomogeneous elastic media*, Calcolo **22** (1985), 249–317.
18. J.E. Santos, J. Douglas, Jr. and A. P. Calderon, *Finite element methods for a composite model in elastodynamics*, SIAM J. Num. Anal. **25** (1988), 513–532.
19. J. E. Santos and O. M. Lovera, *Numerical methods for a model for wave propagation in composite anisotropic media*, Math. Model. Numer. Anal. **21** (1987), 113–128.
20. M. Vogelius, *An analysis of the p-version of the finite element method for nearly incompressible materials, uniformly valid, optimal order error estimates*, Numer. Math. **41** (1983), 39–53.

#	Author/s	Title
1121	Nahum Shimkin & Adam Shwartz,	Asymptotically efficient adaptive strategies in repeated games, part II: Asymptotic optimality
1122	M.E. Bradley,	Well-posedness and regularity results for a dynamic Von Kármán plate
1123	Zhangxin Chen,	Finite element analysis of the 1D full drift diffusion semiconductor model
1124	Gang Bao & David C. Dobson,	Diffraction optics in nonlinear media with periodic structure
1125	Steven Cox & Enrique Zuazua,	The rate at which energy decays in a damped string
1126	Anthony W. Leung,	Optimal control for nonlinear systems of partial differential equations related to ecology
1127	H.J. Sussmann,	A continuation method for nonholonomic path-finding problems
1128	Yung-Jen Guo & Walter Littman,	The null boundary controllability for semilinear heat equations
1129	Q. Zhang & G. Yin,	Turnpike sets in stochastic manufacturing systems with finite time horizon
1130	I. Györi, F. Hartung & J. Turi,	Approximation of functional differential equations with time- and state-dependent delays by equations with piecewise constant arguments
1131	I. Györi, F. Hartung & J. Turi,	Stability in delay equations with perturbed time lags
1132	F. Hartung & J. Turi,	On the asymptotic behavior of the solutions of a state-dependent delay equation
1133	Pierre-Alain Gremaud,	Numerical optimization and quasiconvexity
1134	Jie Tai Yu,	Resultants and inversion formula for N polynomials in N variables
1135	Avner Friedman & J.L. Velázquez,	The analysis of coating flows in a strip
1136	Eduardo D. Sontag,	Control of systems without drift via generic loops
1137	Yuan Wang & Eduardo D. Sontag,	Orders of input/output differential equations and state space dimensions
1138	Scott W. Hansen,	Boundary control of a one-dimensional, linear, thermoelastic rod
1139	Robert Lipton & Bogdan Vernescu,	Homogenization of two phase emulsions with surface tension effects
1140	Scott Hansen & Enrique Zuazua,	Exact controllability and stabilization of a vibrating string with an interior point mass
1141	Bei Hu & Jiongmin Yong,	Pontryagin Maximum principle for semilinear and quasilinear parabolic equations with pointwise state constraints
1142	Mark H.A. Davis,	A deterministic approach to optimal stopping with application to a prophet inequality
1143	M.H.A. Davis & M. Zervos,	A problem of singular stochastic control with discretionary stopping
1144	Bernardo Cockburn & Pierre-Alain Gremaud,	An error estimate for finite element methods for scalar conservation laws
1145	David C. Dobson & Fadil Santosa,	An image enhancement technique for electrical impedance tomography
1146	Jin Ma, Philip Protter, & Jiongmin Yong,	Solving forward-backward stochastic differential equations explicitly — a four step scheme
1147	Yong Liu,	The equilibrium plasma subject to skin effect
1148	Ulrich Hornung,	Models for flow and transport through porous media derived by homogenization
1149	Avner Friedman, Chaocheng Huang, & Jiongmin Yong,	Effective permeability of the boundary of a domain
1150	Gang Bao,	A uniqueness theorem for an inverse problem in periodic diffraction optics
1151	Angelo Favini, Mary Ann Horn, & Irena Lasiecka,	Global existence and uniqueness of regular solutions to the dynamic von Kármán system with nonlinear boundary dissipation
1152	E.G. Kalnins & Willard Miller, Jr.,	Models of q -algebra representations: q -integral transforms and “addition theorems”
1153	E.G. Kalnins, V.B. Kuznetsov & Willard Miller, Jr.,	Quadrics on complex Riemannian spaces of constant curvature, separation of variables and the Gaudin magnet
1154	A. Kersch, W. Morokoff & Chr. Werner,	Selfconsistent simulation of sputtering with the DSMC method
1155	Bing-Yu Zhang,	A remark on the Cauchy problem for the Korteweg-de Vries equation on a periodic domain
1156	Gang Bao,	Finite element approximation of time harmonic waves in periodic structures
1157	Tao Lin & Hong Wang,	Recovering the gradients of the solutions of second-order hyperbolic equations by interpolating the finite element solutions
1158	Zhangxin Chen,	L^p -posteriori error analysis of mixed methods for linear and quasilinear elliptic problems
1159	Todd Arbogast & Zhangxin Chen,	Homogenization of compositional flow in fractured porous media
1160	L. Qiu, B. Bernhardsson, A. Rantzer, E.J. Davison, P.M. Young & J.C. Doyle,	A formula for computation of the real stability radius
1161	Maria Inés Trespavsky,	Adaptive control of linear discrete time systems with external disturbances under inaccurate modelling: A case study
1162	Petr Klouček & Franz S. Rys,	Stability of the fractional step Θ -scheme for the nonstationary Navier-Stokes equations
1163	Eduardo Casas, Luis A. Fernández & Jiongmin Yong,	Optimal control of quasilinear parabolic equations
1164	Darrell Duffie, Jin Ma & Jiongmin Yong,	Black’s consol rate conjecture
1165	D.G. Aronson & J.L. Vazquez,	Anomalous exponents in nonlinear diffusion

- 1166 **Ruben D. Spies**, Local existence and regularity of solutions for a mathematical model of thermomechanical phase transitions in shape memory materials with Landau-Ginzburg free energy
- 1167 **Pu Sun**, On circular pipe Poiseuille flow instabilities
- 1168 **Angelo Favini, Mary Ann Horn, Irena Lasiecka & Daniel Tataru**, Global existence, uniqueness and regularity of solutions to a Von Kármán system with nonlinear boundary dissipation
- 1169 **A. Dontchev, Tz. Donchev & I. Slavov**, On the upper semicontinuity of the set of solutions of differential inclusions with a small parameter in the derivative
- 1170 **Jin Ma & Jiongmin Yong**, Regular-singular stochastic controls for higher dimensional diffusions — dynamic programming approach
- 1171 **Alex Solomonoff**, Bayes finite difference schemes
- 1172 **Todd Arbogast & Zhangxin Chen**, On the implementation of mixed methods as nonconforming methods for second order elliptic problems
- 1173 **Zhangxin Chen & Bernardo Cockburn**, Convergence of a finite element method for the drift-diffusion semiconductor device equations: The multidimensional case
- 1174 **Boris Mordukhovich**, Optimization and finite difference approximations of nonconvex differential inclusions with free time
- 1175 **Avner Friedman, David S. Ross, and Jianhua Zhang**, A Stefan problem for reaction-diffusion system
- 1176 **Alex Solomonoff**, Fast algorithms for micromagnetic computations
- 1177 **Nikan B. Firoozye**, Homogenization on lattices: Small parameter limits, H -measures, and discrete Wigner measures
- 1178 **G. Yin**, Adaptive filtering with averaging
- 1179 **Włodzimierz Byrc and Amir Dembo**, Large deviations for quadratic functionals of Gaussian processes
- 1180 **Ilja Schmelzer**, 3D anisotropic grid generation with intersection-based geometry interface
- 1181 **Alex Solomonoff**, Application of multipole methods to two matrix eigenproblems
- 1182 **A.M. Latypov**, Numerical solution of steady euler equations in streamline-aligned orthogonal coordinates
- 1183 **Bei Hu & Hong-Ming Yin**, Semilinear parabolic equations with prescribed energy
- 1184 **Bei Hu & Jianhua Zhang**, Global existence for a class of Non-Fickian polymer-penetrant systems
- 1185 **Rongze Zhao & Thomas A. Posbergh**, Robust stabilization of a uniformly rotating rigid body
- 1186 **Mary Ann Horn & Irena Lasiecka**, Uniform decay of weak solutions to a von Kármán plate with nonlinear boundary dissipation
- 1187 **Mary Ann Horn, Irena Lasiecka & Daniel Tataru**, Well-posedness and uniform decay rates for weak solutions to a von Kármán system with nonlinear dissipative boundary conditions
- 1188 **Mary Ann Horn**, Nonlinear boundary stabilization of a von Kármán plate via bending moments only
- 1189 **Frank H. Shaw & Charles J. Geyer**, Constrained covariance component models
- 1190 **Tomasz Luczaka**, A greedy algorithm estimating the height of random trees
- 1191 **Timo Seppäläinen**, Maximum entropy principles for disordered spins
- 1192 **Yuandan Lin, Eduardo D. Sontag & Yuan Wang**, Recent results on Lyapunov-theoretic techniques for nonlinear stability
- 1193 **Svante Janson**, Random regular graphs: Asymptotic distributions and contiguity
- 1194 **Rachid Ababou**, Random porous media flow on large 3-D grids: Numerics, performance, & application to homogenization
- 1195 **Moshe Fridman**, Hidden Markov model regression
- 1196 **Petr Klouček, Bo Li & Mitchell Luskin**, Analysis of a class of nonconforming finite elements for Crystalline microstructures
- 1197 **Steven P. Lalley**, Random series in inverse Pisot powers
- 1198 **Rudy Yaksick**, Expected optimal exercise time of a perpetual American option: A closed-form solution
- 1199 **Rudy Yaksick**, Valuation of an American put catastrophe insurance futures option: A Martingale approach
- 1200 **János Pach, Farhad Shahrokhi & Mario Szegedy**, Application of the crossing number
- 1201 **Avner Friedman & Chaocheng Huang**, Averaged motion of charged particles under their self-induced electric field
- 1202 **Joel Spencer**, The Erdős-Hanani conjecture via Talagrand's inequality
- 1203 **Zhangxin Chen**, Superconvergence results for Galerkin methods for wave propagation in various porous media
- 1204 **Russell Lyons, Robin Pemantle & Yuval Peres**, When does a branching process grow like its mean? Conceptual proofs of $L \log L$ criteria
- 1205 **Robin Pemantle**, Maximum variation of total risk
- 1206 **Robin Pemantle & Yuval Peres**, Galton-Watson trees with the same mean have the same polar sets
- 1207 **Robin Pemantle**, A shuffle that mixes sets of any fixed size much faster than it mixes the whole deck
- 1208 **Itai Benjamini, Robin Pemantle & Yuval Peres**, Martin capacity for Markov chains and random walks in varying dimensions
- 1209 **Włodzimierz Bryc & Amir Dembo**, On large deviations of empirical measures for stationary Gaussian processes
- 1210 **Martin Hildebrand**, Some random processes related to affine random walks
- 1211 **Alexander E. Mazel & Yurii M. Suhov**, Ground states of a Boson quantum lattice model