**EXPECTED OPTIMAL EXERCISE TIME OF A
PERPETUAL AMERICAN OPTION:
A CLOSED-FORM SOLUTION**

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SUMMARY

We address the issue: what is the expected duration of a sequential activity, such as playing roulette or investing in financial securities, when the stopping rule involves the determination of whether and when a key, underlying stochastic process hits some boundary? We obtain, in a computationally simple manner, the classic result: the expected first passage time of a Brownian motion with drift is the ratio of the stopping boundary to the drift of the Brownian motion. Our derivation appears to be novel because of its computational simplicity. That is, we use only the optional sampling theorem of martingale theory and elementary algebra. In contrast, the non-martingale approach requires tedious integration and solution of an ordinary differential equation. Finally, this martingale-based methodology enables us to easily obtain our main financial result: The expected optimal exercise time of a perpetual, dividend-paying American call option contract is the ratio of the time-independent stopping boundary to the risk-adjusted drift of the stock price process. This ratio is an analytical expression.

Key Words and Phrases: *Brownian motion, First passage time, Martingale, American call option, Optimal exercise*

1. INTRODUCTION

Martingale methods simplify the solution of stochastic, sequential decision problems. For example, Chung^{3/} (pp. 327-8) uses martingale methods to demonstrate in two lines, and "without any computation" (his quotes), that the probability of a gambler's ruin is inversely proportional to the gambler's initial capital.

This note has a similar objective. That is, we aim to demonstrate that martingale methods greatly simplify the solution of two problems. The first is the classic problem: What is the expected first passage (stopping) time, to a single, fixed boundary, of a Brownian motion with drift? The first passage time is of interest since the solution of a wide variety of gambling and investment decision problems often involves the determination of whether and when a key, underlying stochastic process hits some boundary. For example, in the classic gambler's ruin problem one of the main issues is: what is the expected duration of the game? This issue can be rephrased in sharper, analytic terms as: how much time will elapse, on average, before the gambler's net gain reaches a boundary level of zero?

In Section 2 we demonstrate the relative ease with which one can derive--via martingale methods--the Laplace transform of the first passage time (to a single boundary) density function of a Brownian motion with drift. The derivation begins with a straightforward application of the optional sampling theorem to martingales associated with Brownian motion. Then, elementary algebra is used to obtain the transform. This is a much less

tedious approach than the common practice (as illustrated in Srinivasan and Mehata,^{21/} section 5.5) of computing the Laplace transforms of the terms of the backward Kolmogorov equation and then solving the resultant linear ordinary differential equation in order to obtain the Laplace transform of the first passage density function.

Having obtained the Laplace transform, we simply differentiate it to obtain our main probabilistic result: the expected first passage time is the ratio of the stopping boundary to the drift of the Brownian motion.

The second problem (which is addressed in Section 3) is of particular interest to finance practitioners in both the investment and corporate sectors. Hedging, speculation, arbitrage, and capital budgeting decisions often hinge on the issue: What is the expected optimal exercise (stopping) time of an American option contract that pays dividends? An American option is a financial security which gives the holder the right, but not the obligation, to exercise the option (and thereby obtain the underlying asset of the option) at any time throughout the life of the option. We preface the analysis of this issue by reviewing McKean's^{17/} seminal analysis of the pricing of a perpetual American warrant; a security closely related to an American option. Then, we restate his results within the context of Karatzas' ^{11,12/} recent martingale-based framework for valuing American options. In this framework, which is a bit more elaborate than the Harrison-Pliska^{10/} seminal model, the key economic idea of the absence of arbitrage opportunities is linked

to the probabilistic concept of a martingale. Moreover, this framework is quite general since it enables market participants to consume as well as invest. Thus, permitting a unified approach to the problems of: option pricing, consumption and investment, and equilibrium in a financial market.

Our main financial result is that (for a perpetual, dividend-paying American call option) the optimal exercise time is the ratio of the time-independent exercise (stopping) boundary (b) to the drift of the transformed stock price process. Here, b equals the product of the inverse of the stock volatility times the log of the ratio of the time-independent, optimal exercise stock price to the initial (time zero) stock price. Thus, the optimal exercise time is easily computable since it is an analytical expression.

We conclude (in Section 4) by suggesting future research tasks that may confirm the computational simplicity of martingale methods under a wider class of stochastic processes as well as stopping problems having more general, curved and time-dependent boundaries. Finally, note that all proofs are contained in Appendix A.

2. PROBABILISTIC FOUNDATIONS AND RESULTS

To make the paper self-contained, we define some basic concepts from the theory of stochastic processes.

2.1. Definitions

1. Brownian Motion. Let: i) $(\Omega, \mathcal{F}, P)$ be a filtered probability space; ii) $B = \{B(t), \mathcal{F}_t; 0 \leq t \leq \infty\}$ be a continuous stochastic process on this space; and, iii) μ and σ denote the drift and dispersion coefficient parameters of B , respectively. We say that B is a one-

dimensional (μ, σ) **Brownian motion** with respect to F if:

- i) B is adapted to the filtration $F = \{F_t, t \in \tau \subseteq [0, \infty]\}$, where τ is the continuous time index set, of the measurable sample space (Ω, F) ,
- ii) $B(t) = B(0) + \mu t + \sigma W(t)$, $t \geq 0$, where W is a standard Brownian motion (Wiener process) and $B(0)$ is independent of W ,
- iii) For $0 \leq s < t$, every increment $B(t+s) - B(s)$ is independent of F_s and is normally distributed with constant mean μt and constant variance $\sigma^2 t$, and iv) $B(0) = 0$, a.s.

2. First Passage Time. Suppose that: i) (Ω, F) is a measurable sample space with a filtration F , and ii) $\omega \in \Omega$ is a sample point in the outcome space Ω . We say that the random variable T_b , where $T_b: \Omega \rightarrow \tau$, is a **first passage time** (with respect to F) to a single boundary $b \in \mathbf{R}$ if: for every $t \in \tau$,

$$T_b(\omega) = \begin{cases} \inf\{t \geq 0; B_t(\omega) = b\}, & \text{if } B_t(\omega) \geq b \text{ and sign } \mu = \text{sign } b, \text{ for some } t \geq 0 \\ \infty, & \text{if } B_t(\omega) < b \text{ and sign } \mu = \text{sign } b, \text{ for all } t \geq 0. \end{cases}$$

2.2. Main Probabilistic Results

We now prove the three probabilistic results. The first is a demonstration of the ease with which one can obtain, via martingale methods, the Laplace transform of the first passage time density of $B(t)$ to a single boundary b . To appreciate the simplicity, the reader is advised to consult Bhattacharya and Waymire^{1/} (section 1.10) for a discussion of the mathematically tedious, non-martingale approach. Finally, the first main result rests on the following four lemmas.

Lemma 2.1: The exponential martingale associated with $B(t)$ is

$$V(t) = \exp(\lambda B(t) - \Theta t), \quad (2.1)$$

where $\Theta \equiv \lambda\mu + \lambda^2\sigma^2/2$, and λ is any arbitrary, real constant.

Lemma 2.2: (Optional Sampling Theorem)

Let $V(t)$ be a martingale and T_b a stopping time. If

- i) $\text{Prob} \{T_b < \infty\} = 1$,
- ii) $E[|V(T_b)|] < \infty$, and
- iii) $\lim_{n \rightarrow \infty} E[V(n)1_{\{T_b > n\}}] = 0$,

where $1_{\{T_b > n\}}$ is an indicator function having a value 1, when $T_b > n$, and zero, otherwise. Then:

$$E[V(T_b)] = E[V(0)]. \quad (2.2)$$

Lemma 2.3: Let $V(t)$ be a martingale and T_b a stopping time. Suppose that the hypotheses of the optional sampling theorem are satisfied. Then:

$$1 = E[V(T_b \wedge t)], \quad (2.3)$$

where $(T_b \wedge t) \equiv \min\{T_b, t\}$.

Lemma 2.4: Let $V(t)$ be a martingale and T_b a stopping time. Suppose that the hypotheses of the optional sampling theorem are satisfied. If: i) λ is sufficiently large such that $\Theta > 0$ and, thus, $0 \leq V(T_b \wedge t) \leq \exp(\lambda b)$; and, ii) the collection of random variables $\{V(T_b \wedge t)\}$ is uniformly integrable, i.e.,

$$\lim_{c \rightarrow \infty} \sup_{t \geq 0} E[|V(T_b \wedge t)| 1_{\{V(T_b \wedge t) > c\}}] = 0.$$

$$\text{Then: } E[e^{-\Theta T_b}] = e^{-\lambda b}. \quad (2.4)$$

With these intermediate results, we now obtain our first main probabilistic result.

Theorem 2.1: Laplace Transform of First Passage Time Density.

Suppose that: i) $B(t)$ is a (μ, σ) Brownian motion; ii) $b \neq 0$ is the single, fixed boundary; and, iii) T_b is the first time, if any, that $B(t)$ reaches the level b . Then, the Laplace transform of the first passage time density of $B(t)$ to a single boundary b is:

$$E^{(\mu)}[e^{-\Theta T_b}] = \exp\{\mu b / \sigma^2 - |b| [(\mu^2 + 2\sigma^2 \Theta)^{.5}] / \sigma^2\}, \quad \Theta > 0. \quad (2.5)$$

The second main result identifies the relationship between the drift and boundary that must be satisfied for T_b to have a finite value with probability one.

Theorem 2.2: Existence of Finite First Passage Time. Retain the hypotheses of Theorem 2.1. Then, $B(t)$ attains $b \neq 0$ with probability one if and only if μ and b have the same sign.

Our final result--the expected first passage time of a Brownian motion process with drift--is obtained by differentiating (2.5), the Laplace transform of the first passage time density function of $B(t)$ to a single boundary.

Theorem 2.3: Expected First Passage Time. Retain the hypotheses of Theorem 2.1. Then, the expected first passage time of $B(t)$ is:

$$E^{(\mu)}[T_b] = \begin{cases} b/\mu, & b \text{ and } \mu \text{ have same sign} \\ \infty, & b \text{ and } \mu \text{ have different signs.} \end{cases} \quad (2.6)$$

2.3. Intuition

The underlying intuition of the result is clearly illustrated in Figure 1. The process $B(t)$ increases deterministically along the straight trend line with slope μ . Thus, μ measures the expected rate of change in B , per unit time. That is, $\mu = (E[B(t)] - B(t_0))/t$. The height of the time-independent boundary (line bb)

indicates the vertical distance--in this case b units--that $B(t)$ must travel in order to reach the boundary. Since $B(t)$ is expected to change (increase) in value by μ units per unit time, the expected time to traverse the height of b units (i.e., reach the boundary) is thus b/μ . Of course, the actual time to first passage to the boundary could be greater or less than $E[T_b]$ since $B(t)$ will fluctuate around the trend line due to the random shocks σdW .

FIG. 1 EXPECTED FIRST PASSAGE TIME ($E^{\mu} [T_b]$)

As a further aid to understanding, the main result can be interpreted in the context of a gambling problem. In that problem, our result indicates that the expected duration of the gambler's game is the ratio of the gambler's initial wealth to the drift of

the net gain process. Finally, to gauge the computational simplicity of the martingale approach to computing the expected duration of a gambling game, the reader is directed to Feller^{7'} (pp. 348-349). In a random walk setting, he employs the more tedious method involving the solution of difference equations.

3. EXPECTED EXERCISE TIME: PERPETUAL AMERICAN CALL OPTION

We now use the preceding results to resolve an issue which is very important to the holder of a dividend-paying, infinitely-lived American option contract. That is: How long, on average, will the holder expect to own this option before optimally exercising it? To address this question in a continuous-time setting, one first solves the option valuation (stopping) problem for the two unknowns: the optimal exercise (stopping) time and the optimal exercise boundary. Then, the first-passage time density function is derived and, subsequently, used to compute the expected optimal exercise time. In contrast to these tedious calculations, as illustrated in Carr-Jarrow-Myneni^{1a'}, we shall demonstrate the computational simplicity obtained via martingale methods. However, we first review McKean's^{17'} seminal results for the American option valuation problem.

3.1. Background: McKean's Optimal Exercise Results

McKean was the first to provide a mathematically complete solution to the problem of valuing a perpetual American warrant in a Markovian setting. He assumed a perfect security market in which three assets are continuously traded and no arbitrage opportunities are present. The first asset is a riskless bond whose price $B(t)$

satisfies the linear differential equation

$$dB(t) = B(t)r dt, \quad (3.1)$$

where $r > 0$ is a constant, continuously compounded interest rate offered to all borrowers and lenders. The underlying asset, on which the perpetual American call option is written, is a stock whose price $S(t)$ satisfies the linear, stochastic differential equation

$$dS(t) = S(t)[(m(t) + \mu)dt + \sigma dW(t)], \quad S(0) > 0 \quad (3.2)$$

where: $m(t)$ is the "drift" or instantaneous expected growth rate of $S(t)$; $\mu \in (0, r)$, the dividend rate, and σ are non-negative constants; and, $dW(t)$ is a Standard Brownian motion under P . Finally, the third asset is a perpetual American call option with an exercise price, $q = \$1$.

3.1.1. Results

In this context McKean obtained a closed-form solution for the two unknowns of the warrant holder's valuation (optimal stopping) problem. These are the optimal exercise time (τ^*) and the time-independent exercise (stopping) boundary (S^*), respectively:

$$i) \quad \tau^* = \inf\{t \geq 0; S(t) \geq S^*\} \quad \text{for all } t \in [0, \infty], \quad (3.3)$$

$$ii) \quad S^* \equiv \Phi / (\Phi - 1) > 1, \quad \text{where} \quad (3.4)$$

$$\Phi = ((\delta^2 + 2r\sigma^2)^{.5} - \delta) / \sigma^2 \quad \text{and} \quad \delta = r - \mu - .5\sigma^2.$$

3.2. Restatement in Karatzas' Framework

Restating McKean's result in Karatzas' framework is not straightforward. As Karatzas^{11/} (p. 54) indicates, to evaluate perpetual options two technical difficulties have to be surmounted. That is, $W(t)$ must now be: i) a Brownian motion on the entire set

$[0, \infty]$, and ii) accompanied by a filtration that measures all of the processes in the model. Then, to ensure that arbitrage (riskless, self-financing) trading profits cannot be reaped, a new, equivalent probability measure (P_0) is constructed which is also a martingale measure. Under this new equivalent martingale measure (which, if unique, ensures market completeness), the original Brownian motion, $W(t)$, is reexpressed, via Girsanov's' change-of-measure theorem, as:

$$W_0(t) = W(t) + \int_0^t \Theta(s) ds \quad (3.5)$$

where: $\Theta(t) \equiv [m(t) + \mu - r]/\sigma$.

Finally, the optimal exercise (stopping) time (τ^*) is reexpressed as:

$$T_g \equiv \tau^* = \inf\{t \geq 0; W_0(t) + vt \geq \beta\}, \quad (3.6)$$

where the exercise boundary (β) is:

$$\beta \equiv (1/\sigma) \log(S^*/S(0)), \quad (3.7)$$

and: i) $S^* \equiv \Phi/(\Phi - 1)$, ii) $\Phi \equiv (\sigma^2)^{-1}[(\delta^2 + 2r\sigma^2)^{.5} - \delta]$,

iii) $\delta \equiv r - \mu - .5\sigma^2$, and iv) $v \equiv \delta/\sigma$

3.2.1. Expected Exercise Time

We now derive the main result and three related lemmas. The first lemma specifies the relationship between the stock price at time zero and the optimal exercise price S^* that excludes the possibility of a zero exercise time; an uninteresting case.

Lemma 3.1: Suppose that there exists a solution to the option holder's optimal exercise (stopping) problem. If $S(0) < S^*$, then

$$T_g = \tau^* > 0. \quad (3.8)$$

Lemma 3.2 is the analogue of Theorem 2.1.

Lemma 3.2: The Laplace transform of T_g is:

$$E[e^{-\alpha T_g}] = \exp\{[(\beta v - |\beta| (v^2 + 2\sigma^2\alpha)^{.5})/\sigma^2]\}, \quad \alpha > 0. \quad (3.9)$$

The third lemma specifies the relationship that must hold between the parameters β and v (as well as the sign of v), for the exercise time to be finite-valued with probability one. Thus, this lemma is the analogue of Theorem 2.2 in section 2.

Lemma 3.3: Suppose that: $S^* > S(0)$, and $v, \beta \neq 0$. Then T_g attains $\beta > 0$ with probability one if and only if v and β have the same sign.

Finally, we obtain our main financial result.

Theorem 3.1: Expected Optimal Exercise Time. Retain the hypotheses of Lemma 3.3. Then, the expected optimal exercise time of a perpetual American call option, that pays continuous dividends is:

$$E^{(v)}[T_g] = \begin{cases} \beta/v, & \text{if } \beta \text{ and } v > 0 \\ \infty, & \text{if } \beta > 0, v \leq 0. \end{cases} \quad (3.10)$$

In sum, this appears to be a novel result. The closest work to this appears to be Garman's^{8/} concept of "fugit" (an abbreviation of the Latin phrase: tempus fugit, or "time flies"). This concept measures the average lifetime of an American option. However, this concept was developed in a discrete-time setting. Thus, it yields only an approximate measure of the expected optimal exercise time.

4. IMPLICATIONS AND EXTENSIONS

These results appear to significantly reduce the computational burden associated with stochastic, sequential decision problems. In particular, these results demonstrate that our martingale-based methodology can be used to compute the expected timing of an optimal decision in a wide variety of

stochastic, dynamic decision problems that can be formulated as a stopping problem.

Consider, for example, Martzoukos and Teplitz-Sembitzky's^{16/} recent analysis of a common capital budgeting problem faced by energy planners in developing countries. When designing rural electrification programs, planners must determine the optimal sequencing of decentralized solutions (involving diesel-powered generators) and centralized, grid-based systems. Initially, generators tend to be economic investments since demand is spatially dispersed and loads are relatively low. However, as power demand grows, grid systems become preferred. Given the uncertainty about demand growth, planners, thus, face the problem of optimally timing an irreversible investment in electricity transmission lines. Martzoukos and Teplitz-Sembitzky used martingale results (similar to those used here) to compute the expected optimal timing of this irreversible, expensive investment in transmission lines, given uncertainty about demand. Another cogent example is Collins'^{4/} use of the first passage density to estimate the expected timing of foreign exchange realignments in the European Monetary System.

Finally, our result is easy to implement since the optimal boundary can be found as a solution to the related free boundary problem for the reward function of the stopping problem. [To understand the link between optimal stopping and free boundary problems, the reader is urged to consult the seminal references: Mikhalevich^{18/}, Chernoff^{2/}, and Lindley^{15/}.] Also, the drift parameter can be estimated via standard statistical methods.

Several extensions to this research might aid in confirming the computational simplicity obtainable via martingale methods. First, the class of stopping problems might be widened to include curved and time-dependent boundaries. This would, in turn, permit the comparison of the simplicity of the martingale approach against, for example, the complexity of: 1) Durbin's^{5'} approximation of the first passage time (FPT) density of a continuous Gaussian process to a general boundary, and 2) Durbin and Williams's^{6'} recent approximation of the FPT density of a Brownian motion to a curved boundary. Their approximation is a convergent series of multiple integrals of increasing dimensionality. Of course, these densities would, subsequently be integrated in order to compute the moments.

Likewise, the mean (and higher moments) for processes other than Brownian motion with drift should also be computed via martingale methods. Then, a basis would exist for comparing the computational effort of this work against the results of Ricciardi and Sato^{19'}. They have derived closed-form expressions for all moments of the first passage time of the unrestricted, conditional Ornstein-Uhlenbeck process with a constant boundary.

APPENDIX

A.1 Section 2 Proofs

We begin by proving four lemmas that are intermediate results required for the proof of Theorem 2.1.

PROOF: Lemma 2.1. The objective of the proof is to demonstrate, via simple computation, that $\exp(\lambda B(t))$ satisfies the martingale property. First, the Markov property implies that

$$\begin{aligned} E[\exp(\lambda B(t)) | F_s] &= \exp(\lambda B(s)) E[\exp(\lambda(B(t) - B(s))) | F_s] \\ &= \exp(\lambda B(s)) \exp\{(\lambda \mu t + \lambda^2 \sigma^2 t/2) - (\lambda \mu s + \lambda^2 \sigma^2 s/2)\} \\ &= \exp(\lambda B(s)) \exp\{-(\lambda \mu s + \lambda^2 \sigma^2 s/2)\} \text{ times} \\ &\quad \exp\{(\lambda \mu t + \lambda^2 \sigma^2 t/2)\} \end{aligned} \tag{A.1}$$

since $E[\exp(\lambda B(t))] = \exp(\lambda \mu t + \lambda^2 \sigma^2 t/2)$ because $B(t) - B(s)$ is independent of F_s and is distributed $N(\mu(t-s), \sigma^2(t-s))$. ■

PROOF: Lemma 2.2. Optional Sampling Theorem. See Karlin and Taylor^{14/} (p. 261). ■

PROOF: Lemma 2.3. First, we know that

$$E[V(T_b \wedge t)] = E[V(T_b)], \tag{A.2}$$

(where $T_b \wedge t \equiv \min\{T_b, t\}$), since $V[(T_b \wedge t)] = V(T_b)$ whenever $T_b < \infty$ and $t > T_b$. Next, by Lemma 1 and $B(0) = 0$, we know that:

$$\begin{aligned} E[V(0)] &= E[\exp(\lambda B(0) - \Theta 0)] \\ &= E[\exp(0)], \text{ so} \\ E[V(0)] &= 1. \end{aligned} \tag{A.3}$$

Also, by Lemma 2.2, we know that:

$$E[V(0)] = E[V(T_b)]. \tag{A.4}$$

Finally, by combining (A.2) through (A.4), we conclude that:

$$1 = E[V(0)] = E[V(T_b)] = E[(T_b \wedge t)]. \quad \blacksquare \tag{A.5}$$

PROOF: Lemma 2.4. Passing to the limit as $t \rightarrow \infty$, we observe from the conclusion of Lemma 2.3, that

$$\begin{aligned} 1 &= \lim_{t \rightarrow \infty} E[V(T_b \wedge t)] \\ &= E[\lim_{t \rightarrow \infty} V(T_b \wedge t)], \end{aligned} \quad (\text{A.6})$$

where the limit can be taken under the expectation operator since $\{V(T_b \wedge t)\}$ is hypothesized to be uniformly integrable and, thus, Theorem 5 of Shiriyayev^{20/} (p. 187) can be invoked. Also,

$$\lim_{t \rightarrow \infty} V(T_b \wedge t) = \begin{cases} \exp\{\lambda b - \Theta T_b\}, & \text{if } T_b < \infty, \\ 0, & \text{if } T_b = \infty. \end{cases} \quad (\text{A.7})$$

since $V(T_b \wedge t) = V(T_b)$ whenever $T_b < \infty$ and $t > T_b$. Since T_b is finite-valued by hypothesis, we can use (A.7) to reexpress (A.6) as:

$$\begin{aligned} 1 &= E[\exp(\lambda b - \Theta T_b)] \\ 1 &= e^{\lambda b} E[\exp(-\Theta T_b)], \text{ or} \\ E[e^{-\Theta T_b}] &= e^{-\lambda b}. \blacksquare \end{aligned} \quad (\text{A.8})$$

PROOF: Theorem 2.1. By Lemma 2.1 we know that the exponential martingale associated with $B(t)$ is:

$$V(t) = \exp\{\lambda B(t) - \Theta t\}, \quad (\text{A.9})$$

where $\Theta \equiv \lambda\mu + \lambda^2\sigma^2/2$, and λ is any real constant. By Lemmas two and three we can conclude that:

$$1 = E^{(\mu)}[V(T_b \wedge t)].$$

Using (A.9), we obtain:

$$1 = E^{(\mu)}[\exp\{\lambda B(T_b \wedge t) - \Theta(T_b \wedge t)\}]. \quad (\text{A.10})$$

Passing to the limit as $t \rightarrow \infty$, and using Lemma 2.4, we can conclude that:

$$1 = \lim_{t \rightarrow \infty} E^{(\mu)}[\exp\{\lambda B(T_b \wedge t) - \Theta(T_b \wedge t)\}], \quad (\text{A.11})$$

$$1 = e^{\lambda b} E^{(\mu)}[e^{-\Theta T_b}], \quad \text{or}$$

$$E^{(\mu)}[e^{-\Theta T_b}] = e^{-\lambda b}. \quad (\text{A.12})$$

The final step is to obtain a relationship between Θ and λ that satisfies the constraint

$$E^{(\mu)}[e^{-\Theta T_b}] = e^{-\lambda b} \leq 1. \quad (\text{A.13})$$

This constraint implies that

$$\lambda \geq 0 \text{ if } b > 0, \text{ and}$$

$$\lambda \leq 0, \text{ if } b < 0. \quad (\text{A.14})$$

Given the initial definition of Θ , we know that

$$1/2\sigma^2\lambda^2 + \mu\lambda - \Theta = 0. \quad (\text{A.15})$$

Applying the quadratic formula, we conclude that

$$\lambda = \begin{cases} [-\mu + (\mu^2 + 2\sigma^2\Theta)^{.5}]/\sigma^2, & b > 0 \\ [-\mu - (\mu^2 + 2\sigma^2\Theta)^{.5}]/\sigma^2, & b < 0. \end{cases} \quad (\text{A.16})$$

Substituting (A.16) into the right-hand side of (A.12), we find that the Laplace transform is:

$$E^{(\mu)}[e^{-\Theta T_b}] = \begin{cases} \exp\{[b\mu - b(\mu^2 + 2\sigma^2\Theta)^{.5}]/\sigma^2\}, & b, \Theta > 0 \\ \exp\{[b\mu + b(\mu^2 + 2\sigma^2\Theta)^{.5}]/\sigma^2\}, & b < 0 \text{ and } \Theta > 0. \end{cases} \quad (\text{A.17})$$

Combining values, we obtain

$$E^{(\mu)}[e^{-\Theta T_b}] = \exp\{[b\mu - |b|(\mu^2 + 2\sigma^2\Theta)^{.5}]/\sigma^2\}, \quad \Theta > 0. \quad \blacksquare \quad (\text{A.18})$$

PROOF: Theorem 2.2. Following Karlin and Taylor^{14/} (p. 362), we know that

$$P^{(\mu)}[T_b < \infty] = \lim_{\Theta \rightarrow 0} E^\mu[\exp(-\Theta T_b)]. \quad (\text{A.19})$$

Using (A.18), we obtain:

$$P^{(\mu)}[T_b < \infty] = \lim_{\Theta \rightarrow 0} \exp\{[b\mu - |b|(\mu^2 + 2\sigma^2\Theta)^{.5}]/\sigma^2\}, \quad \Theta > 0. \quad (\text{A.20})$$

After evaluating the limit, we conclude that:

$$P^{(\mu)}[T_b < \infty] = \exp\{(b\mu - |b\mu|)/\sigma^2\}. \quad (\text{A.21})$$

Hence,

$$P^{(\mu)}[T_b < \infty] = \begin{cases} 1, & \mu b > 0 \\ < 1, & \mu b < 0. \end{cases} \quad (\text{A.22})$$

Thus, whenever μ and b have different signs, the density of T_b under $P^{(\mu)}$ is defective since T_b is infinite with positive probability. That is,

$$P^{(\mu)}[T_b = \infty] = 1 - \exp\{(b\mu - |b\mu|)/\sigma^2\} > 0. \quad (\text{A.23})$$

In conclusion, T_b is not finite-valued if we suppose that b and μ have different signs. ■

PROOF: Theorem 2.3. Differentiating (A.18) with respect to the dummy variable Θ and multiplying each side by -1 , we obtain

$$-E^{(\mu)}_{\Theta}[e^{-\Theta T_b}] = \{b\sigma^2 / [\sigma^2(\mu^2 + 2\sigma^2\Theta)^{.5}]\} \exp\{\mu b/\sigma^2 - |b|[(\mu^2 + 2\sigma^2\Theta)^{.5}]/\sigma^2\}. \quad (\text{A.24})$$

Finally, evaluating the derivative as $\Theta \downarrow 0$, we obtain our main result:

$$E^{(\mu)}[T_b] = b/\mu. \quad \blacksquare \quad (\text{A.25})$$

A.2 Proofs of Section 3

PROOF: Lemma 3.1. Suppose that $S(0) \geq S^*$. Then:

$$\beta = \log(S^*/S(0))/\sigma \leq 0. \quad (\text{A.26})$$

Hence: $T_g \equiv \tau^* = \inf\{t \geq 0; W_o(t) + vt \geq \beta\} = 0. \quad \blacksquare \quad (\text{A.27})$

PROOF: Lemma 3.2: Following the argument of the proof of Theorem 2.1, one can easily demonstrate that the Laplace transform of T_g is:

$$E^{(v)}[\exp(-\alpha T_g)] = \exp\{[\beta v - |\beta|(v^2 + 2\sigma^2\alpha)^{.5}]/\sigma^2\}, \quad \alpha > 0. \quad \blacksquare \quad (\text{A.28})$$

PROOF: Lemma 3.3. Following the argument of Theorem 2.2, one can easily demonstrate that

$$P^{(v)}[T_g < \infty] = \exp\{(\beta v - |\beta v|)/\sigma^2\}. \quad (A.29)$$

Our first conclusion is straightforward: if $v \leq 0$, then

$$P(T_g = \infty) = 1 - P(T_g < \infty) > 0. \quad (A.30)$$

Our second, major conclusion is:

$$P^{(v)}[T_g < \infty] = \begin{cases} 1, & v\beta > 0 \text{ and } v > 0 \\ < 1, & v\beta < 0 \text{ and } v > 0. \end{cases} \quad (A.31)$$

Thus, whenever v and β have different signs, the density of T_g under $P^{(v)}$ is defective since T_g is infinite with positive probability. That is,

$$P^{(v)}[T_g = \infty] = 1 - \exp\{(\beta v - |\beta v|)/\sigma^2\} > 0. \quad (A.32)$$

In conclusion, T_g is not finite-valued if we suppose either that β and v have different signs or $v \leq 0$. ■

Note, for a slightly different starting point in the proof, the reader is urged to consult Karatzas and Shreve^{13/} (pp. 196-197).

PROOF: Theorem 3.1: Differentiating the Laplace transform with respect to α , and multiplying each side by -1 , we obtain

$$E[T_g] = \beta(v^2 + 2\sigma^2\alpha)^{-1/2} E[\exp(-\alpha T_g)]. \quad (A.33)$$

Letting $\alpha \downarrow 0$, we obtain $E[T_g] = \beta/v$, where $\beta, v > 0$. ■

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