POINTWISE POTENTIAL ESTIMATES FOR
ELLIPTIC OBSTACLE PROBLEMS

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IMA Preprint Series # 117
December 1984

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To appear in Proc. Berkeley Symposium on Nonlinear Functional Analysis, 1983.

1. Introduction

In its simplest local form an obstacle problem is the minimization of an energy integral of Dirichlet type

$$\mathcal{E}(u) = \int_{\Omega} |Du|^2 dx$$

on some open domain Ω of $\mathbb{R}^n$, under a unilateral constraint

$$u \geq \psi \quad \text{in } \Omega$$

on the solution u . Here ψ is some given function on $\mathbb{R}^n$, the obstacle of our problem.

In order to include the case of so-called thin obstacles, when the constraint is only prescribed on some lower dimensional subset E of Ω , the value $-\infty$ is also allowed to ψ and, moreover, the inequality $u \geq \psi$ is required to hold pointwise in Ω except possibly a set of harmonic capacity zero in $\mathbb{R}^n$.

The Euler inequality satisfied by any locally minimizing function u can be formally written as

$$(*) \quad \min\{u - \psi, Lu\} = 0 \quad \text{in } \Omega,$$

where $L = -\Delta$ is the Laplace operator in $\mathbb{R}^n$.

More generally, we take L to be a uniformly elliptic second order linear operator in divergence form, with bounded measurable coefficients in $\mathbb{R}^n$.

Our goal in this paper is to describe the behavior of an arbitrary local weak solution u of $(*)$ at a given point x_0 of the domain Ω , in terms of the behavior of ψ at that point. More precisely, we want to find out under what general condition on ψ will u be continuous at x_0 and we also want to estimate the modulus of continuity of u .

Several results in this direction are available, both for regular (i.e., continuous) and irregular ψ .

Global results, in which ψ is assumed to be continuous on $\bar{\Omega}$ and u satisfies some boundary condition on $\partial\Omega$ were obtained in [5] by p.d.e. methods and in [16] by an approach from potential theory. In [16] the case of a thin obstacle ψ , continuous only on a given compact subset E of Ω , was also considered.

Local results for ψ continuous on some ball $\bar{B}_R = \{|x| \leq R\}$ were obtained in [6] by potential methods and in [1] and [10] by p.d.e. techniques. Moreover, the modulus of continuity of u , in particular its Hölder exponent, was also estimated.

For a u.s.c. ψ in Ω , a continuity result was given in [7] by potential methods, which, however, do not provide any estimate.

A classical example of a discontinuous obstacle problem occurs in potential theory. In fact, the equilibrium potential u^E of a subset E in a bounded $\Omega \supset \bar{E}$ is the weak solution $u = u^E$ of (*), vanishing on $\partial\Omega$, for the special obstacle: $\psi \equiv 1$ on E , $\psi \equiv -\infty$ in $\Omega - E$. If E is compact, the continuity of u^E at $x \in \partial E$ is submitted to the Wiener's criterion for the boundary regularity of solutions of Dirichlet problems. For a general L as above, this criterion was established in [17] and the modulus of continuity was estimated in [18], in terms of the rate of divergence of the Wiener integral (see also [19] and [15] for the case of quasilinear operators).

More general irregular obstacle problems were considered in [11] and [12], motivated by the probabilistic interpretation of certain quasivariational inequalities arising in stochastic optimal control.

In [12] - [14] the modulus of continuity of u was estimated in terms of the rate of growth, in capacity, of suitable one-sided level sets of ψ in balls

$B_\rho = \{|x| < \rho\}$ as $\rho \rightarrow 0^+$, namely

$$L_\rho(\omega) = \left\{ x \in B_\rho \mid \psi(x) \geq \sup_{B_\rho} \psi - \omega \right\}, \quad \rho > 0, \omega > 0,$$

where the supremum is taken in the capacity essential sense. The estimate obtained,

however, is not as accurate as the local estimates of [6] and [10] (for instance, they bring a loss in the Hölder exponent) or as the Wiener estimate of [18].

In this paper we shall present an estimate that can be seen as an analogue, for the obstacle problem, of the so-called Saint Venant principle of the theory of linear elasticity, see for instance [23] and [22].

The variation of the "potential" in a ball B_r , as given by the quantity

$$V(r) = \operatorname{osc}_{B_r} u + \left[\int_{B_r} |Du|^2 |x|^{2-n} dx \right]^{1/2}$$

where $n \geq 3$ will be estimated for $0 < r \leq R$ in terms of a function

$$\omega_\sigma(r, R), \quad \sigma > 0,$$

which we call the Wiener modulus of ψ at $x_0 = 0$, see [21].

Roughly speaking, for each fixed σ , to be interpreted as a scaling factor, $\omega_\sigma \equiv \omega_\sigma(r, R)$ is implicitly given for every $0 < r \leq R$ by

$$\omega_\sigma \sim \exp \left[- \int_r^R \rho^{1-n} \operatorname{cap}_\rho L_\rho(\sigma \omega_\sigma) d\rho \right].$$

We shall prove that there exist suitable constants c , k and $\beta > 0$, such that

$$V(r) \leq cV(R)\omega_\sigma(r, R)^\beta + k\sigma\omega_\sigma(r, R),$$

$0 < r \leq R/10$, $B_R \subset \Omega$, $\sigma > 0$. The constant c depends only on n , the constants k and β only on n and on the ellipticity constants of L .

The above result applies both to regular and irregular obstacles and improves the estimates of [6], [10], [18], [13] and [14].

The basic tool of the proof, which will be outlined in Section 6, is an inequality of Caccioppoli-De Giorgi type, as in [12] - [14], together with integration

techniques from [18]. Differently from [14], both terms appearing in the expression of $V(r)$ are now estimated simultaneously.

The results presented here have been announced in [21]. A more detailed proof will appear elsewhere.

Similar estimates can be given for degenerate operators L of the type considered in [9], see [2, 3].

An extension to parabolic obstacle problems has been given in [4].

The connections with the Γ -convergence for quadratic functionals and related Wiener's estimates for operators of the form $Lu + q(x)u$, with highly "irregular" $q(x)$, have been studied in [8].

2. The Wiener Modulus

We recall that the external harmonic capacity of an arbitrary subset $E \subset \mathbb{R}^n$, $n \geq 3$, relative to an open subset B of $\mathbb{R}^n$ containing $\bar{E}$, here denoted

$$\text{cap}(E, B)$$

is obtained by the standard extension procedure from the following definition of $\text{cap}(K, B)$ for K compact $\subset B$:

$$\text{cap}(K, B) = \inf \left\{ \int_B |Dv|^2 dx \mid v \in C_0^1(B), v \geq 1 \text{ on } K \right\}$$

where $Dv = (D_1 v, \dots, D_n v)$, $D_i v = \partial v / \partial x_i$. If a property holds everywhere in $\mathbb{R}^n$ except possibly a set of capacity zero, then we say that it holds quasi-everywhere (q.e.) in $\mathbb{R}^n$.

Let us now suppose that an arbitrary function

$$(1) \quad \psi : \mathbb{R}^n \mapsto [-\infty, +\infty[$$

is given, possibly defined only q.e. in $\mathbb{R}^n$.⁽⁺⁾

We fix a point in a neighborhood of which ψ is q.e. bounded from above:

$$(2) \quad x_0 \in \mathbb{R}^n, \quad \text{q.e. } \sup_{B_{2R_0}} \psi < +\infty \quad (R_0 > 0).$$

We use the notation $B_r(z) = \{x \in \mathbb{R}^n \mid |x - z| < r\}$ for $z \in \mathbb{R}^n$ and $r > 0$ and since x_0 is fixed we simply write B_r in place of $B_r(x_0)$.

We shall now introduce a function

$$\omega_\sigma(r, R), \quad 0 < r \leq R, \quad \sigma > 0,$$

which we call the Wiener modulus of ψ at x_0 .

For arbitrary $\omega > 0$ and $\rho > 0$ we consider, as in [11] and [12], the following one-sided level sets of ψ :

$$L_\rho(\omega) = \left\{ x \in B_\rho \mid \psi(x) \geq \text{q.e. } \sup_{B_\rho} \psi - \omega \right\} \quad (\text{q.e.})$$

and their relative capacities

$$\delta(\omega, \rho) = \frac{\text{cap}(L_\rho(\omega), B_{2\rho})}{\text{cap}(B_\rho, B_{2\rho})} = c_n \rho^{2-n} \text{cap}(L_\rho(\omega), B_{2\rho})$$

where $c_n = \text{cap}(B_1, B_2)^{-1}$ is a constant depending only on n .

We then define $\omega_\sigma(r, R)$ for arbitrary $0 < r \leq R$ and $\sigma > 0$, by setting

$$\omega_\sigma(r, R) = \inf \left\{ \omega > 0 \mid \omega \cdot \exp \int_r^R \delta(\sigma\omega, \rho) \frac{d\rho}{\rho} \geq 1 \right\}.$$

Since $0 \leq \delta(\omega, \rho) \leq 1$ ⁽⁺⁺⁾ for every $\omega > 0$ and $\rho > 0$, and since $\delta(\omega, \rho)$ is not decreasing in ω , the definition above makes sense and we have

(+) We are assuming that ψ cannot be $= +\infty$ on a set of positive capacity.

(++) If $\text{q.e. } \sup_{B_\rho} \psi = +\infty$, we take $\text{cap } L_\rho(\omega)$ to be zero, hence $\delta(\omega, \rho) = 0$, for every $\omega > 0$.

$$r/R \leq \omega_\sigma(r, R) \leq 1$$

for all $0 < r \leq R$ and $\sigma > 0$.

Remark 1. The parameter σ , for fixed $0 < r < R$, has the role of a scaling factor in $\mathbb{R}^n$ for the "height" of the level sets of ψ in the balls B_ρ , $r \leq \rho \leq R$. It may possibly be chosen conveniently depending on r and R . $\square$

Remark 2. We may also introduce a scaling factor $m \geq 1$ for the radius of the balls B_ρ where the level sets $L(\omega)$ are taken, by replacing the above definition of $L_\rho(\omega)$ with

$$L_\rho(\omega) = \left\{ x \in B_\rho \mid \psi(x) \geq \text{q.e.} \sup_{B_{\rho/m}} \psi - \omega \right\} .$$

The parameter m will then be fixed once for all, together with ψ and x_0 . $\square$

For the sake of simplicity in the above notation we have omitted explicitly indicating the dependence of $L_\rho(\omega)$, $\delta(\omega, \rho)$ and $\omega_\sigma(r, R)$ on ψ , x_0 and possibly m .

3. The Local Weak Solutions

We now suppose that the functions a_{ij} , $i, j = 1, \dots, n$, are given in $\mathbb{R}^n$, which are measurable and satisfy the following uniform ellipticity condition

$$(3) \quad \lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \lambda^{-1} |\xi|^2, \quad \lambda > 0,$$

for a.e. $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$.

We then consider in $\mathbb{R}^n$ a set

$$(4) \quad \Omega \text{ open } \ni x_0$$

and in Ω an arbitrary solution u of the problem

$$u \in H^1(\Omega), \quad u \geq \psi \text{ q.e. in } \Omega$$

$$(5) \quad \sum_{i,j=1}^n \int_{\Omega} a_{ij}(x) D_j u D_i (v-u) dx \geq 0 \quad \forall v \in H^1(\Omega), \quad v-u \in H_0^1(\Omega), \quad v \geq \psi \text{ q.e. in } \Omega$$

By $H^1(\Omega)$ we denote the Sobolev space of all functions $v \in L^2(\Omega)$ with distribution derivatives $D_i v \in L^2(\Omega)$, $i = 1, \dots, n$. The functions of $H^1(\Omega)$ are considered as being precise in $\mathbb{R}^n$ up to sets of capacity zero, see, for instance, [20]. $H_0^1(\Omega)$ is the closure of $C_0^1(\Omega)$ in $H^1(\Omega)$.

4. The Potential Estimate

As we said in the Introduction, our main goal is to estimate the quantity

$$V(r) = \operatorname{osc}_{B_r} u + \left[\int_{B_r} |Du|^2 |x - x_0|^{2-n} dx \right]^{1/2},$$

where $\operatorname{osc}_{B_r} u = \sup_{B_r} u - \inf_{B_r} u$, both the supremum and infimum being taken in the essential sense.

We shall use the notation

$$Z(R) = \inf \left\{ \left[R^{-n} \int_{B_R} (u-d)^2 dx \right]^{1/2} \mid d \geq \text{q.e. sup}_{B_R} \psi \right\}$$

for $R > 0$.

Under the assumptions (1) to (5) above, we obtain the following results.

THEOREM 1. There exist constants c , k and $\beta > 0$ such that

$$(6) \quad V(r) \leq cV(R)\omega_{\sigma}(r,R)^{\beta} + k\omega_{\sigma}(r,R)$$

for every $0 < r \leq R/10$, with $B_R \subset \Omega$, and every $\sigma > 0$. The constant c depends
only on n , the constants k and β only on n and λ . $\square$

Theorem 1 is completed by

THEOREM 2. There exists a constant k such that

$$(7) \quad V(R) \leq kZ(2R)$$

for every $R > 0$, $B_{2R} \subset \Omega$. In particular

$$(8) \quad V(R) \leq k \left[R_0^{-n/2} \|u\|_{L^2(B_{2R_0})} + \max_{B_{2R_0}}(\text{q.e. sup } \psi, 0) \right]$$

for every $0 < R \leq R_0$, $B_{2R} \subset \Omega$. The constant k depends only on n and λ .

Remark 3. If a scaling factor $m \geq 1$ is introduced in the definition of $L_\rho(\omega)$ and hence in $\omega_\sigma(r, R)$, see Remark 2, then the same results hold, the constants c , k and β now possibly depending also on m . Let us also point out that all constants in the theorems above do not depend on ψ , x_0 , Ω , u and they depend on the coefficients $a_{ij}(x)$ of the operator L only via the ellipticity constant λ . $\square$

5. Some Special Cases

We apply the results of Section 4 to some important examples.

a) Equations. Let $\psi \equiv -\infty$ q.e. on B_R for some $R > 0$. Then u is a weak solution of the equation $Lu = 0$ in $B_R \cap \Omega$.

Since $L_\rho(\omega) = B_\rho$ (q.e.) for every $\omega > 0$ and $0 < \rho \leq R$, we now have

$$\omega_\sigma(r, R) \equiv r/R$$

for arbitrary $\sigma > 0$. Therefore we obtain

$$(9) \quad V(r) \leq cV(R)(r/R)^\beta$$

In particular, u is Hölder continuous in $\Omega \cap B_R$. Moreover, the energy integral

$$\mathcal{E}(r) = \int_{B_r} |Du|^2 dx$$

satisfies the estimate

$$(10) \quad \mathcal{E}(r) \leq k\mathcal{E}(2R)(r/R)^{n-2+\beta}.$$

Thus, our results reduce in this case to the classical results of De Giorgi, Nash and Moser. We must point out, however, that these results are indirectly used in our proof.

b) Equilibrium potentials. Let E be an arbitrary subset of $\mathbb{R}^n$ and let $\psi \equiv 1$ on E , $\psi \equiv -\infty$ on $\mathbb{R}^n - E$.

We distinguish two cases, according to the position of the given x_0 with respect to E .

If $\text{cap}(B_R \cap E) = 0$ for some $R > 0$, then $\psi \equiv -\infty$ q.e. on B_R and we are back to example a).

Let us then suppose that

$$\text{cap } B_\rho \cap E > 0 \quad \text{for all } \rho > 0.$$

Then $L_\rho(\omega) = B_\rho \cap E$ (q.e.) for every $\omega > 0$ and $\rho > 0$, therefore

$$\omega_\sigma(r, R) = c_n \exp \left[- \int_r^R \rho^{1-n} \text{cap}(B_\rho \cap E, B_{2\rho}) d\rho \right]$$

for every $0 < r \leq R$ and arbitrary $\sigma > 0$.

Thus, we have the estimate

$$(11) \quad V(r) \leq cV(R) \exp \left[-\beta \int_r^R \rho^{1-n} \text{cap}(B_\rho \cap E) d\rho \right]$$

In particular,

$$(12) \quad \text{osc}_{B_r} u \leq C \exp \left[-\beta \int_r^R \rho^{1-n} \text{cap}(B_\rho \cap E) d\rho \right]$$

where $c = c(n, \lambda, R, \|u\|_{L^2(B_{2R})})$. Moreover, it is easily seen that

$$(13) \quad \mathcal{E}(r) \leq c \mathcal{E}(2R) \frac{r^{n-2}}{\text{cap}(B_{2R} \cap E)} \exp \left[-\beta \int_r^R \rho^{1-n} \text{cap}(B_\rho \cap E) d\rho \right]$$

For compact E and $x_0 \in \partial E$, the estimate (12) was first obtained by Maz'ja [18], as a boundary estimate for solutions of the Dirichlet problem. However, the case of a noncompact E and x_0 arbitrarily given in the domain cannot be reduced to a boundary problem.

c) Regular obstacles. The Wiener modulus $\omega_\sigma(r, R)$ can be estimated by the q.e. oscillation of ψ on B_R as follows:

$$(14) \quad r/R \leq \omega_\sigma(r, R) \leq \min \left\{ 1, \max_{B_R} [r/R, \sigma^{-1} \text{q.e. osc } \psi] \right\},$$

for every $0 < r \leq R$ and every $\sigma > 0$.

In fact, since we have already seen in Section 2 that $r/R \leq \omega_\sigma(r, R) \leq 1$, it suffices now to prove that

$$\omega_\sigma(r, R) \leq \max [r/R, \sigma^{-1} \omega]$$

for arbitrary $\omega > \text{q.e. osc } \psi$. For such ω and for every $0 < \rho \leq R$ we clearly have

$L_\rho(\omega) = B_\rho$ (q.e.), hence $\delta(\omega, \rho) = 1$. Then the desired inequality follows from the definition of $\omega_\sigma(r, R)$.

As a consequence of (14), either we have $\text{q.e. osc } \psi = 0$, and then

$\omega_\sigma(r, R) \equiv r/R$ for arbitrary $\sigma > 0$, or we have $\text{q.e. osc } \psi > 0$ and then in this case,

by choosing

$$\sigma = \frac{R}{r} \text{ q.e. osc } \psi_{B_R}$$

again we find

$$\omega_\sigma(r, R) = r/R .$$

Moreover,

$$\sigma \omega_\sigma(r, R) = \text{q.e. osc } \psi_{B_R} .$$

Therefore we get

$$(15) \quad V(r) \leq cV(R)(r/R)^\beta + k \text{ q.e. osc } \psi_{B_R} ;$$

in particular,

$$(16) \quad \text{osc } u_{B_r} \leq C(r/R)^\beta + k \text{ q.e. osc } \psi_{B_R} .$$

From the last inequality, by standard iteration arguments, it follows in particular that if ψ is Hölder continuous at x_0 with exponent $\alpha > 0$, then u is also Hölder continuous at x_0 , with exponent $\gamma = \min(\beta - \epsilon, \alpha)$ for all $\epsilon > 0$.

Estimate (16) improves previous estimates in [13] and [14]. It also generalizes the local results of [6], [1] and [10] to only pointwise continuous ψ .

d) Regular thin obstacles. Estimates (9), (11) and (15) above are all special cases of a general estimate of the same type which is also an immediate consequence of Theorem 1.

We shall use below the following notation

$$F = F_\psi = \{x \in \mathbb{R}^n \mid \psi(x) > -\infty\} \quad (\text{q.e.}) ,$$

$$W_F(r, R) = \exp \left[- \int_r^R \frac{\text{cap}(B_\rho^F, B_{2\rho})}{\text{cap}(B_\rho, B_{2\rho})} \frac{d\rho}{\rho} \right] , \quad 0 < r \leq R ,$$

where

$$B_\rho^F = B_\rho \cap F \text{ if } \text{cap}(B_\rho \cap F) > 0 ,$$

$$B_\rho^F = B_\rho \text{ if } \text{cap}(B_\rho \cap F) = 0 .$$

We then deduce from Theorem 1 the following

COROLLARY OF THEOREM 1. Under the same assumptions and with the same constants of the theorem, we have

$$(17) \quad V(r) \leq cV(R)W_F(r,R)^\beta + k \underset{B_R \cap F}{\text{q.e. osc}} \psi \quad (+)$$

for every $0 < r \leq R/10$, $B_R \subset \Omega$. $\blacksquare$

It suffices to apply the following estimate of $\omega_\sigma(r,R)$.

$$(18) \quad r/R \leq \omega_\sigma(r,R) \leq \min \left\{ 1, \max \left[W_F(r,R), \sigma^{-1} \underset{B_R \cap F}{\text{q.e. osc}} \psi \right] \right\} ,$$

$0 < r \leq R$, $\sigma > 0$, which is proved by refining the argument above leading to (14).

In fact, if $\underset{B_R \cap F}{\text{q.e. osc}} \psi = 0$, then $\omega_\sigma(r,R) \leq W_F(r,R)$ for arbitrary $\sigma > 0$.

Otherwise, we choose in (6)

$$\sigma = W_F(r,R)^{-1} \underset{B_R \cap F}{\text{q.e. osc}} \psi$$

and we observe that for such σ (18) implies

$$\omega_\sigma(r,R) \leq W_F(r,R)$$

and $\sigma \omega_\sigma(r,R) \leq \underset{B_R \cap F}{\text{q.e. osc}} \psi$.

(+) If $\text{cap} B_R \cap F = 0$, we put $\underset{B_R \cap F}{\text{q.e. osc}} \psi = 0$.

It follows in particular from the Corollary that u is continuous at x_0 , provided F is "regular" at x_0 , i.e., $W_F(r, R) \rightarrow 0$ as $r \rightarrow 0^+$, and ψ is q.e. continuous at x_0 on F , i.e., $\text{q.e. osc } \psi \rightarrow 0$ as $R \rightarrow 0^+$. This is the pointwise version of the result given in [16] for $B_R \cap F$ compact and ψ continuous on F .

e) Irregular obstacles. If ψ presents a more irregular behavior at x_0 it may be difficult to separate the contribution to the estimate arising from the "geometry" of the set $F = F_\psi$ and that due to the analytic behavior of ψ on F , as at the right side of (17) in the Corollary above. Then, we have to come back to the general estimate of Theorem 1 and estimate the Wiener modulus of ψ at the given point.

We refer to [11] and [13] for some examples of irregular obstacles to which the above results apply.

We only mention here a simple sufficient condition for the Hölder continuity of u at x_0 , see [11].

If ψ satisfies at x_0 the condition

$$(19) \quad \text{cap}(L_\rho(\omega), B_{2\rho}) \geq \gamma \text{cap}(B_\rho, B_{2\rho}), \quad \gamma > 0,$$

for all small $\rho > 0$, γ independent of ρ , then u is Hölder continuous at x_0 and we have

$$(20) \quad V(r) \leq cV(R)(r/R)^{\gamma\beta}$$

for r and R small enough.

In fact, if (19) holds then it is easily seen that $\omega_\sigma(r, R) \leq (r/R)^\gamma$ for arbitrary $\sigma > 0$.

As observed in [11], a simple case in which (19) is satisfied, with a suitable scaling factor m in the definition of $L_\rho(\omega)$, is that of a monotone ψ in $\mathbb{R}^n$, that is, ψ such that $\psi(x) \leq \psi(y)$ if $x \leq y$ for the partial ordering in $\mathbb{R}^n$. Since in this case both m and γ turn out to depend only on n , we have the rather surprising result that the estimate (20) holds uniformly for all monotone ψ in $\mathbb{R}^n$.

6. Outline of the Proof

The proof of Theorem 1 and Theorem 2 is based on the following Caccioppoli-DeGiorgi inequality on two concentric balls $B_{q\rho} \subset B_\rho \subset \Omega$, $q \in]0, 1[$:

$$(21) \quad V(q\rho)^2 \equiv (\operatorname{osc}_{B_{q\rho}} u)^2 + \int_{B_{q\rho}} |Du|^2 |x|^{2-n} dx \leq c\rho^{-n} \int_{B_\rho - B_{q\rho}} |u-d|^2 dx$$

for every constant

$$(22) \quad d \geq q.e. \sup_{B_\rho} \psi .$$

Inequality (21) can be proved by choosing a suitable test function in (5) and applying standard estimation techniques, see [11] and [13]. The test function used here contains the Green's function of L , whose size must be estimated according to [17]. An alternative proof uses instead suitable equilibrium potentials of balls, whose size can be estimated by the Harnack inequality according to [19].

In particular we prove that u is bounded in B_{R_0} . We then consider the level sets

$$L_{q,\rho}(\omega) = L_\rho(\omega) \cap (B_\rho - B_{q\rho}) , \quad \omega > 0 ,$$

and their relative capacities

$$\delta_q(\omega, \rho) = \frac{\operatorname{cap}(L_{q,\rho}(\omega), B_{2\rho})}{\operatorname{cap}(B_\rho, B_{2\rho})} .$$

If

$$\delta \geq q.e. \inf_{L_{q,\rho}(\omega)} u$$

is an arbitrary constant, then $d = \delta + \omega$ verifies (22), hence it is admissible in (21).

By Poincaré's inequality, (21) implies

$$V(q\rho)^2 \leq \frac{c}{\delta_q(\omega, \rho)} \int_{B_\rho - B_{q\rho}} |Du|^2 |x|^{2-n} dx + c\omega^2 .$$

By adding the term

$$(\operatorname{osc}_{B_\rho} u)^2 + \int_{B_{q\rho}} |Du|^2 |x|^{2-n} dx$$

to both sides of the inequality and integrating conveniently over the interval $0 \leq \rho \leq R$ in $d\rho/\rho$, we find

$$V(qr)^2 \leq cV(R)^2 \exp \left[-\beta |\log q|^{-1} \int_{q^{-1}r}^R \delta_q(\omega, \rho) \frac{d\rho}{\rho} \right] + k\omega^2$$

The integral above can be estimated from below by replacing the capacities $\delta_q(\omega, \rho)$ with the capacities $\delta(\omega, \rho)$ and this leads to

$$V(qr)^2 \leq cV(R)^2 \exp \left[-\beta \int_r^R \delta(\omega, \rho) \frac{d\rho}{\rho} \right] + k\omega^2.$$

Since $\omega > 0$ is arbitrary, this last inequality implies (6) by our definition of $\omega_\sigma(r, R)$.

Acknowledgment

The author wishes to thank D. Adams, M. Biroli, R. Gariepy, L. Hedberg and L. Neřas for the stimulating discussions at the Berkeley AMS Institute in August 1983, which greatly influenced the final draft of this paper.

The author also wishes to thank the Institute for Mathematics and its Applications for its hospitality at the University of Minnesota, where this paper was written.

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Minneapolis, October 1984

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