**L^p -POSTERIORI ERROR ANALYSIS OF MIXED METHODS
FOR LINEAR AND QUASILINEAR ELLIPTIC PROBLEMS**

By

Zhangxin Chen

IMA Preprint Series # 1158

July 1993

L^p-POSTERIORI ERROR ANALYSIS OF MIXED METHODS FOR LINEAR AND QUASILINEAR ELLIPTIC PROBLEMS

ZHANGXIN CHEN*

Abstract. We consider mixed finite element methods for the approximation of linear and quasilinear second-order elliptic problems. A class of postprocessing methods for improving mixed finite element solutions is analyzed. In particular, error estimates in L^p , $1 \leq p \leq \infty$, are given. These postprocessing methods are applicable to all the existing mixed methods, and can be easily implemented.

Key words. finite element, mixed method, postprocessing, linear and quasilinear problems, error estimate.

AMS(MOS) subject classifications. 65N30, 65N22, 41A10, 41A25.

1. Introduction.

Let Ω be a smooth, bounded domain in $\mathbb{R}^2$ and consider the Dirichlet problem

$$(1.1a) \quad L(u) = -\operatorname{div}(a(x)\nabla u) = f \quad \text{in } \Omega,$$

$$(1.1b) \quad u = 0 \quad \text{on } \partial\Omega,$$

where $a(x)$ is assumed to be Lipschitz and bounded below by a positive constant. In many applications the variable $\sigma = -a\nabla u$ is of primary interest, and then mixed finite element methods are designed to approximate it well. With this purpose the problem (1.1) is recast in the mixed form as follows. Let $(\cdot, \cdot)$ and $\|\cdot\|$ denote the standard inner product and norm in $L^2(\Omega)$, and let

$$V = H(\operatorname{div}; \Omega) = \{\tau : \tau \in (L^2(\Omega))^2, \operatorname{div} \tau \in L^2(\Omega)\},$$

$$W = L^2(\Omega),$$

with V equipped with the usual graph norm

$$\|\tau\|_{H(\operatorname{div}; \Omega)} = \left(\sum_{i=1}^2 \|\tau_i\|^2 + \|\operatorname{div} \tau\|^2 \right)^{1/2},$$

where $\tau = (\tau_1, \tau_2)$. Then the mixed form of (1.1) for the pair $(\sigma, u) \in V \times W$ is

$$(1.2a) \quad (a^{-1}\sigma, \tau) - (\operatorname{div} \tau, u) = 0, \quad \forall \tau \in V,$$

$$(1.2b) \quad (\operatorname{div} \sigma, v) = (f, v), \quad \forall v \in W.$$

*Department of Mathematics and Institute for Scientific Computation, Texas A&M University, College Station, TX 77843. Supported in part by the Army Research Office contract number DAAL03-89-C-0038 with the University of Minnesota Army High Performance Computing Research Center.

For the two-dimensional case mixed finite elements for approximating the solution of (1.1) have been introduced in [5, 17]. Let T_h be a quasi-regular decomposition of Ω into triangles or rectangles of diameters bounded by h , $0 < h < 1$, with boundary elements allowed to have a curved side, and let $V_h \times W_h$ denote one of the spaces over T_h defined by Raviart-Thomas (RT) [17] and Brezzi-Douglas-Marini (BDM) [5] of index such that the approximation properties below hold:

$$(1.3a) \quad \inf\{\|\tau - \xi\| : \xi \in V_h\} \leq C \left(\sum_T h_T^{2r} \|\tau\|_{r,T}^2 \right)^{1/2}, \quad 0 \leq r \leq k+1,$$

$$(1.3b) \quad \inf\{\|v - \xi\| : \xi \in W_h\} \leq C \left(\sum_T h_T^{2r} \|v\|_{r,T}^2 \right)^{1/2}, \quad 0 \leq r \leq k^*,$$

for all $(\tau, v) \in V \times W$, where C is a generic constant, $k^* = k+1$ for the RT space with $k \geq 0$ [17], and $k^* = k$ for the BDM space with $k \geq 1$ [5]. Then the mixed finite solution of (1.1) is $(\sigma_h, u_h) \in V_h \times W_h$ satisfying

$$(1.4a) \quad (a^{-1}\sigma_h, \tau) - (\operatorname{div} \tau, u_h) = 0, \quad \forall \tau \in V_h,$$

$$(1.4b) \quad (\operatorname{div} \sigma_h, v) = (f, v), \quad \forall v \in W_h.$$

Much work has been devoted to obtaining error estimates for the solution (σ_h, u_h) of (1.4); see [8, 11, 12] and the bibliography therein. It is well known [2, 5, 6, 18] that appropriate postprocessing techniques can be used to obtain approximate solutions to u which are asymptotically more accurate than the approximation u_h given by (1.4). The objective of this paper is to analyze a class of postprocessing methods for improving mixed finite element solutions of linear and quasi-linear second order elliptic problems introduced in [6, 18]. These postprocessing methods are applicable to all the mixed finite element methods of Raviart-Thomas [17], Nédélec [16], Brezzi-Douglas-Marini [5], Brezzi-Douglas-Durán-Fortin [3], Brezzi-Douglas-Fortin-Marini [4], and Chen-Douglas [7], and can be easily implemented. In this paper special emphasis is placed on obtaining error estimates in L^p , $1 \leq p \leq \infty$, for these postprocessing methods.

In the next section the linear problem (1.1) will be considered. Then, in §3, a quasi-linear problem will be analyzed, and corresponding results will be stated. Finally, in §4, some concluding remarks on the extension of the present techniques to more general problems will be made.

Throughout this paper we shall use the Sobolev spaces $W^{m,p}(\Omega)$ with the usual norm $\|\cdot\|_{m,p}$, where m is a nonnegative integer and $1 \leq p \leq \infty$. When $p = 2$, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and write $\|\cdot\|_{m,2} = \|\cdot\|_m$.

2. The linear problem.

2.1. Known results for mixed methods. In this subsection we review some known L^p -error estimates for the approximation pair (σ_h, u_h) . They can be found in [12, 13], for example.

Let P_h denote the L^2 -projection into W_h , and let $2 \leq p < \infty$ and $k \geq 0$. Then,

$$(2.1a) \quad \|\sigma - \sigma_h\|_{0,p} \leq C_p \|\sigma\|_{k+1,p} h^{k+1},$$

$$(2.1b) \quad \|u - u_h\|_{0,p} \leq \begin{cases} C_p^2 \|u\|_{k^*,p} h^{k^*}, & \text{for } k \geq 1, \\ C_p^2 \|u\|_{2,p} h, & \text{for the RT space: } k = 0, \end{cases}$$

$$(2.1c) \quad \|P_h u - u_h\| \leq \begin{cases} C \|u\|_{k+2} h^{k+2}, & \text{for } k \geq 2, \\ C \|u\|_3 h^2, & \text{for the BDM space: } k = 1, \\ C \|u\|_3 h^{k+2}, & \text{for the RT space: } k = 0, 1, \end{cases}$$

where $C_p = C_p$. It should be remarked that the BDM spaces are of smaller dimensions than the RT spaces of the same index. While the former provide asymptotic error estimates for the flux variable σ_h of the same order as the latter, u_h obtained by the latter is asymptotically more accurate than that produced by the former. However, if appropriate postprocessing techniques are introduced for improving the variable u_h , superconvergent approximations of this variable are asymptotically of the same order for both families except for the BDM space of the lowest order.

2.2. Postprocessing methods. In order to fix our ideas, let us concentrate on the case where T_h consists only of triangles; the same analysis can be made for the rectangular case with minor modifications.

Let $R(T) = R_{k+1}(T)$ with $k \geq 0$ for the RT space, or $R(T) = R_k(T)$ with $k \geq 1$ for the BDM space, where k is the index in (1.3) and P_k denotes the restriction of polynomials of degree not greater than k to the set T . Set

$$W_h^* = \{v : v|_T \in R(T), T \in T_h\}.$$

Then, for each $T \in T_h$, we define $u_h^* \in W_h^*$ as the solution of the equations [6, 18]

$$(2.2a) \quad (u_h^*, 1)_T = (u_h, 1)_T,$$

$$(2.2b) \quad (a \nabla u_h^*, \nabla v)_T = (f, v)_T - \langle \sigma_h \cdot n_T, v \rangle_{\partial T}, \quad \forall v \in R(T),$$

where (σ_h, u_h) is the solution of (1.4), $\langle \cdot, \cdot \rangle_{\partial T}$ denotes the inner product in $L^2(\partial T)$, and n_T indicates the exterior unit normal to T . Notice that the scheme (2.2) is local and thus fully parallelizable. We have the following error estimates for (2.2) [6]

$$(2.3) \quad \|u - u_h^*\| \leq \begin{cases} C \|u\|_{k+2} h^{k+2}, & \text{for } k \geq 2, \\ C \|u\|_3 h^2, & \text{for the BDM space: } k = 1, \\ C \|u\|_3 h^{k+2}, & \text{for the RT space: } k = 0, 1. \end{cases}$$

Hence we see that u_h^* indeed approximates u with a higher order of accuracy than u_h .

2.3. L^p -posteriori error analysis. To carry out an error analysis in L^p , we shall also need a family $\{U_h\}_{0 < h < 1}$ of continuous function spaces in $\bar{\Omega}$, which are piecewise polynomials in $P_{k+1}(T)$, vanish on $\partial\Omega$, and satisfy

$$(2.4) \quad \inf_{\xi \in U_h} \{ \|v - \xi\|_{0,p} + h \|\nabla(v - \xi)\|_{0,p} \} \leq C \|v\|_{s,p} h^s,$$

if $2 \leq s \leq k+2$ and $v \in W^{s,p}(\Omega)$.

THEOREM 2.1. *Let $1 < p < 2$, and let $u \in H^{k+2}(\Omega)$ and u_h^* be defined by (2.2). Then,*

$$(2.5) \quad \|u - u_h^*\|_{0,p} \leq \begin{cases} C \|u\|_{k+2} h^{k+2}, & \text{for } k \geq 2, \\ C \|u\|_3 h^2, & \text{for the BDM space: } k = 1, \\ C \|u\|_3 h^{k+2}, & \text{for the RT space: } k = 0, 1. \end{cases}$$

Proof. Let $q = p/(p-1)$ and $g \in L^q(\Omega)$, and let $v \in W_0^{1,q}(\Omega)$ be such that $L^*v = g$, where L^* is the adjoint of L in (1.1a). Then, by the elliptic regularity result and Sobolev's inequality, we have

$$(2.6) \quad |v|_{1,\infty} \leq C \|v\|_{2,q} \leq C \|g\|_{0,q}.$$

Assume that $\tilde{u}_h$ and v_h in U_h interpolate u and v , respectively, and that P_T denotes the L^2 -projection onto $P_0(T)$ for each $T \in \mathcal{T}_h$. Then, by integration by parts, (1.1), and (2.2), we have

$$(2.7) \quad \begin{aligned} ((I - P_T)(\tilde{u}_h - u_h^*), g)_T &= (a \nabla(\tilde{u}_h - u_h^*), \nabla v)_T - \left\langle (I - P_T)(\tilde{u}_h - u_h^*), a \frac{\partial v}{\partial n} \right\rangle_{\partial T} \\ &= (a \nabla(\tilde{u}_h - u_h^*), \nabla(v - v_h))_T + (a \nabla(\tilde{u}_h - u), \nabla v_h)_T \\ &\quad + \langle (\sigma_h - \sigma) \cdot n_T, v_h \rangle_{\partial T} - \left\langle (I - P_T)(\tilde{u}_h - u_h^*), a \frac{\partial v}{\partial n} \right\rangle_{\partial T} \\ &= (a \nabla(\tilde{u}_h - u_h^*), \nabla(v - v_h))_T - (a \nabla(\tilde{u}_h - u), \nabla(v - v_h))_T \\ &\quad + (\tilde{u}_h - u, g)_T + \left\langle \tilde{u}_h - u, a \frac{\partial v}{\partial n} \right\rangle_{\partial T} \\ &\quad + \langle (\sigma_h - \sigma) \cdot n_T, v_h \rangle_{\partial T} - \left\langle (I - P_T)(\tilde{u}_h - u_h^*), a \frac{\partial v}{\partial n} \right\rangle_{\partial T} \\ &\leq C \left\{ \|\nabla(\tilde{u}_h - u_h^*)\|_T \|\nabla(v - v_h)\|_T \right. \\ &\quad + \|\nabla(\tilde{u}_h - u)\|_{0,p,T} \|\nabla(v - v_h)\|_{0,q,T} \\ &\quad + \|\tilde{u}_h - u\|_{0,p,T} \|g\|_{0,q,T} + \left\langle \tilde{u}_h - u, a \frac{\partial v}{\partial n} \right\rangle_{\partial T} \\ &\quad + \langle (\sigma_h - \sigma) \cdot n_T, v_h \rangle_{\partial T} \\ &\quad \left. + h_T |v|_{1,\infty} \left(h_T^{-1} \int_{\partial T} |(I - P_T)(\tilde{u}_h - u_h^*)|^2 ds \right)^{1/2} \right\}. \end{aligned}$$

Using a standard scaling argument, note that

$$(2.8) \quad \left(h_T^{-1} \int_{\partial T} |(I - P_T)(\tilde{u}_h - u_h^*)|^2 ds \right)^{1/2} \leq C \|\nabla(\tilde{u}_h - u_h^*)\|_T.$$

Also, as for (5.12) in [6], we see that

$$(2.9) \quad \|\nabla(\tilde{u}_h - u_h^*)\|_T \leq C \left\{ \|\nabla(\tilde{u}_h - u)\|_T + \left(h_T \int_{\partial T} |(\sigma_h - \sigma) \cdot n_T|^2 ds \right)^{1/2} \right\}.$$

Hence, from (2.6), (2.8), and (2.9), the inequality (2.7) implies that

$$(2.10) \quad \|(I - P_h^0)(\tilde{u}_h - u_h^*)\|_{0,p} \leq C \{ \|\tilde{u}_h - u\| + h \|\nabla(\tilde{u}_h - u)\| + h \|\sigma - \sigma_h\| \},$$

where P_h^0 denotes the L^2 -projection onto $\{v : v|_T \in P_0(T), T \in T_h\}$. It can be easily shown (see, e.g., [6]) that

$$\|P_h^0(\tilde{u}_h - u_h^*)\|_{0,p} \leq \|\tilde{u}_h - u\| + \|P_h u - u_h\|,$$

so that, by (2.10),

$$(2.11) \quad \|\tilde{u}_h - u_h^*\|_{0,p} \leq C \{ \|\tilde{u}_h - u\| + h \|\nabla(\tilde{u}_h - u)\| + h \|\sigma - \sigma_h\| + \|P_h u - u_h\| \}.$$

Finally, apply (2.1a), (2.1c), (2.4), (2.11), and the triangle inequality to obtain the desired result (2.5), and this completes the proof. $\square$

We now consider the case: $2 < p < \infty$. As an example, we focus on the RT space of the lowest order, i.e., the case of $k = 0$; the extension of the error analysis below to the general case is straightforward. In this case, W_h^* is defined by

$$W_h^* = \{v : v|_T \in P_1(T), T \in T_h\}.$$

THEOREM 2.2. *Let $2 < p < \infty$ and u_h^* be the solution of (2.2). Then, if $u \in W^{2,p}(\Omega)$, we have*

$$(2.12) \quad \|u - u_h^*\|_{0,p} \leq C_p \|u\|_{2,p} h^2.$$

Proof. For each $T \in T_h$, by (1.1) and (2.2), we see that

$$\begin{aligned} C \|\nabla(\tilde{u}_h - u_h^*)\|_{0,p,T}^p &\leq (a \nabla(\tilde{u}_h - u_h^*), \nabla(\tilde{u}_h - u_h^*) |\nabla(\tilde{u}_h - u_h^*)|^{p-2})_T \\ &= (a \nabla(\tilde{u}_h - u), \nabla(\tilde{u}_h - u_h^*) |\nabla(\tilde{u}_h - u_h^*)|^{p-2})_T \\ &\quad + \langle (\sigma_h - \sigma) \cdot n_T, \nabla(\tilde{u}_h - u_h^*) |\nabla(\tilde{u}_h - u_h^*)|^{p-2} \rangle_{\partial T}. \end{aligned}$$

Thus, using the same techniques as in Theorem 2.1, we obtain

$$(2.13) \quad \|\nabla(\tilde{u}_h - u_h^*)\|_{0,p,T} \leq C \left\{ \|\nabla(\tilde{u}_h - u)\|_{0,p,T} + \|\sigma - \sigma_h\|_{0,p,T} \right\}.$$

To estimate $\|\tilde{u}_h - u_h^*\|_{0,p}$, we apply a standard duality argument. Let w be determined by

$$\begin{aligned} -\Delta w &= \text{sign}(\tilde{u}_h - u_h^*) |\tilde{u}_h - u_h^*|^{p-1} && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then, with $q = p/(p-1)$ again,

$$(2.14) \quad \|w\|_{2,q} \leq C \|\Delta w\|_{0,q} \leq C \|\tilde{u}_h - u_h^*\|_{0,p}^{p-1}.$$

Hence, if we denote by w_h the interpolation of w in U_h , it can be shown in the same way as in (2.11) that

$$\begin{aligned} \|\tilde{u}_h - u_h^*\|_{0,p,T}^p &= (\tilde{u}_h - u_h^*, -\Delta w)_T \\ &= (\nabla(\tilde{u}_h - u_h^*), \nabla(w - w_h))_T - \left\langle \tilde{u}_h - u_h^*, \frac{\partial}{\partial n}(w - w_h) \right\rangle_{\partial T} \\ &\leq h \|\nabla(\tilde{u}_h - u_h^*)\|_{0,p,T} \|w\|_{2,q}, \end{aligned}$$

which together with (2.13)-(2.14) yields that

$$(2.15) \quad \|\tilde{u}_h - u_h^*\|_{0,p} \leq Ch (\|\nabla(\tilde{u}_h - u)\|_{0,p} + \|\sigma - \sigma_h\|_{0,p}).$$

Consequently, the result (2.12) follows from (2.1a), (2.4), and the triangle inequality. The proof is complete. $\square$

THEOREM 2.3. *Let $p = \infty$ and u_h^* be defined by (2.2). Then, if $u \in W^{2,\infty}(\Omega)$, we have*

$$(2.16) \quad \|u - u_h^*\|_{0,\infty} \leq C |\log h| \|u\|_{2,\infty} h^2.$$

Proof. Apply (2.1a), (2.4), and (2.5) to get

$$\|\tilde{u}_h - u_h^*\|_{0,p} \leq C_p \|u\|_{2,p} h^2.$$

Then, the inverse inequality implies that

$$(2.17) \quad \|\tilde{u}_h - u_h^*\|_{0,\infty} \leq C_p h^{-2/p} \|u\|_{2,p} h^2.$$

Now, following [14], take $p = \log h$ in (2.17) to obtain

$$\|\tilde{u}_h - u_h^*\|_{0,\infty} \leq C |\log h| \|u\|_{2,\infty} h^2,$$

which together with (2.4) yields the inequality (2.16). The proof is finished. $\square$

We close this section with a remark that when $p = 1$, we can show by the usual duality that

$$\|u - u_h^*\|_{0,1} \leq C |\log h| \|u\|_{2,1} h^2.$$

3. A quasi-linear problem.

3.1. The Dirichlet problem. In this section, we treat the quasi-linear problem

$$(3.1a) \quad -\operatorname{div}(a(u)\nabla u) = f \quad \text{in } \Omega,$$

$$(3.1b) \quad u = 0 \quad \text{on } \partial\Omega.$$

The coefficient $a(u) = a(x, u) : \bar{\Omega} \times \mathbf{R} \rightarrow \mathbf{R}$ is assumed to be twice continuously differentiable with bounded derivatives through second order and $a(x, u) \geq a_0 > 0$. We also assume that the problem (3.1) has a unique solution $u \in H^{2+\varepsilon}(\Omega)$ for some ε , $0 < \varepsilon < 1$. Note that then $u \in W^{1,\infty}(\Omega)$.

The weak formulation analogous to (1.2) is stated as follows:

$$(3.2a) \quad (a^{-1}(u)\sigma, \tau) - (\operatorname{div} \tau, u) = 0, \quad \forall \tau \in V,$$

$$(3.2b) \quad (\operatorname{div} \sigma, v) = (f, v), \quad \forall v \in W.$$

Again, let $V_h \times W_h$ be one of the RT and BDM spaces. Then the mixed finite element solution $(\sigma_h, u_h) \in V_h \times W_h$ of (3.1) satisfies

$$(3.3a) \quad (a^{-1}(u_h)\sigma_h, \tau) - (\operatorname{div} \tau, u_h) = 0, \quad \forall \tau \in V_h,$$

$$(3.3b) \quad (\operatorname{div} \sigma_h, v) = (f, v), \quad \forall v \in W_h.$$

The existence and uniqueness of solution to (3.3) was proved in [9, 15] for the RT space, and in [10] for the BDM space for sufficiently small h .

3.2. Postprocessing methods. Again, we define W_h^* as

$$W_h^* = \{v : v|_T \in R(T), T \in T_h\}.$$

Then our postprocessing method is defined in the following. For each $T \in T_h$, $u_h^* \in W_h^*$ is determined as the solution of the system

$$(3.4a) \quad (u_h^*, 1)_T = (u_h, 1)_T,$$

$$(3.4b) \quad (a(u_h^*)\nabla u_h^*, \nabla v)_T = (f, v)_T - \langle \sigma_h \cdot n_T, v \rangle_{\partial T}, \quad \forall v \in R(T),$$

where (σ_h, u_h) is the solution of (3.3).

To see that there exists at least one solution u_h^* to (3.4), let us consider the map $S : W_h^* \rightarrow W_h^*$ defined by

$$(3.5a) \quad (Sy, 1)_T = (u_h, 1)_T,$$

$$(3.5b) \quad (a(y)\nabla Sy, \nabla v)_T = (f, v)_T - \langle \sigma_h \cdot n_T, v \rangle_{\partial T}, \quad \forall v \in R(T),$$

for $y \in W_h^*$. Note that, by (3.3b),

$$(f, v)_T - \langle \sigma_h \cdot n_T, v \rangle_{\partial T} = 0, \quad \forall v \in P_0(T),$$

so that the finite set of the linear equations (3.5) defines S uniquely. Now, choosing $v = Sy$ in (3.5b) shows that the range of S is contained in a ball. Since S is clearly continuous, the Brower fixed point theorem implies that (3.4) has a solution.

3.3. Posteriori error analysis. We now present an error analysis in L^p for the postprocessing scheme (3.4). We only derive an L^2 -error estimate; other results can be obtained using the techniques in §2.3 and known L^p -error results for the mixed method (3.3) in [10, 12].

As in the last section, we also need a family $\{U_h\}_{0 < h < 1}$ of finite element spaces of piecewise polynomials of degree not greater than $k + 1$ such that

$$(3.6) \quad \inf_{\xi \in U_h} \{ \|v - \xi\| + h \|\nabla(v - \xi)\| + h^2 \|v - \xi\|_{1,6} \} \leq C \|v\|_s h^s,$$

if $2 \leq s \leq k + 2$ and $v \in H^s(\Omega)$.

Our main result in this section is the following theorem.

THEOREM 3.1. *Let $u \in H^{2+\varepsilon}(\Omega)$ be the solution of (3.1) and u_h^* be defined by (3.4). Then,*

$$(3.7) \quad \|u - u_h^*\| \leq \begin{cases} C \|u\|_{k+2} h^{k+2}, & \text{for } k \geq 2, \\ C \|u\|_3 h^2, & \text{for the BDM space: } k = 1, \\ C \|u\|_3 h^{k+2}, & \text{for the RT space: } k = 0, 1. \end{cases}$$

Proof. Because of the finite dimensionality of each U_h the infimum in (3.6) is achieved. Let $\tilde{u}_h \in U_h$ be such that $\|u - \tilde{u}_h\| + h \|\nabla(u - \tilde{u}_h)\| + h^2 \|u - \tilde{u}_h\|_{1,6}$ is minimal. Note that, by (3.1) and the relation $\sigma = -a(u)\nabla u$,

$$(3.8) \quad (a(u)\nabla u, \nabla v)_T = (f, v)_T - \langle \sigma \cdot n_T, v \rangle_{\partial T}, \quad \forall v \in R(T).$$

Consequently, subtracting (3.4b) from (3.8) yields the error equation

$$(3.9) \quad (a(u)\nabla u - a(u_h^*)\nabla u_h^*, \nabla v)_T = \langle (\sigma_h - \sigma) \cdot n_T, v \rangle_{\partial T}, \quad \forall v \in R(T).$$

Thus,

$$\begin{aligned} a_0 \|\nabla(\tilde{u}_h - u_h^*)\|_T^2 &= a_0 \|\nabla(I - P_T)(\tilde{u}_h - u_h^*)\|_T^2 \\ &\leq (a(u_h^*)\nabla(I - P_T)(\tilde{u}_h - u_h^*), \nabla(I - P_T)(\tilde{u}_h - u_h^*))_T \\ &= (a(u)\nabla(\tilde{u}_h - u), \nabla(\tilde{u}_h - u_h^*))_T \\ &\quad + (\{a(u_h^*) - a(u)\}\nabla u_h^*, \nabla(\tilde{u}_h - u_h^*))_T \\ &\quad + \langle (\sigma_h - \sigma) \cdot n_T, (I - P_T)(\tilde{u}_h - u_h^*) \rangle_{\partial T} \\ &\leq C \|\nabla(\tilde{u}_h - u)\|_T \|\nabla(\tilde{u}_h - u_h^*)\|_T \\ &\quad + \|a(u_h^*) - a(u)\|_{0,3,T} \|\nabla \tilde{u}_h\|_{0,6,T} \|\nabla(\tilde{u}_h - u_h^*)\|_T \\ &\quad + \left(h_T \int_{\partial T} |(\sigma_h - \sigma) \cdot n_T|^2 ds \right)^{1/2} \\ &\quad \times \left(h_T^{-1} \int_{\partial T} |(I - P_T)(\tilde{u}_h - u_h^*)|^2 ds \right)^{1/2}, \end{aligned}$$

so that, by (2.8),

(3.10)

$$\begin{aligned} \|\nabla(\tilde{u}_h - u_h^*)\|_T \leq C & \left\{ \|a(u_h^*) - a(u)\|_{0,3,T} \|\nabla \tilde{u}_h\|_{0,6,T} \right. \\ & \left. + \|\nabla(\tilde{u}_h - u)\|_T + \left(h_T \int_{\partial T} |(\sigma_h - \sigma) \cdot n_T|^2 ds \right)^{1/2} \right\}. \end{aligned}$$

Using the fact that a is uniformly Lipschitz continuous with respect to u and the interpolation result

$$\|\phi\|_{0,3,T} \leq C \|\phi\|_T^{1/2} \|\nabla \phi\|_T^{1/2},$$

it follows from (3.10) that

$$\begin{aligned} (3.11) \quad \|\nabla(\tilde{u}_h - u_h^*)\|_T \leq C & \left\{ \|\nabla(\tilde{u}_h - u)\|_T + \|u_h^* - u\|_T \|\nabla \tilde{u}_h\|_{0,6,T} \right. \\ & + \|\nabla(\tilde{u}_h - u)\|_T \|\nabla \tilde{u}_h\|_{0,6,T} \\ & \left. + \left(h_T \int_{\partial T} |(\sigma_h - \sigma) \cdot n_T|^2 ds \right)^{1/2} \right\}. \end{aligned}$$

For each $\phi \in (I - P_T)P_{k+1}(T)$, scaling arguments imply that

$$\|\phi\|_T \leq Ch_T \|\nabla \phi\|_T.$$

Hence, by (3.11), we find that

$$\begin{aligned} (3.12) \quad \|\tilde{u}_h - u_h^*\|_T \leq & \|P_T(\tilde{u}_h - u_h^*)\|_T + Ch_T \left\{ \|\nabla(\tilde{u}_h - u)\|_T \right. \\ & + (\|u_h^* - u\|_T + \|\nabla(\tilde{u}_h - u)\|_T) \|\nabla \tilde{u}_h\|_{0,6,T} \\ & \left. + \left(h_T \int_{\partial T} |(\sigma_h - \sigma) \cdot n_T|^2 ds \right)^{1/2} \right\}. \end{aligned}$$

We now estimate $\|P_T(\tilde{u}_h - u_h^*)\|_T$. Since P_T is bounded, it follows by (3.4a) that

$$\begin{aligned} \|P_T(\tilde{u}_h - u_h^*)\|_T & \leq \|P_T(\tilde{u}_h - u)\|_T + \|P_T(u - u_h^*)\|_T \\ & \leq \|\tilde{u}_h - u\|_T + \|P_h u - u_h\|_T, \end{aligned}$$

which together with (3.12) implies that

$$\begin{aligned} \|\tilde{u}_h - u_h^*\|_T \leq & \|\tilde{u}_h - u\|_T + \|P_h u - u_h\|_T \\ & + Ch_T \left\{ \|\nabla(\tilde{u}_h - u)\|_T + \|\sigma_h - \sigma\|_T \right. \\ & \left. + (\|u_h^* - u\|_T + \|\nabla(\tilde{u}_h - u)\|_T) \|\nabla \tilde{u}_h\|_{0,6,T} \right\}. \end{aligned}$$

Summing this expression over all $T \in T_h$ and applying the fact that

$$\|\nabla \tilde{u}_h\|_{0,6} \leq C\|u\|_{1,6} \leq C\|u\|_{2+\varepsilon},$$

we obtain

$$\begin{aligned} \|\tilde{u}_h - u_h^*\| \leq C \Big\{ & \|\tilde{u}_h - u\| + \|P_h u - u_h\| \\ & + h(\|\nabla(\tilde{u}_h - u)\| + \|\sigma_h - \sigma\| + \|u_h^* - u\|) \Big\}. \end{aligned}$$

Thus, for h sufficiently small, it follows that

$$\begin{aligned} \|\tilde{u}_h - u_h^*\| \leq C \Big\{ & \|\tilde{u}_h - u\| + \|P_h u - u_h\| \\ & + h(\|\nabla(\tilde{u}_h - u)\| + \|\sigma_h - \sigma\|) \Big\}, \end{aligned}$$

where C depends on $\|u\|_{2+\varepsilon}$. Finally, by similar results to (2.1c) and (2.1a) for the quasi-linear case [10, 12] and (3.6), we obtain the desired result (3.7). The proof has been completed. $\square$

4. Extensions.

The postprocessing methods studied in the last two sections can be extended to the following problem:

$$(4.1a) \quad \operatorname{div} \sigma + c(u) = f \quad \text{in } \Omega,$$

$$(4.1b) \quad \sigma = -(a(u)\nabla u + b(u)) \quad \text{in } \Omega,$$

$$(4.1c) \quad u = -g \quad \text{on } \Gamma_1,$$

$$(4.1d) \quad \sigma \cdot n = 0 \quad \text{on } \Gamma_2,$$

where $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, where $a(x, u)$ is a uniformly positive definite, bounded, symmetric tensor, and $b(u)$ is a bounded vector. Let

$$\begin{aligned} \beta &= a^{-1}(u)b(u), \\ V &= \{v \in H(\operatorname{div}; \Omega) : v \cdot n = 0 \text{ on } \Gamma_2\}. \end{aligned}$$

Then, in the present case, the weak formulation for $(\sigma, u) \in V \times W$ is stated as

$$(4.2a) \quad (a^{-1}(u)\sigma, \tau) - (\operatorname{div} \tau, u) + (\beta(u), \tau) = \langle g, \tau \cdot n \rangle_{\Gamma_1}, \quad \forall \tau \in V,$$

$$(4.2b) \quad (\operatorname{div} \sigma, v) + (c(u), v) = (f, v), \quad \forall v \in W.$$

With the same notation as before, the discrete version of (4.2) for $(\sigma_h, u_h) \in V_h \times W_h$ is as follows:

$$(4.3a) \quad (a^{-1}(u_h)\sigma_h, \tau) - (\operatorname{div} \tau, u_h) + (\beta(u_h), \tau) = \langle g, \tau \cdot n \rangle_{\Gamma_1}, \quad \forall \tau \in V_h,$$

$$(4.3b) \quad (\operatorname{div} \sigma_h, v) + (c(u_h), v) = (f, v), \quad \forall v \in W_h.$$

The existence, uniqueness, and error estimates for the solution (σ_h, u_h) have been studied in [15]. For the corresponding linear problem, we refer to [1].

The postprocessing solution $u_h^* \in W_h^*$ is now determined as the solution of the system

$$(4.4a) \quad (u_h^*, 1)_T = (u_h, 1)_T,$$

$$(4.4b) \quad (a(u_h^*)\nabla u_h^* + b(u_h^*), \nabla v)_T + (c(u_h^*), v)_T = (f, v)_T - \langle \sigma_h \cdot n_T, v \rangle_{\partial T},$$

$$\forall v \in R(T),$$

for each $T \in T_h$, where (σ_h, u_h) is the solution of (4.3). The L^p -error analysis for (4.4) can be carried out as in §2 and §3, and the same results remain valid. Finally, while the postprocessing methods have been analyzed only for the two-dimensional linear and quasi-linear elliptic problems, the extension to the corresponding three-dimensional problems and time-dependent problems is straightforward.

REFERENCES

- [1] T. ARBOGAST AND Z. CHEN, *Implementation of rectangular and other mixed methods as nonconforming methods for second order elliptic problems*, SIAM J. Numer. Anal. (1993 submitted).
- [2] D. N. ARNOLD AND F. BREZZI, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, Modélisation Mathématique et Analyse Numérique, 19 (1985), pp. 7–32.
- [3] F. BREZZI, J. DOUGLAS, JR., R. DURÁN, AND M. FORTIN, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987), pp. 237–250.
- [4] F. BREZZI, J. DOUGLAS, JR., M. FORTIN, AND L. D. MARINI, *Efficient rectangular mixed finite elements in two and three space variables*, RAIRO Modél. Math. Anal. Numer., 21 (1987), pp. 581–604.
- [5] F. BREZZI, J. DOUGLAS, JR., AND L. D. MARINI, *Two families of mixed finite elements for second order elliptic problems*, Numerische Mathematik, 47 (1985), pp. 217–235.
- [6] Z. CHEN, *Unified analysis of the hybrid form of mixed finite elements for second order elliptic problems*, J. Engng. Math., 8 (1991), pp. 91–102.
- [7] Z. CHEN AND J. DOUGLAS, JR., *Prismatic mixed finite elements for second order elliptic problems*, Calcolo, 26 (1989), pp. 135–148.
- [8] Z. CHEN, *Analysis of mixed methods using conforming and nonconforming finite element methods*, RAIRO Modél. Math. Anal. Numer., 27 (1993), pp. 9–34.
- [9] Z. CHEN, *On the existence, uniqueness and convergence of nonlinear mixed finite element methods*, Mat. Aplic. Comput., 8 (1989), pp. 241–258.
- [10] Z. CHEN, *BDM mixed methods for a nonlinear elliptic problem*, J. Comp. Appl. Math. (1993 to appear).

- [11] J. DOUGLAS, JR. AND J. ROBERTS, *Global estimates for mixed methods for second order elliptic problems*, Math. Comp., 45 (1985), pp. 39–52.
- [12] R. G. DURÁN, *Error analysis in L^p , $1 \leq p \leq \infty$, for mixed finite element methods for linear and quasi-linear elliptic problems*, RAIRO Modél. Math. Anal. Numer., 22 (1988), pp. 371–387.
- [13] L. GASTALDI AND R. NOCHETTO, *Optimal L^∞ -error estimates for nonconforming and mixed finite element methods of lowest order*, Numer. Math., 50 (1987), pp. 587–611.
- [14] C. JOHNSON AND V. THOMÉE, *Error estimates for some mixed finite element methods for parabolic type problems*, RAIRO Modél. Math. Anal. Numer., 15 (1981), pp. 41–78.
- [15] F. A. MILNER, *Mixed finite element methods for quasilinear second-order elliptic problems*, Math. Comp., 44 (1985), pp. 303–320.
- [16] J. C. NEDELEC, *Mixed finite elements in $\mathbf{R}^3$* , Numer. Math., 35 (1980), pp. 315–341.
- [17] P. A. RAVIART AND J. M. THOMAS, *A mixed finite element method for 2nd order elliptic problems*, in *Mathematical Aspects of the Finite Element Method*, Lecture Notes in Mathematics 606, Springer-Verlag, Berlin, 1977, pp. 292–315.
- [18] R. STENBERG, *A family of mixed finite elements for the elasticity problem*, Numer. Math., 53 (1988), pp. 513–538.

#	Author/s	Title
1078	Michael Böhm & Mario Taboada,	Global existence and regularity of solutions of the nonlinear string equation
1079	Zhangxin Chen,	BDM mixed methods for a nonlinear elliptic problem
1080	J.J.L. Velázquez,	On the dynamics of a closed thermosyphon
1081	Frédéric Bonnans & Eduardo Casas,	Some stability concepts and their applications in optimal control problems
1082	Hong-Ming Yin,	$\mathcal{L}^{2,\mu}(Q)$ -estimates for parabolic equations and applications
1083	David L. Russell & Bing-Yu Zhang,	Smoothing and decay properties of solutions of the Korteweg-de Vries equation on a periodic domain with point dissipation
1084	J.E. Dunn & K.R. Rajagopal,	Fluids of differential type: Critical review and thermodynamic analysis
1085	Mary Elizabeth Bradley & Mary Ann Horn,	Global stabilization of the von Kármán plate with boundary feedback acting via bending moments only
1086	Mary Ann Horn & Irena Lasiecka,	Global stabilization of a dynamic von Kármán plate with nonlinear boundary feedback
1087	Vilmos Komornik,	Decay estimates for a petrovski system with a nonlinear distributed feedback
1088	Jesse L. Barlow,	Perturbation results for nearly uncoupled Markov chains with applications to iterative methods
1089	Jong-Shenq Guo,	Large time behavior of solutions of a fast diffusion equation with source
1090	Tongwen Chen & Li Qiu,	$\mathcal{H}_\infty$ design of general multirate sampled-data control systems
1091	Satyanad Kichenassamy & Walter Littman,	Blow-up surfaces for nonlinear wave equations, I
1092	Nahum Shimkin,	Asymptotically efficient adaptive strategies in repeated games, Part I: certainty equivalence strategies
1093	Caroline Fabre, Jean-Pierre Puel & Enrique Zuazua,	On the density of the range of the semigroup for semilinear heat equations
1094	Robert F. Stengel, Laura R. Ray & Christopher I. Marrison,	Probabilistic evaluation of control system robustness
1095	H.O. Fattorini & S.S. Sritharan,	Optimal chattering controls for viscous flow
1096	Kathryn E. Lenz,	Properties of certain optimal weighted sensitivity and weighted mixed sensitivity designs
1097	Gang Bao & David C. Dobson,	Second harmonic generation in nonlinear optical films
1098	Avner Friedman & Chaocheng Huang,	Diffusion in network
1099	Xinfu Chen, Avner Friedman & Tsuyoshi Kimura,	Nonstationary filtration in partially saturated porous media
1100	Walter Littman & Baisheng Yan,	Rellich type decay theorems for equations $P(D)u = f$ with f having support in a cylinder
1101	Satyanad Kichenassamy & Walter Littman,	Blow-up surfaces for nonlinear wave equations, II
1102	Nahum Shimkin,	Extremal large deviations in controlled I.I.D. processes with applications to hypothesis testing
1103	A. Narain,	Interfacial shear modeling and flow predictions for internal flows of pure vapor experiencing film condensation
1104	Andrew Teel & Laurent Praly,	Global stabilizability and observability imply semi-global stabilizability by output feedback
1105	Karen Rudie & Jan C. Willems,	The computational complexity of decentralized discrete-event control problems
1106	John A. Burns & Ruben D. Spies,	A numerical study of parameter sensitivities in Landau-Ginzburg models of phase transitions in shape memory alloys
1107	Gang Bao & William W. Symes,	Time like trace regularity of the wave equation with a nonsmooth principal part
1108	Lawrence Markus,	A brief history of control
1109	Richard A. Brualdi, Keith L. Chavey & Bryan L. Shader,	Bipartite graphs and inverse sign patterns of strong sign-nonsingular matrices
1110	A. Kersch, W. Morokoff & A. Schuster,	Radiative heat transfer with quasi-monte carlo methods
1111	Jianhua Zhang,	A free boundary problem arising from swelling-controlled release processes
1112	Walter Littman & Stephen Taylor,	Local smoothing and energy decay for a semi-infinite beam pinned at several points and applications to boundary control
1113	Srdjan Stojanovic & Thomas Svobodny,	A free boundary problem for the Stokes equation via nonsmooth analysis
1114	Bronislaw Jakubczyk,	Filtered differential algebras are complete invariants of static feedback
1115	Boris Mordukhovich,	Discrete approximations and refined Euler-Lagrange conditions for nonconvex differential inclusions
1116	Bei Hu & Hong-Ming Yin,	The profile near blowup time for solution of the heat equation with a nonlinear boundary condition
1117	Jin Ma & Jiongmin Yong,	Solvability of forward-backward SDEs and the nodal set of Hamilton-Jacobi-Bellman Equations
1118	Chaocheng Huang & Jiongmin Yong,	Coupled parabolic and hyperbolic equations modeling age-dependent epidemic dynamics with nonlinear diffusion

- 1119 **Jiongmin Yong**, Necessary conditions for minimax control problems of second order elliptic partial differential equations
- 1120 **Eitan Altman & Nahum Shimkin**, Worst-case and Nash routing policies in parallel queues with uncertain service allocations
- 1121 **Nahum Shimkin & Adam Schwartz**, Asymptotically efficient adaptive strategies in repeated games, part II: Asymptotic optimality
- 1122 **M.E. Bradley**, Well-posedness and regularity results for a dynamic Von Kármán plate
- 1123 **Zhangxin Chen**, Finite element analysis of the 1D full drift diffusion semiconductor model
- 1124 **Gang Bao & David C. Dobson**, Diffractive optics in nonlinear media with periodic structure
- 1125 **Steven Cox & Enrique Zuazua**, The rate at which energy decays in a damped string
- 1126 **Anthony W. Leung**, Optimal control for nonlinear systems of partial differential equations related to ecology
- 1127 **H.J. Sussmann**, A continuation method for nonholonomic path-finding problems
- 1128 **Yung-Jen Guo & Walter Littman**, The null boundary controllability for semilinear heat equations
- 1129 **Q. Zhang & G. Yin**, Turnpike sets in stochastic manufacturing systems with finite time horizon
- 1130 **I. Györi, F. Hartung & J. Turi**, Approximation of functional differential equations with time- and state-dependent delays by equations with piecewise constant arguments
- 1131 **I. Györi, F. Hartung & J. Turi**, Stability in delay equations with perturbed time lags
- 1132 **F. Hartung & J. Turi**, On the asymptotic behavior of the solutions of a state-dependent delay equation
- 1133 **Pierre-Alain Gremaud**, Numerical optimization and quasiconvexity
- 1134 **Jie Tai Yu**, Resultants and inversion formula for N polynomials in N variables
- 1135 **Avner Friedman & J.L. Velázquez**, The analysis of coating flows in a strip
- 1136 **Eduardo D. Sontag**, Control of systems without drift via generic loops
- 1137 **Yuan Wang & Eduardo D. Sontag**, Orders of input/output differential equations and state space dimensions
- 1138 **Scott W. Hansen**, Boundary control of a one-dimensional, linear, thermoelastic rod
- 1139 **Robert Lipton & Bogdan Vernescu**, Homogenization of two phase emulsions with surface tension effects
- 1140 **Scott Hansen & Enrique Zuazua**, Exact controllability and stabilization of a vibrating string with an interior point mass
- 1141 **Bei Hu & Jiongmin Yong**, Pontryagin Maximum principle for semilinear and quasilinear parabolic equations with pointwise state constraints
- 1142 **Mark H.A. Davis**, A deterministic approach to optimal stopping with application to a prophet inequality
- 1143 **M.H.A. Davis & M. Zervos**, A problem of singular stochastic control with discretionary stopping
- 1144 **Bernardo Cockburn & Pierre-Alain Gremaud**, An error estimate for finite element methods for scalar conservation laws
- 1145 **David C. Dobson & Fadil Santosa**, An image enhancement technique for electrical impedance tomography
- 1146 **Jin Ma, Philip Protter, & Jiongmin Yong**, Solving forward-backward stochastic differential equations explicitly — a four step scheme
- 1147 **Yong Liu**, The equilibrium plasma subject to skin effect
- 1148 **Ulrich Hornung**, Models for flow and transport through porous media derived by homogenization
- 1149 **Avner Friedman, Chaocheng Huang, & Jiongmin Yong**, Effective permeability of the boundary of a domain
- 1150 **Gang Bao**, A uniqueness theorem for an inverse problem in periodic diffractive optics
- 1151 **Angelo Favini, Mary Ann Horn, & Irena Lasiecka**, Global existence and uniqueness of regular solutions to the dynamic von Kármán system with nonlinear boundary dissipation
- 1152 **E.G. Kalnins & Willard Miller, Jr.**, Models of q -algebra representations: q -integral transforms and “addition theorems”
- 1153 **E.G. Kalnins, V.B. Kuznetsov & Willard Miller, Jr.**, Quadrics on complex Riemannian spaces of constant curvature, separation of variables and the Gaudin magnet
- 1154 **A. Kersch, W. Morokoff & Chr. Werner**, Selfconsistent simulation of sputtering with the DSMC method
- 1155 **Bing-Yu Zhang**, A remark on the Cauchy problem for the Korteweg-de Vries equation on a periodic domain
- 1156 **Gang Bao**, Finite element approximation of time harmonic waves in periodic structures
- 1157 **Tao Lin & Hong Wang**, Recovering the gradients of the solutions of second-order hyperbolic equations by interpolating the finite element solutions
- 1158 **Zhangxin Chen**, L^p -posteriori error analysis of mixed methods for linear and quasilinear elliptic problems
- 1159 **Todd Arbogast & Zhangxin Chen**, Homogenization of compositional flow in fractured porous media
- 1160 **L. Qiu, B. Bernhardsson, A. Rantzer, E.J. Davison, P.M. Young & J.C. Doyle**, A formula for computation of the real stability radius
- 1161 **Maria Inés Trespovsky**, Adaptive control of linear discrete time systems with external disturbances under inaccurate modelling: A case study
- 1162 **Petr Klouček & Franz S. Rys**, Stability of the fractional step Θ -scheme for the nonstationary Navier-Stokes equations
- 1163 **Eduardo Casas, Luis A. Fernández & Jiongmin Yong**, Optimal control of quasilinear parabolic equations
- 1164 **Darrell Duffie, Jin Ma & Jiongmin Yong**, Black’s consol rate conjecture
- 1165 **D.G. Aronson & J.L. Vazquez**, Anomalous exponents in nonlinear diffusion