**RECOVERING THE GRADIENTS OF THE SOLUTIONS OF
SECOND-ORDER HYPERBOLIC EQUATIONS BY INTERPOLATING
THE FINITE ELEMENT SOLUTIONS**

By

Tao Lin

and

Hong Wang

IMA Preprint Series # 1157

July 1993

Recovering the gradients of the solutions of second-order hyperbolic equations by interpolating the finite element solutions *

Tao Lin [†]

Hong Wang [‡]

Abstract. We present a post-processing technique to generate superconvergent approximations to the gradients of the solutions of second-order hyperbolic equations. The basic idea of the technique is to interpolate a standard (bi)quadratic finite element solution by a higher degree piecewise polynomial. The post-processed numerical solution can approximate the exact gradient with a convergence rate of one order higher than the gradient of the quadratic finite element solution can. This technique can be applied either locally or globally depending on the applications. It is completely vectorizable and parallelizable. Numerical examples are presented to show the properties of this technique as well as to confirm our theoretical error estimates.

Key words. Finite element method, superconvergence, post-processing, second order hyperbolic equation.

AMS(MOS) subject classifications. 65M12, 65M15, 65M60.

1 Introduction

The gradients of the solutions of second-order hyperbolic equations are directly related to important physical quantities such as the energy of the vibrating systems, which need to be approximated very accurately in many applications. However, the well-known error estimates [8, 12, 27, 30] show that the gradient of a finite element solution is generally one order less accurate than the finite element solution itself. For example, the gradient of the quadratic finite element solution is only second-order accurate while the finite element solution itself is third-order accurate. The conventional approach to generate an accurate approximation to the gradient of the exact solution is by increasing the degree of the piecewise polynomial in the finite element space or decreasing the mesh size. Both of these two ways end up with a significant increase of the computation cost.

*This research was supported in part by Office of Naval Research Contract No. 0014-88-K-0370, by National Science Foundation Grants No. DMS-8922865, by funding from the Institutes for Scientific Computation at Texas A&M University, by funding from DOE, DE-ACO5-84OR21400, Martin Marietta, Subcontract, SK966V, and by the funding from the Norwegian Research Council for Science and the Humanities.

[†]Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U. S. A.

[‡]Institute for Scientific Computation and Department of Mathematics, Texas A&M University, College Station, Texas, 77843-3404, U.S.A.

Since 1970's it has been observed that the gradients of finite element solutions may have superconvergence. For example, at certain points of the domain (such as mid-points of the element edges, Gaussian points of the elements, etc.) the numerical solutions may have better convergence rates than the global convergence rate. There has been intensive research in this area [2, 3, 5, 13, 17, 22, 23, 25, 31, 32, 33]. Recently, much attention has also been given to another kind of superconvergence, in which one can obtain better approximations to the gradients by post-processing the finite element solutions [1, 4, 5, 7, 10, 11, 13, 14, 15, 16, 17, 18, 28, 29]. The advantage of these post-processing techniques is that they can generate better approximations to the gradients, not only at certain special points of the domain, but also on the boundaries, in a subdomain, or even in the whole domain. Moreover, post-processing techniques are usually very cheap in the sense that their computational cost is much less than that for finding the finite element solution itself. However, most of the post-processing techniques are for the approximation of elliptic or parabolic problems. As pointed out in [13], relatively few papers are devoted to superconvergence phenomena in finite element methods for initial-boundary value problems of hyperbolic equations, especially for approximations to the gradients.

There are mainly three types of post-processing techniques for the approximation of the gradient. The first type is based on using special test functions [4, 5, 28]. These techniques are useful for approximating the flux on the boundary. The second type of techniques post-process the gradients of the finite element solutions by some kinds of averaging [1, 10, 11, 13, 14, 17, 29]. These techniques can usually generate better approximations to the gradient almost anywhere. The third type of techniques post-process the finite element solutions directly [15, 16, 18]. These techniques use a higher-degree interpolation to post-process a lower-degree finite element solution so that higher-order global/local approximations to the gradient can be obtained.

The main purpose of this paper is to study how the third type of post-processing techniques work for quadratic finite elements for second-order hyperbolic equations, because of the popularity of quadratic finite elements in practical computations. Without loss of generality, we consider the following model initial-boundary value problems for second-order hyperbolic equations:

$$\Phi(\mathbf{x})u_{tt}(\mathbf{x}, t) = \nabla \cdot (D(\mathbf{x})\nabla u(\mathbf{x}, t)) + R(\mathbf{x})u(\mathbf{x}, t) + f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times [0, T],$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = u_1(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (1.1)$$

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T].$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded, polygonal domain. $\Phi(\mathbf{x})$ is bounded from below and above by two positive constants C_1 and C_2 . $D(\mathbf{x}) = [D_{ij}(\mathbf{x})]_{i,j=1}^2$ is a symmetric and positive definite matrix, whose spectrum is bounded from below and above by two positive constants C_3 and C_4 . $R(\mathbf{x})$ and $f(\mathbf{x}, t)$ are bounded functions. ∇ is the gradient operator. The results in this paper can be extended to more complicated problems such as those considered in [26, 27, 30].

The remainder of this paper is organized as follows: In Section 2 we introduce a technique to post-process the quadratic finite element solution. The result of this post-processing procedure is called the *interpolated quadratic finite element solution* (IQFES). In Section 3 we prove that the gradients of IQFES are third-order accurate, which is one-order higher than that of the gradient of the quadratic finite element solution itself. In Section 4 we present numerical results to reveal some important properties of the IQFES.

2 The Interpolated Quadratic Finite Element Solutions

In this section we first discuss some preliminaries and the finite element schemes. Then we introduce the post-processing technique to generate the superconvergent IQFES.

2.1 Finite Element Schemes

As preparation we introduce the following Sobolev spaces for later use in this paper:

$$H^m(\Omega) = \left\{ v(\mathbf{x}) \in L^2(\Omega) \left| \frac{\partial^{\alpha_1+\alpha_2} v(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \in L^2(\Omega), \quad 0 \leq \alpha_1 + \alpha_2 \leq m \right. \right\}, \quad (2.1)$$

$$H_0^1(\Omega) = \left\{ v(\mathbf{x}) \in H^1(\Omega) \mid v(\mathbf{x})|_{\partial\Omega} = 0, \text{ in the sense of traces} \right\},$$

with the norm and semi-norm defined by

$$\begin{aligned} \|v(\mathbf{x})\|_{H^m(\Omega)} &= \left\{ \sum_{0 \leq \alpha_1 + \alpha_2 \leq m} \int_{\Omega} \left[\frac{\partial^{\alpha_1+\alpha_2} v(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right]^2 d\Omega \right\}^{1/2}, \\ |v(\mathbf{x})|_{H^m(\Omega)} &= \left\{ \sum_{\alpha_1 + \alpha_2 = m} \int_{\Omega} \left[\frac{\partial^{\alpha_1+\alpha_2} v(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right]^2 d\Omega \right\}^{1/2}. \end{aligned} \quad (2.2)$$

For $v(\mathbf{x}, t)$ defined on $\Omega \times (t_1, t_2)$, we use the following norms:

$$\|v(\mathbf{x}, t)\|_{L^2(t_1, t_2; H^m(\Omega))} = \left\{ \int_{t_1}^{t_2} \|v(\mathbf{x}, t)\|_{H^m(\Omega)}^2 dt \right\}^{1/2}, \quad (2.3)$$

$$\|v(\mathbf{x}, t)\|_{L^\infty(t_1, t_2; H^m(\Omega))} = \operatorname{ess\,sup}_{t \in (t_1, t_2)} \|v(\mathbf{x}, t)\|_{H^m(\Omega)}.$$

To solve (1.1) with the finite element method, we rewrite (1.1) in the following equivalent variational form: Find $u(\mathbf{x}, t) \in H_0^1(\Omega)$, $\forall t \in [0, T]$, such that the following variational formulation holds:

$$\langle \Phi(\mathbf{x})u_{tt}, w \rangle + \mathcal{A}(u, w) = \langle R(\mathbf{x})u + f(\mathbf{x}, t), w \rangle, \quad \forall w(\mathbf{x}) \in H_0^1(\Omega), \quad t \in [0, T], \quad (2.4)$$

$$\mathcal{A}(u(\mathbf{x}, 0), w) = \mathcal{A}(u_0, w), \quad \mathcal{A}(u_t(\mathbf{x}, 0), w) = \mathcal{A}(u_1, w), \quad \forall w(\mathbf{x}) \in H_0^1(\Omega),$$

where $\mathcal{A}(v, w)$ and $\langle v, w \rangle$ are given by

$$\begin{aligned} \mathcal{A}(v, w) &= \int_{\Omega} \nabla w \cdot (D(\mathbf{x}) \nabla v) \, d\Omega, \\ \langle v, w \rangle &= \int_{\Omega} v(\mathbf{x}) w(\mathbf{x}) d\Omega. \end{aligned} \quad (2.5)$$

In the numerical schemes we use the finite difference in the time direction and the finite element in space. Let N be a positive integer. We define the partition in time as follows:

$$\tau = \frac{T}{N}, \quad t^n = n\tau, \quad n = 0, 1, \dots, N. \quad (2.6)$$

We use the following finite difference operators:

$$\begin{aligned} U^{n+1/2} &\equiv \frac{U^{n+1} + U^n}{2}, \quad U^{n, 1/4} \equiv \frac{U^{n+1} + 2U^n + U^{n-1}}{4} \\ \partial_{tt}U^n &\equiv \frac{U^{n+1} - 2U^n + U^{n-1}}{\tau^2}, \quad U^{n, 1/2} \equiv \frac{U^{n+1} + U^{n-1}}{2}, \\ \partial_t U^{n+1/2} &\equiv \frac{U^{n+1} - U^n}{\tau}, \quad \partial_t U^n \equiv \frac{U^{n+1} - U^{n-1}}{2\tau}. \end{aligned} \quad (2.7)$$

For the notations in (2.7), we have the following obvious relations:

$$\partial_t U^n = \frac{\partial_t U^{n+1/2} + \partial_t U^{n-1/2}}{2} = \frac{U^{n+1/2} - U^{n-1/2}}{\tau},$$

(2.8)

$$\partial_{tt}U^n = \frac{\partial_t U^{n+1/2} - \partial_t U^{n-1/2}}{\tau}.$$

In this paper, we use ε to denote an arbitrary small number, C to denote a general constant, which does not necessarily have the same value at different places, and C_i to denote some definite constants.

After discretizing the time direction, we now discretize the spatial domain. Depending on whether the partition Π_h is a rectangular or a triangular partition, we can use a standard piecewise biquadratic or a piecewise quadratic finite element space $S^h(\Omega)$ defined on the partition Π_h . We can then use one of the following schemes to generate biquadratic or quadratic finite element solutions: For $U^0(\mathbf{x})$, $U^1(\mathbf{x}) \in S^h(\Omega)$ given by (2.11), find $U^{n+1}(\mathbf{x}) \in S^h(\Omega)$, $n = 1, 2, \dots, N-1$, such that

Scheme 1:

$$\langle \Phi \partial_{tt}U^n, w \rangle + \mathcal{A}(U^{n,1/4}, w) = \langle R(\mathbf{x})U^{n,1/4} + f^{n,1/4}(\mathbf{x}), w \rangle, \quad \forall w \in S^h(\Omega). \quad (2.9)$$

or

Scheme 2:

$$\langle \Phi \partial_{tt}U^n, w \rangle + \mathcal{A}(U^{n,1/2}, w) = \langle R(\mathbf{x})U^{n,1/2} + f^n(\mathbf{x}), w \rangle, \quad \forall w \in S^h(\Omega), \quad (2.10)$$

is satisfied.

The initial values $U^0(\mathbf{x})$ and $U^1(\mathbf{x})$ used to start these schemes are given by the following equations:

$$\mathcal{A}(U^0, w) = \mathcal{A}(u_0, w), \quad \mathcal{A}(U^1, w) = \mathcal{A}(u^*, w), \quad \forall w \in S^h(\Omega), \quad (2.11)$$

where $u^*(\mathbf{x}) = u_0(\mathbf{x}) + \tau u_1(\mathbf{x}) + \frac{\tau^2}{2} u_{tt}(\mathbf{x}, 0)$ and $u_{tt}(\mathbf{x}, 0)$ can be obtained from (1.1).

Both Schemes 1 and 2 are implicit schemes, so they are not subject to stability restrictions. Also, it is easy to see that the coefficient matrices for both schemes are symmetric

and positive definite when τ is small, so each scheme has a unique solution. In the one-dimensional case, we have five-banded coefficient matrices for both schemes. We can solve the systems directly. In multiple spatial dimensions, the coefficient matrices of both Schemes 1 and 2 are regularly-structured sparse matrices. Therefore, iterative methods are preferred. To understand the computational efficiency, let us briefly discuss the condition numbers of the coefficient matrices. The left-hand side of (2.9) or (2.10) can be rewritten as

$$\mathcal{B}(U, w) = \langle \Phi U, w \rangle + C^* \tau^2 \mathcal{A}(U, w), \quad (2.12)$$

where $C^* = 1/4$ for Scheme 1 and $C^* = 1/2$ for Scheme 2.

Since both Φ and D have positive lower and upper bounds, we have

$$C_1 \|U\|_{L^2(\Omega)}^2 \leq \langle \Phi U, U \rangle \leq C_2 \|U\|_{L^2(\Omega)}^2 \quad (2.13)$$

and

$$\mathcal{A}(U, U) \leq C_4 \|\nabla U\|_{(L^2(\Omega))^2}^2 \leq C_5 h^{-2} \|U\|_{L^2(\Omega)}^2, \quad (2.14)$$

where we have used the well-known inverse assumption [6]:

$$\|\nabla U\|_{(L^2(\Omega))^2} \leq C h^{-1} \|U\|_{L^2(\Omega)}. \quad (2.15)$$

Combining (2.12)–(2.14), we obtain

$$C_1 \|U\|_{L^2(\Omega)}^2 \leq \mathcal{B}(U, U) \leq \left[C_2 + \frac{C^* C_5 \tau^2}{h^2} \right] \|U\|_{L^2(\Omega)}^2. \quad (2.16)$$

(2.16) shows that the condition numbers of the coefficient matrices of both Scheme 1 and Scheme 2 are bounded by $C(1 + \tau^2/h^2)$ [12]. If we take $\tau = O(h)$, which is reasonable in practical computations, the coefficient matrices are well conditioned, symmetric, and positive definite. Therefore, conjugate gradient methods can be used to solve the system efficiently without preconditioning.

2.2 The Interpolated Quadratic Finite Element Solutions

Once a finite element solution is obtained, we can post-process it to generate a better approximation to the gradient. For simplicity of exposition, we use a two-dimensional rectangular domain Ω to illustrate the ideas and then briefly mention some generalizations.

We start by discussing the case in which we use the third degree polynomial interpolation to post-process the finite element solutions. We consider the biquadratic finite elements first. We introduce two sets of nodes, $x_1 < x_2 < \dots < x_{N_x}$ and $y_1 < y_2 < \dots < y_{N_y}$, in both the x and the y directions, with $N_x = 6k_x + 1$ and $N_y = 6k_y + 1$, for some positive integers k_x, k_y . Here, (x_1, y_1) is the lower left corner and (x_{N_x}, y_{N_y}) is the upper right corner of Ω . We use the nodes $\{(x_{2i+1}, y_{2j+1})\}$ ($i = 0, 1, \dots, (N_x - 1)/2$ and $j = 0, 1, \dots, (N_y - 1)/2$) to form a fine rectangular partition Π_h and use the nodes $\{(x_{3i+1}, y_{3j+1})\}$ ($i = 0, 1, \dots, (N_x - 1)/3$ and $j = 0, 1, \dots, (N_y - 1)/3$) to form a coarse rectangular partition $\Pi_{3h/2}$ (see Figures 1 and 2.) The biquadratic finite element space $S^h(\Omega)$ consists of those functions which are continuous in Ω and are biquadratic polynomials on each rectangular element $[x_{2i+1}, x_{2i+3}] \times [y_{2j+1}, y_{2j+3}]$ ($i = 0, 1, \dots, (N_x - 3)/2$ and $j = 0, 1, 2, \dots, (N_y - 3)/2$) of Π_h with nine nodes (x_{2i+k}, y_{2j+l}) ($k, l = 1, 2, 3$).

Solving either Scheme 1 or Scheme 2 on $S^h(\Omega)$ yields $U^{n+1}(\mathbf{x})$, which provide approximate values to the true solution $u(\mathbf{x}, t^n)$ over all the nodes (x_i, y_j) , ($i = 1, \dots, N_x$, $j = 1, \dots, N_y$). Using these approximate values, we define a new function $\mathcal{I}_{3h/2}^{3b} U^{n+1}(\mathbf{x})$, which is a piecewise bicubic interpolation of $U^{n+1}(\mathbf{x})$ over the partition $\Pi_{3h/2}$. Namely, on each element $[x_{3i+1}, x_{3i+4}] \times [y_{3j+1}, y_{3j+4}]$ of $\Pi_{3h/2}$ ($i = 1, 2, \dots, (N_x - 4)/3$ and $j = 1, 2, \dots, (N_y - 4)/3$), $\mathcal{I}_{3h/2}^{3b} U^{n+1}(\mathbf{x})$ is a bicubic interpolation of $U^{n+1}(\mathbf{x})$ over the sixteen nodes (x_{3i+k}, y_{3j+l}) ($k, l = 1, 2, 3, 4$). The superscript $3b$ and the subscript $3h/2$ in the interpolation operator $\mathcal{I}_{3h/2}^{3b}$ emphasize the fact that the bicubic polynomial is used for the post-processing interpolation and the fact that “one and one half” fine elements in Π_h are used to construct a coarse element in $\Pi_{3h/2}$ (see the Figure 3.) Because of the way we define the fine and coarse partition, $\mathcal{I}_{3h/2}^{3b} U^{n+1}(\mathbf{x})$ is well defined and continuous.

We can also use the bicubic polynomial interpolation to post-process the numerical solutions obtained by the triangular quadratic finite elements. In this case we use the same coarse partition $\Pi_{3h/2}$ as above. As for the fine partition, we start with a fine rectangular partition as previously defined together with $\Pi_{3h/2}$. We then subdivide each fine rectangular element into two triangular elements using its diagonal line with positive (or negative) slope to form a triangular partition Π_h . We use the nodes of the original rectangular elements to define the nodes in the triangular elements. This time, the finite element space $S^h(\Omega)$ consists of continuous and piecewise quadratic polynomials determined by their values on the nodes of the triangles of Π_h . In each coarse element of $\Pi_{3h/2}$, there are still sixteen nodes

Figure 1: A rectangular partion Π_h of Ω by which the finite element space $S^h(\Omega)$ is constructed.

Figure 2: A rectangular partition $\Pi_{3h/2}$ of Ω in which the $\mathcal{I}_{3h/2}^3 U^n(\mathbf{x})$ is constructed. This coarse partition is related to the partition Π_h illustrated in the previous figure.

Figure 3: A typical element of $\Pi_{3h/2}$ and its related nodes in Π_h . $\circ$ indicates the nodes where the approximations to $u(\mathbf{x}, t)$ are provided by the biquadratic finite element solution on Π_h . These approximate values are used to construct a bicubic interpolation polynomial in the element $K_{3h/2}$.

of Π_h . The finite element solution $U^{n+1}(\mathbf{x}) \in S^h(\Omega)$ can supply approximations to the true solution $u(\mathbf{x}, t^{n+1})$ at the sixteen nodes of Π_h in each element of $\Pi_{3h/2}$. Therefore, we can define a new function $\mathcal{I}_{3h/2}^{3b} U^{n+1}(\mathbf{x})$, which is a piecewise bicubic interpolation of $U^{n+1}(\mathbf{x})$ over the partition of $\Pi_{3h/2}$, as before. Again, $\mathcal{I}_{3h/2}^{3b} U^{n+1}(\mathbf{x})$ is well defined and continuous.

Now we discuss how to use the quartic polynomial interpolation to do post-processing. We introduce two sets of nodes $x_1 < x_2 < \dots < x_{N_x}$ and $y_1 < y_2 < \dots < y_{N_y}$ in both the x and the y directions with $N_x = 4k_x + 1$ and $N_y = 4k_y + 1$ for some positive integers k_x, k_y . Here (x_1, y_1) is the lower left corner and (x_{N_x}, y_{N_y}) is the upper right corner of Ω . We use the nodes $\{(x_{2i+1}, y_{2j+1})\}$ ($i = 0, 1, \dots, (N_x - 1)/2$ and $j = 0, 1, \dots, (N_y - 1)/2$) to form either a fine rectangular or a fine triangular partition Π_h such that the nodes of its elements are formed by (x_i, y_j) , $i = 1, \dots, N_x$, $j = 1, \dots, N_y$ (see Figures 7-10.) The finite element space $S^h(\Omega)$ can then be the set of functions which are continuous, either piecewise biquadratic or quadratic on each element of Π_h . To use the biquartic polynomial interpolation in the post-processing, we use the node (x_{4i+1}, y_{4j+1}) , ($i = 0, \dots, k_x$, $j = 0, \dots, k_y$) to form a coarse rectangular partition Π_{2h} such that each rectangle of Π_{2h} contains twenty five nodes defined by (x_i, y_j) , $i = 1, \dots, N_x$, $j = 1, \dots, N_y$. Here $2h$ in Π_{2h} indicates that the sides of two fine rectangular elements form one side of a rectangular element in Π_{2h} . Using the finite element solution $U^{n+1}(\mathbf{x}) \in S^h(\Omega)$ provided by either Scheme 1 or Scheme 2, we define a new function $\mathcal{I}_{2h}^{4b} U^{n+1}(\mathbf{x})$ to be the piecewise biquartic interpolation of $U^{n+1}(\mathbf{x})$ over

Figure 4: A triangular partition Π_h of Ω by which the triangular quadratic finite element space $S^h(\Omega)$ is constructed.

Figure 5: A rectangular partition $\Pi_{3h/2}$ of Ω in which $\mathcal{I}_{3h/2}^{3b} U^n(\mathbf{x})$ is constructed. This coarse partition is related with the partition Π_h illustrated in the previous figure.

Figure 6: A typical element of $\Pi_{3h/2}$ and its related nodes in Π_h . $\circ$ indicates the nodes where the approximations to $u(\mathbf{x}, t)$ are provided by the quadratic finite element solution on Π_h . These approximate values are used to construct a bicubic interpolation polynomial in the element $K_{3h/2}$.

each element of the partition Π_{2h} . Here, $4b$ in the interpolation operator emphasizes that the biquartic interpolation is used in the post-processing. To use the quartic polynomial interpolation in the post-processing, we can start with a coarse partition which consists of (25 nodes) rectangles as before. Cutting each coarse rectangle into two triangles along its diagonal line with positive slope, we obtain a coarse triangular partition Π_{2h} . Each triangle of Π_{2h} then contains fifteen nodes, and all the nodes of Π_{2h} are again defined by (x_i, y_j) , $i = 1, \dots, N_x$, $j = 1, \dots, N_y$. Then we define a new function $\mathcal{I}_{2h}^{4t} U^{n+1}(\mathbf{x})$ to be the piecewise quartic interpolation of $U^{n+1}(\mathbf{x})$ over each element of Π_{2h} . Here, $4t$ in $\mathcal{I}_{2h}^{4t}$ indicates the triangular quartic interpolation is used. Obviously, both $\mathcal{I}_{2h}^{4b} U^{n+1}(\mathbf{x})$ and $\mathcal{I}_{2h}^{4t} U^{n+1}(\mathbf{x})$ are well defined and continuous.

Since $\mathcal{I}_{3h/2}^{3b} U^n(\mathbf{x})$, $\mathcal{I}_{2h}^{4b} U^n(\mathbf{x})$, and $\mathcal{I}_{2h}^{4t} U^n(\mathbf{x})$ are generated by piecewise interpolations of the (bi-)quadratic finite element solution with higher-degree polynomials, we call them the *Interpolated Quadratic Finite Element Solution* (IQFES).

As we will see in the next two sections, all the IQFES introduced have the same asymptotic convergence rates in H^1 norm. This sounds as if the quartic interpolation has no advantage over the cubic interpolation in the post-processing. Nevertheless, the quartic interpolation has some special properties. One of them is that $\mathcal{I}_{2h}^{4t}$ can be applied when the Ω is a general polygonal domain. This time we first introduce a coarse triangular partition

Figure 7: The related rectangular partitions Π_{2h} and Π_h

Π_{2h} over Ω , then form a fine triangulation Π_h by subdividing each triangle in Π_{2h} into four triangles, say, uniformly through the midpoints of the sides. We introduce nine nodes in each triangle of Π_h by its vertices and midpoints, which completely determines a quadratic polynomial. Consequently, each triangle of Π_{2h} contains fifteen nodes of Π_h (see Figures 11 and 12.) Define the finite element space $S^h(\Omega)$ to be the space of the continuous piecewise quadratic polynomials through the partition Π_h . Then we can define a new function $\mathcal{I}_{2h}^{4t} U^{n+1}(\mathbf{x})$ by piecewise quartic interpolation of the finite element solution $U^{n+1}(\mathbf{x}) \in S^h(\Omega)$ over the partition Π_{2h} . Evidently, $\mathcal{I}_{2h}^{4t} U^{n+1}(\mathbf{x})$ is well defined and continuous.

For the simplicity of explanation, we used the coarse partition in the construction of the IQFES. This is not necessary as long as each element in Π_h can form a larger element together with some other elements of Π_h so that we can interpolate the (bi-)quadratic finite element solution with a higher degree polynomial on the larger element. For example, assuming that Π_h is formed by rectangular elements, to define the IQFES by the biquartic interpolation, we just have to make sure that each element of Π_h can form a larger rectangle together with three other elements of Π_h (see Figure 13.) Obviously, this can be realized by using a partition parameter h which is small enough. The only technical difficulty here is in defining the IQFES uniquely, since an element in Π_h may belong to different larger elements as does K_h^4 in Figure 14. For more detailed discussions about these issues and the related error estimates please see [21].

Figure 8: A typical element of Π_{2h} and its related rectangular elements in Π_h . $\circ$ indicates the nodes where the approximations to $u(\mathbf{x}, t)$ are provided by the biquadratic finite element solution on Π_h . These approximate values are used to form a bi-4th degree interpolation polynomial in the element K_{2h} .

Figure 9: A rectangular partition Π_{2h} and its related fine triangular partition Π_h .

Figure 10: A typical element in Π_{2h} and its related triangular elements in Π_h . $\circ$ indicates the nodes where the approximations to $u(\mathbf{x}, t)$ are provided by the triangular quadratic finite element solution on Π_h . These approximate values are used to form a bi-4th degree interpolation polynomial in the element K_{2h} .

Figure 11: The triangulation Π_{2h} of a polygonal domain Ω and its related fine partition Π_h .

Figure 12: A typical element K_{2h} of Π_{2h} and 4 of its related elements in Π_h .

Figure 13: The elements of the partition Π_h here do not form a partition Π_{2h} , but each element of Π_h belongs to a larger rectangular element which is formed by 4 elements of Π_h . The IQFES can still be defined using the biquadratic finite element solution on Π_h .

A portion of Π_h

K_h^1	K_h^3	K_h^5
K_h^2	K_h^4	K_h^6

Figure 14: K_h^4 belongs to two larger rectangles formed by $K_h^1, K_h^2, K_h^3, K_h^4$ and $K_h^3, K_h^4, K_h^5, K_h^6$, respectively.

The post-processing procedure introduced in this paper has many noticeable features. Among them are the following: (1) the gradient of the IQFES is a piecewise biquadratic or (bi-)cubic polynomial in each element of the coarse partition and is a better approximation of the gradient of the true solution globally over the whole domain, as proved in the next section. (2) However, it is easy to see that the post-processing techniques described above can also be applied regionally to a subset of Ω , including its boundary where a better approximation to the gradient is needed. In this situation, the post-processing can be carried out only in those elements of the coarse partition which intersect the subset. This not only reduces the computation cost, but also is very useful for some applications such as inverse problems in which the extra data are provided by the flux on the boundaries [9, 19, 20]. (3) When the IQFES is constructed, the higher-degree interpolation in each element of the coarse partition is independent of each other, and can be carried out simultaneously to take advantage of modern computer structures. (4) The post-processing procedure to form the IQFES is completely independent of the procedure to obtain the finite element solution. Therefore, it can be incorporated easily into almost any existing finite element solver that uses either the popular triangular quadratic or biquadratic finite elements. (5) Even though we restricted ourselves to a rectangular domain in the discussion, the idea of using a higher-degree interpolation polynomial to do post-processing can be generated easily

Figure 15: The elements of the partition Π_h do not form a partition Π_{2h} , but each element of Π_h belongs to a larger rectangular element which is formed by 4 elements of Π_h . The IQFES can still be defined using the biquadratic finite element solution on Π_h .

to more complicated domains. For example, $\mathcal{I}_{3h/2}^{3b}$ and $\mathcal{I}_{2h}^{4b}$ can be used for any domain which is the union of axis-parallel rectangles, while $\mathcal{I}_{2h}^{4t}$ can be applied to any polygonal domain.

3 Error Estimates for the IQFES

In this section we prove that the gradient of the IQFES has superconvergence. We assume that the fine partition Π_h used to compute the finite element solution is regular and quasiuniform. We start with some auxiliary lemmas which support the proof of the error estimation results. First we prove the boundedness of the interpolation operators used in Section 2. We assume that a coarse partition is used in constructing the IQFES, and we give a complete proof for $\mathcal{I}_{2h}^{4b}$. The results for other interpolation operators will be stated. For their proofs and the discussion about the general cases in which the coarse partitions are not used, please see [21].

Consider the rectangular reference element $\hat{K} = [-1, 1] \times [-1, 1]$ which has twenty five nodes $(\hat{x}_i, \hat{y}_j)$, $i = 1, \dots, 5$, $j = 1, \dots, 5$ such that

$$\begin{aligned} -1 &= \hat{x}_1 < \hat{x}_2 < \hat{x}_3 < \hat{x}_4 < \hat{x}_5 = 1, \\ -1 &= \hat{y}_1 < \hat{y}_2 < \hat{y}_3 < \hat{y}_4 < \hat{y}_5 = 1. \end{aligned}$$

The lines parallel to the axes passing through $\hat{x}_1, \hat{x}_3, \hat{x}_5$ and $\hat{y}_1, \hat{y}_3, \hat{y}_5$ define a “fine” rectan-

Figure 16: A typical element K in Π_{2h} and its reference element $\hat{K}$.

gular (or triangular) partition on $\hat{K}$ that has four smaller rectangles (or eight triangles). The nodes $(\hat{x}_i, \hat{y}_j)$, $i = 1, \dots, 5$, $j = 1, \dots, 5$ are used to define the biquadratic (or triangular quadratic) finite elements on this fine partition. The biquartic interpolation operator $\hat{\mathcal{I}}_{2h}^{4b}$ is well-defined with the nodes $(\hat{x}_i, \hat{y}_j)$, $i, j = 1, 2, \dots, 5$ (see Figure 16.) The following lemma gives the H^1 boundedness of this operator over the reference element $\hat{K}$.

Lemma 3.1 *There exists a constant C such that*

$$\left| \hat{\mathcal{I}}_{2h}^{4b} v \right|_{H^1(\hat{K})} \leq C |v|_{H^1(\hat{K})}, \quad \forall v \in S^h(\hat{K}).$$

Proof: Let $\hat{\psi}_{ij}(\hat{\mathbf{x}})$, $(i, j = 1, 2, 3, 4, 5)$ be the nodal basis functions of $S^h(\hat{K})$ such that

$$\hat{\psi}_{ij}(\hat{\mathbf{x}}_{kl}) = \delta_{ik} \delta_{jl},$$

where $\hat{\mathbf{x}}_{kl} = (\hat{x}_k, \hat{y}_l)$ and $S^h(\hat{K})$ is the biquadratic (or triangular quadratic) finite element space defined on the fine partition of $\hat{K}$. Then any $\hat{v}(\hat{\mathbf{x}}) \in S^h(\hat{K})$ has the following expression:

$$\hat{v}(\hat{\mathbf{x}}) = \sum_{i,j=1}^5 \hat{v}(\hat{\mathbf{x}}_{ij}) \hat{\psi}_{ij}(\hat{\mathbf{x}}).$$

Let $\hat{\phi}_{ij}(\hat{\mathbf{x}})$, $(i, j = 1, 2, 3, 4, 5)$ be the biquartic nodal basis polynomials such that

$$\hat{\phi}_{ij}(\hat{\mathbf{x}}_{kl}) = \delta_{ik} \delta_{jl}.$$

Then

$$\hat{\mathcal{I}}_{2h}^{4b} \hat{v}(\hat{\mathbf{x}}) = \sum_{i,j=1}^5 \hat{v}(\hat{\mathbf{x}}_{ij}) \hat{\phi}_{ij}(\hat{\mathbf{x}}).$$

Let $\vec{\xi}$ be the 25-dimensional vector whose component are $\hat{v}(\hat{\mathbf{x}}_{ij})$, $(i, j = 1, 2, 3, 4, 5)$ and

$$e_1(\vec{\xi}) \equiv \left| \hat{\mathcal{I}}_{2h}^{4b} \hat{v} \right|_{H^1(\hat{K})},$$

$$\begin{aligned}
e_2(\vec{\xi}) &\equiv \left| \hat{v} \right|_{H^1(\hat{K})}, \\
e_3(\vec{\xi}) &\equiv \frac{e_1(\vec{\xi})}{e_2(\vec{\xi})}, \quad \vec{\xi} \neq 0.
\end{aligned}$$

Note that $e_3(\vec{\xi})$ is a continuous function of $\vec{\xi}$ and is homogeneous of degree zero, i.e.,

$$e_3(r\vec{\xi}) = e_3(\vec{\xi}), \quad \forall r \in \mathbb{R}, \quad r \neq 0.$$

Thus, the maximum of $e_3(\vec{\xi})$ in $\mathbb{R}^{25}$ is equal to the maximum of $e_3(\vec{\xi})$ on the unit ball of $\mathbb{R}^{25}$. Since the unit ball is a compact set and $e_3(\vec{\xi})$ is continuous, there exists a constant C such that

$$\left| e_3(\vec{\xi}) \right| \leq C.$$

Combining this result with the definitions of e_i ($i = 1, 2, 3$) gives the result of this lemma. □

With the help of Lemma 3.1 we can prove the same result for $\mathcal{I}_{2h}^{4b}$.

Lemma 3.2 *There exists a constant $C > 0$ such that*

$$\left| \mathcal{I}_{2h}^{4b} v \right|_{H^1(\Omega)} \leq C |v|_{H^1(\Omega)}, \quad v \in S^h(\Omega). \quad (3.1)$$

Here $S^h(\Omega)$ is either the biquadratic or the quadratic finite element space defined on a rectangular or triangular partition Π_h .

Proof: Let K_{2h} be a typical element of Π_{2h} that contains 25 nodes of the partition Π_h :

$$\mathbf{x}_{ij} = (x_i, y_j), \quad i, j = 1, 2, 3, 4, 5,$$

with

$$x_1 < x_2 < x_3 < x_4 < x_5,$$

$$y_1 < y_2 < y_3 < y_4 < y_5,$$

(see Figure 16). Let $S^h(K_{2h})$ be the set of functions of $S^h(\Omega)$ restricted to K . For any function $v(\mathbf{x})$ in $S^h(K_{2h})$ we define a function $\hat{v}(\hat{\mathbf{x}})$ in $S^h(\hat{K})$ by

$$\hat{v}(\hat{\mathbf{x}}) \equiv v(\mathbf{x}),$$

where

$$\begin{aligned}\mathbf{x} &= F_{K_{2h}}(\hat{\mathbf{x}}) \equiv B_{K_{2h}}\hat{\mathbf{x}} + \mathbf{b}, \\ B_{K_{2h}} &= \begin{pmatrix} \frac{1}{2}(x_5 - x_1) & 0 \\ 0 & \frac{1}{2}(y_5 - y_1) \end{pmatrix}, \\ \mathbf{b} &= \begin{pmatrix} \frac{1}{2}(x_5 + x_1) \\ \frac{1}{2}(y_5 + y_1) \end{pmatrix}.\end{aligned}$$

According to [6] and Lemma 3.1,

$$\begin{aligned}\left| \mathcal{I}_{2h}^{4b} v \right|_{H^1(K_{2h})} &\leq C \left\| B_{K_{2h}}^{-1} \right\| |\det B_{K_{2h}}|^{\frac{1}{2}} \left| \hat{\mathcal{I}}_{2h}^{4b} v \right|_{H^1(\hat{K})}, \\ &\leq C \left\| B_{K_{2h}}^{-1} \right\| |\det B_{K_{2h}}|^{\frac{1}{2}} \left| \hat{\mathcal{I}}_{2h}^{4b} \hat{v} \right|_{H^1(\hat{K})} \\ &\leq C \left\| B_{K_{2h}}^{-1} \right\| |\det B_{K_{2h}}|^{\frac{1}{2}} |\hat{v}|_{H^1(\hat{K})} \\ &\leq C \left\| B_{K_{2h}}^{-1} \right\| |\det B_{K_{2h}}|^{\frac{1}{2}} \|B_{K_{2h}}\| |\det B_{K_{2h}}|^{-\frac{1}{2}} |v|_{H^1(K_{2h})} \\ &\leq C |v|_{H^1(K_{2h})},\end{aligned}$$

where in the last step we have used the fact that the partition Π_h is regular. Finally, adding the above inequality over all the elements of the coarse partition yields the result of this lemma. $\square$

The following lemma states similar results for $\mathcal{I}_{2h}^{4t}$ and $\mathcal{I}_{3h/2}^{3b}$. The proof is similar to the previous lemma, therefore is omitted.

Lemma 3.3 $\mathcal{I}_{2h}^{4t}$ and $\mathcal{I}_{3h/2}^{3b}$ satisfy the following inequalities:

$$\left| \mathcal{I}_{2h}^{4t} v \right|_{H^1(\Omega)} \leq C |v|_{H^1(\Omega)}, \quad v \in S^h(\Omega), \quad (3.2)$$

$$\left| \mathcal{I}_{3h/2}^{3b} v \right|_{H^1(\Omega)} \leq C |v|_{H^1(\Omega)}, \quad v \in S^h(\Omega), \quad (3.3)$$

where $S^h(\Omega)$ is a biquadratic (or quadratic) finite element space defined on a rectangular (or triangular) partition Π_h . $\square$

With the help of the above lemmas we are now in the position to prove the superconvergence for the IQFES. For any $v(\mathbf{x}) \in H_0^1(\Omega)$, we introduce its Ritz projection $\mathcal{P}_h^2 v(\mathbf{x}) \in S^h(\Omega)$ by

$$\mathcal{A}(\mathcal{P}_h^2 v(\mathbf{x}), w) = \mathcal{A}(v(\mathbf{x}), w), \quad \forall w \in S^h(\Omega). \quad (3.4)$$

Here the finite element space $S^h(\Omega)$ is either a biquadratic finite element space defined on a rectangular partition Π_h or a quadratic finite element space defined on a triangular partition Π_h . The subscript h in $\mathcal{P}_h^2$ emphasizes that the “fine” partition Π_h is used, and the superscript 2 indicates that a quadratic/biquadratic finite element is used. The following two lemmas indicate that the post-processing technique introduced in this paper can improve the convergence rate of the gradient of the Ritz projection.

Lemma 3.4 *Let $S^h(\Omega)$ be a quadratic/biquadratic finite element space defined on Π_h . Then we have the following estimate:*

$$\|\nabla(\mathcal{I}_{3h/2}^{3b} \mathcal{P}_h^2 v - v)\|_{(L^2(\Omega))^2} \leq Ch^3 \|v\|_{H^4(\Omega)} \quad \forall v(\mathbf{x}) \in H^4(\Omega) \cap H_0^1(\Omega). \quad (3.5)$$

Proof: The following estimate comes from the approximation property of the bicubic interpolation operator

$$\|\nabla(\mathcal{I}_{3h/2}^{3b} v - v)\|_{(L^2(\Omega))^2} \leq Ch^3 \|v\|_{H^4(\Omega)}, \quad \forall v \in H^4(\Omega) \cap H_0^1(\Omega). \quad (3.6)$$

According to the well-known fact that the Ritz projection $\mathcal{P}_h^2 v$ is a superconvergent approximation to the interpolation of v in $S^h(\Omega)$ globally over the whole domain (see Lemma 2 of [4] and Lemma 4 of [10]), we have

$$\|\nabla(\mathcal{I}_h^2 v - \mathcal{P}_h^2 v)\|_{(L^2(\Omega))^2} \leq Ch^3 \|v\|_{H^4(\Omega)}, \quad \forall v \in H^4(\Omega) \cap H_0^1(\Omega), \quad (3.7)$$

where $\mathcal{I}_h^2$ is the interpolation operator. From the construction of the partitions $\Pi_{3h/2}$ and Π_h , we have the following important relation

$$\mathcal{I}_{3h/2}^{3b} \mathcal{I}_h^2 v(\mathbf{x}) = \mathcal{I}_{3h/2}^{3b} v(\mathbf{x}). \quad (3.8)$$

Applying these and Lemma 3.3, we immediately obtain the estimate as follows

$$\begin{aligned} & \|\nabla(\mathcal{I}_{3h/2}^{3b} \mathcal{P}_h^2 v - v)\|_{(L^2(\Omega))^2} \\ & \leq \|\nabla(\mathcal{I}_{3h/2}^{3b} \mathcal{P}_h^2 v - \mathcal{I}_{3h/2}^{3b} \mathcal{I}_h^2 v)\|_{(L^2(\Omega))^2} + \|\nabla(\mathcal{I}_{3h/2}^{3b} \mathcal{I}_h^2 v - v)\|_{(L^2(\Omega))^2} \\ & \leq \|\nabla(\mathcal{P}_h^2 v - \mathcal{I}_h^2 v)\|_{(L^2(\Omega))^2} + \|\nabla(\mathcal{I}_{3h/2}^{3b} v - v)\|_{(L^2(\Omega))^2} \\ & \leq Ch^3 \|v\|_{H^4(\Omega)}. \end{aligned} \quad (3.9)$$

□

Similar results hold for other cases as stated in the following lemma.

Lemma 3.5 *Let $S^h(\Omega)$ be either a quadratic or a biquadratic finite element space defined on Π_h . Then we have the following estimates*

$$\left\| \nabla(\mathcal{I}_{2h}^{4b} \mathcal{P}_h^2 v - v) \right\|_{(L^2(\Omega))^d} \leq Ch^3 \|v\|_{(H^4(\Omega))}, \quad \forall v \in H^4(\Omega) \cap H_0^1(\Omega), \quad (3.10)$$

$$\left\| \nabla(\mathcal{I}_{2h}^{4t} \mathcal{P}_h^2 v - v) \right\|_{(L^2(\Omega))^d} \leq Ch^3 \|v\|_{(H^4(\Omega))}, \quad \forall v \in H^4(\Omega) \cap H_0^1(\Omega). \quad (3.11)$$

□

Remark: Before we proceed to prove the superconvergence results for the IQFES for problem (1.1), we briefly discuss the motivation of the IQFES, which comes from the proofs of Lemmas 3.4 and 3.5. It has been long noticed that the (quadratic) Ritz projection $\mathcal{P}_h^2 v$ of a smooth function v is globally superconvergent to the (quadratic) interpolation $\mathcal{I}_h^2 v$ of v . However, $\mathcal{I}_h^2 v$ is superconvergent to v only at certain special points (e.g, Gaussian points). Consequently, we can obtain the natural superconvergence results only at these special points. To obtain the global superconvergence one has to conduct some post-processing. In this paper, we use higher degree polynomial interpolation to do the post-processing. The boundedness of the higher degree interpolations in Lemmas 3.2 and 3.3 retains the global superconvergent estimate of the error between $\mathcal{P}_h^2 v$ and $\mathcal{I}_h^2 v$. Then the invariant property (3.8) implies that $\mathcal{I}_{3h/2}^3 \mathcal{I}_h^2 v$ has the same convergence rate to v as $\mathcal{I}_{3h/2}^2 v$ does. Thus, we can finally obtain the global superconvergence of the post-processed numerical solution.

□

Now we turn to the study of the superconvergence of the IQFES for problem (1.1). Namely we want to prove that $\nabla(\mathcal{I}_{3h/2}^{3b} U^{n+1}(\mathbf{x}) - u(\mathbf{x}, t^{n+1}))$ has third-order convergence rate. We decompose the error $\mathcal{I}_{3h/2}^{3b} U^{n+1}(\mathbf{x}) - u(\mathbf{x}, t^{n+1})$ as the sum of $\mathcal{I}_{3h/2}^{3b}(U^{n+1}(\mathbf{x}) - \mathcal{P}_h^2 u(\mathbf{x}, t^{n+1}))$ and $\mathcal{I}_{3h/2}^{3b} \mathcal{P}_h^2 u(\mathbf{x}, t^{n+1}) - u(\mathbf{x}, t^{n+1})$. The previous lemmas have established the superconvergent error bound for $\mathcal{I}_{3h/2}^{3b} \mathcal{P}_h^2 u(\mathbf{x}, t^{n+1}) - u(\mathbf{x}, t^{n+1})$. Thus, we only have to estimate $\mathcal{I}_{3h/2}^{3b}(U^{n+1}(\mathbf{x}) - \mathcal{P}_h^2 u(\mathbf{x}, t^{n+1}))$. The boundedness of $\mathcal{I}_{3h/2}^{3b}$ indicates that we only need to estimate $(U^n(\mathbf{x}) - \mathcal{P}_h^2 u(\mathbf{x}, t^{n+1}))$. This is the result of the next lemma. For convenience, we introduce the following auxiliary functions:

$$\xi^n(\mathbf{x}) \equiv U^n(\mathbf{x}) - \mathcal{P}_h^2 u(\mathbf{x}, t^n), \quad \eta(\mathbf{x}, t) \equiv \mathcal{P}_h^2 u(\mathbf{x}, t) - u(\mathbf{x}, t),$$

(3.12)

$$e^n(\mathbf{x}) \equiv U^n(\mathbf{x}) - u(\mathbf{x}, t^n), \quad \mathbf{x} \in \Omega, \quad t \in [0, T], \quad n = 0, 1, \dots, N.$$

As mentioned before, we will prove a superconvergence estimate for $\xi^{n+1}(\mathbf{x}) = U^{n+1}(\mathbf{x}) - \mathcal{P}_h^2 u(\mathbf{x}, t^{n+1})$, $n = 0, 1, \dots, N-1$. We only derive the error estimates for Scheme 1 in this lemma. The estimate for Scheme 2 can be obtained similarly.

Lemma 3.6 *There exists a constant C depending on $u(\mathbf{x}, t)$ such that*

$$\sup_{n=1,2,\dots,N} \left\{ \left\| \partial_t \xi^{n-1/2}(\mathbf{x}) \right\|_{L^2(\Omega)} + \left\| \xi^{n-1/2}(\mathbf{x}) \right\|_{H^1(\Omega)} \right\} \leq C \{h^3 + \tau^2\}. \quad (3.13)$$

Proof: We can rewrite (2.4) at time $t = t^n$ as follows:

$$\langle \Phi \partial_{tt} u^n, w \rangle + \mathcal{A}(u^{n,1/4}, w) = \langle R(\mathbf{x}) u^{n,1/4} + f^{n,1/4}(\mathbf{x}), w \rangle + \langle \gamma^n, w \rangle, \quad \forall w \in S^h(\Omega), \quad (3.14)$$

where $\gamma^n = \Phi(\mathbf{x})[\partial_{tt} u^n - (u_{tt}(\mathbf{x}, t))^{n,1/4}]$. From [8, 26, 27], we have the estimate

$$\|\gamma^n(\mathbf{x})\|_{L^2(\Omega)}^2 \leq C \tau^3 \left\| \frac{\partial^4 u(\mathbf{x}, t)}{\partial t^4} \right\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2. \quad (3.15)$$

Subtracting (3.14) from Scheme 1, we have

$$\langle \Phi \partial_{tt} e^n, w \rangle + \mathcal{A}(e^{n,1/4}, w) = \langle R(\mathbf{x}) e^{n,1/4}, w \rangle - \langle \gamma^n, w \rangle, \quad \forall w \in S^h(\Omega). \quad (3.16)$$

Equation (3.16) can be rewritten as follows:

$$\begin{aligned} & \langle \Phi \partial_{tt} \xi^n, w \rangle + \mathcal{A}(\xi^{n,1/4}, w) \\ &= \langle R(\mathbf{x}) \xi^{n,1/4}, w \rangle + \langle R(\mathbf{x}) \eta^{n,1/4}, w \rangle - \langle \Phi \partial_{tt} \eta^n, w \rangle - \langle \gamma^n, w \rangle, \end{aligned} \quad (3.17)$$

$$\forall w \in S^h(\Omega),$$

where we have used the property of the Ritz projection:

$$\mathcal{A}(\eta^{n,1/4}, w) = 0, \quad \forall w \in S^h(\Omega). \quad (3.18)$$

If we take $w = \partial_t \xi^n$ in (3.18), we have

$$\begin{aligned} & \frac{1}{2\tau} \left\{ \langle \Phi \partial_t \xi^{n+1/2}, \partial_t \xi^{n+1/2} \rangle - \langle \Phi \partial_t \xi^{n-1/2}, \partial_t \xi^{n-1/2} \rangle \right. \\ & \quad \left. + \mathcal{A}(\xi^{n+1/2}, \xi^{n+1/2}) - \mathcal{A}(\xi^{n-1/2}, \xi^{n-1/2}) \right\} \\ &= \langle R \xi^{n,1/4}, \partial_t \xi^n \rangle + \langle R \eta^{n,1/4}, \partial_t \xi^n \rangle - \langle \Phi \partial_{tt} \eta^n, \partial_t \xi^n \rangle - \langle \gamma^n, \partial_t \xi^n \rangle, \end{aligned} \quad (3.19)$$

$$n = 1, 2, \dots, N-1.$$

Multiplying (3.19) by 2τ , and summing the result for $n = 1, 2, \dots, N_1 - 1$, we obtain

$$\begin{aligned} & \langle \Phi \partial_t \xi^{N_1-1/2}, \partial_t \xi^{N_1-1/2} \rangle + \mathcal{A}(\xi^{N_1-1/2}, \xi^{N_1-1/2}) \\ & \leq C \left\{ \left\| \partial_t \xi^{1/2}(\mathbf{x}) \right\|_{L^2(\Omega)}^2 + \left\| \xi^{1/2}(\mathbf{x}) \right\|_{H^1(\Omega)}^2 \right\} \\ & + 2\tau \left| \sum_{n=1}^{N_1-1} \left\{ \langle R \xi^{n,1/4}, \partial_t \xi^n \rangle + \langle R \eta^{n,1/4}, \partial_t \xi^n \rangle - \langle \Phi \partial_{tt} \eta^n, \partial_t \xi^n \rangle - \langle \gamma^n, \partial_t \xi^n \rangle \right\} \right| \end{aligned} \quad (3.20)$$

We now estimate the first two terms on the right-hand side of (3.20). From (2.11) and (3.4), we have the following estimates:

$$\xi^0(\mathbf{x}) = 0, \quad (3.21)$$

$$\begin{aligned} \left\| \xi^1(\mathbf{x}) \right\|_{H^1(\Omega)}^2 & \leq C \mathcal{A}(\xi^1, \xi^1) = C \mathcal{A}(u^* - u(\mathbf{x}, t^1), \xi^1) \\ & \leq C \left\| u^*(\mathbf{x}) - u(\mathbf{x}, t^1) \right\|_{H^1(\Omega)} \left\| \xi^1(\mathbf{x}) \right\|_{H^1(\Omega)}; \end{aligned} \quad (3.22)$$

hence

$$\left\| \xi^1(\mathbf{x}) \right\|_{H^1(\Omega)} \leq C \left\| u^*(\mathbf{x}) - u(\mathbf{x}, t^1) \right\|_{H^1(\Omega)} \leq C\tau^3, \quad (3.23)$$

and

$$\left\| \xi^{1/2}(\mathbf{x}) \right\|_{H^1(\Omega)} \leq C\tau^3, \quad \left\| \partial_t \xi^{1/2}(\mathbf{x}) \right\|_{H^1(\Omega)} \leq C\tau^2. \quad (3.24)$$

Now we estimate the remaining terms on the right-hand side of (3.20). By (2.8) and (3.15), we have the following estimate:

$$2\tau \sum_{n=1}^{N_1-1} |\langle \gamma^n, \partial_t \xi^n \rangle| \leq C\tau \sum_{n=1}^{N_1} \left\| \partial_t \xi^{n-1/2}(\mathbf{x}) \right\|_{L^2(\Omega)}^2 + C\tau^4 \left\| \frac{\partial^4 u(\mathbf{x}, t)}{\partial t^4} \right\|_{L^2(0, T; L^2(\Omega))}^2. \quad (3.25)$$

Applying the summation by parts formula to the next-to-the-last term on the right-hand side of (3.20),

$$\begin{aligned} & \tau \left| \sum_{n=1}^{N_1-1} \langle \Phi \partial_{tt} \eta^n, \partial_t \xi^n \rangle \right| \\ & = \tau \left| - \sum_{n=2}^{N_1} \langle \Phi \partial_{tt}(\eta^n - \eta^{n-1}), \xi^{n-1/2} \rangle + \langle \Phi \partial_{tt} \eta^{N_1-1}, \xi^{N_1-1/2} \rangle - \langle \Phi \partial_{tt} \eta^1, \xi^{1/2} \rangle \right| \end{aligned}$$

$$\begin{aligned}
&\leq C \left\| \xi^{1/2} \right\|_{H^1(\Omega)}^2 + \varepsilon \left\| \xi^{N_1-1/2} \right\|_{H^1(\Omega)}^2 + C\tau \sum_{n=1}^{N_1-1} \left\| \xi^{n-1/2} \right\|_{H^1(\Omega)}^2 \\
&\quad + C \left\| \partial_{tt} \eta^{N_1-1} \right\|_{L^2(\Omega)}^2 + C \left\| \partial_{tt} \eta^1 \right\|_{L^2(\Omega)}^2 + C\tau \sum_{n=1}^{N_1-1} \left\| \partial_{tt} \left(\partial_t \eta^{n-1/2} \right) \right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.26}$$

By the Taylor expansion, we have

$$\begin{aligned}
\eta^{n+1}(\mathbf{x}) &= \eta^n(\mathbf{x}) + \eta_t^n(\mathbf{x})\tau + \int_{t^n}^{t^{n+1}} \eta_{tt}(\mathbf{x}, t)(t^{n+1} - t)dt, \\
\eta^{n-1}(\mathbf{x}) &= \eta^n(\mathbf{x}) - \eta_t^n(\mathbf{x})\tau + \int_{t^{n-1}}^{t^n} \eta_{tt}(\mathbf{x}, t)(t - t^{n-1})dt.
\end{aligned} \tag{3.27}$$

Thus, from (3.27) and the error estimates for the Ritz projection [24] we have

$$\left\| \partial_{tt} \eta^n \right\|_{L^2(\Omega)} \leq C \left\| \eta_{tt} \right\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch^3 \left\| u_{tt}(\mathbf{x}, t) \right\|_{L^\infty(0,T;H^3(\Omega))}. \tag{3.28}$$

Similarly, we have

$$\begin{aligned}
\tau \sum_{n=1}^{N_1-1} \left\| \partial_{tt} \left(\partial_t \eta^{n-1/2} \right) \right\|_{L^2(\Omega)}^2 &\leq C \left\| \eta_{ttt}(\mathbf{x}, t) \right\|_{L^2(0,T;L^2(\Omega))}^2 \\
&\leq Ch^6 \left\| u_{ttt}(\mathbf{x}, t) \right\|_{L^2(0,T;H^3(\Omega))}^2.
\end{aligned} \tag{3.29}$$

Combining (3.24) and (3.28),(3.29), we obtain

$$\begin{aligned}
&\tau \left| \sum_{n=1}^{N_1-1} \langle \Phi \partial_{tt} \eta^n, \partial_t \xi^n \rangle \right| \\
&\leq C \{h^6 + \tau^4\} + \varepsilon \left\| \xi^{N_1-1/2} \right\|_{H^1(\Omega)}^2 + C\tau \sum_{n=1}^{N_1-1} \left\| \xi^{n-1/2} \right\|_{H^1(\Omega)}^2.
\end{aligned} \tag{3.30}$$

In a similar way we can show that

$$\begin{aligned}
&2\tau \left| \sum_{n=1}^{N_1-1} \langle R\eta^{n,1/4}, \partial_t \xi^n \rangle \right| \\
&\leq C \{h^6 + \tau^4\} + \varepsilon \left\| \xi^{N_1-1/2} \right\|_{H^1(\Omega)}^2 + C\tau \sum_{n=1}^{N_1-1} \left\| \xi^{n-1/2} \right\|_{H^1(\Omega)}^2.
\end{aligned} \tag{3.31}$$

For the remaining term on the right-hand side of (3.20), it is easy to see that

$$2\tau \left| \sum_{n=1}^{N_1-1} \langle R\xi^{n,1/4}, \partial_t \xi^n \rangle \right| \leq C\tau \sum_{n=1}^{N_1-1} \left\{ \left\| \partial_t \xi^{n-1/2} \right\|_{L^2(\Omega)}^2 + \left\| \xi^{n-1/2} \right\|_{H^1(\Omega)}^2 \right\}. \tag{3.32}$$

Putting all the estimates above into (3.20), we obtain the following estimate:

$$\left\| \partial_t \xi^{N_1-1/2}(\mathbf{x}) \right\|_{L^2(\Omega)}^2 + \left\| \xi^{N_1-1/2}(\mathbf{x}) \right\|_{H^1(\Omega)}^2 \quad (3.33)$$

$$\begin{aligned} &\leq \varepsilon \left\| \xi^{N_1-1/2}(\mathbf{x}) \right\|_{H^1(\Omega)}^2 + C\tau \left\{ \left\| \partial_t \xi^{N_1-1/2}(\mathbf{x}) \right\|_{L^2(\Omega)}^2 + \left\| \xi^{N_1-1/2}(\mathbf{x}) \right\|_{H^1(\Omega)}^2 \right\} \\ &+ C\tau \sum_{n=1}^{N_1-1} \left\{ \left\| \partial_t \xi^{n-1/2}(\mathbf{x}) \right\|_{L^2(\Omega)}^2 + \left\| \xi^{n-1/2}(\mathbf{x}) \right\|_{H^1(\Omega)}^2 \right\} + C \{h^6 + \tau^4\}. \end{aligned} \quad (3.34)$$

Taking ε and τ small enough such that $\varepsilon = 1/4$ and $C\tau \leq 1/4$, and then applying Gronwall's inequality, we obtain

$$\sup_{n=1,2,\dots,N} \left\{ \left\| \partial_t \xi^{n-1/2}(\mathbf{x}) \right\|_{L^2(\Omega)} + \left\| \xi^{n-1/2}(\mathbf{x}) \right\|_{H^1(\Omega)} \right\} \leq C \{h^3 + \tau^2\}, \quad (3.35)$$

which shows the superconvergence of the gradient and the first-order temporal difference quotient of $\xi^{n-1/2}$.

□

Theorem 3.1 *Let $u(\mathbf{x}, t)$ be the exact solution of problem (1.1), $S^h(\Omega)$ be either the bi-quadratic or triangular quadratic finite element space defined on the partition Π_h , and assume that $u'(\mathbf{x}, t) \in L^\infty(0, T; H^4(\Omega))$, $u_t(\mathbf{x}, t)$ and $u_{tt}(\mathbf{x}, t) \in L^\infty(0, T; H^3(\Omega))$, $u_{ttt}(\mathbf{x}, t) \in L^2(0, T; H^3(\Omega))$, and $u_{tttt}(\mathbf{x}, t) \in L^2(0, T; L^2(\Omega))$. Then we have*

$$\sup_{n=1,2,\dots,N} \left\| \mathcal{I}_{3h/2}^{3b} U^{n-1/2}(\mathbf{x}) - u(\mathbf{x}, t^{n-1/2}) \right\|_{H^1(\Omega)} \leq C \{h^3 + \tau^2\}, \quad (3.36)$$

for $U^n(\mathbf{x})$ generated by Scheme 1, and

$$\sup_{n=0,1,\dots,N} \left\| \mathcal{I}_{3h/2}^{3b} U^n(\mathbf{x}) - u(\mathbf{x}, t^n) \right\|_{H^1(\Omega)} \leq C \{h^3 + \tau^2\}, \quad (3.37)$$

for $U^n(\mathbf{x})$ generated by Scheme 2.

Proof: Because of the similarity, we only present the proof for Scheme 1. Notice that

$$\begin{aligned} \nabla \left(\mathcal{I}_{3h/2}^{3b} U^{n-1/2}(\mathbf{x}) - u(\mathbf{x}, t^{n-1/2}) \right) &= \nabla \mathcal{I}_{3h/2}^{3b} \left(U^{n-1/2}(\mathbf{x}) - \mathcal{P}_h^2 u(\mathbf{x}, t^{n-1/2}) \right) \\ &+ \nabla \left(\mathcal{I}_{3h/2}^{3b} \mathcal{P}_h^2 u(\mathbf{x}, t^{n-1/2}) - u(\mathbf{x}, t^{n-1/2}) \right). \end{aligned} \quad (3.38)$$

For the first term on the right hand side of (3.38), using the results of Lemma 3.3 and Lemma 3.6, we have

$$\begin{aligned} &\left\| \nabla \mathcal{I}_{3h/2}^{3b} (U^{n-1/2}(\mathbf{x}) - \mathcal{P}_h^2 u(\mathbf{x}, t^{n-1/2})) \right\|_{(L^2(\Omega))^d} \\ &\leq C \left\| \nabla (U^{n-1/2}(\mathbf{x}) - \mathcal{P}_h^2 u(\mathbf{x}, t^{n-1/2})) \right\|_{(L^2(\Omega))^d} \leq C \{h^3 + \tau^2\}. \end{aligned} \quad (3.39)$$

By Lemma 3.4,

$$\left\| \nabla (\mathcal{I}_{3h/2}^{3b} \mathcal{P}_h^2 u^{n-1/2}(\mathbf{x}) - u(\mathbf{x}, t^{n-1/2})) \right\|_{(L^2(\Omega))^d} \leq Ch^3. \quad (3.40)$$

Then we obtain the estimate (3.36) by using (3.39) and (3.40) in (3.38).

□

Theorem 3.2 *Let $u(\mathbf{x}, t)$ be the exact solution of problems (1.1), $S^h(\Omega)$ be either the bi-quadratic or triangular quadratic finite element space defined on the partition Π_h , and assume that $u(\mathbf{x}, t) \in L^\infty(0, T; H^4(\Omega))$, $u_t(\mathbf{x}, t)$ and $u_{tt}(\mathbf{x}, t) \in L^\infty(0, T; H^3(\Omega))$, $u_{ttt}(\mathbf{x}, t) \in L^2(0, T; H^3(\Omega))$, and $u_{tttt}(\mathbf{x}, t) \in L^2(0, T; L^2(\Omega))$. Then, we have*

$$\sup_{n=1,2,\dots,N} \left\| \mathcal{I}_{2h}^{4i} U^{n-1/2}(\mathbf{x}) - u(\mathbf{x}, t^{n-1/2}) \right\|_{H^1(\Omega)} \leq C \{h^3 + \tau^2\}, \quad (3.41)$$

for $U^n(\mathbf{x})$ generated by Scheme 1, and

$$\sup_{n=0,1,\dots,N} \left\| \mathcal{I}_{2h}^{4i} U^n(\mathbf{x}) - u(\mathbf{x}, t^n) \right\|_{H^1(\Omega)} \leq C \{h^3 + \tau^2\}, \quad (3.42)$$

for $U^n(\mathbf{x})$ generated by Scheme 2. Here $i = b$ or $i = t$.

Proof: The argument is similar to that for Theorem 3.1 except that we should use Lemma 3.5 instead of Lemma 3.4.

□

Remark: Our theoretical analyses above show that $\mathcal{I}_{2h}^{4i} U^n(\mathbf{x})$ and $\mathcal{I}_{3h/2}^3 U^n(\mathbf{x})$ have the same asymptotic superconvergence rates. The numerical results in the next section verify the theoretical results and suggest that the theoretical estimates are sharp. This indicates that the quartic interpolation does not have any advantage over the cubic interpolation in the post-processing from the point of view of improving the asymptotic convergence rate in the gradient. However, the quartic interpolation does have some special properties such as: (1) $\mathcal{I}_{2h}^{4i}$ can be applied to post-process finite element solutions in polygonal domains, since the quartic interpolation is well-defined over a triangular partition Π_{2h} ; (2) our numerical experiments show that the gradient of $\mathcal{I}_{2h}^{4b} U^n(\mathbf{x})$ is usually more accurate than that of $\mathcal{I}_{3h/2}^{3b} U^n(\mathbf{x})$ in the sense of the absolute error, especially when $U^n(\mathbf{x})$ is obtained by the biquadratic finite elements. Of course, $\mathcal{I}_{2h}^{4b}$ is more expensive to implement than $\mathcal{I}_{3h/2}^{3b}$.

4 Numerical Experiments

In this section we present some numerical results to observe the performance of the IQFES for the following initial-boundary value problems for a second-order hyperbolic equation:

$$\begin{aligned}
u_{tt}(\mathbf{x}, t) &= (a(x, y)u_x(\mathbf{x}, t))_x + (a(x, y)u_y(\mathbf{x}, t))_y + f(\mathbf{x}, t), \\
u(\mathbf{x}, 0) &= x^\alpha(1 - x)\sin(\pi y), \quad \mathbf{x} \in \Omega, \\
u_t(\mathbf{x}, 0) &= 0, \quad \mathbf{x} \in \Omega, \\
u(\mathbf{x}, t) &= 0, \quad \mathbf{x} \in \partial\Omega \times (0, 1],
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
a(x, y) &= 1 + \sin(\pi x) + \sin(\pi y), \\
\Omega &= (0, 1) \times (0, 1), \\
\mathbf{x} &= (x, y).
\end{aligned}$$

$f(\mathbf{x}, t)$ was chosen such that

$$u(\mathbf{x}, t) = \cos(\pi t)x^\alpha(1 - x)\sin(\pi y), \quad \alpha = 4.75, \tag{4.2}$$

is the analytical solution for problem (4.1). We use a uniform rectangular partition Π_h with h_x and h_y being the mesh size in x and y , respectively. $h = \max\{h_x, h_y\}$. The triangular partition Π_h is obtained by subdividing each rectangular element in the partition above into two triangles along the diagonal line with positive slope. Then we can construct either a biquadratic or a quadratic finite element space $S^h(\Omega)$ depending on whether a rectangular partition or a triangular partition is used.

To observe the convergence rates of different IQFES introduced in this paper, we perform the numerical experiments using different space steps. The time step τ is always chosen to be $\tau = h^2$. We measure the error in semi- H^1 norm at $t = 1$. We present the data in Tables 1 - 4. Table 1 contains the semi- H^1 error of $U^N(\mathbf{x}) - u(\mathbf{x}, 1)$, where $U^N(\mathbf{x})$ is computed by the biquadratic finite elements. Linear regression shows a second-order convergence rate as predicted by the error estimates for biquadratic finite elements:

$$\left| U^N(\mathbf{x}) - u(\mathbf{x}, 1) \right|_{H^1} = 0.45 h^{1.99}.$$

Table 2 contains the semi- H^1 error of $\mathcal{I}_{3h/2}^{3b} U^N(\mathbf{x}) - u(\mathbf{x}, 1)$, where $U^N(\mathbf{x})$ is still computed by the biquadratic finite elements. Using linear regression we obtain third-order

convergence as predicted by the theorems in the previous section:

$$\left| \mathcal{I}_{3h/2}^3 U^n(\mathbf{x}) - u(\mathbf{x}, 1) \right|_{H^1} = 0.18 h^{2.97}.$$

Table 3 contains the semi- H^1 error of $\mathcal{I}_{2h}^{4b} U^N(\mathbf{x}) - u(\mathbf{x}, 1)$, where $U^N(\mathbf{x})$ is computed by the triangular quadratic finite elements. This time we also obtain third-order convergence as predicted by the theorems in the previous section:

$$\left| \mathcal{I}_{2h}^4 U^n(\mathbf{x}) - u(\mathbf{x}, 1) \right|_{H^1} = 0.39 h^{2.95} \quad (\text{triangular quadratic finite elements}).$$

Table 4 contains the semi- H^1 error of $\mathcal{I}_{2h}^{4b} U^N(\mathbf{x}) - u(\mathbf{x}, 1)$, where $U^N(\mathbf{x})$ is computed by the rectangular biquadratic finite elements. We again obtain third-order convergence as predicted by our theorems:

$$\left| \mathcal{I}_{2h}^{4b} U^n(\mathbf{x}) - u(\mathbf{x}, 1) \right|_{H^1} = 0.18 h^{3.18} \quad (\text{biquadratic finite elements}).$$

Tables 1 - 4 show that we can only obtain third-order asymptotic convergence rates for the gradients of the IQFES, no matter whether cubic or quartic interpolation is used in the post-processing. This observation coincides with our theoretical analyses in the previous section. In contrast, the gradient of $U^n(\mathbf{x})$ is only second-order accurate as is already well-known. Thus, the IQFES has a superconvergence rate of only one order higher than the convergence rate of the quadratic finite element solution.

Probably it is also interesting for us to look at the absolute errors in Tables 1 - 4. If we compare the digits in these tables, we have the following observations. First, all the three IQFES have significant improvements compared with the quadratic finite element solutions. Second, the numerical experiments in Tables 3 - 4 indicate that the gradient of $\mathcal{I}_{2h}^{4b} U^n(\mathbf{x})$ is more accurate if $U^n(\mathbf{x})$ is obtained by using biquadratic finite elements than if $U^n(\mathbf{x})$ is obtained by using the triangular quadratic finite elements. These two tables show the strong effect of the numerical solutions $U^n(\mathbf{x})$ on the final accuracy when the same kind of post-processing techniques are used. In fact, Tables 2 and 3 show that $\mathcal{I}_{3h/2}^{3b} U^n(\mathbf{x})$, with $U^n(\mathbf{x})$ computed by the biquadratic finite elements, is even more accurate than $\mathcal{I}_{2h}^{4b} U^n(\mathbf{x})$ with $U^n(\mathbf{x})$ computed by the quadratic finite elements. This indicates that we should use biquadratic finite elements whenever possible. Of course, the triangular finite elements can be used for general polygonal domain Ω .

To have a better understanding of the performance of different IQFES, we plot the pictures of the x component of the gradients of the (exact and numerical) solutions in Figures

17 - 20 at time $t = 1$. Here $h = 1/6$ and $\tau = 1/36$ are used. Figure 17 is the plot of $u_x(\mathbf{x}, 1)$. Figure 18 is the plot of $U_x^N(\mathbf{x})$, where $U^N(\mathbf{x})$ is the biquadratic finite element solution. We can see the obvious discontinuity from the picture. Moreover, the maximum of the $U_x^N(\mathbf{x})$ is about 0.9 while the maximum of $u_x(\mathbf{x}, 1)$ is 1. Figure 19 is the plot of $\frac{\partial \mathcal{I}_{3h/2}^{3b} U^N(\mathbf{x})}{\partial x}$. Now we can see that the discontinuity of $\frac{\partial \mathcal{I}_{3h/2}^{3b} U^N(\mathbf{x})}{\partial x}$ is much smaller than $U_x^N(\mathbf{x})$. The maximum of $\frac{\partial \mathcal{I}_{3h/2}^{3b} U^N(\mathbf{x})}{\partial x}$ is quite close to 1 now. Figure 20 is the plot of $\frac{\partial \mathcal{I}_{2h}^{4b} U^N(\mathbf{x})}{\partial x}$. The discontinuity of $\frac{\partial \mathcal{I}_{3h/2}^{4b} U^N(\mathbf{x})}{\partial x}$ is negligible. The maximum of $\frac{\partial \mathcal{I}_{3h/2}^{4b} U^N(\mathbf{x})}{\partial x}$ is almost 1. From these plots as well as the tables, we see that even though $\mathcal{I}_{2h}^{4b} U^N(\mathbf{x})$ and $\mathcal{I}_{3h/2}^{3b} U^N(\mathbf{x})$ have the same asymptotic convergence rates in their gradients, $\mathcal{I}_{2h}^{4b} U^N(\mathbf{x})$ performs much better in practice. We observed similar properties in the y component of the IQFES and omit the related plots here.

Before we conclude this section, we want to observe how the IQFES works for the initial boundary value problems whose solutions are not so smooth as required in the theorems. For this purpose we applied the IQFES to (4.1) with the $f(\mathbf{x}, t)$ chosen such that

$$u(\mathbf{x}, t) = \cos(\pi t) x^\alpha (1 - x) \sin(\pi y), \quad \alpha = 2.75,$$

is the analytical solution. It is easy to see that $u(\mathbf{x}, t)$ is in $H^3(\Omega)$ for fixed $t > 0$, but not in $H^4(\Omega)$. The errors are presented in Tables 5 and 6. Linear regression shows the following convergence rates:

$$\left| \mathcal{I}_{2h}^{4b} U^N(\mathbf{x}) - u(\mathbf{x}, 1) \right|_{H^1} = 0.56 \ h^{3.12}$$

and

$$\left| U^N(\mathbf{x}) - u(\mathbf{x}, 1) \right|_{H^1} = 0.24 \ h^{2.0}.$$

This suggests that the regularity requirement on the exact solutions in the theorems might be eased. The authors are currently working in this direction.

For simplicity of the presentations, we only present two computational examples in this section. We want to point out that the examples are representative. We have observed similar behavior from our other experiments not presented here.

In conclusion, we see that the IQFES introduced in this paper can significantly accelerate the convergence rates of the gradients of the numerical solutions. It is cheap to

implement and can be easily incorporated with the existing standard element solvers. It can be applied to improve the accuracy, either globally or locally. It is completely vectorizable and parallelizable. Our theoretical analyses guarantee the asymptotic superconvergence rates. The numerical experiments show the strong potential of the IQFES.

References

- [1] A.B. ANDREEV, and R.D. LAZAROV, Superconvergence of the gradient for quadratic triangular finite elements, *Numerical Methods for Partial Differential Equations*, 4, 15-32, 1988.
- [2] J. BARLOW, Optimal stress location in finite element models, *Int. J. Numer. Mech. Eng.*, 10, 243-251 (1976).
- [3] C.M CHEN, Superconvergence of finite element solutions and their derivatives, *Numer. Math. J. Chinese Univ.*, 3(1981), 118-125.
- [4] S.-S. CHOW and R. D. LAZAROV, Superconvergence analysis of flux computations for nonlinear problems, *Bull. Austral. Math. Soc.*, Vol. 40(1989), 465-479.
- [5] S.S. CHOW, G.F. CARAY, and R.D. LAZAROV, Natural and postprocessed superconvergence in semilinear problems, *Numerical Methods for Partial Differential Equations*, 7(1991), 245-259.
- [6] P. G. CIARLET, The finite element method for elliptic problems, North-Holland, Amsterdam, 1978.
- [7] J. DOUGLAS, T. DUPONT and M. WHEELER, A Galerkin procedure for approximating the flux on the boundary for elliptic and parabolic boundary value problems, *RAIRO Anal. Numer.* 2(1974), 47-59.
- [8] T. DUPONT, L^2 -estimates for Galerkin methods for second order hyperbolic equations. *SIAM J. Numer. Anal.* 10, (1973), 880-889.
- [9] R. EWING, T. LIN and R. FALK, Inverse and ill-posed problems in reservoir simulation, *Inverse and Ill-Posed Problems*, Academic Press, 1987, 483-497.
- [10] G. GOODSELL and J.R. WHITEMAN, Superconvergence of recovered gradients of piecewise quadratic finite element approximations. Part I: L_2 -error estimates, *Numerical Methods for Partial Differential Equations*, 1, 61-83, 1991.
- [11] G. GOODSELL and J.R. WHITEMAN, Superconvergence of recovered gradients of piecewise quadratic finite element approximations. Part II: L_∞ -error estimates, *Numerical Methods for Partial Differential Equations*, 7, 85-99, 1991.

- [12] C. JOHNSON, (1987): Numerical solutions of partial differential equations by the finite element method, Cambridge University Press, Cambridge.
- [13] M. KŘÍŽEK and R. NEITTAANMÄKI, On superconvergence techniques, *Acta Applicandae Mathematicae* 9(1987), 175-198.
- [14] N.D. LEVINE, Superconvergent recovery of the gradient from piecewise linear finite element approximations, *IMA J. Numer. Anal.*, 5(1985), 407-472.
- [15] Q. LIN, Global error expansion and superconvergence for higher order interpolation of finite elements, *J. Comp. Math.* 4, (1991).
- [16] Q. LIN, Interpolated Finite Element Methods and Global Error Expansions. (to appear).
- [17] Q. LIN, T. LÜ, and S. SHEN, Maximum norm estimate, extrapolation and optimal point of stresses for the finite element methods on the strongly regular triangulations, *J. Comput. Math.*, 1(1983), 376-383.
- [18] Q. LIN, H. WANG, and T. LIN, Interpolated finite element methods for second order hyperbolic equations and their global super-convergence, (accepted by *System Sciences and Mathematics*).
- [19] T. LIN and R. EWING, A direct numerical method for an inverse problem of hyperbolic equations, *Numerical Methods for Partial Differential Equations*, 8(1992), 551-574.
- [20] T. LIN and H. WANG, Solving a source term inverse problem of the diffusion equations by the interpolated quadratic finite element method, *Computational Methods in Water Resources IX*, Vol. I: Numerical Methods in Water Resources, 413-420.
- [21] T. LIN, H. WANG, and Q. LIN, A superconvergent post-processing technique for approximating the gradient of the solution to an elliptic boundary value problem (in preparation.)
- [22] L.A. OGANESJAN and L.A. RUHOVEC, An investigation of the rate of convergence of variational difference schemes for second order elliptic equations in a two-dimensional region with smooth boundary, *U.S.S.R. Computational Mathematics and Mathematical Physics*, Vol. 9, 5 (1969), 158-183.
- [23] G. STRANG AND G.J. FIX, *An analysis of the finite element method*, Prentice-Hall, Englewood Cliffs, 1973.

- [24] V. THOMÉE, Galerkin finite element methods for parabolic problems. *Lecture Notes in Math.* **1054**, Springer-Verlag, 1984.
- [25] V. THOMÉE, J.C. XU, and N.Y. ZHANG, Superconvergence of the gradient in piecewise linear finite element approximation to a parabolic problem, *SIAM J. Numer. Anal.*, Vol. 26, No. 3, 553-573, 1989.
- [26] H. WANG, Superconvergence of finite element solution and its derivatives for quasilinear second order hyperbolic equations. *Northeastern Math. J.*, **2**, (1986), 205–214.
- [27] H. WANG, Stability and convergence of the finite element methods for nonlinear second order hyperbolic equations. *Mathematica Numerica Sinica*, **9**, (1987), 87–99.
- [28] M.F. WHEELER, Galerkin procedure for estimating the flux two-point boundary value problems, *SIAM J. Numer. Anal.*, 11(1974), 764-768.
- [29] J.R. WHITEMAN and G. GOODSELL, Some gradient superconvergence results in the finite element method, *Numerical Analysis and Parallel Processing*, Lecture Notes in Mathematics, No. 1397, 182-258, Springer-Verlag, Berlin, 1987.
- [30] Y. YUAN and H. WANG, Error estimates for the finite element method for nonlinear hyperbolic equations. *J. Sys. Sci. and Math. Sci.* **5**, (1985), 161–171.
- [31] Q.D. ZHU, Uniform superconvergence estimates of derivatives for the finite element method, *Numer. Math. J. Chinese Univ.*, 5(1983), 311-318.
- [32] M. ZLÁMAL, Some superconvergence results in the finite element method, *Mathematical aspects of finite element method*, Lecture Notes in Mathematics, Vol. 606, Springer-Verlag, New York, 1977, 353-362.
- [33] M. ZLÁMAL, Superconvergence and reduced integration in the finite element method, *Math. Comp.* 32(1978), 663-685.

h	$ U^N(\mathbf{x}) - u(\mathbf{x}, 1) _{H^1}$	h	$ U^N(\mathbf{x}) - u(\mathbf{x}, 1) _{H^1}$
1/6	0.0127981	1/21	0.0010730
1/9	0.0057808	1/24	0.0008220
1/12	0.0032699	1/27	0.0006497
1/15	0.0020985	1/30	0.0005264
1/18	0.0014592	1/33	0.0004351

Table 1: Errors of the $U^N(\mathbf{x})$ in semi- H_1 norm, where $U^N(\mathbf{x})$ is computed by the biquadratic finite elements.

h	$ \mathcal{I}_{3h/2}^{3b}U^n(\mathbf{x}) - u(\mathbf{x}, 1) _{H^1}$	h	$ \mathcal{I}_{3h/2}^{3b}U^N(\mathbf{x}) - u(\mathbf{x}, 1) _{H^1}$
1/6	0.0008915	1/21	0.0000215
1/9	0.0002624	1/24	0.0000143
1/12	0.0001056	1/27	0.0000101
1/15	0.0000617	1/30	0.0000074
1/18	0.0000326	1/33	0.0000055

Table 2: Errors of $\mathcal{I}_{3h/2}^{3b}U^n(\mathbf{x})$ in semi- H_1 norm, where $U^N(\mathbf{x})$ is computed by the biquadratic finite elements.

h	$ \mathcal{I}_{2h}^{4b}U^N(\mathbf{x}) - u(\mathbf{x}, 1) _{H^1}$	h	$ \mathcal{I}_{2h}^{4b}U^N(\mathbf{x}) - u(\mathbf{x}, 1) _{H^1}$
1/6	0.0019759	1/26	0.0000264
1/10	0.0004454	1/30	0.0000178
1/14	0.0001631	1/34	0.0000118
1/18	0.0000769	1/38	0.0000087
1/22	0.0000439	1/42	0.0000063

Table 3: Errors of $\mathcal{I}_{2h}^{4b}U^N(\mathbf{x})$ in semi- H_1 norm, where $U^N(x)$ is obtained by the triangular quadratic finite elements

h	$ \mathcal{I}_{2h}^{4b}U^N(\mathbf{x}) - u(\mathbf{x}, 0.125) _{H^1}$	h	$ \mathcal{I}_{2h}^{4b}U^N(\mathbf{x}) - u(\mathbf{x}, 0.125) _{H^1}$
1/6	0.0006444	1/26	0.0000055
1/10	0.0001134	1/30	0.0000036
1/14	0.0000421	1/34	0.0000024
1/18	0.0000166	1/38	0.0000018
1/22	0.0000092	1/42	0.0000013

Table 4: Errors of $\mathcal{I}_{2h}^{4b}U^N(\mathbf{x})$ in semi- H_1 norm, where $U^N(x)$ is obtained by the biquadratic finite elements.

h	$ \mathcal{I}_{2h}^{4b}U^N(\mathbf{x}) - u(\mathbf{x}, 1) _{H^1}$	h	$ \mathcal{I}_{2h}^{4b}U^N(\mathbf{x}) - u(\mathbf{x}, 1) _{H^1}$
1/6	0.0022919	1/26	0.0000211
1/10	0.0004067	1/30	0.0000138
1/14	0.0001333	1/34	0.0000092
1/18	0.0000643	1/38	0.0000068
1/22	0.0000368	1/42	0.0000049

Table 5: Errors of the $\mathcal{I}_{2h}^{4b}U^N(\mathbf{x})$ in the semi- H_1 norm. Here $U^N(x)$ is the quadratic finite element solution and the exact solution is only in $H^3(\Omega)$.

h	$ U^N(\mathbf{x}) - u(\mathbf{x}, 1) _{H^1}$	h	$ U^N(\mathbf{x}) - u(\mathbf{x}, 1) _{H^1}$
1/6	0.0066243	1/26	0.0003333
1/10	0.0022776	1/30	0.0002504
1/14	0.0011529	1/34	0.0001949
1/18	0.0006960	1/38	0.0001561
1/22	0.0004656	1/42	0.0001278

Table 6: Errors of the $U^N(\mathbf{x})$ in the semi- H_1 norm, where $U^N(x)$ is the quadratic finite element solution and exact solution is only in $H^3(\Omega)$.

Figure 17: Plot of the x component of the true gradient.

Figure 18: The approximation of the x-derivative of the solution by the quadratic finite elements. Notice the obvious discontinuity in the x-direction.

Figure 19: The approximation of the x-derivative of the solution by the $\mathcal{I}_{3h/2}^{3b}U^n(\mathbf{x})$. The discontinuity in the x-direction is smaller.

Figure 20: The approximation of the x-derivative of the solution by the $\mathcal{I}_{2h}^{4b}U^n(\mathbf{x})$. The discontinuity in the x-direction is negligible.

#	Author/s	Title
1078	Michael Böhm & Mario Taboada,	Global existence and regularity of solutions of the nonlinear string equation
1079	Zhangxin Chen,	BDM mixed methods for a nonlinear elliptic problem
1080	J.J.L. Velázquez,	On the dynamics of a closed thermosyphon
1081	Frédéric Bonnans & Eduardo Casas,	Some stability concepts and their applications in optimal control problems
1082	Hong-Ming Yin,	$\mathcal{L}^{2,\mu}(Q)$ -estimates for parabolic equations and applications
1083	David L. Russell & Bing-Yu Zhang,	Smoothing and decay properties of solutions of the Korteweg-de Vries equation on a periodic domain with point dissipation
1084	J.E. Dunn & K.R. Rajagopal,	Fluids of differential type: Critical review and thermodynamic analysis
1085	Mary Elizabeth Bradley & Mary Ann Horn,	Global stabilization of the von Kármán plate with boundary feedback acting via bending moments only
1086	Mary Ann Horn & Irena Lasiecka,	Global stabilization of a dynamic von Kármán plate with nonlinear boundary feedback
1087	Vilmos Komornik,	Decay estimates for a petrovski system with a nonlinear distributed feedback
1088	Jesse L. Barlow,	Perturbation results for nearly uncoupled Markov chains with applications to iterative methods
1089	Jong-Shenq Guo,	Large time behavior of solutions of a fast diffusion equation with source
1090	Tongwen Chen & Li Qiu,	$\mathcal{H}_\infty$ design of general multirate sampled-data control systems
1091	Satyanad Kichenassamy & Walter Littman,	Blow-up surfaces for nonlinear wave equations, I
1092	Nahum Shimkin,	Asymptotically efficient adaptive strategies in repeated games, Part I: certainty equivalence strategies
1093	Caroline Fabre, Jean-Pierre Puel & Enrique Zuazua,	On the density of the range of the semigroup for semilinear heat equations
1094	Robert F. Stengel, Laura R. Ray & Christopher I. Marrison,	Probabilistic evaluation of control system robustness
1095	H.O. Fattorini & S.S. Sritharan,	Optimal chattering controls for viscous flow
1096	Kathryn E. Lenz,	Properties of certain optimal weighted sensitivity and weighted mixed sensitivity designs
1097	Gang Bao & David C. Dobson,	Second harmonic generation in nonlinear optical films
1098	Avner Friedman & Chaocheng Huang,	Diffusion in network
1099	Xinfu Chen, Avner Friedman & Tsuyoshi Kimura,	Nonstationary filtration in partially saturated porous media
1100	Walter Littman & Baisheng Yan,	Rellich type decay theorems for equations $P(D)u = f$ with f having support in a cylinder
1101	Satyanad Kichenassamy & Walter Littman,	Blow-up surfaces for nonlinear wave equations, II
1102	Nahum Shimkin,	Extremal large deviations in controlled I.I.D. processes with applications to hypothesis testing
1103	A. Narain,	Interfacial shear modeling and flow predictions for internal flows of pure vapor experiencing film condensation
1104	Andrew Teel & Laurent Praly,	Global stabilizability and observability imply semi-global stabilizability by output feedback
1105	Karen Rudie & Jan C. Willems,	The computational complexity of decentralized discrete-event control problems
1106	John A. Burns & Ruben D. Spies,	A numerical study of parameter sensitivities in Landau-Ginzburg models of phase transitions in shape memory alloys
1107	Gang Bao & William W. Symes,	Time like trace regularity of the wave equation with a nonsmooth principal part
1108	Lawrence Markus,	A brief history of control
1109	Richard A. Brualdi, Keith L. Chavey & Bryan L. Shader,	Bipartite graphs and inverse sign patterns of strong sign-nonsingular matrices
1110	A. Kersch, W. Morokoff & A. Schuster,	Radiative heat transfer with quasi-monte carlo methods
1111	Jianhua Zhang,	A free boundary problem arising from swelling-controlled release processes
1112	Walter Littman & Stephen Taylor,	Local smoothing and energy decay for a semi-infinite beam pinned at several points and applications to boundary control
1113	Srdjan Stojanovic & Thomas Svobodny,	A free boundary problem for the Stokes equation via nonsmooth analysis
1114	Bronislaw Jakubczyk,	Filtered differential algebras are complete invariants of static feedback
1115	Boris Mordukhovich,	Discrete approximations and refined Euler-Lagrange conditions for nonconvex differential inclusions
1116	Bei Hu & Hong-Ming Yin,	The profile near blowup time for solution of the heat equation with a nonlinear boundary condition
1117	Jin Ma & Jiongmin Yong,	Solvability of forward-backward SDEs and the nodal set of Hamilton-Jacobi-Bellman Equations
1118	Chaocheng Huang & Jiongmin Yong,	Coupled parabolic and hyperbolic equations modeling age-dependent epidemic dynamics with nonlinear diffusion

- 1119 **Jiongmin Yong**, Necessary conditions for minimax control problems of second order elliptic partial differential equations
- 1120 **Eitan Altman & Nahum Shimkin**, Worst-case and Nash routing policies in parallel queues with uncertain service allocations
- 1121 **Nahum Shimkin & Adam Shwartz**, Asymptotically efficient adaptive strategies in repeated games, part II: Asymptotic optimality
- 1122 **M.E. Bradley**, Well-posedness and regularity results for a dynamic Von Kármán plate
- 1123 **Zhangxin Chen**, Finite element analysis of the 1D full drift diffusion semiconductor model
- 1124 **Gang Bao & David C. Dobson**, Diffractive optics in nonlinear media with periodic structure
- 1125 **Steven Cox & Enrique Zuazua**, The rate at which energy decays in a damped string
- 1126 **Anthony W. Leung**, Optimal control for nonlinear systems of partial differential equations related to ecology
- 1127 **H.J. Sussmann**, A continuation method for nonholonomic path-finding problems
- 1128 **Yung-Jen Guo & Walter Littman**, The null boundary controllability for semilinear heat equations
- 1129 **Q. Zhang & G. Yin**, Turnpike sets in stochastic manufacturing systems with finite time horizon
- 1130 **I. Györi, F. Hartung & J. Turi**, Approximation of functional differential equations with time- and state-dependent delays by equations with piecewise constant arguments
- 1131 **I. Györi, F. Hartung & J. Turi**, Stability in delay equations with perturbed time lags
- 1132 **F. Hartung & J. Turi**, On the asymptotic behavior of the solutions of a state-dependent delay equation
- 1133 **Pierre-Alain Gremaud**, Numerical optimization and quasiconvexity
- 1134 **Jie Tai Yu**, Resultants and inversion formula for N polynomials in N variables
- 1135 **Avner Friedman & J.L. Velázquez**, The analysis of coating flows in a strip
- 1136 **Eduardo D. Sontag**, Control of systems without drift via generic loops
- 1137 **Yuan Wang & Eduardo D. Sontag**, Orders of input/output differential equations and state space dimensions
- 1138 **Scott W. Hansen**, Boundary control of a one-dimensional, linear, thermoelastic rod
- 1139 **Robert Lipton & Bogdan Vernescu**, Homogenization of two phase emulsions with surface tension effects
- 1140 **Scott Hansen & Enrique Zuazua**, Exact controllability and stabilization of a vibrating string with an interior point mass
- 1141 **Bei Hu & Jiongmin Yong**, Pontryagin Maximum principle for semilinear and quasilinear parabolic equations with pointwise state constraints
- 1142 **Mark H.A. Davis**, A deterministic approach to optimal stopping with application to a prophet inequality
- 1143 **M.H.A. Davis & M. Zervos**, A problem of singular stochastic control with discretionary stopping
- 1144 **Bernardo Cockburn & Pierre-Alain Gremaud**, An error estimate for finite element methods for scalar conservation laws
- 1145 **David C. Dobson & Fadil Santosa**, An image enhancement technique for electrical impedance tomography
- 1146 **Jin Ma, Philip Protter, & Jiongmin Yong**, Solving forward-backward stochastic differential equations explicitly — a four step scheme
- 1147 **Yong Liu**, The equilibrium plasma subject to skin effect
- 1148 **Ulrich Hornung**, Models for flow and transport through porous media derived by homogenization
- 1149 **Avner Friedman, Chaocheng Huang, & Jiongmin Yong**, Effective permeability of the boundary of a domain
- 1150 **Gang Bao**, A uniqueness theorem for an inverse problem in periodic diffractive optics
- 1151 **Angelo Favini, Mary Ann Horn, & Irena Lasiecka**, Global existence and uniqueness of regular solutions to the dynamic von Kármán system with nonlinear boundary dissipation
- 1152 **E.G. Kalnins & Willard Miller, Jr.**, Models of q -algebra representations: q -integral transforms and “addition theorems”
- 1153 **E.G. Kalnins, V.B. Kuznetsov & Willard Miller, Jr.**, Quadrics on complex Riemannian spaces of constant curvature, separation of variables and the Gaudin magnet
- 1154 **A. Kersch, W. Morokoff & Chr. Werner**, Selfconsistent simulation of sputtering with the DSMC method
- 1155 **Bing-Yu Zhang**, A remark on the Cauchy problem for the Korteweg-de Vries equation on a periodic domain
- 1156 **Gang Bao**, Finite element approximation of time harmonic waves in periodic structures
- 1157 **Tao Lin & Hong Wang**, Recovering the gradients of the solutions of second-order hyperbolic equations by interpolating the finite element solutions
- 1158 **Zhangxin Chen**, L^p -posteriori error analysis of mixed methods for linear and quasilinear elliptic problems
- 1159 **Todd Arbogast & Zhangxin Chen**, Homogenization of compositional flow in fractured porous media
- 1160 **L. Qiu, B. Bernhardsson, A. Rantzer, E.J. Davison, P.M. Young & J.C. Doyle**, A formula for computation of the real stability radius
- 1161 **Maria Inés Trespavsky**, Adaptive control of linear discrete time systems with external disturbances under inaccurate modelling: A case study
- 1162 **Petr Klouček & Franz S. Rys**, Stability of the fractional step Θ -scheme for the nonstationary Navier-Stokes equations
- 1163 **Eduardo Casas, Luis A. Fernández & Jiongmin Yong**, Optimal control of quasilinear parabolic equations
- 1164 **Darrell Duffie, Jin Ma & Jiongmin Yong**, Black’s consol rate conjecture
- 1165 **D.G. Aronson & J.L. Vazquez**, Anomalous exponents in nonlinear diffusion