**FINITE ELEMENT APPROXIMATION OF TIME
HARMONIC WAVES IN PERIODIC STRUCTURES**

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Abstract

Consider the diffraction of a time harmonic wave incident on a periodic surface of some inhomogeneous material. It is shown that the scattering (diffraction) problem may be modeled by a Helmholtz equation with transparent boundary conditions. The diffraction problem may be solved by a finite element method. In this paper, optimal error estimates for the finite element method are established.

Key words. diffractive optics, periodic structure, finite element method, error estimates, Helmholtz equations.

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Abbreviated title. Time Harmonic Waves in Periodic Structures

1 Introduction

Consider a time-harmonic electromagnetic plane wave incident on a slab of some optical material in $\mathbb{R}^3$, often referred to as grating material, which is periodic in the x_1 direction. Throughout, we assume that the material is invariant in the x_3 direction. The material which is characterized by its dielectric coefficient or index of refraction can be very general. In fact, the dielectric coefficient in this work needs only to be a bounded measurable function. However, the dielectric coefficients below and above the material are assumed to be fixed constants. Usually, it is air above the material and the dielectric coefficient below the material has an imaginary part that accounts for absorption [4]. The scattering problem is to study the electromagnetic field distributions.

Recently, the scattering of electromagnetic waves in a periodic (or grating) structure, or the direct scattering problem, has been studied extensively by either integral equation methods or variational approaches. We refer to [6], [9], and [15] for some interesting existence and uniqueness results. Numerical methods based on both an integral equation method and a variational (finite element) method have been developed. Computationally, compared with the integral equation method the variational approach seems to be superior: it is easier to be implemented; it may be used to handle more complicated materials. The goal of this paper is to study the convergence of a finite element method that is currently being used to solve the scattering (diffraction) problem. This is done by establishing some optimal error estimates. Related problems regarding error estimates for solving Helmholtz equations by finite element

methods have been studied by many researchers during the last two decades, see for example [13] (waveguides), [3] (boundary elements), and references therein. One should also consult [7] for an extensive account of finite element analysis of general elliptic equations.

The scattering theory in periodic structures has many applications in micro-optics, where periodic structures are often called *diffraction gratings*. We refer the reader to the book [12] for a description of this and other mathematical problems which arise in these applications. A good introduction to the problem of electromagnetic diffraction through periodic structures, along with some numerical methods, can be found in a collection of papers in [17].

The outline of this paper is as follows. In the next section, we describe the scattering problem. The approach is adopted from [9]. The problem can be reduced to a “box” with transparent boundary conditions. We also state some recent results on existence and uniqueness for the model problem. Section 3 is devoted to the finite element analysis. The main results are two optimal error estimates. In Section 4, we obtain error estimates in the case of truncated boundary operators.

We conclude this introduction by some general remarks on our method:

The general mathematical framework in our convergence analysis fashions roughly after Goldstein [13] in his study of waveguides in a homogeneous medium (constant index of refraction). However, the complication of the presence of inhomogeneous (variable index of refraction) materials with periodic structures requires not only a different formulation, but also new results of regularity, as well as error estimates presented here.

Our method enjoys a great generality in the sense that it allows very general diffraction structures. In particular, the dielectric coefficient (or index of refractive) can be a general spatially varying bounded measurable function.

In some sense, this paper may be served as our first attempt to understand various convergence properties of finite element methods for solving diffraction problems of periodic structures. A closely related but more difficult problem is to study the diffractive optics in a bi-periodic structure, *i.e.* a structure that is periodic in two orthogonal directions. Diffraction problems of bi-periodic structures are extremely important in many optical applications. In order to study the diffraction of bi-periodic structures, one has to deal with Maxwell equations of vector form rather than the scalar Helmholtz equation as in the singly periodic structure case. Results on existence and uniqueness were obtained in [11] by using an integral equation approach, and in [10] by using a variational method. Numerical schemes based on both the integral equation approach and the method of finite elements were implemented in [8] and [10]. It appears that once again the variational approach is favorable compared to the integral equation approach. The convergence analysis of this interesting problem is in progress and will be addressed elsewhere.

The method reported here may also be generalized to study convergence properties for numerical methods to solve scattering problems of nonlinear diffractive optics in periodic structures. Some preliminary results on the existence and uniqueness may be found in [2].

Figure 1: Problem geometry.

2 The scattering problem

Let us first specify the problem geometry. As shown in Figure 1, S_1 and S_2 are two simple curves imbedded in the strip

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -b < x_2 < b\},$$

where b is some positive constant. The medium in the region Ω between S_1 and S_2 is inhomogeneous. Above the surface S_1 and below the surface S_2 , the media are assumed to be homogeneous. The entire structure is taken to be periodic in the x_1 -direction. More precisely, we assume that S_1 and S_2 are periodic of period 2π with respect to the integers $Z = \{0, \pm 1, \pm 2, \dots\}$ in the sense that $(S_i \cap \{(x_1, x_2)\}) + (2\pi n, 0) = S_i \cap \{(x_1 + 2\pi n, x_2)\}$ for all $n \in Z$, $(x_1, x_2) \in \mathbb{R}^2$, and $i = 1, 2$.

Let $\Omega_1 = \{x \in \mathbb{R}^2 : x_2 > b\}$, $\Omega_2 = \{x \in \mathbb{R}^2 : x_2 < -b\}$. Define the boundaries $\Gamma_1 = \{x_2 = b\}$, $\Gamma_2 = \{x_2 = -b\}$. The curves S_1 and S_2 divide Ω into three connected components. Denote the component which meets Γ_1 by Ω_1^+ ; the component which meets Γ_2 by Ω_2^+ ; and let $\Omega_0 = \Omega - \bar{\Omega}_1^+ - \bar{\Omega}_2^+$.

Suppose that the whole space is filled with material with index of refraction $k = \omega c^{-1} \sqrt{\epsilon}$ where c is speed of light, ω is angular frequency. Suppose also that the media are nonmagnetic (with constant magnetic permeability). The magnetic permeability constant is assumed to

be one everywhere for convenience. Further, the index of refraction k satisfies

$$k(x) = \begin{cases} k_1, & \text{in } \Omega_1^+ \cup \bar{\Omega}_1, \\ k_0(x), & \text{in } \Omega_0, \\ k_2, & \text{in } \Omega_2^+ \cup \bar{\Omega}_2, \end{cases}$$

where $k_0 \in L^\infty$, k_1 and k_2 are constants, k_1 is real and positive, and $\operatorname{Re} k_2 > 0$, $\operatorname{Im} k_2 \geq 0$. The case $\operatorname{Im} k_2 > 0$ accounts for materials which absorb energy (see e.g. [4]).

We want to solve the Helmholtz equation

$$(\Delta + k^2)u = 0, \quad \text{in } \mathbb{R}^2, \quad (2.1)$$

when an incoming plane wave

$$u_I = e^{i\alpha x_1 - i\beta_1 x_2}$$

is incident on S from Ω_1 ,

$$\alpha = k_1 \sin \theta, \quad \beta_1 = k_1 \cos \theta,$$

and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ is the angle of incidence.

We are interested in “quasiperiodic” solutions u , that is, solutions u such that $ue^{-i\alpha x_1}$ are 2π -periodic. Define $u_\alpha = ue^{-i\alpha x_1}$. It is easily seen that if u satisfies (2.1) then u_α satisfies

$$(\Delta_\alpha + k^2)u_\alpha = 0, \quad \text{in } \mathbb{R}^2, \quad (2.2)$$

where the operator Δ_α is defined by

$$\Delta_\alpha = \Delta + 2i\alpha\partial_1 - |\alpha|^2.$$

We expand u_α in a Fourier series:

$$u_\alpha(x_1, x_2) = \sum_{n \in \mathbb{Z}} u_\alpha^{(n)}(x_2) e^{in x_1}, \quad (2.3)$$

where

$$u_\alpha^{(n)}(x_2) = \frac{1}{2\pi} \int_0^{2\pi} u_\alpha(x_1, x_2) e^{-in x_1} dx_1.$$

Introduce the sets

$$\Gamma'_1 = \{x \in \mathbb{R}^2 : x_2 = b_1\}, \quad \Gamma'_2 = \{x_2 = -b_1\},$$

with $0 < b_1 < b$, such that $\Omega_0 \subseteq \{-b_1 < x_2 < b_1\}$. Let $D_1 = \{x \in \mathbb{R}^2 : x_2 > b_1\}$, $D_2 = \{x \in \mathbb{R}^2 : x_2 < -b_1\}$.

Define for $j = 1, 2$ the coefficients

$$\beta_j^n(\alpha) = \begin{cases} \sqrt{k_j^2 - (n + \alpha)^2}, & k_j^2 > (n + \alpha)^2, \\ i\sqrt{(n + \alpha)^2 - k_j^2}, & k_j^2 < (n + \alpha)^2. \end{cases}$$

We assume that $k_j^2 \neq (n + \alpha)^2$ for all $n \in \mathbb{Z}$, $j = 1, 2$. This condition excludes “resonance” and ensures that a fundamental solution for (2.2) exists inside D_1 and D_2 . It then follows

from knowledge of the fundamental solution (see e. g. [6] and [11]) that inside D_1 and D_2 , u_α can be expressed as a sum of plane waves:

$$u_\alpha|_{D_j} = \sum_{n \in \mathbb{Z}} a_j^n e^{\pm i\beta_j^n(\alpha)x_2 + inx_1}, \quad j = 1, 2, \quad (2.4)$$

where the a_j^n are complex scalars.

We next impose a radiation condition on the scattering problem. Since β_j^n is real for at most finitely many n , there are only a finite number of propagating plane waves in the sum (2.4), the remaining waves are exponentially damped (or unbounded) as $|x_2| \rightarrow \infty$. We will insist that u_α is composed of bounded outgoing plane waves in D_1 and D_2 , plus the incident incoming wave u_I in D_1 .

From (2.3) and (2.4) we then have the condition that

$$u_\alpha^{(n)}(x_2) = \begin{cases} u_\alpha^{(n)}(b)e^{i\beta_1^n(\alpha)(x_2-b)}, & n \neq 0, \text{ in } D_1, \\ u_\alpha^{(0)}(b)e^{i\beta_1(x_2-b)} + e^{-i\beta_1 x_2} - e^{i\beta_1(x_2-2b)}, & n = 0, \text{ in } D_1, \\ u_\alpha^{(n)}(-b)e^{-i\beta_2^n(\alpha)(x_2+b)}, & \text{in } D_2. \end{cases} \quad (2.5)$$

From (2.5) we can then calculate the derivative of $u_\alpha^{(n)}(x_2)$ with respect to ν , the unit normal, on Ω :

$$\left. \frac{\partial u_\alpha^{(n)}}{\partial \nu} \right|_{\Gamma_j} = \begin{cases} i\beta_1^n(\alpha)u_\alpha^{(n)}(b), & n \neq 0, \text{ on } \Gamma_1, \\ i\beta_1 u_\alpha^{(0)}(b) - 2i\beta_1 e^{-i\beta_1 b}, & n = 0, \text{ on } \Gamma_1, \\ i\beta_2^n(\alpha)u_\alpha^{(n)}(-b), & \text{on } \Gamma_2. \end{cases} \quad (2.6)$$

Thus from (2.4), (2.6),

$$\left. \frac{\partial u_\alpha}{\partial \nu} \right|_{\Gamma_1} = \sum_{n \in \mathbb{Z}} i\beta_1^n(\alpha)u_\alpha^{(n)}(b)e^{inx_1} - 2i\beta_1 e^{-i\beta_1 b}, \quad (2.7)$$

$$\left. \frac{\partial u_\alpha}{\partial \nu} \right|_{\Gamma_2} = \sum_{n \in \mathbb{Z}} i\beta_2^n(\alpha)u_\alpha^{(n)}(-b)e^{inx_1}. \quad (2.8)$$

In particular, the above discussion yields the following simple result.

Lemma 2.1 *Suppose that $(n + \alpha)^2 > k_1^2$. Then*

$$u_\alpha^{(n)}(b) = u_\alpha^{(n)}(b_1)e^{-(b-b_1)\sqrt{(n+\alpha)^2 - k_1^2}}.$$

Similarly, if $(n + \alpha)^2 > k_2^2$, then

$$u_\alpha^{(n)}(-b) = u_\alpha^{(n)}(-b_1)e^{-(b-b_1)\sqrt{(n+\alpha)^2 - k_2^2}}.$$

Since the fields u_α are 2π -periodic in x_1 , we can move the problem from $\mathbb{R}^2$ to the quotient space (cylinder) $\mathbb{R}^2/(2\pi\mathbb{Z} \times \{0\})$. For the remainder of the paper, we shall identify Ω with the cylinder $\Omega/(2\pi\mathbb{Z} \times \{0\})$, and similarly for the boundaries $\Gamma_j \equiv \Gamma_j/2\pi\mathbb{Z}$. Thus from now on, all functions defined on Ω and Γ_j are implicitly 2π -periodic in the x_1 variable.

For functions $f \in H^{\frac{1}{2}}(\Gamma_j)$ (the Sobolev space of complex valued functions on the circle), define the operator T_j^α by

$$(T_j^\alpha f)(x_1) = \sum_{n \in \mathbb{Z}} i\beta_j^n(\alpha) f^{(n)} e^{inx_1}, \quad (2.9)$$

where $f^{(n)} = \frac{1}{2\pi} \int_0^{2\pi} f(x_1) e^{-inx_1}$, and equality is taken in the sense of distributions.

It is necessary in our study to understand the continuity properties of the above “Dirichlet-Neumann” maps. Fortunately, this is trivial by observing that T_j^α is a standard pseudodifferential operator (a convolution operator) of order one from the definition of $\beta_j^n(\alpha)$. Thus the standard theory on pseudodifferential operators (see e.g. [16]) applies.

Lemma 2.2 *For $j = 1, 2$, the operator $T_j^\alpha : H^{\frac{1}{2}}(\Gamma_j) \rightarrow H^{-\frac{1}{2}}(\Gamma_j)$ is continuous.*

The scattering problem can be formulated as follows: find $u_\alpha \in H^1(\Omega)$ such that

$$(\Delta_\alpha + k^2)u_\alpha = 0, \text{ in } \Omega, \quad (2.10)$$

$$(T_1^\alpha - \frac{\partial}{\partial \nu})u_\alpha = 2i\beta_1 e^{-i\beta_1 b}, \text{ on } \Gamma_1, \quad (2.11)$$

$$(T_2^\alpha - \frac{\partial}{\partial \nu})u_\alpha = 0, \text{ on } \Gamma_2. \quad (2.12)$$

The scattering problem has an equivalent variational form: find $u_\alpha \in H^1(\Omega)$ such that

$$a(u_\alpha, \phi) = (f, \phi), \quad \forall \phi \in H^1(\Omega), \quad (2.13)$$

here the sesquilinear form is defined by

$$\begin{aligned} a(w_1, w_2) &= \int_\Omega \nabla w_1 \cdot \nabla \overline{w_2} - \int_\Omega (k^2 - \alpha^2) w_1 \overline{w_2} - 2i\alpha \int_\Omega (\partial_1 w_1) \overline{w_2} \\ &\quad - \int_{\Gamma_1} (T_1^\alpha w_1) \overline{w_2} - \int_{\Gamma_2} (T_2^\alpha w_1) \overline{w_2}, \end{aligned}$$

and

$$(f, \phi) = - \int_{\Gamma_1} 2i\beta_1 e^{-i\beta_1 b} \overline{\phi},$$

where $\int_{\Gamma_j}$ represents the dual pairing of $H^{-\frac{1}{2}}(\Gamma_j)$ with $H^{\frac{1}{2}}(\Gamma_j)$.

Theorem 2.1 *(Existence and uniqueness) Suppose that $k \in L^\infty$. Then for all but a discrete set of frequencies ω , there exists a unique solution $u_\alpha \in H^2(\Omega)$ to the variational problem (2.13).*

Remarks on Theorem 2.1. The proof of a slightly different version of Theorem 2.1 may be found in [9] or [2]. In fact, it was proved that under the assumption of Theorem 2.1, there exists a unique solution $u_\alpha \in H^1(\Omega)$. An application of the elliptic regularity theory will then deduce further that $u_\alpha \in H^2(\Omega)$. However for $k \in L^\infty$, it seems that one can not expect better regularity for u_α . It would be interesting to study the improvement of regularity on u_α when the index of refraction k is more regular.

It was also shown in [9] that (2.13) admits a unique solution $u_\alpha \in H^1(\Omega)$ for a given range of incidence angles, provided that the frequency ω is sufficiently small. Thus in this case the problem is well-posed without excluding a discrete set of frequencies.

3 Finite element analysis

To simplify the notations, from now on, we will use β_j^n to represent $\beta_j^n(\alpha)$, and u to represent u_α . For $s \in \mathbb{R}$, H^s stands for the standard Sobolev space of complex valued functions, whose norm is denoted by $\|\cdot\|_{H^s}$. For an integer m , the semi-norm $|\cdot|_m$ is defined by $|\cdot|_m = \max_{q=m} \|D^q \cdot\|_{L^2}$. See [1] or [14] for a complete account of Sobolev spaces.

Let $\{S^h : h \in (0, 1]\}$ denote a family of finite dimensional subspaces of H^1 , where h stands for the maximum mesh size after partitioning Ω into simple spaces. We make the following assumption on the subspaces S^h : for $v \in H^l(\Omega)$, $l \geq 2$,

$$\inf_{\psi \in S^h} (\|v - \psi\|_{L^2(\Omega)} + h\|v - \psi\|_{H^1(\Omega)} + h^{1/2}\|v - \psi\|_{L^2(\Gamma_1)} + h^{1/2}\|v - \psi\|_{L^2(\Gamma_2)} + h\|v - \psi\|_{H^{1/2}(\Gamma_1)} + h\|v - \psi\|_{H^{1/2}(\Gamma_2)}) \leq Ch^{l'}\|v\|_{H^{l'}(\Omega)}, \quad (3.1)$$

where the constant C is independent of h and v , and for any integer $l' \in [2, l]$. The assumption (3.1) is somewhat standard and satisfied by the usual finite element spaces in practice, we refer the reader to the excellent book [7] for more discussions and references on approximation properties of various types of finite element spaces.

We define the finite element approximation of the solution u of (2.13) by the following equation, for each $v^h \in S^h$,

$$a(u^h, v^h) = (f, v^h). \quad (3.2)$$

In fact, this provides the basic idea for solving our model equation, *i.e.* one first chooses a basis of S^h , $\{\phi_1, \phi_2, \dots, \phi_k\}$, which is a finite set according to the definition of S^h . Substituting the expression of

$$u^h = c_1\phi_1 + c_2\phi_2 + \dots + c_k\phi_k$$

into the equation (3.2), by choosing $v^h = \phi_i$, $i = 1, \dots, k$, one gets a system of linear equations. Solving this system for $\{c_j\}$ then leads to the approximation of u in S^h .

We begin by establishing the well-posedness of the problem (3.2).

Theorem 3.1 *Suppose that $u \in H^l(\Omega)$ ($l \geq 2$) satisfies (2.13) with $k \in L^\infty$. Suppose also that the family of finite element spaces $\{S^h\}$ satisfies the assumption (3.1). Then there exists $h_0 \in (0, 1]$ such that for $h \in (0, h_0)$ the problem (3.2) attains a unique solution u^h . Moreover, the following estimates hold*

$$\begin{aligned} \|u - u^h\|_{L^2(\Omega)} &\leq Ch^l\|u\|_{H^l(\Omega)}, \\ \|u - u^h\|_{H^1(\Omega)} &\leq Ch^{l-1}\|u\|_{H^l(\Omega)}, \end{aligned}$$

where C depends on $\|k\|_{L^\infty}$ but is independent of h and u .

Note that because of the finite dimensionality of S^h , the uniqueness implies the well-posedness. Further, it is easy to show that the uniqueness follows from the estimates in the statement of Theorem 3.1 together with the well-posedness of the continuous problem. Thus, in order to establish the theorem, it suffices to prove the following two lemmas.

To simplify the proofs, we will employ a positive constant C as a generalized constant whose precise value is not required and may change line by line. Of course, we must point out any dependence of C on various parameters.

The following simple ϵ -inequality will be used frequently in our proofs.

Proposition 3.1 For $a, b \in \mathbb{R}$, there exist positive constants ϵ and $C(\epsilon)$ such that

$$ab \leq \epsilon a^2 + C(\epsilon)b^2.$$

Denote $e_h = u - u^h$.

Lemma 3.1 There exists a constant $h_1 \in (0, 1]$, such that for $h \in (0, h_1)$ the estimate

$$\|e_h\|_{H^1(\Omega)}^2 \leq C(k)(h^{2l-2}\|u\|_{H^l(\Omega)}^2 + \|e_h\|_{L^2(\Omega)}^2)$$

holds, where the constant $C(k)$ depends on $\|k\|_{L^\infty}$ but is independent of h and u .

Proof. Because of the periodicity of e_h , we have

$$a(e_h, e_h) = \|\nabla e_h\|_{L^2(\Omega)}^2 - \int_{\Omega} (k^2 - \alpha^2)|e_h|^2 - \int_{\Gamma_1} (T_1^\alpha e_h) \bar{e}_h - \int_{\Gamma_2} (T_2^\alpha e_h) \bar{e}_h. \quad (3.3)$$

Recall that for $j = 1, 2$

$$T_j^\alpha e_h|_{\Gamma_j} = \sum_{n \in Z} i\beta_j^n (e_h|_{\Gamma_j})^{(n)} e^{inx_1}.$$

Thus

$$\begin{aligned} \operatorname{Re} a(e_h, e_h) &= |e_h|_1^2 + \int_{\Omega} \alpha^2 |e_h|^2 - \int_{\Omega} \operatorname{Re}(k^2) |e_h|^2 \\ &\quad - \operatorname{Re} \left(\int_{\Gamma_1} \sum_{n \in Z} i\beta_1^n (\alpha) (e_h)^{(n)} e^{inx_1} \bar{e}_h + \int_{\Gamma_2} \sum_{n \in Z} i\beta_2^n (\alpha) (e_h)^{(n)} e^{inx_1} \bar{e}_h \right). \end{aligned}$$

Introduce

$$\Lambda_j^+(\alpha) = \{n \in Z, \operatorname{Im} \beta_j^n(\alpha) = 0\}, \quad \Lambda_j^-(\alpha) = Z - \Lambda_j^+(\alpha),$$

then

$$\begin{aligned} \operatorname{Re} \int_{\Gamma_j} T_j^\alpha e_h \bar{e}_h &= \operatorname{Re} \int_{\Gamma_j} \sum_{n \in Z} i\beta_j^n (\alpha) (e_h)^{(n)} e^{inx_1} \bar{e}_h \\ &= 2\pi \sum_{n \in \Lambda_j^-(\alpha)} i\beta_j^n (\alpha) |e_h^{(n)}|_{\Gamma_j}|^2. \end{aligned}$$

Note that $i\beta_j^n(\alpha) < 0$ when $n \in \Lambda_j^-(\alpha)$. Thus,

$$|e_h|_1^2 + \int_{\Omega} \alpha^2 |e_h|^2 + 2\pi \sum_{n \in \Lambda_j^-(\alpha), j=1,2} (-i\beta_j^n) |e_h^{(n)}|_{\Gamma_j}|^2 = \int_{\Omega} \operatorname{Re}(k^2) |e_h|^2 + \operatorname{Re} a(e_h, e_h). \quad (3.4)$$

It is easy to see that $\forall \psi \in S^h$,

$$a(u - u^h, u - u^h) = a(u - u^h, u - \psi),$$

in fact, this follows from knowing that

$$a(u^h, \psi - u^h) = (f, \psi - u^h) \text{ and } a(u, \psi - u^h) = (f, \psi - u^h).$$

Due to the finite dimensionality of S^h , the infimum in (3.1) is actually assumed. We choose particularly the function ψ such that the infimum attains for $v = u$ in (3.1).

Therefore, set $C_0 = \|k^2 - \alpha^2\|_{L^\infty}$, Proposition 3.1 yields that, for some small positive constant ϵ ,

$$\begin{aligned}
|a(e_h, e_h)| &= |a(e_h, u - \psi)| \\
&= \left| \int_{\Omega} \nabla e_h \cdot \nabla (\overline{u - \psi}) - \int_{\Omega} (k^2 - \alpha^2) e_h (\overline{u - \psi}) - 2i\alpha \int_{\Omega} \frac{\partial}{\partial x_1} e_h (\overline{u - \psi}) \right| \\
&\quad - \int_{\Gamma_1} (T_1^\alpha e_h) (\overline{u - \psi}) - \int_{\Gamma_2} (T_2^\alpha e_h) (\overline{u - \psi}) \\
&\leq \epsilon |e_h|_1^2 + C(\epsilon) |u - \psi|_1^2 + h^2 C_0^2 |e_h|^2 / 2 + 3h^{-2} \|u - \psi\|^2 / 2 + \alpha^2 h^2 |e_h|_1^2 \\
&\quad + \left| \int_{\Gamma_1} (T_1^\alpha e_h) (\overline{u - \psi}) + \int_{\Gamma_2} (T_2^\alpha e_h) (\overline{u - \psi}) \right|. \tag{3.5}
\end{aligned}$$

The continuity property of the operator T_j^α , Lemma 2.2, and an application of the trace theorem and Proposition 3.1 give

$$\begin{aligned}
\left| \int_{\Gamma_j} (T_j^\alpha e_h) (\overline{u - \psi}) \right| &\leq \|T_j^\alpha e_h\|_{H^{-1/2}(\Gamma_j)} \cdot \|u - \psi\|_{H^{1/2}(\Gamma_j)} \\
&\leq C \|e_h\|_{H^{1/2}(\Gamma_j)} \|u - \psi\|_{H^{1/2}(\Gamma_j)} \\
&\leq \epsilon \|e_h\|_{H^1(\Omega)}^2 + h^{-2} C(\epsilon) h^2 \|u - \psi\|_{H^{1/2}(\Gamma_j)}^2. \tag{3.6}
\end{aligned}$$

It follows from the estimate (3.1) that

$$|a(e_h, e_h)| \leq 3\epsilon \|e_h\|_{H^1(\Omega)}^2 + \alpha^2 h^2 |e_h|_1^2 + C(\epsilon) h^{2l-2} \|u\|_{H^l(\Omega)}^2. \tag{3.7}$$

Now let h_1 be a constant that satisfies $\alpha^2 h_1^2 + 3\epsilon \leq 1$. Then for $h \in (0, h_1)$, we obtain from the estimates (3.4) and (3.7) that

$$\|e_h\|_{H^1(\Omega)}^2 \leq C(k) (h^{2l-2} \|u\|_{H^l(\Omega)}^2 + \|e_h\|_{L^2(\Omega)}^2).$$

□

Next, we want to estimate the L^2 -norm of e_h .

Lemma 3.2 *There exists a constant $h_2 \in (0, 1]$, such that the following estimate holds*

$$\|e_h\|_{L^2(\Omega)} \leq C(k) h \|e_h\|_{H^1(\Omega)},$$

where the constant $C(k)$ depends on $\|k\|_{L^\infty}$ but is independent of h and u .

Proof. We use a duality technique. From the definition,

$$\|e_h\|_{L^2(\Omega)} = \sup_{\phi \in C_0^\infty(\Omega)} \frac{|(e_h, \phi)|}{\|\phi\|_{L^2(\Omega)}}. \tag{3.8}$$

For $\alpha_1 = -\alpha$, introduce a new 2π -periodic function w that satisfies

$$-(\Delta_{\alpha_1} + k^2)w = \bar{\phi} \text{ in } \Omega, \quad (3.9)$$

$$(T_1^{\alpha*} - \frac{\partial}{\partial \nu})w = 0 \text{ on } \Gamma_1, \quad (3.10)$$

$$(T_2^{\alpha*} - \frac{\partial}{\partial \nu})w = 0 \text{ on } \Gamma_2, \quad (3.11)$$

where the superscript $*$ stands for the adjoint operator. Since T_j^α for $j = 1, 2$ are pseudodifferential operators of order one, the standard calculus of pseudodifferential operators [16] yields that the existence of $T_j^{\alpha*}$, and moreover they remain to be pseudodifferential operators of order one. Actually, in this simple case, explicit expressions of the adjoint operators are available.

Thus, a simple integration by parts reveals that

$$\begin{aligned} |(e_h, \phi)| &= |(e_h, -(\Delta_\alpha + k^2)w)| \\ &= |a(e_h, \bar{w})| \\ &= |a(e_h, \bar{w} - \psi)| \end{aligned}$$

for any $\psi \in S^h$. In particular, ψ can be chosen in such a way that the infimum is assumed for $v = w$ in the estimate (3.1). Similar to the arguments as in the proof of Lemma 3.1, we have by applying the Cauchy-Schwarz inequality to (3.5) and (3.6) that

$$|a(e_h, \bar{w} - \psi)| \leq Ch \|\bar{w}\|_{H^2(\Omega)} (|e_h|_1 + C(k)h \|e_h\|_{L^2(\Omega)} + C \|e_h\|_{H^1(\Omega)}) . \quad (3.12)$$

The standard elliptic estimates give that

$$\|w\|_{H^2(\Omega)} \leq C \|\phi\|_{L^2(\Omega)} ,$$

therefore, we can find a positive constant $h_2 \leq 1$ such that for $h \in (0, h_2)$ the estimates (3.8) and (3.12) yield

$$\|e_h\|_{L^2(\Omega)} \leq C(k)h \|e_h\|_{H^1(\Omega)} .$$

□

Finally, the proof of Theorem 3.1 will be completed if one chooses $h_0 = \min\{h_1, h_2\}$ and combines the results in Lemma 3.1 and Lemma 3.2.

4 Approximation of the boundary operators

It is evident that, in practice, one cannot compute the operators T_j^α from the (full) infinite series expansions. Thus it is necessary to obtain appropriate error estimates when truncations of these boundary operators take place.

To simplify the notations, from now on, T_j will stand for the operator T_j^α for $j = 1, 2$. We consider the approximation of the boundary operators by T_j^N defined as for $j = 1, 2$

$$T_j^N f(x_1) = \sum_{|n| < N} i\beta_j^n(\alpha) f^{(n)} e^{inx_1} .$$

Define u_N^h , the finite element approximation to the solution u by the equation

$$a^N(u_N^h, v^h) = (f, v^h), \quad \forall v^h \in S^h, \quad (4.1)$$

where

$$\begin{aligned} a^N(w_1, w_2) = & \int_{\Omega} \nabla w_1 \cdot \nabla \bar{w}_2 - \int_{\Omega} (k^2 - \alpha^2) w_1 \bar{w}_2 - 2i\alpha \int_{\Omega} (\partial_1 w_1) \bar{w}_2 \\ & - \int_{\Gamma_1} (T_1^N w_1) \bar{w}_2 - \int_{\Gamma_2} (T_2^N w_1) \bar{w}_2. \end{aligned}$$

Denote

$$e_{h,N} = u - u_N^h.$$

It is our goal in this section to show that the problem (4.1) is well-posed for N sufficiently large and h sufficiently small.

Similar to the discussions in Section 3, we define

$$\Lambda_j^+ = \{n \in Z, |n| < N, \operatorname{Im} \beta_j(\alpha) = 0\}, \quad \Lambda_j^- = \{n \in Z, |n| < N\} - \Lambda_j^+,$$

Recall that

$$\Gamma'_1 = \{x_2 = b_1\}, \text{ and } \Gamma'_2 = \{x_2 = -b_1\},$$

where $b_1 < b$, and $\Omega_0 \subseteq \{-b_1 < x_2 < b_1\}$.

Theorem 4.1 *Suppose that $u \in H^l(\Omega)$ ($l \geq 2$) satisfies (2.13) with $k \in L^\infty$. Suppose also that the family of finite element spaces $\{S^h\}$ satisfies the assumption (3.1). Then there exist $h_0 \in (0, 1]$ and an integer N_0 independent of h_0 , such that for $h \in (0, h_0)$ and $N \geq N_0$, the problem (3.2) attains a unique solution u_N^h . Moreover, the following estimates hold, for $0 \leq s \leq 1/2$,*

$$\|e_{h,N}\|_{L^2(\Omega)} \leq Ch \sum_{j=1,2} e^{-(b-b_1)\sqrt{(N-|\alpha|)^2-k_j^2}} \|u\|_{H^{1/2}(\Gamma'_j)} + (C(k)h + CN^{-s}) \|e_{h,N}\|_{H^1(\Omega)}. \quad (4.2)$$

and

$$\begin{aligned} \|e_{h,N}\|_{H^1(\Omega)} &+ \left(\sum_{n \in \Lambda_j^-, j=1,2} (-i\beta_j^n) e_{h,N}^{(n)}|_{\Gamma_j} \right)^{1/2} \\ &\leq C(k)h^{l-1} \|u\|_{H^l(\Omega)} + C \sum_{j=1,2} e^{-\sqrt{(b-b_1)\sqrt{(N-|\alpha|)^2-k_j^2}}} \|u\|_{H^{1/2}(\Gamma'_j)}. \end{aligned} \quad (4.3)$$

where C depends on $\|k\|_{L^\infty}$, but is independent of h , N , and u .

Proof. As discussed in Section 3, it suffices to prove the estimates (4.2) and (4.3) in order to establish the wellposedness of the discrete problem.

Clearly by taking the real parts in the expression of $a^N(e_{h,N}, e_{h,N})$, we get

$$|e_{h,N}|_1^2 + \int_{\Omega} \alpha^2 |e_{h,N}|^2 + 2\pi \sum_{n \in \Lambda_j^-, j=1,2} (-i\beta_j^n) |e_{h,N}^{(n)}|_{\Gamma_j}|^2 = \int_{\Omega} \operatorname{Re}(k^2) |e_{h,N}|^2 + \operatorname{Re} a^N(e_{h,N}, e_{h,N}). \quad (4.4)$$

Since u and u_N^h are solutions to the model problem and its approximation respectively, for $\psi \in S^h$ we have

$$\begin{aligned}
a^N(e_{h,N}, e_{h,N}) &= a^N(e_{h,N}, u - \psi) + a^N(u - u_N^h, \psi - u_N^h) \\
&= a^N(e_{h,N}, u - \psi) + a^N(u, \psi - u_N^h) - a^N(u_N^h, \psi - u_N^h) \\
&= a^N(e_{h,N}, u - \psi) + a^N(u, \psi - u_N^h) - a(u, \psi - u_N^h) \\
&= a^N(e_{h,N}, u - \psi) - \sum_{j=1,2} \int_{\Gamma_j} \sum_{|n| \geq N} i\beta_j^n u^{(n)}|_{\Gamma_j} e^{inx_1} \overline{(\psi - u_N^h)}|_{\Gamma_j} \\
&= a^N(e_{h,N}, u - \psi) - \sum_{|n| \geq N, j=1,2} i\beta_j^n u^{(n)}|_{\Gamma_j} \overline{(\psi - u_N^h)}^{(n)}|_{\Gamma_j}. \tag{4.5}
\end{aligned}$$

In particular, we again choose $\psi \in S^h$ so that the infimum is assumed in (3.1) for $v = u$. Similar to Section 3, for sufficiently small h , the first term on the right hand side may be estimated as follows

$$|a^N(e_{h,N}, u - \psi)| \leq 2\epsilon \|e_{h,N}\|_{H^1(\Omega)}^2 + C(\epsilon) h^{2l-2} \|u\|_{H^l(\Omega)}^2 + \alpha^2 h^2 |e_h|_1^2 + \epsilon \sum_{n \in \Lambda_j^-, j=1,2} (-i\beta_j^n) |e_{h,N}^{(n)}|_{\Gamma_j}|^2. \tag{4.6}$$

Actually, this can be done by establishing the following estimate, for $j = 1, 2$,

$$\begin{aligned}
& \left| \int_{\Gamma_j} (T_j^N e_{h,N}) \overline{(u - \psi)} \right| = \left| \sum_{|n| < N} \int_{\Gamma_j} i\beta_j^n e_{h,N}^{(n)} \overline{(u - \psi)} e^{inx_1} \right| \\
& \leq C \|e_{h,N}\|_{L^2(\Omega)} \cdot \|\overline{u - \psi}\|_{L^2(\Omega)} + 2\pi \sum_{n \in \Lambda_j^-} (-i\beta_j^n) |e_{h,N}^{(n)}|_{\Gamma_j} \overline{(u - \psi)}^{(n)}|_{\Gamma_j}| \\
& \leq C \|e_{h,N}\|_{L^2(\Omega)} \cdot \|\overline{u - \psi}\|_{L^2(\Omega)} + \epsilon \sum_{n \in \Lambda_j^-} (-i\beta_j^n) |e_{h,N}^{(n)}|_{\Gamma_j}|^2 + C(\epsilon) \sum_{n \in \Lambda_j^-} (-i\beta_j^n) |\overline{(u - \psi)}^{(n)}|_{\Gamma_j}|^2 \\
& \leq C \|e_{h,N}\|_{L^2(\Omega)} \cdot \|u - \psi\|_{L^2(\Omega)} + \epsilon \sum_{n \in \Lambda_j^-} (-i\beta_j^n) |e_{h,N}^{(n)}|_{\Gamma_j}|^2 + C(\epsilon) \|u - \psi\|_{H^{1/2}(\Gamma_j)}^2,
\end{aligned}$$

where to obtain the last estimate we have used a well known fact, see *e. g.* [5] and [13], concerning the equivalence of the discrete norm and the continuous norm of $H^{1/2}$. The estimate (4.6) follows from the estimate (3.1).

We next examine the second term of (4.5).

According to Lemma 2.1,

$$u^{(n)}|_{\Gamma_1} = e^{-(b-b_1)\sqrt{(n+\alpha)^2 - k_1^2}} u^{(n)}|_{\Gamma'_1}, \quad u^{(n)}|_{\Gamma_2} = e^{-(b-b_1)\sqrt{(n+\alpha)^2 - k_2^2}} u^{(n)}|_{\Gamma'_2}.$$

Let N_0 be a sufficiently large integer, such that $N_0 > |\alpha| + |k_j|$. Then for $N \geq N_0$

$$\begin{aligned}
& \left| \sum_{|n| \geq N} i\beta_j^n u^{(n)}|_{\Gamma_j} \overline{(\psi - u_N^h)}^{(n)}|_{\Gamma_j} \right| \\
& = \left| \sum_{|n| \geq N} e^{-(b-b_1)\sqrt{(n+\alpha)^2 - k_j^2}} i\beta_j^n u^{(n)}|_{\Gamma'_j} \overline{(\psi - u_N^h)}^{(n)}|_{\Gamma_j} \right|
\end{aligned}$$

$$\begin{aligned}
&\leq C e^{-(b-b_1)\sqrt{(N-|\alpha|)^2-k_j^2}} \left(\sum_{|n|\geq N} [(n+\alpha)^2 - k_j^2]^{1/2} |u^{(n)}|_{\Gamma_j'}|^2 \right)^{1/2} \\
&\quad \left(\sum_{|n|\geq N} [(n+\alpha)^2 - k_j^2]^{1/2} |(\psi - u_N^h)^{(n)}|_{\Gamma_j}|^2 \right)^{1/2} \\
&\leq C e^{-(b-b_1)\sqrt{(N-|\alpha|)^2-k_j^2}} \|u\|_{H^{1/2}(\Gamma_j')} \|\psi - u_N^h\|_{H^{1/2}(\Gamma_j)} \\
&\leq C e^{-(b-b_1)\sqrt{(N-|\alpha|)^2-k_j^2}} (\|u\|_{H^{1/2}(\Gamma_j')}^2 + \|e_{h,N}\|_{H^1(\Omega)}^2 + h^{2(l-1)} \|u\|_{H^l(\Omega)}^2),
\end{aligned}$$

where to derive the last estimate, we have made use of the trace theorem and the estimate (3.1).

Therefore,

$$\begin{aligned}
\|e_{h,N}\|_{H^1(\Omega)}^2 + C \sum_{n \in \Lambda_j^-} (-i\beta_j^n) |e_{h,N}^{(n)}|_{\Gamma_j}|^2 &\leq C(k) \|e_{h,N}\|_{L^2(\Omega)}^2 + C(\epsilon) h^{2(l-1)} \|u\|_{H^l(\Omega)}^2 \\
&\quad + C \sum_{j=1,2} e^{-(b-b_1)\sqrt{(N-|\alpha|)^2-k_j^2}} \|u\|_{H^{1/2}(\Gamma_j')}^2. \quad (4.7)
\end{aligned}$$

Thus it suffices to estimate $\|e_{h,N}\|_{L^2(\Omega)}$, which may be done once again by using a duality technique.

In fact

$$\|e_{h,N}\|_{L^2(\Omega)} = \sup_{\phi \in C_0^\infty(\Omega)} \frac{|(e_{h,N}, \phi)|}{\|\phi\|_{L^2(\Omega)}},$$

and

$$|(e_{h,N}, \phi)| = |a(e_{h,N}, \bar{w})|,$$

where w solves the problem (3.9-3.11).

Next, choosing $\psi \in S^h$ as in the proof of Lemma 3.1, we have

$$\begin{aligned}
a(e_{h,N}, \bar{w}) &= a(e_{h,N}, \bar{w} - \psi) + a(e_{h,N}, \psi) \\
&= a(e_{h,N}, \bar{w} - \psi) - \sum_{j=1,2} \int_{\Gamma_j} \sum_{|n|\geq N} i\beta_j^n (u_N^h)^{(n)} \bar{\psi} e^{inx_1}.
\end{aligned}$$

Also, the same estimates as in the proof of Lemma 3.1 give rise to

$$|a(e_{h,N}, \bar{w} - \psi)| \leq Ch \|\bar{w}\|_{H^2(\Omega)} [\|e_{h,N}\|_1 + C(k)h \|e_{h,N}\|_{L^2(\Omega)} + C \|e_{h,N}\|_{H^1(\Omega)}].$$

Since $u_N^h = u - e_{h,N}$, for sufficiently large N , we deduce by using the estimate (3.1) that

$$\begin{aligned}
\left| \sum_{|n|\geq N} i\beta_j^n (u_N^h)^{(n)} \bar{\psi}^{(n)}|_{\Gamma_j} \right| &\leq \left[\sum_{|n|\geq N} \sqrt{(n+\alpha)^2 - k_j^2} |u^{(n)}|_{\Gamma_j}|^2 \right]^{1/2} [\|w - \psi\|_{H^{1/2}(\Gamma_j)} + \|w\|_{H^{1/2}(\Gamma_j)}] \\
&\quad + \left[\sum_{|n|\geq N} |e_{h,N}^{(n)}|_{\Gamma_j}|^2 \right]^{1/2} [\|w - \psi\|_{H^1(\Gamma_j)} + \|w\|_{H^1(\Gamma_j)}] \\
&\leq C e^{-(b-b_1)\sqrt{(N-|\alpha|)^2-k_j^2}} \|u\|_{H^{1/2}(\Gamma_j')} h \|w\|_{H^2(\Omega)} \\
&\quad + CN^{-s} \|e_{h,N}\|_{H^s(\Gamma_j)} \|\phi\|_{L^2(\Omega)}.
\end{aligned}$$

Once again, to get the above estimate we have used the equivalence of the discrete norm and the continuous norm of $H^{1/2}$.

Further, one may assume that $0 \leq s \leq 1/2$, the trace theorem then implies that

$$\|e_{h,N}\|_{H^s(\Gamma_j)} \leq C \|e_{h,N}\|_{H^1(\Omega)} .$$

We combine the above estimates with the trace theorem to conclude

$$\|e_{h,N}\|_{L^2(\Omega)} \leq Ch \sum_{j=1,2} e^{-(b-b_1)\sqrt{(N-|\alpha|)^2-k_j^2}} \|u\|_{H^{1/2}(\Gamma'_j)} + (C(k)h + CN^{-s}) \|e_{h,N}\|_{H^1(\Omega)} . \quad (4.8)$$

Consequently, combining (4.7) and (4.8) yields

$$\begin{aligned} \|e_{h,N}\|_{H^1(\Omega)} &+ \left(\sum_{n \in \Lambda_j^-, j=1,2} (-i\beta_j^n) |e_{h,N}^n|_{\Gamma_j}|^2 \right)^{1/2} \\ &\leq C(k)h^{l-1} \|u\|_{H^1(\Omega)} + C \sum_{j=1,2} e^{-\sqrt{(b-b_1)\sqrt{(N-|\alpha|)^2-k_j^2}}} \|u\|_{H^{1/2}(\Gamma'_j)} , \end{aligned}$$

which completes the proof of Theorem 4.1. $\square$

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References

- [1] Adams, R., *Sobolev Spaces*, Academic Press, New York, 1975.
- [2] Bao, G. and Dobson, D., *Diffraction optics in nonlinear media with periodic structure*, preprint.
- [3] Bendali, A., *Numerical analysis of the exterior boundary value problem for the time-harmonic Maxwell equations by a boundary finite element method*, Math. Comp., **43** (1984), Part I: the continuous problem, 29-46, Part II: the discrete problem, 47-68.
- [4] Born, M. and Wolf, E., *Principles of Optics*, sixth edition, Pergamon Press, Oxford, 1980.
- [5] Bramble, J. and Thomée, V., *Discrete time Galerkin methods for a parabolic boundary value problem*, Ann. Mat. Pura Appl., **101** (1974), 115-152.
- [6] Chen, X. and Friedman, A., *Maxwell's equations in a periodic structure*, Trans. Amer. Math. Soc., **323** (1991), 465-507.

- [7] Ciarlet, P.G., *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
- [8] Cox, J. A. and Dobson, D., *An integral equation method for biperiodic diffraction structures*, in J. Lerner and W. McKinney, ed., International Conference on the Application and Theory of Periodic Structures, Proc. SPIE **1545** (1991), pp. 106-113.
- [9] Dobson, D., *Optimal design of periodic antireflective structures for the Helmholtz equation*, Euro. J. Appl. Math., **4** (1993), to appear.
- [10] Dobson, D., A variational method for electromagnetic diffraction in biperiodic structures, preprint.
- [11] Dobson, D. and Friedman, A., *The time-harmonic Maxwell equations in a doubly periodic structure*, J. Math. Anal. Appl., **166** (1992), 507-528.
- [12] Friedman, A., *Mathematics in Industrial Problems*, Part 3, Springer-Verlag, Heidelberg, 1990.
- [13] Goldstein, C., *A finite element method for solving Helmholtz type equations in waveguides and other unbounded domains*, Math. Comp., **39** (1982), 309-324.
- [14] Lions, J. L. and Magenes, E., *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I, Springer-Verlag, New York, 1972.
- [15] Nédélec, J.C., and Starling, F., *Integral equation methods in a quasi-periodic diffraction problem for the time-harmonic Maxwell's equations*, SIAM J. Math. Anal., **22** (1991), 1679-1701.
- [16] Taylor, M., *Pseudodifferential Operators*, Princeton University Press, Princeton, N.J., 1981.
- [17] *Electromagnetic Theory of Gratings*, Topics in Current Physics, Vol. 22, edited by R. Petit, Springer-Verlag, Heidelberg, 1980.

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