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OF A DOMAIN**

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Abstract. Let u be a solution of the Laplace equation in a bounded domain Ω in $\mathbb{R}^n$ whose boundary consists of two disjoint surfaces Γ_0 and Γ_1 . On Γ_1 u satisfies a Dirichlet boundary condition, whereas on Γ_0 u satisfies the impermeability condition $\partial u / \partial n = 0$ everywhere except on m_ε small “holes” $\Gamma_{0,i}^\varepsilon$ separated from one another by distance of order of magnitude ε , and $u = 0$ on the holes. In this paper we study the effective permeability of the surface Γ_0 as $\varepsilon \rightarrow 0$.

§1. The problem. Consider a bounded domain G in $\mathbb{R}^n$ surrounded by shell, or “thick” boundary, Ω . Thus, Ω is also a domain in $\mathbb{R}^n$ whose boundary consists of inner boundary Γ_0 , which is the boundary of G , and outer boundary Γ_1 . The problem to be considered is motivated by the physical situation whereby G contains fluid and $\overline{G} \cup \overline{\Omega}$ is surrounded by air. The boundary Γ_0 has some “holes” $\Gamma_{0,i}^\varepsilon$ ($i = 1, \dots, m_\varepsilon$) through which fluid penetrates from G into Ω . The concentration of fluid in G is kept at constant level Q_0 .

Denote by $Q_0 - u^\varepsilon$ the concentration of fluid in Ω . We consider the stationary case and assume the standard diffusion law for the fluid concentration in Ω , i.e.,

$$(1.1) \quad \Delta u^\varepsilon = 0 \quad \text{in} \quad \Omega.$$

Since the fluid evaporate from Γ_1 into the surrounding air, we have

$$(1.2) \quad u^\varepsilon = Q_0 \quad \text{on} \quad \Gamma_1.$$

Finally,

$$(1.3) \quad u^\varepsilon = 0 \quad \text{on} \quad \Gamma_0^\varepsilon \equiv \bigcup_i \Gamma_{0,i}^\varepsilon$$

and

$$(1.4) \quad \frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on} \quad \Gamma_0 \setminus \Gamma_0^\varepsilon.$$

This model arises in plastic fuel tank. In order to reduce permeation of fuel, the inner boundary of the container is coated with thin barrier layer. This is done by blow molding process. The resulting thin layer Γ_0 , however, typically has flaws: it leaves many small

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patches uncovered. These are the “holes” $\Gamma_{0,i}^\varepsilon$, where the parameter ε indicate the scale: roughly, the distance between adjacent holes. The rough size of the holes will be denoted by σ_ε . This model is described in more details in [8] [6; chap. 12].

The question that arises is: What is the effect on the total permeation of fuel through the boundary of the container,

$$\int_{\Gamma_0^\varepsilon} \frac{\partial u^\varepsilon}{\partial n} ,$$

for small ε ? This of course can be calculated approximately once we have determined the limit of u^ε as $\varepsilon \rightarrow 0$.

The problem is amenable to methods of homogenization. A related problem was studied by Allaire [2] (for the Navier–Stokes equation) whereby in a domain Ω there is a periodic distribution of small spherical holes with centers in a fixed hyperplane. The present problem, where the “holes” are actually patches on the boundary of the domain, is technically quite different, but the basic approach is along the same outline as in [2] (see also [1]); in fact this approach was already used by Cioranescu and Murat [5]. It is based, in our case, on the obvious relation

$$(1.5) \quad \int_{\Omega} (\nabla u^\varepsilon \cdot \nabla W^\varepsilon) \varphi = \int_{\Omega} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon - \int_{\Omega} u^\varepsilon \nabla W^\varepsilon \cdot \nabla \varphi$$

where $\varphi \in C^\infty(\overline{\Omega})$. The main effort is the construction of suitable functions W^ε which will capture the behavior of u^ε near the “holes.” We shall construct such functions and prove that, for $n \geq 3$, if $\varphi = 0$ on Γ_1 then, as $\varepsilon \rightarrow 0$,

$$(1.6) \quad \int_{\Omega} (\nabla u^\varepsilon \cdot \nabla W^\varepsilon) \varphi \rightarrow - \int_{\Omega} \nabla u \cdot \nabla \varphi$$

and

$$(1.7) \quad \int_{\Omega} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon \rightarrow \int_{\Gamma_0} \mu_n u \varphi$$

where u is the weak limit u^ε as $\varepsilon \rightarrow 0$ (in $H^1(\Omega)$), and

$$(1.8) \quad \int_{\Omega} u^\varepsilon \nabla W^\varepsilon \cdot \nabla \varphi \rightarrow 0 .$$

Here μ_n is a nonnegative function depending on the distribution of the holes as well as their shape.

From (1.5)–(1.8) it follows that

$$(1.9) \quad \frac{\partial u}{\partial n} + \mu_n u = 0 \quad \text{on} \quad \Gamma_0$$

is the “homogenized” boundary condition for (1.3), (1.4) as $\varepsilon \rightarrow 0$. The basic assumption under which the assertions (1.6)–(1.8) will be established is that

$$(1.10) \quad \beta_n \equiv \lim_{\varepsilon \rightarrow 0} \frac{\sigma_\varepsilon}{\varepsilon^{\frac{n-1}{n-2}}} \quad \text{is finite} \quad (n \geq 3) .$$

For $n = 2$ the assertion (1.9) is still true provided

$$(1.11) \quad \beta_2 \equiv \lim_{\varepsilon \rightarrow 0} \left(\varepsilon \log \frac{1}{\sigma_\varepsilon} \right)^{-1} \quad \text{is finite} ,$$

but the proof is somewhat different.

When $\beta_n = \infty$, (1.9) is replaced by

$$(1.12) \quad u = 0 \quad \text{on} \quad \Gamma_0 ;$$

in this case the inner coating is totally ineffective.

In §2 we shall state the main results of the paper more precisely, and outline the structure of the paper. Special cases of the subcases $\beta_n = 0$ and $\beta_n = \infty$ were established by Chechkin [4]. For general literature and references to homogenization, mostly in domains with two phase material arranged periodically, see [3] [7] [9].

§2. The main results. We denote by $B_\rho(z)$ the ball in $\mathbb{R}^n$ with center z and radius ρ .

Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ consist of two disjoint surfaces: Γ_0 in C^3 and Γ_1 in $C^{0,1}$. We shall need several assumptions:

(A₀) For any small $\varepsilon > 0$ there are given disjoint domains T_i^ε ($i = 1, 2, \dots, m_\varepsilon$) and points z_i^ε in $\Gamma_{0,i}^\varepsilon \equiv T_i^\varepsilon \cap \Gamma_0$ such that

$$T_i^\varepsilon \subset B_{\sigma_\varepsilon^i}(z_i^\varepsilon) , \quad B_{\varepsilon_i}(z_i^\varepsilon) \cap B_{\varepsilon_j}(z_j^\varepsilon) = \emptyset \quad \text{if} \quad i \neq j ,$$

where $\sigma_\varepsilon^i < \varepsilon_i$ and

$$c\varepsilon \leq \varepsilon_i \leq C\varepsilon , \quad c_0\sigma_\varepsilon \leq \sigma_\varepsilon^i \leq C_0\sigma_\varepsilon$$

for some positive constants c, C, c_0, C_0 .

For most of the results of this paper we shall require, in addition:

$$(A_1) \quad \varepsilon_i = \varepsilon , \quad \sigma_\varepsilon^i = \sigma_\varepsilon \quad \text{for all } i, \text{ and } \quad |z_i^\varepsilon - z_j^\varepsilon| \geq 2\varepsilon \quad \text{if} \quad i \neq j ;$$

furthermore,

$$T_i^\varepsilon = z_i^\varepsilon + \sigma_\varepsilon T$$

where T is convex domain with C^3 boundary ∂T ,

$$B_{\delta_0}(0) \subset T \subset B_1(0) \quad \text{for some } \delta_0 > 0.$$

(A_2) There exists a function $P(x)$ in $L^\infty(\Gamma_0)$ such that $P \geq 0$, $P \not\equiv 0$ and

$$(2.1) \quad \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{m_\varepsilon} \varepsilon^{n-1} \varphi(z_i^\varepsilon) = \int_{\Gamma_0} P(x) \varphi(x) dS_x$$

for any $\varphi \in C(\Gamma_0)$.

Remark 2.1. Setting

$$P_0 = \int_{\Gamma_0} P(x) dS_x \quad (P_0 > 0)$$

and taking $\varphi \equiv 1$ in (2.1) we get

$$(2.2) \quad m_\varepsilon \sim \frac{P_0}{\varepsilon^{n-1}} \quad \text{as } \varepsilon \rightarrow 0.$$

Since Γ_0 is in C^3 , (2.1) implies that

$$(2.3) \quad \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{m_\varepsilon} |B_\varepsilon(z_i^\varepsilon) \cap \Gamma_0| \varphi(z_i^\varepsilon) = \omega_{n-1} \int_{\Gamma_0} P(x) \varphi(x) dS_x$$

where ω_m is the volume of the unit ball in $\mathbf{R}^m$.

Assume that (A_0) holds, and let g_1 be a nonnegative function in $H^1(\Omega)$. Consider the problem

$$(2.4) \quad \Delta u^\varepsilon = 0 \quad \text{in } \Omega,$$

$$(2.5) \quad u^\varepsilon = g_1 \quad \text{on } \Gamma_1,$$

$$(2.6) \quad u^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \equiv \bigcup_{i=1}^{m_\varepsilon} \Gamma_{0,i}^\varepsilon,$$

$$(2.7) \quad \frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_0 \setminus \Gamma_0^\varepsilon$$

where n is the outward unit normal to Γ_0 . It is well known that the function u^ε is uniquely defined as the solution of the problem:

$$\min \left\{ \int_{\Omega} |\nabla v|^2, v \in V_\varepsilon \right\}$$

where

$$V_\varepsilon = \{v \in H^1(\Omega), v = g_1 \text{ on } \Gamma_1, v = 0 \text{ on } \Gamma_0^\varepsilon\} ;$$

hence

$$(2.8) \quad \int \nabla u^\varepsilon \cdot \nabla \psi = 0 \quad \text{if } \psi \in H^1(\Omega), \psi = 0 \text{ on } \Gamma_1 \cup \Gamma_0^\varepsilon .$$

LEMMA 2.1. *The solution u^ε of (2.4)–(2.7) satisfies*

$$(2.9) \quad 0 \leq u^\varepsilon \leq \sup g_1 ,$$

$$(2.10) \quad \int_{\Omega} |\nabla u^\varepsilon|^2 \leq C$$

where C is a constant independent of ε .

Proof. The estimates in (2.9) follow by the maximum principle, whereas (2.10) follows by applying (2.8) with $\psi = u^\varepsilon - \varphi g_1$ where $\varphi \in C^\infty(\Omega)$, $\varphi = 1$ on Γ_1 and $\varphi = 0$ on Γ_0 .

Denote by u^D and u^N the solutions of (2.4), (2.5) satisfying

$$u^D = 0 \text{ on } \Gamma_0 , \quad \frac{\partial u^N}{\partial n} = 0 \text{ on } \Gamma_0 .$$

Let $\{\Gamma_{0,i}^\delta\}$ be another family of “holes” such that

$$(2.11) \quad \bigcup_{i=1}^{m_\varepsilon} \Gamma_{0,i}^\varepsilon \subset \bigcup_{i=1}^{m_\delta} \Gamma_{0,i}^\delta .$$

LEMMA 2.2. *Under the assumption (2.11), the solutions u^ε and u^δ satisfy:*

$$(2.12) \quad u^D \leq u^\delta \leq u^\varepsilon \leq u^N .$$

Indeed this easily follows by the maximum principle.

LEMMA 2.3. *There exists an extension $\tilde{u}^\varepsilon$ of u^ε to $H^1(\mathbb{R}^n)$ such that*

$$0 \leq \tilde{u}^\varepsilon \leq \sup g_1 ,$$

$$\|\nabla \tilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla u^\varepsilon\|_{L^2(\Omega)}$$

where C is constant independent of ε .

Indeed the extension is obtained by locally flattening the boundary, then using reflection, and finally multiplying a cut-off function.

From Lemma 2.1 we know that every sequence $\tilde{\varepsilon} \rightarrow 0$ has a subsequence $\varepsilon' \rightarrow 0$ such that

$$u^{\varepsilon'} \rightarrow u \quad \text{weakly in } H^1(\Omega) ,$$

and $0 \leq u \leq \sup g_1$. Our objective is to prove that u is uniquely determined; then the entire family u^ε will converge to u . For simplicity we shall henceforth denote the sequence ε' by ε .

Our main result is the characterization of u . This is described in terms of the numbers β_n introduced in (1.10), (1.11).

THEOREM 2.4. *If $(A_0) - (A_2)$ hold and the limits in (1.10), (1.11) exist, and if $0 \leq \beta_n < \infty$, then*

$$(2.13) \quad \Delta u = 0 \quad \text{in } \Omega ,$$

$$(2.14) \quad u = g_1 \quad \text{on } \Gamma_1$$

and

$$(2.15) \quad \frac{\partial u}{\partial n} + \mu_n u = 0 \quad \text{on } \Gamma_0 ,$$

where

$$(2.16) \quad \mu_2 = \beta_2 \pi P(x) ,$$

$$(2.17) \quad \mu_n = \frac{1}{2} \gamma(T) (\beta_n)^{n-2} s_n P(x) \quad \text{if } n \geq 3 ;$$

s_n is the surface area of $\partial B_1(0)$ and $\gamma(T)$ is a constant depending only on the domain T .

The definition of $\gamma(T)$ is given in Lemma 3.3. For the case $\beta_n = 0$, Theorem 2.4 can be strengthened:

THEOREM 2.4'. *If (A_0) holds and $\beta_n = 0$, then $u = u^N$.*

In case $\beta_n = \infty$ we require the following condition:

(A_3) for any $x \in \Gamma_0$ and $\lambda > 0$ there exists a constant $C(x, \lambda) > 0$ independent of ε such that

$$\text{meas} \left\{ \bigcup_{i=1}^{m_\varepsilon} (B_\lambda(x) \cap B_{c\varepsilon}(z_i^\varepsilon)) \cap \Gamma_0 \right\} \geq C(x, \lambda)$$

if ε is sufficiently small; here c is as in (A_0) .

Remark 2.2. If $(A_0) - (A_2)$ hold and if $P(x) > 0$ for all $x \in \Gamma_0$, then (A_3) holds.

THEOREM 2.5. *If $(A_0), (A_3)$ hold and $\beta_n = \infty$, then $u = u^D$.*

In §3 we construct the auxiliary functions W^ε mentioned in §1 and derive some estimates on their first derivatives. Section 4 establishes a result which converts the integral of $u^\varepsilon \varphi$ (φ a test function) over $\bigcup_i \partial B_\varepsilon(z_i^\varepsilon) \cap \Omega$ into an integral over $\bigcup_i B_\varepsilon(z_i^\varepsilon) \cap \Gamma_0$. In §5,6 we prove Theorem 2.4 for $n \geq 3$, and in §7 we give the proof for $n = 2$. Theorem 2.4' is briefly proved in §8 and Theorem 2.5 is proved in §9.

Theorem 2.4 establishes weak convergence in $H^1(\Omega)$. Setting

$$(2.18) \quad \rho_\varepsilon = (u^\varepsilon - u) + (1 - W^\varepsilon)u$$

we have:

THEOREM 2.6. *If $(A_0) - (A_2)$ hold and $0 \leq \beta_n < \infty$, $n \geq 2$, then*

$$(2.19) \quad \int_{\Omega} |\nabla \rho_\varepsilon|^2 \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0.$$

This means that by adding to u^ε the *corrector* $(1 - W^\varepsilon)u$ we get strong convergence in $H^1(\Omega)$ to u .

THEOREM 2.6'. *If $(A_0), (A_3)$ hold and $\beta_n = \infty$ then $\int_{\Omega} \nabla |u^\varepsilon - u|^2 \rightarrow 0$ if $\varepsilon \rightarrow 0$.*

Theorems 2.6 and 2.6' are proved in §10.

§3. Construction of W^ε . Let T be the domain introduced in (A_1) ; $B_{\delta_0}(0) \subset T \subset B_1(0)$. Set $B_R = \{x \in \mathbb{R}^n, |x| < R\}$, $B_R^+ = B_R \cap \{x_n > 0\}$, $T^+ = T \cap \{x_n > 0\}$ and let $D = T \cap \mathbb{R}^{n-1}$ where $\mathbb{R}^{n-1} = \mathbb{R}^n \cap \{x_n = 0\}$. For any $R \geq 2$ consider the problem

$$(3.1) \quad \Delta W_R = 0 \quad \text{in } B_R^+.$$

$$(3.2) \quad W_R = 1 \quad \text{on } \partial B_R^+ \cap \{x_n > 0\},$$

$$(3.3) \quad W_R = 0 \quad \text{on } D,$$

$$(3.4) \quad \frac{\partial W_R}{\partial n} = 0 \quad \text{on } \partial B_R^+ \cap (\mathbb{R}^{n-1} \setminus D).$$

LEMMA 3.1. Let $n \geq 3$. There exists a unique solution W_R of (3.1)–(3.4), and

$$(3.5) \quad 0 \leq 1 - W_R(x) \leq \frac{1}{|x|^{n-2}} \quad \text{if } 1 \leq |x| \leq R ,$$

$$(3.6) \quad |\nabla W_R(x)| \leq \frac{C}{|x|^{n-1}} \quad \text{if } 2 \leq |x| \leq R ,$$

$$(3.7) \quad \|\nabla W_R\|_{L^2(B_R^+)} \leq C ,$$

$$(3.8) \quad \|W_R\|_{C^\alpha(B_R^+)} \leq C$$

where α is any positive constant $< \frac{1}{2}$ and C is a constant independent of R . (In (3.8), C depends on α).

Proof. Existence and uniqueness are standard, and (3.5) follows by the maximum principle. We extend $W_R(x)$ by reflection to harmonic function in $B_R \setminus D$, which we again denote by W_R . For any harmonic function v in a ball $B_\rho(x^0)$,

$$(3.9) \quad |\nabla v(x^0)| \leq \frac{C}{\rho} \sup_{\partial B_\rho(x^0)} |v| .$$

Applying this to $v = 1 - W_R$ with $2 \leq |x^0| \leq \frac{R}{2}$, $\rho = \frac{1}{2} |x^0|$, and using (3.5), the assertion (3.6) follows for $2 \leq |x| \leq \frac{R}{2}$. Next we rescale the function $v = 1 - W_R$ by $v_R(x) = v(Rx)$. Then v_R is harmonic in $\frac{1}{3} < |x| < 1$ and $v_R = 0$ on $|x| = 1$. By the boundary estimates for harmonic functions and (3.5) we obtain

$$|\nabla v_R(x)| \leq \frac{C}{R^{n-2}} \quad \text{for } \frac{1}{2} \leq |x| \leq 1 ,$$

and this yields (3.6) for $\frac{R}{2} \leq |x| \leq R$ by rescaling back.

The estimate (3.7), with C depending on R , follows from the variational formulation of (3.1)–(3.4). To establish (3.7) with C independent of R we take a function $\psi \in C_0^\infty$ such that $\psi = 1$ in B_4 and use $\psi^2 W_R$ as a test function (assuming $R > 4$). We obtain

$$\int_{B_4^+} |\nabla W_R|^2 \leq \int_{B_R^+} |\nabla W_R|^2 \psi^2 \leq C \int_{B_R^+} |\nabla \psi|^2 .$$

Using also (3.6), the assertion (3.7) follows.

Finally, W_R is obviously uniformly in C^α (or even C^1) outside any neighborhood of ∂D . By a result of Shamir [10; Th. 5.3]

$$\|W_R\|_{W^{s,p}(B_2)} \leq C \|W_R\|_{L^p(B_2)} \leq C'$$

for any $1 < p < \infty$ and $s < \frac{1}{2} + \frac{2}{p}$. Taking p sufficiently large and using the Sobolev embedding, (3.8) follows.

LEMMA 3.2. *Let $n \geq 3$. Then $W_R \downarrow W$ pointwise and strongly in $H^1(B_\rho)$ for any B_ρ , as $R \uparrow \infty$, and W is the unique solution of*

$$(3.10) \quad \Delta W = 0 \quad \text{in} \quad \mathbb{R}_+^n \equiv \mathbb{R}^n \cap \{x_n > 0\} ,$$

$$(3.11) \quad W = 0 \quad \text{on} \quad D ,$$

$$(3.12) \quad \frac{\partial W}{\partial n} = 0 \quad \text{on} \quad \mathbb{R}^{n-1} \setminus D ,$$

$$(3.13) \quad \lim_{|x| \rightarrow \infty} W(x) = 1 ;$$

furthermore,

$$(3.14) \quad 0 \leq 1 - W(x) \leq \frac{1}{|x|^{n-2}}$$

and

$$(3.15) \quad |\nabla W(x)| \leq \frac{C}{|x|^{n-1}} \quad \text{if} \quad |x| \geq 2 .$$

Proof. By the maximum principle, if $R_1 < R_2$ then

$$0 \leq W_{R_1}(x) - W_{R_2}(x) \leq \frac{1}{R_1^{n-2}} \quad \text{in} \quad B_{R_1}^+ .$$

It follows that $W_R(x)$ decreases monotonically to a limit $W(x)$. Clearly the convergence is uniform in each B_ρ^+ , $\rho > 0$, and the assertions (3.10)–(3.15) follow from the corresponding relations for W_R . Finally, as $R \rightarrow \infty$

$$(3.16) \quad \int_{B_R^+} |\nabla(W_R - W)|^2 = \int_{\partial B_R^+ \cap \{x_n > 0\}} \frac{\partial(W_R - W)}{\partial n} (1 - W) \leq \frac{C}{R^{n-2}} \rightarrow 0 ,$$

by the estimates (3.6), (3.14), (3.15).

LEMMA 3.3. *Let $n \geq 3$. Then, if $R \geq 2$,*

$$(3.17) \quad \left| \nabla W_R(x) - \frac{\gamma(T)x}{|x|^n} \right| \leq \frac{\eta(R)}{|x|^{n-1}} + \frac{C}{|x|^n} \quad \text{if} \quad 2 \leq |x| \leq R ,$$

$$(3.18) \quad |\nabla W_R(x)| \leq C[\text{dist}(x, D)]^{\alpha-1} \quad \text{if} \quad |x| \leq R$$

where $\eta(R) \rightarrow 0$ if $R \rightarrow \infty$, C and α are positive constants independent of R , and

$$(3.19) \quad \gamma(T) = \frac{2}{n\omega_n} \int_{\mathbb{R}_+^n} |\nabla W|^2, \quad \gamma(T) > 0;$$

α can be taken to be any number $< \frac{1}{2}$.

Proof. Choose a cut-off function $\varphi \in C_0^\infty$ radially symmetric such that $0 \leq \varphi \leq 1$ and

$$\varphi = 0 \quad \text{in} \quad B_{3/2}, \quad \varphi = 1 \quad \text{in} \quad \mathbb{R}^n \setminus B_2$$

and set $v_R = \varphi W_R$. Then

$$\Delta v_R = W_R \Delta \varphi + 2 \nabla \varphi \cdot \nabla W_R \equiv f_R$$

and v_R satisfies the boundary conditions (3.2)–(3.4).

Extending both v_R and f_R by reflection, we can represent the extended v_R in terms of Green's functions:

$$(3.20) \quad v_R(x) = 1 - \int_{B_2 \setminus B_{3/2}} f_R(y) G_R(x, y) dy$$

where

$$G_R(x, y) = \frac{1}{n(2-n)\omega_n} \left[\frac{1}{|x-y|^{n-2}} - \left(\frac{R}{|y|} \frac{1}{|x-\bar{y}|} \right)^{n-2} \right] \quad \left(\bar{y} = \frac{R^2}{|y|^2} y \right)$$

is the Green function in B_R . Note that

$$(3.21) \quad \nabla_x G_R(x, y) = \frac{1}{n\omega_n} \left(\frac{x-y}{|x-y|^n} - \frac{R^{n-2}}{|y|^{n-2}} \frac{x-\bar{y}}{|x-\bar{y}|^n} \right)$$

From the representation (3.20) and from (3.21) we deduce that, for $|x| \geq 2$,

$$\nabla W_R(x) = \nabla v_R(x) = \int_{B_2 \setminus B_{3/2}} \frac{(x-y)f_R(y)}{n\omega_n|x-y|^n} dy + F_R(x)$$

where $|F_R(x)| \leq CR^{3-2n}$. Letting

$$\gamma_R = \frac{1}{n\omega_n} \int_{B_2 \setminus B_{3/2}} f_R(y) dy$$

we then have,

$$(3.22) \quad \nabla W_R(x) = \gamma_R \frac{x}{|x|^n} + \tilde{F}_R(x), \quad |\tilde{F}_R(x)| \leq \frac{C}{R^{2n-3}} + \frac{C}{|x|^n}.$$

By integration by parts

$$\begin{aligned} \int_{B_2 \setminus B_{3/2}} f_R(y) &= \int_{B_R} \Delta v_R = \int_{\partial B_R} \frac{\partial v_R}{\partial n} = 2 \int_{\partial B_R^+ \cap \{x_n > 0\}} \frac{\partial W_R}{\partial n} \\ &= 2 \int_{\partial B_R^+ \cap \{x_n > 0\}} W_R \frac{\partial W_R}{\partial n} = 2 \int_{B_R^+} |\nabla W_R|^2 \rightarrow 2 \int_{\mathbb{R}_+^n} |\nabla W|^2 \end{aligned}$$

where (3.16) was used. It follows that $\gamma_R \rightarrow \gamma(T)$ as $R \rightarrow \infty$, and (3.17) is then a consequence of (3.22).

To prove (3.18) let $x \in B_R^+$, $r = \text{dist}(x, D)$ and apply the estimate (3.9) to $v = W_R$ with $x^0 = x$, $\rho = r$. Since for any $y \in \partial B_r(x)$ there is a point z in D such that $|y - z| \leq 2r$, and since $W_R(z) = 0$,

$$|W_R(y)| = |W_R(y) - W_R(z)| \leq Cr^\alpha$$

by the uniform C^α continuity of W_R , the assertion (3.18) follows.

Remark 3.1. The estimates (3.5)–(3.7) and (3.17) do not require that T is convex and $\partial T \in C^3$; they are valid uniformly for all $T \subset B_1$.

LEMMA 3.4. *Let $n \geq 3$. Then*

$$(3.23) \quad \int_{\partial T \cap \mathbb{R}_+^n} \left| \frac{\partial W_R}{\partial n} \right| \leq C$$

where C is a constant independent of R .

Proof. Since T is a convex domain containing $B_{\delta_0}(0)$, ∂T is transversal to $\mathbb{R}^{n-1}$ at $\partial T \cap \mathbb{R}^{n-1}$. For any $x^0 \in \partial T \cap \mathbb{R}^{n-1}$ we can then represent $\mathbb{R}_+^n \cap \partial T$ in a neighborhood N of x^0 in the form

$$\{(x_1, \dots, x_{n-1}, x_n) ; x_1 = \Psi(x_2, \dots, x_n)\}$$

where $\Psi \in C^3$. Hence, by (3.18),

$$\begin{aligned} \int_{\partial T \cap \mathbb{R}_+^n \cap N} \left| \frac{\partial W_R}{\partial n} \right| &\leq C \int_{x_2^2 + \dots + x_n^2 < r^2, x_n > 0} |x_n|^{\alpha-1} \sqrt{1 + |\nabla \Psi|^2} dx_2 \dots dx_n \\ &\leq C \int_{x_2^2 + \dots + x_{n-1}^2 < r^2} dx_2 \dots dx_{n-1} \int_0^r \frac{dx_n}{|x_n|^{1-\alpha}} \leq C. \end{aligned}$$

We can now use partition of unity to complete the proof of the lemma.

Construction of W^ε . Let (A_0) hold. In each $B_{\varepsilon_i}(z_i^\varepsilon)$ we introduce coordinate axes such that $\mathbb{R}^{n-1}$ is the tangent to $\Gamma_{0,i}^\varepsilon$ at z_i^ε , z_i^ε is at the origin, and the inner normal at z_i^ε points in the x_n -direction. Set $D_i^\varepsilon = T_i^\varepsilon \cap \mathbb{R}^{n-1}$ and let $\tilde{w}_i^\varepsilon$ be the solution of

$$(3.24) \quad \begin{aligned} \Delta \tilde{w}_i^\varepsilon &= 0 \quad \text{in} \quad B_{\varepsilon_i}^+(z_i^\varepsilon) \\ \tilde{w}_i^\varepsilon &= 1 \quad \text{on} \quad \partial B_{\varepsilon_i}^+(z_i^\varepsilon) \cap \mathbb{R}_+^n, \\ \tilde{w}_i^\varepsilon &= 0 \quad \text{on} \quad D_i^\varepsilon, \\ \frac{\partial \tilde{w}_i^\varepsilon}{\partial n} &= 0 \quad \text{on} \quad \partial B_{\varepsilon_i}^+ \cap (\mathbb{R}^{n-1} \setminus D_i^\varepsilon). \end{aligned}$$

Extend $\tilde{w}_i^\varepsilon$ by reflection to $B_{\varepsilon_i}(z_i^\varepsilon)$ and define

$$w_i^\varepsilon = \tilde{w}_i^\varepsilon|_{B_{\varepsilon_i}(z_i^\varepsilon) \cap \Omega}.$$

Finally set, for $n \geq 3$,

$$(3.25) \quad W^\varepsilon(x) = \begin{cases} w_i^\varepsilon(x) & \text{if } x \in B_{\varepsilon_i}(z_i^\varepsilon) \cap \Omega \\ 1 & \text{otherwise in } \Omega. \end{cases}$$

For $n = 2$ we define, more simply,

$$(3.26) \quad W^\varepsilon(x) = \begin{cases} \frac{\log |x - z_i^\varepsilon| - \log \sigma_\varepsilon^i}{\log \varepsilon_i - \log \sigma_\varepsilon^i} & \text{if } x \in (B_{\varepsilon_i}(z_i^\varepsilon) \cap \Omega) \setminus B_{\sigma_\varepsilon^i}(z_i^\varepsilon) \\ 0 & \text{if } x \in \Omega \cap B_{\sigma_\varepsilon^i}(z_i^\varepsilon) \\ 1 & \text{otherwise in } \Omega. \end{cases}$$

Observe that $W^\varepsilon \in H^1(\Omega)$. Since $W^\varepsilon = 1$ outside $(C\varepsilon)$ -neighborhood of $\partial\Omega$,

$$(3.27) \quad W^\varepsilon \rightarrow 1 \text{ strongly in } L^2(\Omega) \text{ as } \varepsilon \rightarrow 0.$$

Note that, for $n \geq 3$,

$$(3.28) \quad W^\varepsilon(x) = W_R \left(\frac{x - z_i^\varepsilon}{\sigma_\varepsilon^i} \right) \quad \text{in } B_\varepsilon(z_i^\varepsilon) \cap \Omega$$

where W_R is the function studied previously in this section, with

$$(3.29) \quad R = \frac{\varepsilon_i}{\sigma_\varepsilon^i}, \quad D = \frac{T_i^\varepsilon - z_i^\varepsilon}{\sigma_\varepsilon^i}$$

LEMMA 3.5. Set $R_\varepsilon^i = \varepsilon_i / \sigma_\varepsilon^i$. For $n \geq 3$, if $\sup_i R_\varepsilon^i \rightarrow \infty$ as $\varepsilon \rightarrow 0$ then

$$(3.30) \quad \int_{\Omega} |\nabla W^\varepsilon|^2 \leq C \sum_{i=1}^{m_\varepsilon} (\sigma_\varepsilon^i)^{n-2},$$

$$(3.31) \quad \frac{\partial W^\varepsilon}{\partial n} = \gamma \left(\frac{T_\varepsilon^i - z_\varepsilon^i}{\sigma_\varepsilon^i} \right) \frac{(\sigma_\varepsilon^i)^{n-2}}{(\varepsilon_i)^{n-1}} (1 + o(1)), \quad x \in \partial B_{\varepsilon_i}(z_\varepsilon^i)$$

where $o(1) \rightarrow 0$ uniformly as $\varepsilon \rightarrow 0$. For $n = 2$,

$$(3.32) \quad \int_{\Omega} |\nabla W^\varepsilon|^2 \leq C \sum_{i=1}^{m_\varepsilon} \frac{1}{\log \varepsilon_i - \log \sigma_\varepsilon^i},$$

$$(3.33) \quad \frac{\partial W^\varepsilon}{\partial n} = \sum_{i=1}^{m_\varepsilon} \frac{1}{\varepsilon_i (\log \varepsilon_i - \log \sigma_\varepsilon^i)}, \quad x \in \partial B_{\varepsilon_i}(z_\varepsilon^i).$$

Proof. For $n \geq 3$, by (3.28) and Lemma 3.1,

$$\begin{aligned} \int_{\Omega} |\nabla W^\varepsilon|^2 &= \sum_{i=1}^{m_\varepsilon} \int_{B_{\sigma_\varepsilon^i}(z_\varepsilon^i)} \left| \nabla \left(W_{R_\varepsilon^i} \left(\frac{x - z_\varepsilon^i}{\sigma_\varepsilon^i} \right) \right) \right|^2 \\ &\leq \sum_{i=1}^{m_\varepsilon} \int_{B_{R_\varepsilon^i}} (\sigma_\varepsilon^i)^{n-2} |\nabla W_{R_\varepsilon^i}^\varepsilon(y)|^2 \leq C \sum_{i=1}^{m_\varepsilon} (\sigma_\varepsilon^i)^{n-2}. \end{aligned}$$

The proof of (3.31) follows similarly by using Lemma 3.3. Finally (3.32) and (3.33) follow by direct calculation.

COROLLARY 3.6. Under the assumptions $(A_0) - (A_2)$, if $\beta_n < \infty$ then, for all small $\varepsilon > 0$,

$$(3.34) \quad \|\nabla W^\varepsilon\|_{L^2(\Omega)}^2 \leq C(\beta_n)^{n-2},$$

$$(3.35) \quad \frac{\partial W^\varepsilon}{\partial n} \Big|_{\partial B_\varepsilon(z_\varepsilon^i)} = \gamma(T)(\beta_n)^{n-2}(1 + o(1))$$

for $n \geq 3$ ($o(1) \rightarrow 0$ if $\varepsilon \rightarrow 0$), and

$$(3.36) \quad \|\nabla W^\varepsilon\|_{L^2(\Omega)}^2 \leq C\beta_2,$$

$$(3.37) \quad \frac{\partial W^\varepsilon}{\partial n} \Big|_{\partial B_\varepsilon(z_\varepsilon^i)} = \frac{1}{\varepsilon(\log \varepsilon - \log \sigma_\varepsilon)} (1 + o(1))$$

for $n = 2$.

§4. The average of u^ε near the holes. In this section we prove:

LEMMA 4.1. *Let $n \geq 2$ and assume that $(A_0) - (A_2)$ hold. Then for any $\varphi \in C^\infty(\overline{\Omega})$,*

$$(4.1) \quad \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \Omega} u^\varepsilon \varphi - \frac{s_n}{2\omega_{n-1}} \sum_{i=1}^{m_\varepsilon} \int_{B_\varepsilon(z_i^\varepsilon) \cap \Gamma_0} u \varphi \rightarrow 0$$

as $\varepsilon \rightarrow 0$.

Recall that we are taking here the sequence of ε 's such that $u^\varepsilon \rightarrow u$ weakly in $H^1(\Omega)$. In (4.1) s_n is the surface area of ∂B_1 .

COROLLARY 4.2. *Let $n \geq 2$ and assume that $(A_0) - (A_2)$ hold. Then, for any φ in $C^\infty(\overline{\Omega})$,*

$$(4.2) \quad \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \Omega} u^\varepsilon(x) \varphi(x) dx \rightarrow \frac{s_n}{2} \int_{\Gamma_0} P(x) u(x) \varphi(x) dS_x$$

Indeed, approximating u by continuous functions in both $L^2(\Gamma)$ and $L^2(\Gamma_0)$ and using (2.3) we get

$$\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{m_\varepsilon} \int_{B_\varepsilon(z_i^\varepsilon) \cap \Gamma_0} u \varphi dS_x = \omega_{n-1} \int_{\Gamma_0} P(x) u \varphi dS_x ,$$

so that (4.2) is a consequence of (4.1).

To prove Lemma 4.1 we need the following result:

LEMMA 4.3. *Let $q_n = (s_n + 2\omega_{n-1})/\omega_n$ where ω_n is the volume of B_1 in $\mathbb{R}^n$ ($n \geq 2$) and $\omega_1 = 2$. Then there exists a unique solution of*

$$(4.3) \quad \begin{aligned} \Delta v &= q_n \quad \text{in} \quad B_1^+ , \\ \frac{\partial v}{\partial n} &= 1 \quad \text{on} \quad \partial B_1^+ , \end{aligned}$$

and it can be extended into harmonic function v in B_1 which belongs to $C^{1+\alpha}(\overline{B}_1)$ for any $0 < \alpha < 1$.

Proof. Observe that q_n is such that the consistency condition

$$\int_{B_1^+} \Delta v = \int_{\partial B_1^+} \frac{\partial v}{\partial n}$$

is satisfied. Hence there exists a solution v to (4.3) (unique up to a constant). Set $\tilde{v} = v + x_n$ and extend it by reflection into B_1 . The extended function is harmonic in B_1 since

$\partial\tilde{v}/\partial x_n = 0$ at $x_n = 0+$. Since, further, $\partial\tilde{v}/\partial n$ is Lipschitz continuous on ∂B_1 , it follows that $\tilde{v} \in C^{1+\alpha}(\overline{B}_1)$ for any $0 < \alpha < 1$. The desired extension of v is given by

$$v(x', x_n) = v(x', -x_n) - 2x_n, \quad x' = (x_1, \dots, x_{n-1}).$$

Proof of Lemma 4.1. Set

$$\Omega_i^\varepsilon = B_\varepsilon(z_i^\varepsilon) \cap \Omega.$$

In each $B_\varepsilon(z_i^\varepsilon)$ define

$$v^\varepsilon(x) = \varepsilon v\left(\frac{x - z_i^\varepsilon}{\varepsilon}\right).$$

Then

$$(4.4) \quad \Delta v^\varepsilon = \frac{q_n}{\varepsilon} \quad \text{in} \quad B_\varepsilon(z_i^\varepsilon),$$

$$(4.5) \quad \frac{\partial v^\varepsilon}{\partial n} = 1 \quad \text{on} \quad \partial B_\varepsilon^+(z_i^\varepsilon).$$

By Green's formula,

$$(4.6) \quad \sum_{i=1}^{m_\varepsilon} \int_{\partial\Omega_i^\varepsilon} \frac{\partial v^\varepsilon}{\partial n} u^\varepsilon \varphi = \sum_{i=1}^{m_\varepsilon} \int_{\Omega_i^\varepsilon} \nabla v^\varepsilon \cdot \nabla(u^\varepsilon \varphi) + \frac{q_n}{\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{\Omega_i^\varepsilon} u^\varepsilon \varphi.$$

Since

$$\begin{aligned} \sum_{i=1}^{m_\varepsilon} \int_{\Omega_i^\varepsilon} |\nabla v^\varepsilon \cdot \nabla(u^\varepsilon \varphi)| &\leq \left\{ \int_{\Omega} |\nabla(u^\varepsilon \varphi)|^2 \right\}^{1/2} \left\{ \sum_{i=1}^{m_\varepsilon} \int_{\Omega_i^\varepsilon} |\nabla v^\varepsilon|^2 \right\}^{1/2} \\ &\leq C \left\{ m_\varepsilon \int_{B_1} |\nabla v|^2 \varepsilon^n \right\}^{1/2} \leq C\sqrt{\varepsilon} \quad \text{by (2.2),} \end{aligned}$$

(4.6) yields

$$\begin{aligned} (4.7) \quad \sum_{i=1}^{m_\varepsilon} \int_{\partial\Omega_i^\varepsilon \cap \Omega} \frac{\partial v^\varepsilon}{\partial n} u^\varepsilon \varphi &= \frac{q_n}{\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{\Omega_i^\varepsilon} u^\varepsilon \varphi - \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)} \frac{\partial v^\varepsilon}{\partial n} u^\varepsilon \varphi + O(\sqrt{\varepsilon}) \\ &\equiv I_1 + I_2 + O(\sqrt{\varepsilon}). \end{aligned}$$

To evaluate I_2 note that

$$\frac{\partial v^\varepsilon}{\partial n} = 1 \quad \text{on} \quad \partial B_\varepsilon^+(z_i^\varepsilon).$$

However we also have to evaluate $\partial v^\varepsilon / \partial n$ on the rest of $\partial B_\varepsilon(z_i) \cap \Omega \equiv \partial \Omega_i^\varepsilon \cap \Omega$. To do this we parametrize $\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)$ by a C^2 function Ψ such that $\Psi(0) = 0$, $\nabla \Psi(0) = 0$ where 0 corresponds to z_i^ε . Then for $x \in \Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)$,

$$(4.8) \quad \text{dist}(x, \mathbb{R}^{n-1}) \leq |\Psi(x_1, \dots, x_{n-1})| \leq C\varepsilon^2.$$

Hence the area of the portion of $\partial B_\varepsilon(z_i^\varepsilon)$ between Γ_0 and $\mathbb{R}^{n-1}$ is bounded by $Cm_\varepsilon \varepsilon^n \leq C\varepsilon$. Since $|\partial v^\varepsilon / \partial n| \leq C$, we conclude that

$$(4.9) \quad \sum_{i=1}^{m_\varepsilon} \int_{\partial \Omega_i^\varepsilon \cap \Omega} \frac{\partial v^\varepsilon}{\partial n} u^\varepsilon \varphi = \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \Omega} u^\varepsilon \varphi + O(\varepsilon).$$

Next we estimate I_2 in (4.7). For $x \in \Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)$,

$$\frac{\partial v^\varepsilon(x)}{\partial n} = \frac{\partial v}{\partial n} \left(\frac{x - z_i^\varepsilon}{\varepsilon} \right) = \frac{(\nabla \Psi, -1)}{(1 + |\nabla \Psi|^2)^{1/2}} \cdot \nabla v \left(\frac{x - z_i^\varepsilon}{\varepsilon} \right).$$

Take $y \in \mathbb{R}^{n-1} \cap B_\varepsilon(z_i^\varepsilon)$ to be the projection of x on $\mathbb{R}^{n-1}$. By (4.8), $|x - y| \leq C\varepsilon^2$. Also

$$\frac{\partial v}{\partial n} \left(\frac{y - z_i^\varepsilon}{\varepsilon} \right) = -\frac{\partial v}{\partial x_n} = 1.$$

Since further $|\nabla \Psi(x')| \leq C|x'| \leq C\varepsilon$ ($x' = (x_1, \dots, x_{n-1})$), we get

$$\left| \frac{\partial v^\varepsilon(x)}{\partial n} - 1 \right| \leq C\varepsilon + \left| \frac{\partial v}{\partial x_n} \left(\frac{x - z_i^\varepsilon}{\varepsilon} \right) - \frac{\partial v}{\partial x_n} \left(\frac{y - z_i^\varepsilon}{\varepsilon} \right) \right| \leq C\varepsilon^\alpha$$

for any $0 < \alpha < 1$, where in the last inequality we used the fact that $v \in C^{1+\alpha}(\overline{B_1})$. It follows that

$$(4.10) \quad I_2 = - \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)} u^\varepsilon \varphi + O(\varepsilon^\alpha).$$

Finally we evaluate I_1 in (4.7). We can write

$$\begin{aligned} I_1 &= \frac{q_n}{\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{|x'| \leq \varepsilon} dx' \int_{\Psi(x')}^{\sqrt{\varepsilon^2 - |x'|^2}} u^\varepsilon \varphi(x', x_n) dx_n \\ &= \frac{q_n}{\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{|x'| \leq \varepsilon} u^\varepsilon \varphi(x', \Psi(x')) [\sqrt{\varepsilon^2 - |x'|^2} - \Psi(x')] dx' \\ &\quad + \frac{q_n}{\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{|x'| \leq \varepsilon} dx' \int_{\Psi(x')}^{\sqrt{\varepsilon^2 - |x'|^2}} dx_n \int_{\Psi(x')}^{x_n} \frac{\partial}{\partial x_n} (u^\varepsilon \varphi) dx_n \\ &\equiv I_{11} + I_{12}. \end{aligned}$$

Then

$$\begin{aligned}
|I_{12}| &\leq \frac{C}{\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{\Omega_i^\varepsilon} |\sqrt{\varepsilon^2 - |x'|^2} - \Psi(x')| |\nabla(u^\varepsilon \varphi)| \\
&\leq C \sum_{i=1}^{m_\varepsilon} \int_{\Omega_i^\varepsilon} |\nabla(u^\varepsilon \varphi)| \leq C \left[\int_{\Omega} |\nabla(u^\varepsilon \varphi)|^2 \right]^{1/2} \left| \bigcup_{i=1}^{m_\varepsilon} \Omega_i^\varepsilon \right|^{1/2} \leq C \sqrt{\varepsilon} .
\end{aligned}$$

By (4.8) we have

$$I_{11} = \frac{q_n}{\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)} u^\varepsilon \varphi \sqrt{\varepsilon^2 - |x - z_i^\varepsilon|^2} + O(\varepsilon) .$$

Combining these estimates we get

$$(4.11) \quad I_1 = \frac{q_n}{\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)} u^\varepsilon \varphi \sqrt{\varepsilon^2 - |x - z_i^\varepsilon|^2} + O(\sqrt{\varepsilon}) .$$

Since $u^\varepsilon \rightarrow u$ weakly in $H^1(\Omega)$, by compactness of the imbedding $H^{\frac{1}{2}}(\Gamma_0) \rightarrow L^2(\Gamma_0)$ (note that the trace of $H^1(\Omega)$ on Γ_0 is in $H^{\frac{1}{2}}(\Gamma_0)$)

$$u^\varepsilon \rightarrow u \quad \text{strongly in } L^2(\Gamma_0) .$$

Therefore (4.11) yields

$$(4.12) \quad I_1 = \frac{q_n}{\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)} u \varphi \sqrt{\varepsilon^2 - |x - z_i^\varepsilon|^2} + o(1)$$

where $o(1) \rightarrow 0$ if $\varepsilon \rightarrow 0$.

To evaluate the last expression for I_1 suppose first that $u|_{\Gamma_0}$ is in C^1 . By (4.8)

$$\begin{aligned}
\int_{\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)} \sqrt{\varepsilon^2 - |x - z_i^\varepsilon|^2} &= \int_{B_\varepsilon(z_i^\varepsilon) \cap \mathbb{R}^{n-1}} \sqrt{\varepsilon^2 - |x - z_i^\varepsilon|^2} + O(\varepsilon^{n+1}) \\
&= \int_{B_\varepsilon^+(z_i^\varepsilon)} dx + O(\varepsilon^{n+1}) = \frac{1}{2} \omega_n \varepsilon^n + O(\varepsilon^{n+1}) .
\end{aligned}$$

It follows that

$$I_1 = \frac{q_n}{\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)} [u\varphi(x) - u\varphi(z_i^\varepsilon)] \sqrt{\varepsilon^2 - |x - z_i^\varepsilon|^2} \\ + \sum_{i=1}^{m_\varepsilon} \varepsilon^{n-1} (\omega_{n-1} + \frac{1}{2} s_n) u\varphi(z_i^\varepsilon) + O(\varepsilon)$$

where we have used the definition of q_n (given in Lemma 4.2).

Using the assumption that u is in $C^1(\Gamma_0)$ we conclude that

$$(4.13) \quad I_1 = \frac{\omega_{n-1} + \frac{1}{2} s_n}{\omega_{n-1}} \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)} u\varphi + o(1) .$$

In the general case where u is only in $L^2(\Gamma_0)$, we approximate u by C^1 functions to which the above argument applies. This yields, for u in $L^2(\Gamma_0)$, the estimate (4.13). Combining this estimate with (4.10), (4.9), and (4.7), the assertion (4.2) follows.

§5. Proof of (1.6) for $n \geq 3$, $\beta_n < \infty$. In this section we prove:

LEMMA 5.1. *Let $(A_0) - (A_2)$ hold with $n \geq 3$, and suppose that $\beta_n < \infty$. Then, for any $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi = 0$ on Γ_1 ,*

$$(5.1) \quad \int_{\Omega} (\nabla u^\varepsilon \cdot \nabla W^\varepsilon) \varphi dx \rightarrow - \int_{\Omega} \nabla u \cdot \nabla \varphi dx \quad \text{if } \varepsilon \rightarrow 0$$

To prove the lemma we shall need the following result:

LEMMA 5.2. *For any $\delta \in (0, \sigma_\varepsilon)$ let*

$$\Omega_i^{\varepsilon, \delta} = \{x \in \Omega, \text{dist}(x, \partial\Gamma_{0,i}^\varepsilon) < \delta\} .$$

Then

$$(5.2) \quad \int_{\partial\Omega_i^{\varepsilon, \delta} \cap \Omega} |\nabla u^\varepsilon| \leq C_\varepsilon (\sigma_\varepsilon)^{n-2} \delta^{\alpha_\varepsilon}$$

for some constants $C_\varepsilon > 0$, $0 < \alpha_\varepsilon < 1$.

Proof. By the proof of (3.18),

$$(5.3) \quad \int_{\partial\Omega_i^{\varepsilon, \delta} \cap \Omega} |\nabla u^\varepsilon| \leq C_\varepsilon \int_{\partial\Omega_i^{\varepsilon, \delta} \cap \Omega} \text{dist}(x, \Gamma_0)^{\alpha_\varepsilon - 1} \\ \leq C_\varepsilon \int_{\partial\Gamma_{0,i}^\varepsilon} \int_{\partial B_\delta^*(x)} \text{dist}(x, \Gamma_0)^{\alpha_\varepsilon - 1}$$

where $B_\delta^*(x)$ is a disc centered at $x \in \partial\Gamma_{0,i}^\epsilon$ with radius δ , lying in the 2-dimensional plane orthogonal to $\partial\Gamma_{0,i}^\epsilon$ at x . Choose coordinates $z_1, \dots, z_n$ centered at x so that $B_\delta^*(x)$ is given by $z_1^2 + z_2^2 \leq \delta^2$ and Γ_0 is given by $z_n = \Psi(z_1, \dots, z_{n-1})$ with $\Psi(0) = 0$, $\nabla\Psi(0) = 0$. Then

$$ds = \sqrt{dz_1^2 + dz_2^2} \leq \frac{C\delta}{\sqrt{\delta^2 - z_1^2}} \quad \text{along } \partial B_\delta^*(x) .$$

Since $|\Psi(z')| \leq C|z'|^2$,

$$\begin{aligned} \int_{\partial B_\delta^*(x)} \text{dist}(x, \Gamma_0)^{\alpha_\epsilon - 1} &\leq C \int_0^\delta \delta \frac{|\sqrt{\delta^2 - z_1^2} - \Psi|^{\alpha_\epsilon - 1}}{\sqrt{\delta^2 - z_1^2}} dz_1 \\ &\leq C\delta^{\alpha_\epsilon} \int_0^1 \frac{(\sqrt{1 - y^2} - C\delta y^2)^{\alpha_\epsilon - 1}}{\sqrt{1 - y^2}} dy \leq C_\epsilon \delta^{\alpha_\epsilon} . \end{aligned}$$

Substituting this into (5.3), the assertion (5.2) follows.

To prove Lemma 5.1 we need also the following:

LEMMA 5.3. *Under the assumption of Lemma 5.1*

$$(5.4) \quad \int \nabla u^\epsilon \cdot \nabla(W^\epsilon \varphi) dx \rightarrow 0 \quad \text{if } \epsilon \rightarrow 0 .$$

Proof. Write

$$(5.5) \quad \int_{\Omega} \nabla u^\epsilon \cdot \nabla(W^\epsilon \varphi) = \int_{\Omega \setminus \bigcup_i \Omega_i^{\epsilon, \delta}} \nabla u^\epsilon \cdot (\nabla W^\epsilon \varphi) + \int_{\bigcup_i \Omega_i^{\epsilon, \delta}} \nabla u^\epsilon \cdot \nabla(W^\epsilon \varphi) \equiv I_1 + I_2 .$$

By integration by parts,

$$\begin{aligned} (5.6) \quad I_1 &= \sum_{i=1}^{m_\epsilon} \int_{\partial\Omega_i^{\epsilon, \delta} \cap \Omega} \frac{\partial u^\epsilon}{\partial n} W^\epsilon \varphi + \sum_{i=1}^{m_\epsilon} \int_{\Gamma_{0,i}^\epsilon \setminus \Omega_i^{\epsilon, \delta}} \frac{\partial u^\epsilon}{\partial n} W^\epsilon \varphi \\ &\equiv I_{11} + I_{12} \end{aligned}$$

where we have used the fact that $\partial u^\epsilon / \partial n$ vanishes on $\Gamma_0 \setminus \Gamma_0^\epsilon$. By Lemma 5.2 and (2.2)

$$(5.7) \quad |I_{11}| \leq C_\epsilon (\beta_n)^{n-2} \delta^{\alpha_\epsilon} .$$

To estimate I_{12} we introduce coordinate system about z_i^ε such that

$$\Gamma_0 \cap B_{\varepsilon_i}(z_i^\varepsilon) = \{(x_1, \dots, x_n); x_n = \Psi(x_1, \dots, x_{n-1}), x_1^2 + \dots + x_{n-1}^2 < \varepsilon^2\}$$

and $\Psi(0) = 0$, $\nabla \Psi(0) = 0$. Then, for $x \in \Gamma_{0,i}^\varepsilon$,

$$(5.8) \quad \text{dist}(x, \mathbb{R}^{n-1} \cap B_{\delta_0 \sigma_\varepsilon}(z_i^\varepsilon)) \leq C|x_n| \leq C(x_1^2 + \dots + x_{n-1}^2) \leq C\sigma_\varepsilon^2.$$

Since W_R is in C^α and W^ε vanishes on $\mathbb{R}^{n-1} \cap T_i^\varepsilon$, which contains $\mathbb{R}^{n-1} \cap B_{\delta_0 \sigma_\varepsilon}(z_i^\varepsilon)$, we have, by (3.28)

$$|W^\varepsilon(x)| \leq C \left[\frac{\text{dist}(x, \mathbb{R}^{n-1} \cap B_{\delta_0 \sigma_\varepsilon})}{\sigma_\varepsilon} \right]^\alpha \leq C(\sigma_\varepsilon)^\alpha,$$

where (5.8) was used. It follows that

$$(5.9) \quad |I_{12}| \leq C(\sigma_\varepsilon)^\alpha \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_{0,i}^\varepsilon \setminus \overline{\Omega_i^{\varepsilon,\delta}}} \left(-\frac{\partial u^\varepsilon}{\partial n} \right)$$

since $\partial u^\varepsilon / \partial n < 0$ on $\Gamma_{0,i}^\varepsilon$ (u^ε takes minimum on $\Gamma_{0,i}^\varepsilon$).

Now take any $\varphi_1 \in C^\infty(\overline{\Omega})$ such that $\varphi_1 = 1$ on Γ_0 and $\varphi_1 = g_1$ on Γ_1 . Then

$$\begin{aligned} \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_{0,i}^\varepsilon \setminus \overline{\Omega_i^{\varepsilon,\delta}}} \left(-\frac{\partial u^\varepsilon}{\partial n} \right) &= \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_{0,i}^\varepsilon \setminus \overline{\Omega_i^{\varepsilon,\delta}}} \frac{\partial u^\varepsilon}{\partial n} (u^\varepsilon - \varphi_1) \\ &= \sum_{i=1}^{m_\varepsilon} \int_{\Omega \setminus \bigcup_i \Omega_i^{\varepsilon,\delta}} \nabla u^\varepsilon \cdot \nabla (u^\varepsilon - \varphi_1) - \sum_{i=1}^{m_\varepsilon} \int_{\partial \Omega_i^{\varepsilon,\delta} \cap \Omega} \frac{\partial u^\varepsilon}{\partial n} (u^\varepsilon - \varphi_1) \\ &\leq \int_{\Omega} [|\nabla u^\varepsilon|^2 + |\nabla u^\varepsilon| |\nabla \varphi_1|] + C_\varepsilon (\beta_n)^{n-2} \delta^{\alpha_\varepsilon} \end{aligned}$$

where in the last inequality we used an estimate similar to (5.7). Substituting this into (5.9) we get

$$|I_{12}| \leq (\sigma_\varepsilon)^\alpha [C + C_\varepsilon (\beta_n)^{n-2} \delta^{\alpha_\varepsilon}].$$

Combining this estimate with (5.7) we get, by (5.6),

$$|I_1| \leq C(\sigma_\varepsilon)^\alpha + C_\varepsilon \delta^{\delta_\varepsilon}.$$

Next

$$\begin{aligned}
|I_2| &\leq \left[\int_{\bigcup_i \Omega_i^{\varepsilon, \delta}} |\nabla u^\varepsilon|^2 \right]^{1/2} \left[\int_{\Omega} |\nabla(W^\varepsilon \varphi)|^2 \right]^{1/2} \\
&\leq C \left[\int_{\bigcup_i \Omega_i^{\varepsilon, \delta}} |\nabla u^\varepsilon|^2 \right]^{1/2}.
\end{aligned}$$

Taking $\delta \rightarrow 0$ we then get

$$|I_1 + I_2| \leq C(\sigma_\varepsilon)^\alpha$$

and, in view of (5.5), the assertion (5.4) follows.

Proof of Lemma 5.1. Writing

$$\int_{\Omega} \nabla u^\varepsilon \cdot \nabla(W^\varepsilon \varphi) = \int_{\Omega} (\nabla u^\varepsilon \cdot \nabla W^\varepsilon) \varphi + \int_{\Omega} (\nabla u^\varepsilon \cdot \nabla \varphi) W^\varepsilon$$

and using Lemma 5.3 and the fact that $W^\varepsilon \rightarrow 1$ strongly in L^2 , the assertion (5.1) follows.

§6. Proof of Theorem 2.4 for $n \geq 3$. By Lemma 2.3 we may assume that u^ε has been extended to $\mathbb{R}^n$, $0 \leq u^\varepsilon \leq \sup g_1$, and

$$\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C,$$

C independent of ε . It is clearly enough to establish just the boundary condition for u on Γ_0 . Let $\varphi \in C^\infty(\overline{\Omega})$ be such that $\varphi|_{\Gamma_1} = 0$, and write

$$(6.1) \quad \int_{\Omega} (\nabla u^\varepsilon \cdot \nabla W^\varepsilon) \varphi = \int_{\Omega} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon - \int_{\Omega} u^\varepsilon \nabla W^\varepsilon \cdot \nabla \varphi.$$

By (3.27) and (3.34), $\nabla W^\varepsilon \rightarrow 0$ weakly in $L^2(\Omega)$. Hence

$$(6.2) \quad \int_{\Omega} u^\varepsilon \nabla W^\varepsilon \cdot \nabla \varphi \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0.$$

Introduce coordinates about each z_i^ε such that $\{x_n = 0\}$ is tangent to Γ_0 at z_i^ε , and the inner normal to Γ_0 at z_i^ε is in the x_n -direction. Set $B_\varepsilon^\pm(z_i^\varepsilon) = \{x; |x - z_i^\varepsilon| < \varepsilon, x_n \gtrless 0\}$.

Since $\nabla W^\varepsilon = 0$ in $\Omega \setminus \cup \Omega_i^\varepsilon$, we can write

$$\begin{aligned}
(6.3) \quad & \int_{\Omega} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon = \sum_{i=1}^{m_\varepsilon} \int_{\Omega_i^\varepsilon} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon = \sum_{i=1}^{m_\varepsilon} \int_{Q_i^{\varepsilon-}} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon \\
& - \sum_{i=1}^{m_\varepsilon} \int_{Q_i^{\varepsilon+}} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon + \sum_{i=1}^{m_\varepsilon} \int_{(\Omega_i^\varepsilon \setminus Q_i^{\varepsilon-}) \cup Q_i^{\varepsilon+}} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon \equiv I_1 + I_2 + I_3
\end{aligned}$$

where

$$\begin{aligned}
Q_i^{\varepsilon+} &= B_\varepsilon^+(z_i^\varepsilon) \setminus (\Omega \cup T_i^\varepsilon), \\
Q_i^{\varepsilon-} &= [B_\varepsilon^-(z_i^\varepsilon) \cap \Omega] \setminus T_i^\varepsilon
\end{aligned}$$

By (3.28)

$$(6.4) \quad \int_{Q_i^{\varepsilon\pm}} |\nabla W^\varepsilon|^2 = \frac{1}{\sigma_\varepsilon^2} \int_{Q_i^{\varepsilon\pm}} |(\nabla W_R) \left(\frac{x - z_i^\varepsilon}{\sigma_\varepsilon} \right)|^2 = \sigma_\varepsilon^{n-2} \int_{Q_i^{\varepsilon\pm}/\sigma_\varepsilon} |\nabla W_R|^2$$

where $Q_i^{\varepsilon\pm}/\sigma_\varepsilon$ is the region in $B_R^\pm(0)$ bounded by $(\Gamma_0 \cap B_\varepsilon^\pm(z_i^\varepsilon) - z_i^\varepsilon)/\sigma_\varepsilon$ and ∂T , and $R = \varepsilon/\sigma_\varepsilon \rightarrow \infty$ since $\beta_n < \infty$. Representing Γ_0 near z_i^ε by $x_n = \Psi(x')$, where $\Psi(0) = 0$, $\nabla \Psi(0) = 0$, we have

$$(\Gamma_0 \cap B_\varepsilon^\pm(z_i^\varepsilon) - z_i^\varepsilon)/\sigma_\varepsilon = \left\{ (x', x_n); \quad x_n = \frac{1}{\sigma_\varepsilon} \Psi(\sigma_\varepsilon x'), \quad |x'| < R \right\} \cap B_R(0).$$

For any fixed $r < R$,

$$(6.5) \quad |B_r(0) \cap (Q_i^{\varepsilon\pm}/\sigma_\varepsilon)| \leq \int_{|x'| < r} \frac{1}{\sigma_\varepsilon} |\Psi(\sigma_\varepsilon x')| dx' \leq C \sigma_\varepsilon$$

since $|\Psi(y')| \leq C|y'|^2$. Writing

$$\int_{Q_i^{\varepsilon\pm}/\sigma_\varepsilon} |\nabla W_R|^2 \leq \int_{(Q_i^{\varepsilon\pm}/\sigma_\varepsilon) \cap B_r(0)} |\nabla W_R|^2 + \int_{\mathbb{R}^n \setminus B_r(0)} |\nabla W_R|^2$$

and recalling (see (3.16)) that $\nabla W_R \rightarrow \nabla W$ strongly in $L^2(\mathbb{R}^n)$, we can choose r large enough so that the second integral on the right-hand side is arbitrarily small, and then

use (6.5) to deduce that the first integral goes to zero as $\varepsilon \rightarrow 0$. From (6.4) it then follows that

$$\int_{Q_i^{\varepsilon \pm}} |\nabla W^\varepsilon|^2 \leq (\sigma_\varepsilon)^{n-2} \eta(\varepsilon), \quad \eta(\varepsilon) \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0.$$

This enables us to estimate $I_1 + I_2$ in (6.3):

$$\begin{aligned} |I_1 + I_2| &\leq \sum_{i=1}^{m_\varepsilon} \int_{Q_i^{\varepsilon -} \cup Q_i^{\varepsilon +}} |\nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon| \\ (6.6) \quad &\leq \left\{ \int_{\bigcup_{i=1}^{m_\varepsilon} (Q_i^{\varepsilon -} \cup Q_i^{\varepsilon +})} |\nabla W^\varepsilon|^2 \right\}^{1/2} \left\{ \int_{\Omega} |\nabla(u^\varepsilon \varphi)|^2 \right\}^{1/2} \\ &\leq C(\beta_n)^{\frac{n-2}{2}} \eta(\varepsilon) \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0. \end{aligned}$$

Next we can write

$$(6.7) \quad I_3 = \sum_{i=1}^{m_\varepsilon} \int_{\partial T_i^\varepsilon \cap (Q_i^{\varepsilon -} \cup Q_i^{\varepsilon +})} u^\varepsilon \varphi \frac{\partial W^\varepsilon}{\partial n} + \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \mathbb{R}_+^n} u^\varepsilon \varphi \frac{\partial W^\varepsilon}{\partial n}$$

since $u^\varepsilon = 0$ on $\Gamma_{0,i}^\varepsilon$, $\partial W^\varepsilon / \partial n = 0$ on $x_n = 0$ outside T_i^ε , and $\partial W^\varepsilon / \partial n$ is integrable on ∂T_i^ε by Lemma 3.4.

We estimate the first term on the right-hand side as follows:

$$\begin{aligned} \sum_{i=1}^{m_\varepsilon} \int_{\partial T_i^\varepsilon \cap Q_i^{\varepsilon \pm}} |u^\varepsilon \varphi \frac{\partial W^\varepsilon}{\partial n}| &\leq C \sum_{i=1}^{m_\varepsilon} \int_{\partial T_i^\varepsilon \cap Q_i^{\varepsilon \pm}} \left| \frac{\partial W^\varepsilon}{\partial n} \right| \\ &\leq C \sum_{i=1}^{m_\varepsilon} \frac{1}{\sigma_\varepsilon} \int_{\partial T_i^\varepsilon \cap Q_i^{\varepsilon \pm}} \left| \frac{\partial W_R}{\partial n} \right| \left(\frac{x - z_i^\varepsilon}{\sigma_\varepsilon} \right) \leq C \sum_{i=1}^{m_\varepsilon} \sigma_\varepsilon^{n-2} \int_{(\partial T_i^\varepsilon \cap Q_i^{\varepsilon \pm}) / \sigma_\varepsilon} \left| \frac{\partial W_R}{\partial n} \right|. \end{aligned}$$

For $(x', x_n) \in (\partial T_i^\varepsilon \cap Q_i^{\varepsilon \pm}) / \sigma_\varepsilon$ we have $|x'| \leq 1$ and

$$|x_n| \leq \frac{1}{\sigma_\varepsilon} |\Psi(\sigma_\varepsilon x')| \leq C \sigma_\varepsilon \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0.$$

By the proof of Lemma 3.4, each integral on the right-hand side must then converge to zero as $\varepsilon \rightarrow 0$ (uniformly in ε). Recalling (2.2) and the assumption that $\beta_n < \infty$, we conclude that the entire sum converges to zero if $\varepsilon \rightarrow 0$. Hence (6.7) yields

$$(6.8) \quad I_3 = \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \mathbb{R}_+^n} u^\varepsilon \varphi \frac{\partial W^\varepsilon}{\partial n} + \eta(\varepsilon), \quad \eta(\varepsilon) \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0.$$

Next, by (3.17), if $|x - z_i^\varepsilon| = \varepsilon$,

$$\begin{aligned} \frac{\partial W^\varepsilon}{\partial n}(x) &= \frac{1}{\sigma_\varepsilon} \left(\frac{\partial W_R}{\partial n} \right) \left(\frac{x - z_i^\varepsilon}{\sigma_\varepsilon} \right) = \frac{1}{\sigma_\varepsilon} \gamma(T) \frac{1}{R^{n-1}} (1 + o(1)) \\ &= \gamma(T) \frac{\sigma_\varepsilon^{n-2}}{\varepsilon^{n-1}} (1 + o(1)) \quad (o(1) \rightarrow 0 \text{ if } \varepsilon \rightarrow 0). \end{aligned}$$

Substituting this into (6.8) we get

$$\begin{aligned} (6.9) \quad I_3 &= \gamma(T) (\beta_n)^{n-2} \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \mathbb{R}_+^n} u^\varepsilon \varphi + o(1) \\ &\rightarrow \gamma(T) (\beta_n)^{n-2} \frac{s_n}{2} \int_{\Gamma_0} P u \varphi \quad \text{if } \varepsilon \rightarrow 0, \end{aligned}$$

where the last step we used Corollary 4.2. Combining this with (6.6) we see that the left-hand side of (6.3) converges to the right-hand side of (6.9) if $\varepsilon \rightarrow 0$. From (6.1), (6.2) we then get

$$(6.10) \quad \int_{\Omega} (\nabla u^\varepsilon \cdot \nabla W^\varepsilon) \varphi \rightarrow \int_{\Gamma_0} \mu_n u \varphi \quad \text{if } \varepsilon \rightarrow 0.$$

Using (6.10), (6.2) and (5.1) in (6.1) as $\varepsilon \rightarrow 0$, we conclude that

$$\int_{\Omega} \nabla u \cdot \nabla \varphi + \int_{\Omega_0} \mu_n u \varphi = 0.$$

Since φ is arbitrary, the proof of Theorem 2.4 is complete in case $n \geq 3$.

§7. Proof of Theorem 2.4 for $n = 2$. We first assume that $T = B_1$. Let $\varphi \in C^\infty(\overline{\Omega})$ be such that $\varphi = 0$ on Γ_1 and let W^ε be the function defined in (3.26). Since $W^\varepsilon \varphi$ belongs to $H^1(\Omega)$ and vanishes on $\Gamma_1 \cup \Gamma_0^\varepsilon$,

$$\int_{\Omega} \nabla u^\varepsilon \cdot \nabla (W^\varepsilon \varphi) = 0.$$

It follows that

$$\begin{aligned} (7.1) \quad \int_{\Omega} \nabla (u^\varepsilon \varphi) \cdot \nabla W^\varepsilon &= \int_{\Omega} (\nabla u^\varepsilon \cdot \nabla W^\varepsilon) \varphi + \int_{\Omega} (\nabla \varphi \cdot \nabla W^\varepsilon) u^\varepsilon \\ &= - \int_{\Omega} (\nabla u^\varepsilon \cdot \nabla \varphi) W^\varepsilon + \int_{\Omega} (\nabla \varphi \cdot \nabla W^\varepsilon) u^\varepsilon \rightarrow - \int_{\Omega} \nabla u \cdot \nabla \varphi \end{aligned}$$

since $W^\varepsilon \rightarrow 0$ weakly in $H^1(\Omega)$.

On the other hand, setting

$$\Omega_i^\varepsilon = [B_\varepsilon(z_i^\varepsilon) \setminus B_{\sigma_\varepsilon}(z_i^\varepsilon)] \cap \Omega$$

we can write

$$(7.2) \quad \begin{aligned} \int_{\Omega} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon &= \sum_{i=1}^{m_\varepsilon} \int_{\Omega_i^\varepsilon} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon = \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \Omega} \frac{\partial W^\varepsilon}{\partial n} u^\varepsilon \varphi \\ &+ \sum_{i=1}^{m_\varepsilon} \int_{\partial B_{\sigma_\varepsilon}(z_i^\varepsilon) \cap \Omega} \frac{\partial W^\varepsilon}{\partial n} u^\varepsilon \varphi + \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)} \frac{\partial W^\varepsilon}{\partial n} u^\varepsilon \varphi \equiv I_1 + I_2 + I_3. \end{aligned}$$

As before we introduce coordinates about z_i^ε so that $\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)$ is represented by $x_2 = \Psi(x_1)$ where $\Psi(0) = \Psi'(0) = 0$. Then along $\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)$ the normal is $(1 + (\Psi')^2)^{-1/2}(\Psi', -1)$, so that

$$\frac{\partial W^\varepsilon}{\partial n} = \frac{1}{\log \varepsilon - \log \sigma_\varepsilon} \frac{1}{r^2} \frac{1}{\sqrt{1 + (\Psi')^2}} (x_1 \Psi' - x_2).$$

Hence

$$\left| \frac{\partial W^\varepsilon}{\partial n} \right| \leq \frac{C}{|\log \varepsilon - \log \sigma_\varepsilon|} \leq C \beta_2 \varepsilon$$

and, therefore,

$$(7.3) \quad I_3 \rightarrow 0 \quad \text{if} \quad \varepsilon \rightarrow 0.$$

Next, on $\partial B_{\sigma_\varepsilon}(z_i^\varepsilon)$,

$$\frac{\partial W^\varepsilon}{\partial n} = -\frac{\partial W^\varepsilon}{\partial r} = -\frac{1}{\log \varepsilon - \log \sigma_\varepsilon} \frac{1}{\sigma_\varepsilon}.$$

Hence

$$(7.4) \quad I_2 = \frac{-1}{\varepsilon(\log \varepsilon - \log \sigma_\varepsilon)} \sum_{i=1}^{m_\varepsilon} \frac{\varepsilon}{\sigma_\varepsilon} \int_{\partial B_{\sigma_\varepsilon}(z_i^\varepsilon) \cap \Omega} u^\varepsilon \varphi$$

Let

$$v^\varepsilon(x) = \sigma_\varepsilon v\left(\frac{x - z_i^\varepsilon}{\sigma_\varepsilon}\right)$$

where v is defined in Lemma 4.3. Since $T = B_1$, $\partial v^\varepsilon / \partial n = 0$ on $\partial B_{\sigma_\varepsilon}^+(z_i^\varepsilon)$. Also $u^\varepsilon = 0$ on $\Gamma_0 \cap B_{\sigma_\varepsilon}(z_i^\varepsilon)$. By Green's formula and an estimate analogous to (4.9) we therefore have

$$(7.5) \quad \begin{aligned} \sum_{i=1}^{m_\varepsilon} \frac{\varepsilon}{\sigma_\varepsilon} \int_{\partial B_{\sigma_\varepsilon}(z_i^\varepsilon) \cap \Omega} u^\varepsilon \varphi &= \frac{\varepsilon}{\sigma_\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{B_{\sigma_\varepsilon}(z_i^\varepsilon) \cap \Omega} \nabla v^\varepsilon \cdot \nabla(u^\varepsilon \varphi) \\ &+ \frac{q_2 \varepsilon}{\sigma_\varepsilon^2} \sum_{i=1}^{m_\varepsilon} \int_{B_{\sigma_\varepsilon}(z_i^\varepsilon) \cap \Omega} u^\varepsilon \varphi \equiv I_{21} + I_{22} + O(\varepsilon). \end{aligned}$$

Clearly

$$\begin{aligned}
|I_{21}| &\leq \frac{\varepsilon}{\sigma_\varepsilon} \left(\sum_{i=1}^{m_\varepsilon} \int_{B_{\sigma_\varepsilon}(z_i^\varepsilon) \cap \Omega} |\nabla v^\varepsilon|^2 \right)^{1/2} C \\
&\leq \frac{C\varepsilon}{\sigma_\varepsilon} \left(\frac{1}{\varepsilon} \sigma_\varepsilon^2 \int_{B_1(z_1^\varepsilon)} |\nabla v|^2 \right)^{1/2} \leq C\sqrt{\varepsilon} .
\end{aligned}$$

Since $u^\varepsilon = 0$ on Γ_0^ε , we can apply the Poincaré inequality to get

$$\begin{aligned}
|I_{22}| &\leq \frac{C\varepsilon}{\sigma_\varepsilon^2} \left[\sum_{i=1}^{m_\varepsilon} \int_{B_{\sigma_\varepsilon}(z_i^\varepsilon) \cap \Omega} |u^\varepsilon \varphi|^2 \right]^{1/2} \frac{\sigma_\varepsilon}{\sqrt{\varepsilon}} \\
&\leq \frac{C\sqrt{\varepsilon}}{\sigma_\varepsilon} \left[\sigma_\varepsilon^2 \int_{\bigcup_{i=1}^{m_\varepsilon} (B_{\sigma_\varepsilon}(z_i^\varepsilon) \cap \Omega)} |\nabla(u^\varepsilon \varphi)|^2 \right]^{1/2} \leq C\sqrt{\varepsilon} .
\end{aligned}$$

Using these estimates in (7.5) we get, from (7.4),

$$(7.6) \quad I_2 \rightarrow 0 \quad \text{if} \quad \varepsilon \rightarrow 0 .$$

Finally, by Lemma 4.2,

$$\begin{aligned}
I_1 &= \frac{1}{\varepsilon \log \varepsilon - \varepsilon \log \sigma_\varepsilon} \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \Omega} u^\varepsilon \varphi \\
&\rightarrow \frac{1}{2} \beta_2 s_2 \int_{\Gamma_0} P u \varphi .
\end{aligned}$$

Using this and (7.6), (7.3) in the equation (7.2), and comparing with (7.1), we arrive at the equation

$$\int_{\Omega} \nabla u \cdot \nabla \varphi + \int_{\Gamma_0} \mu_2 u \varphi = 0 .$$

Since φ is arbitrary function in $C^\infty(\overline{\Omega})$ which vanishes on Γ_1 , the proof of the theorem is complete in case $T = B_1$.

One can easily check that the proof extends to the case where $T = B_{\delta_0}$ and u^ε converges to the same limit u (in fact, $I_2 \rightarrow 0$, $I_3 \rightarrow 0$). In the general case where $B_{\delta_0} \subset T \subset B_1$, we use comparison (i.e., Lemma 2.2) to complete the proof.

§8. Proof of Theorem 2.4'. Let $\bar{u}^\varepsilon$ denote the solution to (2.4)–(2.7) with holes $\bar{T}_i = z_i^\varepsilon + \bar{\sigma}_\varepsilon B_1(0)$ where $\bar{\sigma}_\varepsilon = \max_i \sigma_\varepsilon^i$. By comparison,

$$\bar{u}^\varepsilon \leq u^\varepsilon \leq u^N .$$

By the proof of Theorem 2.4 $\bar{u}^\varepsilon \rightarrow u^N$ weakly in $H^1(\Omega)$, and therefore the same is true of u^ε .

§9. Proof of Theorem 2.5. By Lemma 2.2,

$$u^D \leq u ;$$

furthermore we may shrink the holes if necessary and thus assume that $T_i^\varepsilon = B_{\sigma_\varepsilon}(z_i^\varepsilon)$ for some σ_ε for which we still have $\beta_n = \infty$ as well as $\sigma_\varepsilon/\varepsilon \rightarrow 0$.

Consider the case $n \geq 3$. As in §6 (see (6.3))

$$(9.1) \quad \begin{aligned} \int_{\Omega} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon &= \sum_{i=1}^{m_\varepsilon} \int_{Q_i^{\varepsilon-}} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon - \sum_{i=1}^{m_\varepsilon} \int_{Q_i^{\varepsilon+}} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon \\ &+ \sum_{i=1}^{m_\varepsilon} \int_{(\Omega_i^\varepsilon \setminus Q_i^{\varepsilon-}) \cup Q_i^{\varepsilon+}} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon = I_1 + I_2 + I_3 . \end{aligned}$$

The estimate (6.6) remains valid, namely,

$$(9.2) \quad |I_1 + I_2| \leq C(\beta_n^\varepsilon)^{n-2} \eta(\varepsilon) \quad \left(\beta_n^\varepsilon = \frac{\sigma_\varepsilon}{\varepsilon^{\frac{n-1}{n-2}}} \right)$$

where $\eta(\varepsilon) \rightarrow 0$ if $\varepsilon \rightarrow 0$. Next, by integration by parts we find, as in the derivation of (6.9), that

$$(9.3) \quad I_3 = \gamma(T)(\beta_n^\varepsilon)^{n-2} \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \Omega} u^\varepsilon \varphi + (\beta_n^\varepsilon)^{n-2} \eta(\varepsilon), \quad \eta(\varepsilon) \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0 ;$$

we have used here the fact that $R = \frac{\varepsilon}{\sigma_\varepsilon} \rightarrow \infty$.

Finally, the left-hand side of (9.1) can be estimated using Corollary 3.6:

$$(9.4) \quad \int_{\Omega} |\nabla(u^\varepsilon \varphi)| |\nabla W^\varepsilon| \leq C \left(\int_{\Omega} |\nabla W^\varepsilon|^2 \right)^{1/2} \leq C(\beta_n^\varepsilon)^{\frac{n-2}{2}} .$$

Using (9.2)-(9.4) in (9.1) we get

$$\gamma(T) \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \Omega} u^\varepsilon \varphi = (\beta_n^\varepsilon)^{-\frac{n-2}{2}} O(1) + \eta(\varepsilon) \rightarrow 0$$

if $\varepsilon \rightarrow 0$, and by Lemma 4.1 it then follows that

$$(9.5) \quad \sum_{i=1}^{m_\varepsilon} \int_{B_\varepsilon(z_i^\varepsilon) \cap \Gamma_0} u \varphi \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0 .$$

We claim that $u = 0$ a.e. on Γ_0 ; by elliptic regularity this will then imply that u is continuous up to Γ_0 and $u = 0$ everywhere on Γ_0 . If the assertion is not true then there exists a positive constant μ and a point $x_0 \in \Gamma_0$ such that

$$\frac{1}{\lambda^{n-1}} \text{meas}\{x \in \Gamma_0 \cap B_\lambda(x_0) ; u(x) \geq \mu\} \geq c , \quad c > 0 \quad \text{if } \lambda \rightarrow 0 .$$

By (A_3) , if λ is small enough then

$$(9.6) \quad \text{meas} \left\{ x \in \Gamma_0 \cap B_\lambda(x_0) \cap \left(\bigcup_{i=1}^{m_\varepsilon} B_\varepsilon(z_i^\varepsilon) \right) ; u(x) \geq \mu \right\} \geq c_0 > 0 .$$

for all sufficiently small ε . For such λ , choose $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi = 0$ on Γ_1 and on $\Gamma_0 \setminus B(x_0)$, $\varphi = 1$ on $\Gamma_0 \cap B_{\lambda/2}(x_0)$, and $\varphi \geq 0$. Then (9.5) implies that the left-hand side of (9.6) converges to zero if $\varepsilon \rightarrow 0$, a contradiction.

For $n = 2$ we use (7.2). As in §7,

$$\left| \int_{\Omega} \nabla(u^\varepsilon \varphi) \cdot \nabla W^\varepsilon \right| \leq C(\beta_2^\varepsilon)^{1/2} ,$$

$$|I_2| = \left| \sum_{i=1}^{m_\varepsilon} \int_{\partial B_{\sigma_\varepsilon}(z_i^\varepsilon) \cap \Omega} \frac{\partial W^\varepsilon}{\partial n} u^\varepsilon \varphi \right| \leq C\beta_2^\varepsilon \sqrt{\varepsilon} ,$$

$$|I_3| = \left| \sum_{i=1}^{m_\varepsilon} \int_{\Gamma_0 \cap B_\varepsilon(z_i^\varepsilon)} \frac{\partial W^\varepsilon}{\partial n} u^\varepsilon \varphi \right| \leq C\beta_n^\varepsilon \varepsilon ,$$

and

$$\sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \Omega} \frac{\partial W^\varepsilon}{\partial n} u^\varepsilon \varphi = \beta_2^\varepsilon (1 + \eta(\varepsilon)) \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \Omega} u^\varepsilon \varphi .$$

It follows that

$$\left| \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \Omega} u^\varepsilon \varphi \right| \leq C(\beta_2^\varepsilon)^{-\frac{1}{2}} + C\sqrt{\varepsilon} \rightarrow 0$$

if $\varepsilon \rightarrow 0$. We can now proceed as in the case $n \geq 3$.

§10. **Proof of Theorem 2.6.** Since $\rho_\varepsilon = u^\varepsilon - W^\varepsilon u$, we can write

$$(10.1) \quad \int_{\Omega} |\nabla \rho_\varepsilon|^2 = \int_{\Omega} \nabla u^\varepsilon \cdot \nabla (u^\varepsilon - W^\varepsilon u) - \int_{\Omega} \nabla (W^\varepsilon u) \cdot \nabla (u^\varepsilon - W^\varepsilon u) .$$

Consider the case $n \geq 3$. Using the fact that the function $u^\varepsilon - W^\varepsilon u$ vanishes on Γ_1 , the arguments used to prove Lemma 5.3 can be slightly modified to derive the relation

$$(10.2) \quad \int_{\Omega} \nabla u^\varepsilon \cdot \nabla (u^\varepsilon - W^\varepsilon u) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 .$$

Next

$$(10.3) \quad \begin{aligned} \int_{\Omega} \nabla (W^\varepsilon u) \cdot \nabla (u^\varepsilon - W^\varepsilon u) &= \int_{\Omega} u \nabla W^\varepsilon \cdot \nabla u^\varepsilon - \int_{\Omega} u \nabla W^\varepsilon \cdot \nabla (W^\varepsilon u) \\ &+ \int_{\Omega} W^\varepsilon \nabla u \cdot \nabla (u^\varepsilon - W^\varepsilon u) \equiv J_1 - J_2 + J_3 . \end{aligned}$$

Since $W^\varepsilon \rightarrow 1$ weakly in $H^1(\Omega)$,

$$(10.4) \quad J_3 \rightarrow \int \nabla u \cdot \nabla (u - u) = 0 .$$

Take $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi = 0$ on Γ_1 and $\varphi = 1$ in some $\overline{\Omega}$ -neighborhood of Γ_0 . Since $\nabla W^\varepsilon = 0$ outside $\overline{\Omega}$ -neighborhood of Γ_0 ,

$$\nabla W^\varepsilon = \varphi \nabla W^\varepsilon$$

and (since u is smooth in $\overline{\Omega}$) by (6.10)

$$(10.5) \quad J_1 = \int_{\Omega} (\varphi u) \nabla W^\varepsilon \cdot \nabla u^\varepsilon \rightarrow \int_{\Gamma_0} \mu_n u (\varphi u) = \int_{\Gamma_0} \mu_n u^2 .$$

Consider next

$$(10.6) \quad J_2 = \int_{\Omega} \nabla W^\varepsilon \cdot \nabla (W^\varepsilon u^2) - \int_{\Omega} W^\varepsilon u \nabla u \cdot \nabla W^\varepsilon = J_{21} + J_{22} .$$

Using the C^α continuity of W^ε in $B_\varepsilon(z_i^\varepsilon)$, we can proceed as in the derivation of (6.8) to get

$$J_{21} = \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \mathbb{R}_+^n} \frac{\partial W^\varepsilon}{\partial n} W^\varepsilon u^2 + \eta(\varepsilon)$$

where $\eta(\varepsilon) \rightarrow 0$ if $\varepsilon \rightarrow 0$. Using Lemmas 3.3 and 4.1 we find that

$$\begin{aligned} J_{21} &= \gamma(T)(\beta_n)^{n-2} \sum_{i=1}^{m_\varepsilon} \int_{\partial B_\varepsilon(z_i^\varepsilon) \cap \mathbb{R}_+^n} u^2 + \eta(\varepsilon) \\ &\rightarrow \int_{\Gamma_0} \mu_n u^2 \quad \text{if } \varepsilon \rightarrow 0. \end{aligned}$$

Since $W^\varepsilon \rightarrow 1$ weakly in $H^1(\Omega)$, $J_{22} \rightarrow 0$ if $\varepsilon \rightarrow 0$ and from (10.6) we get

$$J_2 \rightarrow \int_{\Gamma_0} \mu_n u^2.$$

Combining this with (10.4) and (10.3), we conclude that the left-hand side of (10.3) converges to zero if $\varepsilon \rightarrow 0$. Together with (10.2) and (10.1), the assertion (2.19) then follows in case $n \geq 3$.

The proof for $n = 2$ is similar, using (7.2) and (7.5).

The proof of Theorem 2.6' is immediate: Since $u^\varepsilon - u = 0$ on $\Gamma_1 \cup \Gamma_0^\varepsilon$,

$$\int_{\Omega} \nabla u^\varepsilon \cdot \nabla (u^\varepsilon - u) = 0;$$

hence

$$\int_{\Omega} |\nabla (u^\varepsilon - u)|^2 = - \int_{\Omega} \nabla u \cdot \nabla (u^\varepsilon - u) \rightarrow - \int_{\Omega} \nabla u \cdot \nabla (u - u) = 0$$

as $\varepsilon \rightarrow 0$. One can also show that $W^\varepsilon u \rightarrow u$ strongly in $H^1(\Omega)$.

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