THE NULL BOUNDARY CONTROLLABILITY FOR SEMILINEAR HEAT EQUATIONS

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Abstract. We consider the null boundary controllibility for one-dimensional semilinear heat equations. We obtain null boundary controllability results for semilinear equations when the initial data is bounded continuous and sufficiently small. In this work, we also prove a version of the nonlinear Cauchy-Kowalevski Theorem

1. Introduction. The aim of this work is to study the null boundary controllability problem for one-dimensional semilinear heat equations. First, we consider the following initial boundary value problem for a semilinear heat equation with Dirichlet boundary conditions: find $w(x,t)$ such that

$$w_t - w_{xx} = f(w,x) \quad \text{on } (0,1)\times(0,\infty) \tag{1.1}$$

$$w(0,t) = 0 \quad \text{for } t \geq 0 \tag{1.2}$$

$$w(x,0) = w_0(x) \quad \text{for } x \in (0,1] \tag{1.3}$$

$$w(1,t) = h(t) \quad \text{for } t \geq 0 \tag{1.4}$$

where $f(s,x)$ is an analytic function in both arguments in a neighborhood of the origin and belongs locally to Gevrey class 2 in its first argument and is Hölder continuous in its second argument such that $f(0,x) = 0$ and $D_1 f(0,x) = 0$ for all $x \in [0,1]$. We also consider the semilinear equation

$$w_t - w_{xx} = f(w, w_x, x) \quad \text{on } (0,1)\times(0,\infty) \tag{1.5}$$

instead of (1.1) where f is analytic in all arguments near $(0,0,0)$ and belongs locally to Gevrey class 2 in the first two arguments and is Hölder continuous in x such that $f(0,0,x) = 0$, $D_1 f(0,0,x) = 0$, $D_2 f(0,0,x) = 0$ for all $x \in [0,1]$. The problem of null boundary controllability for (1.1)–(1.4) or (1.2)-(1.5) which we are going to solve is: given

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$T > 0$, is it possible, for every initial data w_0 which is sufficient small and in an appropriate space, to find a corresponding controller $h(t)$ so that the solution of the resulting problem vanishes for $t \geq T$?

The method we use here is, to some extent, based on the work of W. Littman and L. Markus[**17**].

Our method proceeds roughly as follows:

(1) Extend the domain of the initial data w_0 to be $[0, 2]$ so that the extended w_0 is still small. Also extend the domain of f to the interval $[0, 2]$ such that all properties of f are preserved.

(2) With the modified initial data w_0, solve the initial-boundary value problem (1.1)-(1.3) or (1.5) & (1-2)-(1-3) with $w(2, t) = 0$ for $x \in [0, 2]$ and $t \in [0, \infty)$.

(3) Let ψ be a cut-off function satisfying $\psi(t) = 1$ for $t \leq T/2$ and $\psi(t) = 0$ for $t \geq T$. Set

$$g(t) = w_x(0, t)\psi(t)$$

where w is the solution of Step 2.

(4) Solve the Cauchy problem

$$u_t - u_{xx} = f(u, x) \quad \text{on } (0, 2) \times (0, \infty)$$
$$u(0, t) = 0, \quad u_x(0, t) = g(t) \quad \text{for } t \geq 0$$

in the x-direction to get a solution which vanishes for $t \geq T$ and equals the solution w for $t \leq T/2$.

(5) The control function is then obtained by setting $h(t) = u(1, t)$.

Steps 1-3 are more or less standard. The main difficulty is Step 4. To solve the Cauchy problem in Step 4, we use a nonlinear Ovcyannikov Theorem (or called nonlinear Cauchy-Kowalevski Theorem, see, e.g., [**1,10,17,21,22**]). To apply this theorem, we need g to be of Gevrey class 2 in t for $t > 0$. What this means is that there exist positive constants C, H such that

$$\left|\frac{\partial^n g}{\partial t^n}(t)\right| \leq CH^n(2n)! \quad \text{for } t > 0, n \geq 0.$$

By direct computation, the set of these Gevrey functions forms an algebra which is closed under differentiation with respect to t. Furthermore, it is possible to choose ψ in Step 3 to be a Gevrey class 2 function (such function can be explicitly written, see[4]). Thus g may be obtained as a Gevrey class 2 function as long as the solution of Step 2 is a Gevrey class 2 function.

We note that the definition of Gevrey functions which we use here differs a little from that of the definition used in [**17, 24**] but it is the same as the one used in [**3**,**11**].

We remark that the controller $h(t)$ is not necessarily unique. The null boundary controllability may also be obtained by other continuous controllers.

The paper is organized as follows. Section 2 is devoted to state the modified nonlinear Ovcyannikov Theorem for system of differential equations and its proof is given in Section 3. Because the way this theorem is used in this paper, the roles of the variables x and t are reversed from those in the usual statements, making x the "time variable". In Section 4, we define a scale of Banach spaces of Gevrey class which will be used to solve the problem in Step 4; and in Section 5, we actually solve the problem under the assumption that $g(t)$ is of Gevrey class 2 in $t > 0$. Because we need the existence in the whole unit x-interval (not just a small part of it), it is necessary to keep track of all constants and to check the proof carefully to ensure that by making $g(t)$ small, we get existence in the whole unit x-interval. Finally, we obtain the null controllability result for (1.1)–(1.4) in Section 6. The "sufficient small" assumption on the initial data can be eliminated under certain conditions. See remark after the proof of Theorem 6.1. In section 7, we use the same method to obtain null boundary controllability for (1.2)-(1.5).

A great many decisive developments in the controllability theory of the linear heat equation were initiated by H. O. Fattorini and D. L. Russell. These have been presented in numerous articles (see e.g.[**6-8**,23]).

There are some other methods for proving the boundary controllability results. Some methods which have achieved considerable success are the Method of the Multipliers[**13**] and the "Hilbert space uniqueness method" (HUM) due to J. L. Lions[**14**,**15**]. However, it is not clear whether these methods can be modified to obtain the result of this work. The advantage of our method is to transform the control problem to two well-posed problems.

2. Nonlinear Cauchy-Kowalevski Theorem for Systems of Differential Equations. In this section, we shall study the existence and uniqueness of solutions of some abstract Cauchy problems.

We begin by considering a 1-parameter family of Banach spaces X_s where the parameter s is allowed to vary in $[0, 1]$.

Definition 2.1. *$\{X_s\}_{0\leq s\leq 1}$ is a scale of Banach spaces if for any $s \in [0,1]$, X_s is a linear subspace of X_0 and if $s' \leq s$ then $X_s \subset X_{s'}$ and the natural injection of X_s into $X_{s'}$ has norm less than or equal to 1.*

We denote by $\|\cdot\|_s$ the norm of X_s.

A theorem which is originally due to L. V. Ovcyannikov is exploited (see, e.g., [**1, 10, 21, 22, 25, 26**]) in a number of ways to obtain results in the study of the nonlinear abstract Cauchy problem of the form

$$\frac{du}{dx} = F(u,x), \quad |x| < \eta,\ \eta > 0$$
$$u(0) = u_0.$$

Here the solutions are sought, as functions of the variable x, in a scale of Banach spaces $\{X_s\}$. The standard condition for Ovcyannikov's Theorem on F is that there exists a positive constant C such that for every pair of numbers s, s', $0 \leq s' < s \leq 1$, for all $u, v \in X_s$, and for all x in a prescribed interval, we have

$$\|F(u,x) - F(v,x)\|_{s'} \leq \frac{C}{s-s'}\|u-v\|_s. \tag{2.1}$$

This is the key condition needed to prove existence and uniqueness of the solution of the above Cauchy problem. The assumption (2.1) which holds for all $u, v \in X_s$ is more restrictive. In a paper of T. Kano and T. Nishida[**10**], they allow the condition to be relaxed to hold only for $u, v \in X_s$ with $\|u\|_s < R$ and $\|v\|_s < R$ for some R.

For each i, $i = 1, \cdots, m$, let $\{X_s^i\}_{0\leq s\leq 1}$ be a scale of Banach spaces with norm $\|\cdot\|_s^i$. P. Duchateau and F. Trèves[**1**] consider systems of differential equations of the form

$$\frac{du_i}{dx} = F_i(u_1, u_2, \cdots, u_m, x), \quad |x| < \eta_i,\ \eta_i > 0,\ i = 1, \cdots, m \tag{2.2}$$

$$u_i(0) = u_0^i, \quad i = 1, \cdots, m, \tag{2.3}$$

where the u_i, as functions of the variable x, are in X^i_s, $i = 1, \cdots, m$. They allow each F_i to have different exponents of $(s - s')$, that is, there exist constants $C_i > 0$ such that for every pair of numbers s, s', $0 \le s' < s \le 1$, for all $u_i, v_i \in X^i_s$ and for all x, $|x| < \eta_i$, we have

$$\|F_i(u_1, u_2, \cdots, u_m, x) - F_i(v_1, v_2, \cdots, v_m, x)\|_{s'} \\ \le \frac{C_i}{(s-s')^{\alpha_i}}[\|u_1 - v_1\|^1_s + \cdots + \|u_m - v_m\|^m_s], \quad i = 1, \cdots, m \tag{2.4}$$

for some parameters α_i, $i = 1, \cdots, m$.

We shall discuss the system of differential equations of the kind (2.2) and (2.3). We impose the same conditions on F_i as (2.4) but only for $\|u_i\|^i_s < R_i, \|v_i\|^i_s < R_i$ for some R_i, $i = 1, \cdots, m$. The conditions on α_i are the same as the conditions imposed in [**1**]. We shall omit the index i if there is no confusion.

We need the following assumptions. For the initial data, we assume that:

(H1) $u_{i,0} \in X^i_s$ for every $s \in [0, 1]$ and satisfies

$$\|u_{i,0}\|_s \le R_{i,0}$$

for some $R_{i,0} < \infty$ for $i = 1, \cdots, m$.

Concerning the function F_i, we assume that:

(H2) There are $R_i > R_{i,0} > 0$, $\eta_i > 0$, $i = 1, \cdots, m$, such that for every pair of numbers s, s' with $0 \le s' < s \le 1$ the mapping $F_i(u_1, \cdots, u_m, x)$, $i = 1, \cdots, m$, is continuous from the set

$$\{u_1 \in X^1_s \mid \|u_1\|_s < R_1\} \times \cdots \times \{u_m \in X^m_s \mid \|u_m\|_s < R_m\} \times \{x \mid |x| < \eta_i\}$$

into $X^i_{s'}$.

(H3) There are constants C_i, $i = 1, \cdots, m$, such that for every pair of numbers s, s' with $0 \le s' < s \le 1$, for all $\|u_j\|_s < R_j$, $\|v_j\|_s < R_j$, $j = 1, \cdots, m$, and for all x, $|x| < \eta_i$, we have

$$\|F_i(u_1, u_2, \cdots, u_m, x) - F_i(v_1, v_2, \cdots, v_m, x)\|_{s'} \\ \le \frac{C_i}{(s-s')^{\alpha_i}}[\vartheta^1_i \|u_1 - v_1\|_s + \cdots + \vartheta^m_i \|u_m - v_m\|_s], \quad i = 1, \cdots, m,$$

where the number ϑ_i^j is set to be zero if F_i is independent of u_j and to be one otherwise, for some parameters $\alpha_i \geq 0$, $i = 1, \cdots, m$, such that for any collection of m^2 numbers c_i^j, the degree of $P(\lambda, \mu)$ with respect to λ, μ is at most m, where the expression $P(\lambda, \mu)$ of two variables λ, μ is defined by

$$P(\lambda, \mu) = det(\lambda I - [\mu^{\alpha_i} \vartheta_i^j c_i^j]),$$

with I the $m \times m$ identity matrix and the degree is defined to be the highest degree among all monomials in $P(\lambda, \mu)$.

(H4) $F_i(0, \cdots, 0, x)$ is a continuous function of x, $|x| < \eta_i$, with values in X_s^i for every $s < 1$ and satisfies

$$\|F_i(0, \cdots, 0, x)\|_s \leq \frac{K_i}{(1-s)^{\alpha_i}}, \quad 0 \leq s < 1$$

for some constants K_i, $i = 1, \cdots, m$, with α_i defined in (H3).

We now state the existence and uniqueness theorem for solutions of (2.2) and (2.3) as follows.

Theorem 2.1. *Under the preceding hypotheses (H1)–(H4), there is a positive constant a such that the Cauchy problem (2.2)-(2.3) has a unique solution $\{u_i(x), i = 1, \cdots, m\}$, which are continuously differentiable functions of x, $|x| < a(1-s)$, with values in X_s^i, $\|u_i(x)\|_s < R_i$, for every $s < 1/2$.*

The proof of this theorem will be given in the next section.

A few remarks about Theorem 2.1 are in order.

First, the necessity of the assumption on the degree of $P(\lambda, \mu)$ can be illustrated by the case $m = 1$, $\alpha_1 = 2$ as shown in [**1**].

Secondly, the assumptions (H2) and (H3) are weaker than those of [**1**], because here we only require those conditions to hold for $\|u_j\|_s < R_j$ and $\|v_j\|_s < R_j$ for some R_j, $j = 1, \cdots, m$. Our assumptions are also more flexible than the assumptions in [**10**], because we allow the exponents α_i to be different for each equation. But a small price must be paid for the weaker assumptions. The solution we obtained in Theorem 2.1 has value in X_s^i for $s < \frac{1}{2}$. In fact, the upper bound $\frac{1}{2}$ is not optimal. By inspecting the proof of Theorem 2.1, we see that the solution can have bounded value in X^i_s for $s < 1 - \epsilon$ for any positive small number ϵ.

Finally, when each F_i, $i = 1, \cdots, m$ is analytic in all of its arguments and the Cauchy data is analytic, our theorem yields the classical Cauchy-Kowalevski Theorem.

3. Proof of Theorem 2.1. Our proof modifies the proof of T. Kano and T. Nishida's Theorem in [**10**].

We first transform the problem (2.2) and (2.3) to the integral equation

$$u_i(x) = u_{i,0} + \int_0^x F_i(u_1(y), \cdots, u_m(y), y)dy, \quad i = 1, \cdots, m. \tag{3.1}$$

Let a be a small positive constant to be determined later. For $i = 1, \cdots, m$, let X^i be the Banach space of functions $u(x)$ with values in X^i_s, which are continuous in x for $|x| < a(1-s)$ for every $s \in [0, 1/2)$, and have the norm

$$M_i[u] \equiv \sup_{\substack{0 \leq s < 1/2 \\ |x| < a(1-s)}} \|u(x)\|_s \frac{|a(1-s) - x|^{e_i}}{a(1-s)} < \infty \tag{3.2}$$

where $e_i \geq 1$ will be defined later. We are looking for solutions of integral equations (3.1) whose norms $M_i[u_i]$ are finite with some constant $a > 0$ suitably small.

We will find solutions of (3.1) in $X^i, i = 1, \cdots, m$, via the following successive approximation procedure

$$u_i^{(0)}(x) = u_{i,0} \tag{3.3}$$

$$u_i^{(k+1)}(x) = u_{i,0} + \int_0^x F_i(u_1^{(k)}(y), \cdots, u_m^{(k)}(y), y)dy, \quad k \geq 0, \tag{3.4}$$

where $\|u_i^{(k)}(x)\|_s < R_i$ for $|x| < a(1-s)$, $0 \leq s < 1/2$, $i = 1, \cdots, m$. We shall only deal with the case $0 \leq x < a(1-s)$. The other case is similar.

To ensure that $\|u_i^{(k+1)}(x)\|_s < R_i$, we will require that

$$\sum_{k=1}^{\infty} \|u_i^{(k+1)}(x) - u_i^{(k)}(x)\|_s \leq \frac{R_i - R_{i,0}}{2} \tag{3.5}$$

because by recursion this will imply

$$\begin{aligned}\|u_i^{(k+1)}(x)\|_s &\leq \sum_{j=0}^{k} \|u_i^{(j+1)}(x) - u_i^{(j)}(x)\|_s + \|u_{i,0}\|_s \\ &\leq \frac{R_i - R_{i,0}}{2} + R_{i,0} \\ &< R_i.\end{aligned}$$

We shall prove that (3.5) can be fulfilled for every i if we diminish the x-interval step by step, where we must ensure that the length of the limit interval is positive. The k-th iteration $u_i^{(k)}$ will be defined on

$$0 \leq x < a_k(1-s) \tag{3.6}$$

where the number a_k is defined by

$$a_{k+1} = [1 - \frac{1}{4}(1 + \frac{1}{2} + \cdots + \frac{1}{2^k})]a_0 = a_k - \frac{a_0}{2^{k+2}}, k = 0, 1, \cdots \tag{3.7}$$

so that

$$a = \lim_{k\to\infty} a_k = a_0/2 \tag{3.8}$$

and a_0 will be chosen suitably small later. Corresponding to the x-interval (3.6) we define as (3.2) the quantity

$$M_i^{(k)}[u] \equiv \sup_{\substack{0\leq s<1/2 \\ 0\leq x<a_k(1-s)}} \|u(x)\|_s \frac{[a_k(1-s) - x]^{e_i}}{a_k(1-s)} < \infty.$$

From this definition we immediately get the property

$$M_i^{(k+1)}[u] \leq M_i^{(k)}[u] \quad \text{if} \quad a_{k+1} < a_k. \tag{3.9}$$

Now assume that $u_i^{(j)}$ are determined with $M_i^{(j)}[u_i^{(j)}] < \infty$ and $\|u_i^{(j)}(x)\|_s < R_i$ for $0 \leq x < a_j(1-s)$, $0 \leq s < \frac{1}{2}$, $j = 1, \cdots, k$, $i = 1, \cdots, m$. By assumption (H2), $u_i^{(k+1)}(x)$ is well-defined. Set

$$\lambda_i^{(k)} = M_i^{(k)}[u_i^{(k+1)} - u_i^{(k)}]. \tag{3.10}$$

Then

$$\|u_i^{(k+1)}(x) - u_i^{(k)}(x)\|_s \le \frac{\lambda_i^{(k)}}{(a_k(1-s)-x)^{e_i}/a_k(1-s)}$$
$$\le \frac{\lambda_i^{(k)}}{(1-a_{k+1}/a_k)^{e_i}} \cdot \frac{2^{e_i-1}}{a_k^{e_i-1}}$$

for $0 \le x < a_{k+1}(1-s)$, $0 \le s < 1/2$, and thus, by (3.7),

$$\|u_i^{(k+1)}(x) - u_i^{(k)}(x)\|_s \le (2^{k+2})^{e_i} 2^{e_i-1} a_0^{1-e_i} \lambda_i^{(k)}. \tag{3.11}$$

So if we require that

$$\sum_{k=0}^{\infty} (2^{k+2})^{e_i} 2^{e_i-1} a_0^{1-e_i} \lambda_i^{(k)} \le \frac{R_i - R_{i,0}}{2} \tag{3.12}$$

for suitable a_0, then (3.5) holds and by recursion

$$\|u_i^{(k+1)}(x)\|_s < R_i,$$

and thus $F_i(u_1^{(k+1)}(y), \cdots, u_m^{(k+1)}(y), y)$ is defined, and so is $u_i^{(k+2)}(x)$ for $i = 1, \cdots, m$.

Our aim is to estimate $\lambda_i^{(k)}$, $i = 1, \cdots, m$, so that $\lambda_i^{(k)} \to 0$ as $k \to \infty$ and (3.12) holds for all $k \ge 0$. First of all, let us note that without loss of generality we may assume that the constants K_i, $i = 1, \cdots, m$, in assumption (H4) are zero, i.e., we may assume that

$$F_i(0, \cdots, 0, x) = 0 \quad \text{for } 0 \le x < \eta_i, \quad i = 1, \cdots, m.$$

In fact, we can take $u_{i,0}$ to be $u_{i,0} + \int_0^x F_i(0, \cdots, 0, y)dy$ which is bounded by $R_{i,0} + K_i\eta_i$ for $0 \le x < \eta_i$ and $s \in [0, 1)$.

By (3.4), we have

$$u_i^{(k+2)}(x) - u_i^{(k+1)}(x)$$
$$= \int_0^x [F_i(u_1^{(k+1)}(y), \cdots, u_m^{(k+1)}(y), y) - F_i(u_1^{(k)}(y), \cdots, u_m^{(k)}(y), y)]dy.$$

Thus for $0 \le x < a_{k+1}(1-s)$, it follows from the assumption (H3) that

$$\|u_i^{(k+2)}(x) - u_i^{(k+1)}(x)\|_s \le C_i \int_0^x \frac{1}{(s(y)-s)^{e_i}} \left(\sum_{j=1}^m \vartheta_i^j \|u_j^{(k+1)}(y) - u_j^{(k)}(y)\|_{s(y)} \right) dy$$

for some choice $s(y)$ with $s < s(y) < 1 - y/a_{k+1}$. We may set

$$s(y) = (1 - y/a_{k+1} + s)/2.$$

Then we find by virtue of (3.10),

$$\begin{aligned}
&\|u_i^{(k+2)}(x) - u_i^{(k+1)}(x)\|_s \\
&\le C_i \int_0^x \frac{1}{(s(y)-s)^{\alpha_i}} \sum_{j=1}^m \vartheta_i^j \frac{\lambda_j^{(k)}}{(a_k(1-s(y)) - y)^{e_j}/a_k(1-s(y))} dy \\
&\le C_i \sum_{j=1}^m \vartheta_i^j \lambda_j^{(k)} (2a_{k+1})^{\alpha_i} 2^{e_j - 1} \int_0^x \frac{a_{k+1}(1-s) + y}{[a_{k+1}(1-s) - y]^{\alpha_i + e_j}} dy. \qquad (3.13)
\end{aligned}$$

We define the exponents $e_i, i = 1, \cdots, m$ as follows. By (3.13), to get the finite supremum in the definition of $\lambda_i^{(k)}$, the exponents e_i, $i = 1, \cdots, m$, should be defined so that

$$e_i \ge \alpha_i + e_j - 1 \quad \text{for all } j \text{ such that } \vartheta_i^j = 1 \qquad (3.14)$$

We also know that $\vartheta_i^i = 1$ implies that $\alpha_i \le 1$ from assumption (H5), so we only have to deal with (3.14) for $j \ne i$. By renumbering the variables, we may assume that $\alpha_1 \le \cdots \le \alpha_m$. We define e_i in the reverse order of i, under the following rule:

(1) if e_i has been defined in the previous steps, then we skip to the smaller index;

(2) if $\vartheta_i^j = 0$ for all $j \ne i$, let $e_i = \alpha_i \vee 1$, where $a \vee b = \max(a, b)$, otherwise, let $j_0 = \min\{j \ne i \mid \vartheta_i^j = 1\}$. We let $e_i = (\alpha_i + e_{j_0} - 1) \vee 1$, where $e_{j_0} = \alpha_{j_0} \vee 1$ if e_{j_0} has not been defined in the previous steps.

By considering the restriction on the α_i from assumption (H3), it is easy to check that the e_i, $i = 1, \cdots, m$ satisfy (3.14).

To simplify the calculation of the $\lambda_i^{(k)}$, we consider only $m = 3$ (it is the only case we shall have occasion to use). The proof of the general case is essentially similar, although somewhat more cumbersome. By the definition of e_i, $a_{k+1} < a_0$ and (3.13) we have

$$\begin{aligned}
\lambda_i^{(k+1)} &= M_i^{(k+1)}[u_i^{(k+2)} - u_i^{(k+1)}] \\
&\le \sum_{j=1}^3 C_i \vartheta_i^j \lambda_j^{(k)} (2a_0)^{\alpha_i} 2^{e_j - 1} 2^{(3-(\alpha_i + e_j)) \vee 0} {a_0}^{e_i - (\alpha_i + e_j - 1)} \\
&= \sum_{j=1}^3 \vartheta_i^j M_{ij} {a_0}^{e_i - e_j + 1} \lambda_j^{(k)}
\end{aligned}$$

where

$$\begin{aligned} M_{ij} &= C_i 2^{\alpha_i+e_j-1} 2^{[3-(\alpha_i+e_j)]\vee 0} \\ &= \begin{cases} 4C_i & \text{if } \alpha_i+e_j \le 3 \\ 2^{\alpha_i+e_j-1}C_i & \text{if } \alpha_i+e_j \ge 4. \end{cases} \end{aligned}$$

Hence for $i=1,2,3,\ k=0,1,2,\cdots$,

$$\lambda_i^{(k+1)} \le \sum_{j=1}^{3} \vartheta_i^j M_{ij} {a_0}^{e_i-e_j+1} \lambda_j^{(k)}. \tag{3.15}$$

Now, we compute $\lambda_i^{(0)}$. For $0 \le x < a_0(1-s)$,

$$\begin{aligned} \|u_i^{(1)}(x) - u_i^{(0)}(x)\|_s &= \| \int_0^x F_i(u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, y) dy \|_s \\ &\le C_i \sum_{j=1}^{3} \vartheta_i^j R_{j,0} \int_0^x \frac{1}{(s(y)-s)^{\alpha_i}} dy \\ &\le C_i \sum_{j=1}^{3} \vartheta_i^j R_{j,0} \int_0^x \frac{(2a_0)^{\alpha_i}}{[a_{k+1}(1-s)-y]^{\alpha_i}} dy \end{aligned}$$

and so,

$$\begin{aligned} \lambda_i^{(0)} &= \sup \|u_i^{(1)}(x) - u_i^{(0)}(x)\|_s \frac{[a_0(1-s)-x]^{e_i}}{a_0(1-s)} \\ &\le 2C_i(2a_0)^{\alpha_i} \sum_{j=1}^{3} \vartheta_i^j R_{j,0} < \infty. \end{aligned} \tag{3.16}$$

Let $M = \max_{\substack{1\le i\le 3 \\ 1\le j\le 3}} M_{ij}$. Then by (3.15), for $i=1,2,3,\ k=0,1,2,\cdots$,

$$\begin{aligned} \lambda_i^{(k+1)} &\le \sum_{j=1}^{3} \vartheta_i^j M_{ij} {a_0}^{e_i-e_j+1} \lambda_j^{(k)} \\ &\le M {a_0}^{e_i+1} \Big(\sum_{j=1}^{3} {a_0}^{-e_j} \lambda_j^{(k)} \Big). \end{aligned}$$

So,

$$\sum_{i=1}^{3} {a_0}^{-e_i} \lambda_i^{(k+1)} \le (3Ma_0) \Big(\sum_{j=1}^{3} {a_0}^{-e_j} \lambda_j^{(k)} \Big).$$

Thus if a_0 is chosen so small that

$$3Ma_0 < \left(\frac{1}{3}\right)^{e_i}, \tag{3.17}$$

then

$$\sum_{i=1}^{3} a_0{}^{-e_i}\lambda_i^{(k)} \leq \left[\left(\frac{1}{3}\right)^{e_i}\right]^k \cdot K, \tag{3.18}$$

where

$$K = \sum_{j=1}^{3} a_0{}^{-e_j}\lambda_j^{(0)} < \infty.$$

and hence

$$\sum_{k=0}^{\infty}\sum_{i=1}^{3} a_0{}^{-e_i}\lambda_i^{(k)} \leq 2K. \tag{3.19}$$

Therefore the series $\sum_{k=0}^{\infty}\lambda_i^{(k)}$ converges.

It remains to verify (3.12). According to (3.11), (3.18), we have for $0 \leq x < a_{k+1}(1-s)$, $0 \leq s < 1/2$

$$\begin{aligned}\sum_{k=0}^{\infty}\|u_i^{(k+1)}(x) - u_i^{(k)}(x)\|_s &\leq 2^{3e_i-1}a_0\sum_{k=0}^{\infty}(2^{e_i})^k a_0{}^{-e_i}\lambda_i^{(k)} \\ &\leq 2^{3e_i-1}a_0\sum_{k=0}^{\infty}(2^{e_i})^k\left[\left(\frac{1}{3}\right)^{e_i}\right]^k K \\ &\leq 3a_0 2^{3e_i-1}K.\end{aligned}$$

Thus, if a_0 is taken small enough such that

$$3a_0 2^{3e_i-1}K < \frac{R_i - R_{i,0}}{2}, \quad i = 1,2,3, \tag{3.20}$$

then (3.12) is satisfied and

$$\|u_i^{(k+1)}(x)\|_s < \frac{R_i - R_{i,0}}{2} + R_{i,0} < R_i.$$

We conclude that if we choose a_0 so small that (3.17), (3.20) hold and $a_0 < \eta_i$, $i = 1,2,3$, then the functions $u_i^{(k)}(x)$ are defined for all k and i with

$$\|u_i^{(k)}(x)\|_s < R_i \quad \text{for } 0 \leq x < a_k(1-s), 0 \leq s < 1/2, i = 1,2,3. \tag{3.21}$$

Also, according to (3.10), for $0 \le x < a(1-s) < a_k(1-s)$,

$$\|u_i^{(k+1)}(x) - u_i^{(k)}(x)\|_s \le \frac{\lambda_i^{(k)}}{(a_k(1-s)-x)^{e_i}/a_k(1-s)} < \frac{\lambda_i^{(k)}}{(a(1-s)-x)^{e_i}/a(1-s)},$$

so,

$$M_i[u_i^{(k+1)} - u_i^{(k)}] \le \lambda_i^{(k)}.$$

Since the series $\sum \lambda_i^{(k)}$ converges for each i, the sequence $u_i^{(k)}$ converges to some limit $u_i(x)$ in X^i, $i = 1, 2, 3$, and from (3.21)

$$\|u_i(x)\|_s \le R_i, \quad 0 \le x < a(1-s),\ 0 \le s < 1/2,\ i = 1, 2, 3.$$

These $u_i(x)$ satisfy (3.1). In fact, we have for $0 \le x < a(1-s)$ and $0 \le s' < s < 1/2$ that

$$\begin{aligned}
&\|u_{i,0} + \int_0^x F_i(u_1(y), u_2(y), u_3(y), y)dy - u_i(x)\|_{s'} \\
&\le \int_0^x \|F_i(u_1(y), u_2(y), u_3(y), y) - F_i(u_1^{(k)}(y), u_2^{(k)}(y), u_3^{(k)}(y), y)\|_{s'} dy \\
&\qquad\qquad + \|u_i(x) - u_i^{(k+1)}(x)\|_{s'} \\
&\le \frac{C_i}{(s-s')^{\alpha_i}} \int_0^x [\,\|u_1(y) - u_1^{(k)}(y)\|_s + \|u_2(y) - u_2^{(k)}(y)\|_s + \|u_3(y) - u_3^{(k)}(y)\|_s]dy \\
&\qquad\qquad + \|u_i(x) - u_i^{(k+1)}(x)\|_{s'}
\end{aligned}$$

and both terms on the right-hand side of the last inequality tend to zero as $k \to \infty$. This proves the existence part of Theorem 2.1.

For the uniqueness, we also only consider the case $m = 3$. We suppose that for some a there exist C^1 functions $u_i(x), v_i(x)$ in x for $0 \le x < a(1-s)$, $0 \le s < 1/2$ with values in X^i satisfying (3.1) for $i = 1, 2, 3$. Then $w_i(x) \equiv u_i(x) - v_i(x)$ satisfies

$$w_i(x) = \int_0^x [F_i(u_1(y), u_2(y), u_3(y), y) - F_i(v_1(y), v_2(y), v_3(y), y)]dy.$$

Given any fixed $s_1 < 1/2$, we see that both $N_i[u_i]$ and $N_i[v_i]$ are finite, where the norms N_i are defined by

$$N_i[w] \equiv \sup_{\substack{0 \le s < s_1 \\ 0 \le x < a_k(1-s)}} \|w(x)\|_s \frac{[a(1-s)-x]^{e_i}}{a(1-s)}.$$

Then the assumption (H3) implies that for all $0 \le x < a(1-s)$, $0 \le s < s_1$,

$$\|w_i(x)\|_s \le C_i \int_0^x \frac{1}{(s(y)-s)^{e_i}} \Big(\sum_{j=1}^3 \vartheta_i^j \|w_i(y)\|_{s(y)} \Big) dy$$

with some choice $s(y) \in (s, 1-y/a)$. By the same argument as in the proof of (3.15), we obtain that

$$N_i[w_i] \le \sum_{j=1}^3 M a^{e_i - e_j + 1} N_j[w_j].$$

where M is defined as in (3.17). These together imply that

$$\sum_{i=1}^3 a_0{}^{-e_i} N_i[w_i] \le (3Ma)(\sum_{i=1}^3 a_0{}^{-e_i} N_i[w_i])$$

and thus $N_i[w_i] = 0$ if $3Ma < 1$. Consequently, if we choose a sufficiently small then, $N_i[w_i] = 0, i = 1,2,3$, and hence

$$\|w_i(x)\|_s \equiv 0 \quad \text{for } 0 \le x < a(1-s), 0 \le s < s_1.$$

Since this is true for all s_1, we conclude that $w_i(x) \equiv 0$ for $i = 1,2,3$. This proves the theorem.

4. Scales of Banach spaces of Gevrey functions. We shall define scales of Banach spaces depending on parameters d, β, and s, where $d \ge 1$, $\beta > 2$, and $s \in [0,1]$.

Definition 4.1. *Let K be a compact interval and let θ_0 and θ_1 be two positive constants such that $\theta_0 < \theta_1 < \infty$. Given $d \ge 1$, $\beta > 2$, and $s \in [0,1]$, we define the space $B_s(d,\beta)$ to be the set of all $C^\infty(K)$ functions ϕ which is defined in K satisfying*

$$\|\phi\|_s \equiv \sup_{n \ge 0} \max_{t \in K} \frac{\tilde{n}^\beta \theta(s)^n}{\lambda(dn)!} |\phi^{(n)}(t)| < \infty, \tag{4.1}$$

where $\theta(s)^{-1} = (1-s)\theta_0{}^{-1} + s\theta_1{}^{-1}$, $\tilde{n} = \max(n,1)$, and λ is a positive constant.

It is easy to see that $\|\cdot\|_s$ in (4.1) is a norm on $B_s(d,\beta)$ which makes $B_s(d,\beta)$ into a Banach space. Moreover, for d and β fixed and s ranging in $[0,1]$ the family $\{B_s(d,\beta)\}$ forms a scale of Banach spaces. We shall choose λ so that $B_s(d,\beta)$ is a Banach algebra.

In the following propositions, we shall study the properties of these spaces. First, for any $d \geq 1$, $p \geq q \geq 0$, p, q integers, it is easy to verify the following two inequalities:

$$\binom{p}{q} \leq \binom{dp}{dq}, \quad \text{where } \binom{p}{q} = \frac{p!}{q!(p-q)!} \tag{4.2}$$

$$\frac{(d(p-q)+1)\cdots(d(p-q)+d)(dq+1)\cdots(dq+d)}{(dp+1)\cdots(dp+d)} \leq d^{2d}\frac{(p-q+1)(q+1)}{p+1}. \tag{4.3}$$

Proposition 4.1. *If $\lambda \leq 1/[2+2^\beta \sum_{k=1}^{\infty}(1/k)^\beta]$, then $B_s(d,\beta)$ is a Banach algebra for any $s \in [0,1]$.*

PROOF: It suffices to show that for functions $\phi, \psi \in B_s(d,\beta)$

$$\|\phi\psi\|_s \leq \|\phi\|_s\|\psi\|_s.$$

By Leibnitz's rule, we only need to require that

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\lambda(d(n-k))!}{((n-k)^\sim)^\beta}\frac{\lambda(dk)!}{\tilde{k}^\beta} \leq \frac{\lambda(dn)!}{\tilde{n}^\beta}. \tag{4.4}$$

Now, (4.4) is satisfied if

$$\lambda\sum_{k=0}^{n}\binom{n}{k}\binom{dn}{dk}^{-1}\left(\frac{\tilde{n}}{(n-k)^\sim\tilde{k}}\right)^\beta \leq 1.$$

Since

$$\begin{aligned}\sum_{k=0}^{n}\left(\frac{\tilde{n}}{(n-k)^\sim\tilde{k}}\right)^\beta &\leq 2+\sum_{k=1}^{n-1}\left(\frac{n}{(n-k)k}\right)^\beta \\ &= 2+\sum_{k=1}^{n-1}\left(\frac{1}{n-k}+\frac{1}{k}\right)^\beta \\ &\leq 2+2^\beta\sum_{k=1}^{n-1}(1/k)^\beta \\ &\leq 2+2^\beta\sum_{k=1}^{\infty}(1/k)^\beta\end{aligned}$$

and by (4.2), we see that (4.4) is satisfied if

$$\lambda \leq 1/[2+2^\beta\sum_{k=1}^{\infty}(1/k)^\beta]$$

and hence the proposition is proved. ∎

Remark 4.1. *A change of the number λ changes the norm of the space $B_s(d,\beta)$, but not the space itself. From now on, we shall assume that λ satisfies the above inequality.*

Proposition 4.2. *The partial differentiation $\partial/\partial t$ defines a bounded linear operator from $B_s(d,\beta)$ into $B_{s'}(d,\beta)$ for $0 \le s' < s \le 1$ with norm less than or equal to $C/(s-s')^d$, where C is a positive constant depending only on d, θ_0, and θ_1.*

PROOF: We compute

$$\begin{aligned}
&\|\phi'(t)\|_{s'} \\
&= \sup_{n\ge 0}\max_{t\in K} \frac{\tilde{n}^\beta \theta(s')^n}{\lambda(dn)!}|\phi^{(n+1)}(t)| \\
&\le \sup_{n\ge 0}\max_{t\in K}\left[\theta(s')^{-1}\left(\frac{\theta(s')}{\theta(s)}\right)^{n+1}\left(\frac{\tilde{n}}{n+1}\right)^\beta \frac{(d(n+1))!}{(dn)!}\frac{(n+1)^\beta\theta(s)^{n+1}}{\lambda(d(n+1))!}|\phi^{(n+1)}(t)|\right] \\
&\le \theta(s')^{-1}\sup_{n\ge 0}\{\left(\frac{\theta(s')}{\theta(s)}\right)^{n+1}(dn+1)\cdots(dn+d)\}\|\phi\|_s \\
&\le \theta_0{}^{-1}\sup_{n\ge 0}\{\left(\frac{\theta(s')}{\theta(s)}\right)^{n+1}(dn+d)^d\}\|\phi\|_s \\
&= \theta_0{}^{-1}d^d\|\phi\|_s\sup_{n\ge 1}\{\left(\frac{\theta(s')}{\theta(s)}\right)^{n}n^d\} \\
&\le \theta_0{}^{-1}d^d\|\phi\|_s(d/e)^d(\ln\theta(s)-\ln\theta(s'))^{-d}.
\end{aligned}$$

Since

$$\frac{\theta(s)}{\theta(s')} = 1+\theta(s)(s'-s)(\theta_1^{-1}-\theta_0^{-1}) \ge 1-\theta_1(s-s')(\theta_1^{-1}-\theta_0^{-1}) \ge \exp\left((s-s')\frac{\theta_1-\theta_0}{\theta_0}\right)$$

we obtain

$$\ln\theta(s)-\ln\theta(s') \ge (s-s')\frac{\theta_1-\theta_0}{\theta_0}$$

Hence by taking the constant C to be $(d^2/e)^d\theta_0^{-1}\left((\theta_1-\theta_0)/\theta_0\right)^{-d}$, the proposition is proved. ∎

Remark 4.2. *The constant C in Proposition 4.2 can be made as small as we wish by taking the constant θ_0 sufficiently small while keeping the constant θ_1 fixed in the definition of Banach space $B_s(d,\beta)$ when $d>1$.*

Proposition 4.3. *If $\phi \in B_s(d,\beta)$, then there exists a positive constant C depending only on β, d such that for all positive integers m, n we have*

$$\sup_{t\in K} |D_t^m(\phi_t^n)| \le (C\|\phi\|_s\theta(s)^{-1}\lambda)^{n-1}\|\phi\|_s\theta(s)^{-(m+1)}\lambda\frac{(d(m+1))!}{(m+1)^\beta}.$$

PROOF: Using Proposition 4.2 and the fact that $B_s(d,\beta)$ is a multiplication algebra, we can conclude that $\phi_t^n \in B_{s'}(d,\beta)$ for $s' \le s$ and all positive integer n. We proceed by induction on n. From the definition 4.1, it is obvious that the proposition is true if $n = 1$. We assume that it holds up to level n for some $n \ge 1$. By Leibniz' rule we have

$$|D_t^m(\phi_t^{n+1})| \le \sum_{k=0}^{m}\binom{m}{k}|D_t^{m-k}(\phi_t^n)||D_t^k(\phi_t)|.$$

Then applying the induction hypothesis we get

$$\begin{aligned}|D_t^m(\phi_t^{n+1})| \le \sum_{k=0}^{m}\binom{m}{k}&(C\|\phi\|_s\theta(s)^{-1}\lambda)^{n-1}\|\phi\|_s\theta(s)^{-(m-k+1)}\cdot\\ &\cdot\lambda\frac{(d(m-k+1))!}{(m-k+1)^\beta}\|\phi\|_s\theta(s)^{-(k+1)}\lambda\frac{(d(k+1))!}{(k+1)^\beta}\\ \le (C\|\phi\|_s\theta(s)^{-1}\lambda)^{n-1}&\|\phi\|_s\theta(s)^{-(m+1)}\lambda\|\phi\|_s\theta(s)^{-1}\lambda\frac{(d(m+1))!}{(m+1)^\beta}A_m\end{aligned}$$

where

$$A_m = \sum_{k=0}^{m}\binom{m}{k}\frac{(d(m-k+1))!(d(k+1))!}{(d(m+1))!}\left(\frac{m+1}{(m-k+1)(k+1)}\right)^\beta.$$

But

$$\begin{aligned}&\binom{m}{k}\frac{(d(m-k+1))!(d(k+1))!}{(d(m+1))!}\\ &\le\binom{dm}{dk}\frac{(d(m-k+1))!(d(k+1))!}{(d(m+1))!}\\ &\le\frac{(d(m-k)+1)\cdots(d(m-k)+d)(dk+1)\cdots(dk+d)}{(dm+1)\cdots(dm+d)}\\ &\le d^{2d}\frac{(m-k+1)(k+1)}{m+1}\end{aligned}$$

by (4.2) and (4.3). Thus

$$\begin{aligned}
A_m &\leq d^{2d} \sum_{k=0}^{m} \left(\frac{m+1}{(m-k+1)(k+1)} \right)^{\beta-1} \\
&\leq d^{2d} \sum_{k=0}^{m} \left(\frac{2}{k+1} \right)^{\beta-1} \\
&\leq d^{2d} \sum_{k=0}^{\infty} \left(\frac{2}{k+1} \right)^{\beta-1} \\
&=: A_\infty .
\end{aligned}$$

Since $\beta > 2$, $A_\infty < \infty$. By choosing $C > A_\infty$, the result follows by induction and the proof is complete. ∎

Remark 4.3. *The point in introducing the factor $\tilde{n}^\beta$ in the definition 4.1 is that for $\beta > 1$ the spaces $B_s(d, \beta)$ become Banach algebras and for $\beta > 2$, the estimate in Proposition 4.3 is valid.*

It is clear that there should be some kind of relationship between the spaces $B_s(d, \beta)$ and the Gevrey class 2 functions.

Definition 4.2. *Let Ω be a subset of $\mathbf{R}^n$ and $\delta > 0$. A C^∞ function f in Ω is said to be of Gevrey class δ in Ω (in short, $f \in \gamma^\delta(\Omega)$) if there exist positive constants C and H such that*

$$|D_t^\alpha f(x)| \leq C H^{|\alpha|} (\delta |\alpha|)!$$

for all multi-indices α and for all $x \in \Omega$.

Remark 4.4. *The definition given here of a Gevrey class of functions is different from that in [4] where the fact that that Gevrey class of functions forms an algebra is proved. But, by easy computation, we can also show that, under our definition, the Gevrey class of functions forms as algebra. It is also easy to check that there is an obvious inclusion in both definitions.*

The proof of the following proposition is clear.

Proposition 4.4. *For the case of one dimentional compact interval K,*

(a) *The space $B_s(d,\beta)$ is contained in γ^d, i.e. if $\phi \in B_s(d,\beta)$ then ϕ belongs to Gevrey class d.*

(b) *Suppose $\phi : \mathbf{R} \to \mathbf{R}$ is an infinitely differentiable function defined in K and there are positive constants M and θ such that*

$$|\partial_t^j \phi(t)| \le M\theta^j (dj)!$$

for all t and for all $j = 1, 2, \cdots$. If $\theta < \theta_1$, then $\phi \in B_s(d,\beta)$ for all $s \in [0,1]$, $d \ge 1$, $\beta > 2$.

Proposition 4.5. *Let d, β denote fixed real parameters such that $d > 1$, $\beta > 2$. Suppose that $f(z,x)$ is a real valued function on $\mathbf{R}^2$ which is infinitely differentiable with respect to its first argument and that for every compact z-interval I there exist two constants $M > 0$, $a > 0$ such that for $n = 0, 1, 2, \cdots$ and all $(z,x) \in I \times \{x \in \mathbf{R} \mid |x| \le \delta\}$,*

$$|D_z^n f(z,x)| \le M a^n \frac{(dn)!}{\tilde{n}^\beta} \tag{4.5}$$

where δ is some number > 0 and $\tilde{n} = max(1,n)$, i.e. $f \in \gamma^d$ (it is easy to check that the above statement is equivalent to the definition of Gevrey class d functions).

We define a map F on $B_s(d,\beta) \times \{x \in \mathbf{R} \mid |x| \le \delta\}$ by

$$F(u,x)(t) = f(u(t),x). \tag{4.6}$$

Then F is a map from $B_s(d,\beta) \times \{x \in \mathbf{R} \mid |x| \le \delta\}$ into $B_{s'}(d,\beta)$, $0 \le s' \le s \le 1$.

Proof: First, we remark that since u is a real valued function and f is globally defined, F is well defined.

For functions $u \in B_s(d,\beta)$, u is infinitely differentiable defined in a compact set K, so we can only consider $f(z,x)$ with (z,x) being restricted to a bounded set. According to Hörmander[4], there are functions in γ^d having compact support and forming partitions of unity subordinated to arbitrary locally finite open coverings and consisting of non-negative functions. Therefore, as these functions form multiplication algebras, without loss of generality, we may assume that

$$\text{supp} f \subset \{(z,x) \in \mathbf{R}^2 \mid |z| \le (2aC\lambda)^{-1}, |x| \le \delta\} \tag{4.7}$$

where λ is defined in (4.1), C is the constant in Proposition 4.3 and a is the constant in (4.5).

By the definition of the s'-norm (4.1), we have

$$\|F(u,x)\|_{s'} \leq \sup_{n\geq 0}\sup_{t\in K}\left(\frac{\theta(s')^n\tilde{n}^\beta}{(dn)!\lambda}|D_t^n F(u,x)|\right) \tag{4.8}$$

where D_t^n denotes the n-th derivative of the real valued function $t \to F(u(t),x)$ (x fixed). For $n \geq 1$, we estimate as follows (see [**1**]),

$$|D_t^n F(u,x)| \leq \sum_{j=1}^{n}\binom{n-1}{j-1}|D_z^j f(z,x)|\cdot|D_t^{m-j}(u_t^j)|$$

and apply (4.5) and Proposition 4.3 to get

$$\sup_{t\in K}|D_t^n F(u,x)| \leq \sum_{j=1}^{n}\binom{m-1}{j-1}Ma^j\frac{(dj)!}{j^\beta}[\,\|u\|_s\theta(s)^{-1}\lambda C]^{j-1}\lambda\|u\|_s\cdot$$

$$\cdot\,\theta^{-(n-j+1)}\frac{(d(n-j+1))!}{(n-j+1)^\beta}.$$

Then from (4.8), we have

$$\|F(u,x)\|_{s'} \leq \max\left(M\sup_{n\geq 1}A_n,\ \sup_{t\in K}\frac{1}{\lambda}|F(u,x)|\right),$$

where

$$A_n = \sum_{j=1}^{n}\binom{n-1}{j-1}\left(\frac{(dj)!(d(n-j+1))!}{(dn)!}\right)\left(\frac{n}{j(n-j+1)}\right)^\beta\vartheta^j\frac{1}{\lambda C}$$

with $\vartheta = aC\lambda\|u\|_1$. In view of (4.7) and the definition of the s-norm, it will suffice to consider $\vartheta < \frac{1}{2}$. We observe that

$$\binom{n-1}{j-1}\left(\frac{(dj)!(d(n-j+1))!}{(dn)!}\right) \leq d^{2d}\cdot\frac{j(n-j+1)}{n}$$

and

$$\frac{n}{j(n-j+1)} \leq \frac{1}{j}+\frac{1}{n-j+1} \leq 2$$

for $1 \leq j \leq n$. This then implies

$$A_n \leq \frac{1}{\lambda C}d^{2d}2^{\beta-1}\sum_{j=1}^{n}2^{-j} \leq \frac{1}{\lambda C}d^{2d}2^{\beta-1} < \infty$$

for all n. Therefore

$$\|F(u,x)\|_{s'} < \infty$$

and thus $F(u,x) \in B_{s'}(d,\beta)$, $0 \le s' \le s \le 1$. ∎

5. Solutions of the Cauchy problem in the x-direction. In this section, we shall solve the following problem by using the modified Ovcyannikov Theorem (Theorem 2.1):

$$u_t - u_{xx} = f(u,x) \quad \text{for } x \in (0,2),\ t \ge T_0 \tag{5.1}$$

$$u(0,t) = 0, \quad u_x(0,t) = g(t) \quad \text{for } t \ge T_0, \tag{5.2}$$

where T_0 is a positive constant.

Theorem 5.1. *Let the function $f(u,x)$ belong to Gevrey class 2 locally in its first argument, varying continuously with respect to x and satisfy $f(0,x) = 0$ and $D_1 f(0,x) = 0$ for all $x \in [0,2]$. Let $g(t)$ be a function of Gevrey class 2 in $t \ge T_0$ and $g(t) = 0$ for $t \ge T$ for some $T > T_0$. Then there exists a constant $a > 0$ such that the problem (5.1)–(5.2) has a solution $u(x,t)$ which is twice continuously differentiable with respect to x for $x < a$, infinitely differentiable with respect to t for $t \in [T_0,\infty)$, bounded for $x < a$, $t \in [T_0,\infty)$ and vanishes for $t \ge T$. Moreover, when $g(t)$ is small enough, the x-interval of existence will be greater than 1, i.e. $a > 1$.*

PROOF: First of all, we rewrite the problem (5.1)–(5.2) as the following Cauchy problem

$$u_{xx} = u_t - f(u,x) \quad \text{for } x \in (0,2),\ t \ge T_0 \tag{5.3}$$

$$u(0,t) = 0, \quad u_x(0,t) = g(t) \quad \text{for } t \ge T_0 \tag{5.4}$$

To apply Theorem 2.1, we convert the problem (5.3)–(5.4) to a first order system of differential equations by introducing the variables $v_1 = u$, $v_2 = u_x$, and $v_3 = u_t$. Then (5.3)–(5.4) can be rewritten as

$$\frac{dv_1}{dx}(x,\cdot) = v_2(x,\cdot) \tag{5.5}$$

$$\frac{dv_2}{dx}(x,\cdot) = v_3(x,\cdot) - f(v_1(x,\cdot),x) \tag{5.6}$$

$$\frac{dv_3}{dx}(x,\cdot) = \frac{\partial}{\partial t} v_2(x,\cdot) \tag{5.7}$$

with the Cauchy data

$$v_1(0,\cdot) = 0,\ v_2(0,\cdot) = g(\cdot),\ v_3(0,\cdot) = 0. \tag{5.8}$$

For $s \in [0,1]$, let $X_s = B_s(2,4)$, where $B_s(d,\beta)$ is defined in Section 4 with $K = [T_0, T+\epsilon]$ where ϵ is any finite number, $d = 2$, $\beta = 4$, λ any fixed constant satisfying the assumption of Proposition 4.1, and the constants θ_0 and θ_1 satisfying $0 < \theta_0 < \theta_1 < \infty$. The constant θ_0 will be chosen sufficiently small so that the constant C in Proposition 4.2 is sufficiently small after θ_1 is chosen sufficiently large (see below).

For simplification of notation, we let

$$F_1(v_2) = v_2 \tag{5.9}$$

$$F_2(v_1, v_3, x) = v_3 - f(v_1, x) \tag{5.10}$$

$$F_3(v_2) = \frac{\partial}{\partial t} v_2. \tag{5.11}$$

We use the same notations as in Theorem 2.1. Let $X_s^i = X_s$, $i = 1,2,3$, and $R_{1,0} = R_{3,0} = 0$, $R_{2,0} = \|g\|_1$. Then by Proposition 4.4 it is easy to check that $0 < R_{2,0} < \infty$ if θ_1 is large enough and by Proposition 4.5, assumption (H2) in Theorem 2.1 is satisfied for $R_1 > 0$, $R_2 > R_{2,0}$ and $R_3 > 0$. It is also clear that for any $R_1 > 0$, $R_2 > R_{2,0}$, and $R_3 > 0$ the assumptions (H3) and (H4) of Theorem 2.1 are satisfied with $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 2$. In fact, (H2) follows from Proposition 4.5; (H3) follows from Proposition 4.2 with $P(\lambda,\mu) = \lambda^3 - c_2^3 c_3^2 \lambda \mu^2 - c_1^2 c_2^1 \lambda$; and (H4) is clear.

According to Theorem 2.1, there exist a constant $a > 0$ and a unique solution of the Cauchy problem (5.5)–(5.8) such that $u(x,\cdot) = v_1(x,\cdot)$ is a twice continuously differentiable function of x for $|x| < a$ with values in X_0 and $\|u(x,\cdot)\|_0 < R_1$ for $|x| < a$. Since $u(x,\cdot) \in X_0$ for $|x| < a$, $u(x,t)$ is infinitely differentiable in t for $T_0 \le t \le T+\epsilon$ and $u(x,t)$ is bounded on $|x| < a$, $t \ge T_0$. By using L. Nirenberg's Uniqueness Theorem[**20**] and the fact that the Cauchy data is zero for $T \le t \le T+\epsilon$, the solution $u(x,t)$ obtained above vanishes for $T \le t \le T+\epsilon$. By defining $u(x,t)$ to be zero for $|x| < a$, $t \ge T+\epsilon$, $u(x,t)$ is a solution of (5.1)-(5.2) which is infinitely differentiable with respect to t for $t \ge T_0$, bounded on $|x| < a$, $t \ge T_0$ and vanishes for $t \ge T$.

It is interesting to see how large we can make for the interval of existence. To obtain the estimate, we shall check the proof of Theorem 2.1 and keep track of all constants. First, for any constants $R_1 > 0$, $R_2 > R_{2,0}$, $R_3 > 0$ and for $v_i, \tilde{v}_i \in X_s$, $i = 1, 2, 3$, $s \in [0, 1]$ with $\|v_i\|_s < R_i$, $\|\tilde{v}_i\|_s < R_i$, and $|x| < \eta$, where η can be any large number for our problem, we have for $0 \leq s' < s \leq 1$,

$$\|F_1(v_2) - F_1(\tilde{v}_2)\|_{s'} \leq \|v_2 - \tilde{v}_2\|_s \tag{5.12}$$

$$\begin{aligned}\|F_2(v_1, v_3, x) - F_2(\tilde{v}_1, \tilde{v}_3, x)\|_{s'} &\leq \|v_3 - \tilde{v}_3\|_s + \|f(v_1, x) - f(\tilde{v}_1, x)\|_s \\ &\leq \|v_3 - \tilde{v}_3\|_s + N\|v_1 - \tilde{v}_1\|_s \end{aligned}\tag{5.13}$$

$$\|F_3(v_2) - F_3(\tilde{v}_2)\|_{s'} \leq \frac{C}{(s - s')^2}\|v_2 - \tilde{v}_2\|_s \tag{5.14}$$

where N is a constant depending on R_1 which will become sufficiently small when $R_1 \to 0$ by the assumption $D_1 f(0, x) = 0$; and C is the constant in Proposition 4.2 which can be chosen sufficiently small.

Now, to get a more accurate estimate of the interval of existence, let us go back to check the proof of Theorem 2.1. Since $v_1(0) = v_3(0) = 0$, by using (3.13) with $e_1 = 1$, $e_2 = 1$, $e_3 = 2$ and the inequalities (5.12)–(5.14) we get

$$\begin{aligned}
\lambda_1^{(0)} &\leq 4a_0 R_{2,0} \\
\lambda_2^{(0)} &= 0 \\
\lambda_3^{(0)} &\leq 4C{a_0}^2 R_{2,0} \\
\lambda_1^{(k+1)} &\leq 4a_0 \lambda_2^{(k)} \\
\lambda_2^{(k+1)} &\leq 4(Na_0\lambda_1^{(k)} + \lambda_3^{(k)}) \\
\lambda_3^{(k+1)} &\leq 4C{a_0}^2 \lambda_2^{(k)}
\end{aligned}$$

and thus

$$\begin{aligned}
&\lambda_1^{(2k+1)} = 0,\ \lambda_2^{(2k)} = 0,\ \lambda_3^{(2k+1)} = 0 \\
&\lambda_2^{(1)} \leq \gamma R_{2,0} \\
&\lambda_1^{(2k)} \leq 4a_0\gamma^k R_{2,0} \\
&\lambda_2^{(2k+1)} \leq \gamma^{k+1} R_{2,0} \\
&\lambda_3^{(2k)} \leq 4Ca_0^2\gamma^k R_{2,0}
\end{aligned}$$

for $k \geq 0$, where $\gamma = 16(N+C)a_0^2$. If we take $R_2 > 17R_{2,0}$ and

$$a_0 < \min\{\frac{R_1}{64R_{2,0}}, \frac{1}{16\sqrt{2C}}\left(\frac{R_3}{R_{2,0}}\right)^{1/2}, \eta_i,\, i=1,2,3\} \tag{5.16}$$

so that $\gamma < 1/64$, then (3.5) is satisfied and thus the theorem holds with $a = a_0/2$.

To obtain $a > 1$, we require that

$$\min\{\frac{1}{32\sqrt{N+C}}, \frac{R_1}{64R_{2,0}}, \frac{1}{16\sqrt{2C}}\left(\frac{R_3}{R_{2,0}}\right)^{1/2}\} > 2. \tag{5.17}$$

This can be obtained when the "initial data" $g(t)$ is sufficiently small. Because when this term is small the quantity $R_{2,0} \equiv \|g(t)\|_1$ is small and so we can only consider small R_1 which makes N small.

We remark that the interval $[0, a)$, where $a = a_0/2$ and a_0 is given above, is not necessary the largest length of the interval of existence, because we have not always chosen the best possible constants in our argument. However, the estimate (5.16) is sufficient for our purpose. ▮

Remark. *In that main theorem, we need the analyticity of function f to guarantee that $w_x(0,t)$ belongs to Gevrey class 2 but in Theorem 5.1, we only need to assume that f is of Gevrey class 2 in t to obtain existence fot problem (5.3)-(5.4).*

6. Existence of Boundary Controller. In this section, we shall prove our principal result on the existence of the boundary controller $h(t)$ that steers a prescribed initial data w_0 to the zero for the problem (1.1)–(1.4). The controller $h(t)$ will be continuously differentiable on a finite time duration $0 \leq t \leq T$ with $T > 0$.

Theorem 6.1. *Suppose that $f(s, x)$ is an analytic function in s and x in a neighborhood of the origin and belongs locally to Gevrey class 2 in s and is Hölder continuous in $x \in [0, 1]$ and satisfy $f(0, x) = 0$, $D_1 f(0, x) = 0$ for all $x \in [0, 1]$. Let the initial data $w_0(x)$ be a continuous sufficiently small function*[1] *in $[0, 1]$. Then for any finite time $T > 0$, there*

[1] the "sufficiently small" assumption can be dropped for some cases, see remark after the proof of Theorem 6.1

exists a controller $h(t) \in C^\infty((0,\infty)) \cap C([0,\infty))$ such that the solution $w(x,t)$ of (1.1)–(1.4) satisfies $w(x,T) \equiv 0$ for $x \in [0,1]$.

PROOF: We organize the proof in a series of steps.

Step 1. Extend the domain of the initial data $w_0(x)$ to be $[0,2]$ so that $w_0(x)$ is continuous and the sup norm of the modified initial data is less than or equal to the sup norm of the original initial data. We also extend the domain of f to be $(-\infty,\infty)\times(0,2)$ so that all properties of f are maintained.

Step 2. We solve the initial-boundary value problem with the new modified initial condition:

$$w_t - w_{xx} = f(w,x) \quad \text{on } (0,2)\times(0,\infty) \tag{6.1}$$

$$w(0,t) = 0 \quad \text{for } t \geq 0 \tag{6.2}$$

$$w(2,t) = 0 \quad \text{for } t \geq 0 \tag{6.3}$$

$$w(x,0) = w_0(x) \quad \text{for } x \in (0,2). \tag{6.4}$$

It is well-known that the solution $w(x,t)$ exists locally and it is bounded [**9**]. Let T_1 be any number such that the solution of (6.1)-(6.4) exists for $t \leq T_1$ and $\epsilon < T_1$ be any small positive number so that $f(w,x)$ is an analytic function in the range of values assumed by w, x for $x \in [0,2\epsilon]$. Then it is also clear that the solution $w(x,t)$ is a C^∞ function for $0 \leq x \leq 2\epsilon$ and $\epsilon \leq t \leq T_1$ because the nonlinear term $f(w,x)$ is infinitely differentiable in w and x [**2**,12].

Step 3. We claim that the solution $w(x,t)$ obtained in Step 2 belongs to Gevrey class 2 in t for $t \leq T_1$. Let $u_0(x) = w(x,\epsilon)$, where $\epsilon < T_1$ is any small positive number as in the Step 2. Since $w(x,t)$ is a C^∞ solution of the problem

$$w_t - w_{xx} = f(w,x) \quad \text{on } (0,2\epsilon)\times(\epsilon,T_1]$$

$$w(0,t) = 0 \quad \text{for } \epsilon \leq t \leq T_1$$

$$w(x,\epsilon) = u_0(x) \quad \text{for } x \in (0,2\epsilon),$$

it follows from a theorem of D. Kinderlehrer and L. Nirenberg[**11**] that $w(x,t)$ is real analytic in x and is of Gevrey class 2 in t for $0 \leq x \leq \epsilon$ and $2\epsilon \leq t \leq T_1$. In fact, the derivatives of $w(x,t)$ satisfy

$$|\partial_x^\lambda \partial_t^j w| \le CH^{j+2\lambda}(j+2\lambda)!, \quad \forall \lambda,\ j,$$

for some constants C and H. Thus $w_x(0,t)$ belongs to the Gevrey class 2 in t for $2\epsilon \le t \le T_1$.

One can easily see, by contradiction and compactness, that small initial data $w_0(x)$ in sup norm implies small $w_x(0,t)$ in Gevrey class 2 norm for $t \in [2\epsilon, T]$ when T is a small number less than T_1.

Step 4. Since the initial data $w_0(x)$ is sufficiently small in sup norm, let us assume that the time interval $[2\epsilon, T]$ is small enough for $2\epsilon \le t \le T$, so that $w_x(0,t)$ is sufficiently small for $2\epsilon \le t \le T$.

Next, we modify $w_x(0,t)$ to be a function $w_x(0,t)\psi(t)$ with support in $[0,T]$. Here $\psi(t)$ is C^∞ on $0 \le t < \infty$ satisfying

$$\begin{aligned}&0 \le \psi(t) \le 1\\ &\psi(t) = 0 \quad \text{for } t \ge T\\ &\psi(t) = 1 \quad \text{for } 0 \le t \le (T+2\epsilon)/2\end{aligned}$$

With some care we can take $\psi(t)$ to be of the Gevrey class 2 for $t \ge 2\epsilon$ (see [**4**]). We note that the definition of the Gevrey class 2 in [**4**] is different from the one we use in this paper which is the same as that in [**3**]. But it is easy to check that the Gevrey class 2 functions constructed in [**4**] satisfy our definition.

Let

$$g(t) = \begin{cases} w_x(0,t)\psi(t), & \text{for } 2\epsilon \le t \le T \\ 0, & \text{for } t \ge T. \end{cases}$$

Since the Gevrey class of functions forms an algebra which is closed under multiplication, $g(t) \in \gamma^2$ in t for $t \ge 2\epsilon$ and vanishes for $t \ge T$. When T is small, $g(t)$ will be sufficiently small because $\|w_x(0,t)\|_1$ is sufficiently small.

Step 5. In this step, we solve the Cauchy problem:

$$u_t - u_{xx} = f(u,x) \quad \text{on } (0,2)\times(2\epsilon,\infty) \tag{6.5}$$

$$u(0,t) = 0, \quad u_x(0,t) = g(t) \quad \text{for } t \ge 2\epsilon. \tag{6.6}$$

Since $g(t)$ is small, it follows from Theorem 5.1 that there exist a constant $a > 1$ and a solution $u(x,t)$ of (6.5)–(6.6) which is twice continuously differentiable in x for $x < a$, infinitely differentiable in t for $t \geq 2\epsilon$, bounded for $x < a$ and $t \geq 2\epsilon$ and vanishes for $t \geq T$.

Step 6. Before we give the conclusion, we still need to show that $w(x,t)$ which we obtain in Step 2 and $u(x,t)$ which we get in Step 5 agree in $[2\epsilon, (T+2\epsilon)/2]$. First, we observe that w and u are both bounded functions in $[0,1] \times [2\epsilon, (T+2\epsilon)/2]$. Let $z(x,t) = w(x,t) - u(x,t)$. Then by Mean Value Theorem,

$$|z_t - z_{xx}|^2 = |f(w,x) - f(u,x)|^2 \leq C|z|^2$$

on $[0,1] \times [2\epsilon, (T+2\epsilon)/2]$ for some constant C. On the other hand, $z(0,t) = 0$ and $z_x(0,t) = 0$ for $t \in [2\epsilon, (T+2\epsilon)/2]$, i.e., the Cauchy data are zero. Hence, by L. Nirenberg's Theorem[**20**], $z \equiv 0$ on $[0,1] \times [2\epsilon, (T+2\epsilon)/2]$. This shows that $w(x,t)$ and $u(x,t)$ are identical on $[0,1] \times [2\epsilon, (T+2\epsilon)/2]$.

Step 7. Now comes our final step. We will read off the require boundary controller $h(t)$ through $w(x,t)$ and $u(x,t)$ by defining $h(t) = w(1,t)$ for $0 \leq t \leq 2\epsilon$ and $h(t) = u(1,t)$ for $t \geq 2\epsilon$.

This proves the theorem. ∎

Remark. *If the initial-boundary value problem (6.1)-(6.4) has a global solution $w(x,t)$ and $w(x,t) \to 0$ as $t \to \infty$ for continuous initial data $w_0(x)$, for example if $f(x,u) = u^p$ for $p > 1$ (see* [**18**]*), then we can drop the "sufficiently small" assumption on the initial data and obtain the null boundary controllability for problem (1.1)-(1.4) for time T sufficiently large because when the time is sufficiently large, $w_x(0,t)$ will be sufficiently small and belongs to Gevrey class 2 in t, say $t > T_0$, and thus by substituting T_0 for 2ϵ in steps 4-7 in the proof of Theorem 6.1 and take $T > T_0$, the result follows.*

7. The Null Boundary Controllability For a Semilinear Heat Equation With Nonlinear Term $f(w, w_x, x)$. In this section, we consider the null boundary controllability problem of the following problem:

$$w_t - w_{xx} = f(w, w_x, x) \quad \text{on } (0,1) \times (0,\infty) \tag{7.1}$$

$$w(0,t) = 0 \quad \text{for } t \geq 0 \tag{7.2}$$

$$w(x,0) = w_0(x) \quad \text{for } x \in (0,1] \tag{7.3}$$

$$w(1,t) = h(t) \quad \text{for } t \geq 0 \tag{7.4}$$

where f is an analytic function in all arguments near $(0,0,0)$ and and belongs locally to Gevrey class 2 in the first two arguments and is Hölder continuous with respect to x such that $f(0,0,x) = 0$, $D_1 f(0,0,x) = 0$ and $D_2 f(0,0,x) = 0$ for all $x \in [0,1]$, where D_i is the derivative with respect to i-th argument, $i = 1,2$. The problem is to find a controller $h(t)$ so that the solution of the resulting problem vanishes for $t \geq T$ for finite time $T > 0$. We will use the same method as that in previous section with little modification. We also use the same notations as before.

The main theorem of this section is the following:

Theorem 7.1. *Suppose that $f(y,z,x)$ is an analytic function in y, z and x near $(0,0,0)$, belongs to Gevrey class 2 locally in y and z, is Hölder continuous in $x \in [0,1]$ and satisfies $f(0,0,x) = 0$, $D_1 f(0,0,x) = 0$, $D_2 f(0,0,x) = 0$ for all $x \in [0,1]$. Let the initial data $w_0(x)$ be a sufficiently small C^1 function in $[0,1]$. Then for any finite time $T > 0$, there exists a controller $h(t) \in C^\infty((0,\infty)) \cap C([0,\infty))$ such that the solution $w(x,t)$ of (7.1)–(7.4) satisfies $w(x,T) \equiv 0$ for $x \in [0,1]$.*

Remarks.

(i) *The "sufficiently small" assumption on initial data can be dropped if the following initial boundary value problem*

$$w_t - w_{xx} = f(w, w_x, x) \quad \textit{on } (0,2) \times (0,\infty) \tag{7.5}$$

$$w(0,t) = 0 \quad \textit{for } t \geq 0 \tag{7.6}$$

$$w(2,t) = 0 \quad \textit{for } t \geq 0 \tag{7.7}$$

$$w(x,0) = w_0(x) \quad \textit{for } x \in (0,2) \tag{7.8}$$

has a global solution $w(x,t)$ and $w(x,t) \to 0$ as $t \to \infty$.

(ii) *By Mora's paper ([**19**]), L^p estimate ([**12**]) and the existence theorem ([**2**]), the condition we impose on the initial data gives the local existence of problem (7.5)-(7.8). Other conditions which are not as restricted can also give local existence of (7.5)-(7.8) but they require other regularity on f (see [**12**]).*

(iii) *When we solve our problem and extend the domain of the initial data to be $[0,2]$, besides the smallness of $w_0(x)$, we require $w_0(x) \equiv 0$ in a neighborhood of 2 so that we can obtain local existence for problem (7.5)-(7.8).*

The proof of Theorem 7.1 is similar to the proof of Theorem 6.1. The differences are that we need a proposition similar to Proposition 4.5 to claim that the assumptions of Theorem 2.1 are satisfied and we have to solve the Cauchy problem

$$u_t - u_{xx} = f(u, u_x, x) \quad \text{on } (0,2) \times (2\epsilon, \infty) \tag{7.9}$$

$$u(0,t) = 0, u_x(0,t) = g(t) \quad \text{for } t \geq 2\epsilon \tag{7.10}$$

where ϵ is an small number and f is the extended function of the function f in Theorem 7.1 with domain $[0,2]$ which preserves all properties. We will solve this Cauchy problem in Theorem 7.2.

Now, here is the proposition similar to Proposition 4.5 which will be used to fulfill the assumptions of Theorem 2.1.

Proposition 7.1. *Let d, β denote fixed real parameters such that $d > 1$, $\beta > 2$. Suppose that $f(y,z,x)$ is a real valued function on $\mathbf{R}^2$ which is infinitely differentiable with respect to its first two arguments and that for every compact set I in y,z space, there exist two constants $M > 0$, $a > 0$ such that for $m = 0,1,2,\cdots$, $n = 0,1,2,\cdots$ and all $(y,z) \in I$ and $x \in \{x \in \mathbf{R} \mid |x| \leq \delta\}$,*

$$|D_y^m D_z^n f(y,z,x)| \leq M a^{m+n} \frac{(d(m+n))!}{((m+n)^{\sim})^{\beta}} \tag{7.11}$$

where δ is some number > 0 and $\tilde{n} = sup(1,n)$, i.e. $f \in \gamma^d$ in its first two arguments.

We define a map F on $B_s(d,\beta) \times B_s(d,\beta) \times \{x \in \mathbf{R} \mid |x| \leq \delta\}$ by

$$F(u,v,x)(t) = f(u(t), v(t), x) \tag{7.12}$$

Then F is a map from $B_s(d,\beta)\times B_s(d,\beta)\times\{x\in\mathbf{R}\mid |x|\le\delta\}$ into $B_{s'}(d,\beta)$, $0\le s'\le s\le 1$.

PROOF: With the same reason as in the proof of Proposition 4.5, without loss of generality, we may assume that

$$\mathrm{supp} f\subset\{(y,z,x)\in\mathbf{R}^2\mid |y|\le(2aC\lambda)^{-1},|z|\le(2aC\lambda)^{-1},|x|\le\delta\} \tag{7.13}$$

where λ is defined in (4.1), C is the constant in Proposition 4.3 and a is the constant in (7.11).

By the definition of the s'-norm (4.1), we have

$$\|F(u,v,x)\|_{s'}\le\sup_{n\ge0}\sup_{t\in K}\left(\frac{\tilde{n}^\beta\theta(s')^n}{\lambda(dn)!}|D_t^nF(u,v,x)|\right) \tag{7.14}$$

where D_t^n denotes the n-th derivative of the real valued function $t\to F(u(t),v(t),x)$ (x fixed). For $n\ge1$, we estimate as follows (see [**1**]),

$$|D_t^nF(u,v,x)|\le\sum_{m=1}^{n}\sum_{j=0}^{m}\binom{n-1}{m-1}\binom{m}{j}|D_y^{m-j}D_z^jf(y,z,x)|\cdot|D_t^{n-m}(u_t^{m-j}v_t^j)| \tag{7.15}$$

By Leibniz' rule and Proposition 4.3, we get

$$\begin{aligned}
&\sup_{t\in K}|D_t^{n-m}(u_t^{m-j}v_t^j)|\le\sum_{k=0}^{n-m}\binom{n-m}{k}\sup_{t\in K}|D_t^k(u_t^{m-j})|\cdot\sup_{t\in K}|D_t^{n-m-k}(v_t^j)|\\
&\le\sum_{k=0}^{n-m}\binom{n-m}{k}(C\|u\|_s\theta(s)^{-1}\lambda)^{m-j-1}\|u\|_s\theta(s)^{-(k+1)}\lambda\frac{(d(k+1))!}{(k+1)^\beta}\cdot\\
&\qquad\cdot(C\|v\|_s\theta(s)^{-1}\lambda)^{j-1}\|v\|_s\theta(s)^{-(n-m-k+1)}\lambda\cdot\frac{(d(n-m-k+1))!}{(n-m-k+1)^\beta}\\
&=\sum_{k=0}^{n-m}\binom{n-m}{k}(C\lambda\|u\|_s)^{m-j}(C\lambda\|v\|_s)^j\frac{[\theta(s)^{-1}]^n}{C^2}\cdot\frac{(d(k+1))!}{(k+1)^\beta}\cdot\frac{(d(n-m-k+1)!}{(n-m-k+1)^\beta}
\end{aligned}$$

thus, from (7.11) and (7.15), we get

$$\begin{aligned}
&\sup_{t\in K}\frac{n^\beta[\theta(s')]^n}{\lambda(dn)!}|D_t^nF(u,v,x)|\\
&\le\sum_{m=1}^{n}\sum_{j=0}^{m}\binom{n-1}{m-1}\binom{m}{j}Ma^m\frac{(dm)!}{m^\beta}\sum_{k=0}^{n-m}\binom{n-m}{k}(C\lambda\|u\|_s)^{m-j}(C\lambda\|v\|_s)^j\cdot\\
&\qquad\cdot\frac{[\theta(s)^{-1}]^n}{C^2}\cdot\frac{(d(k+1))!}{(k+1)^\beta}\cdot\frac{(d(n-m-k+1))!}{(n-m-k+1)^\beta}\cdot\frac{[\theta(s)^{-1}]^nn^\beta}{\lambda(dn)!}\\
&\le\frac{M}{\lambda C^2}\sum_{m=1}^{n}\sum_{j=0}^{m}\sum_{k=0}^{n-m}\vartheta_1^{m-j}\vartheta_2^jA_{n-1,m-1}A_{m,j}A_{n-m,k}B_{m,j}
\end{aligned} \tag{7.16}$$

where

$$\vartheta_1 = aC\lambda\|u\|_s, \vartheta_2 = aC\lambda\|v\|_s,$$

$$A_{i,j} = \binom{i}{j} \frac{(d(j+1))!(d(i-j+1))!}{(d(i+1))!} \left[\frac{i+1}{(j+1)(i-j+1)}\right]^\beta,$$

$$B_{m,j} = \binom{m}{j}^{-1} A_{m,j}.$$

In view of (7.13) and the definition of the s-norm, it will suffice to consider $\vartheta_1 < \frac{1}{2}$ and $\vartheta_2 < \frac{1}{2}$. According to the proof of Proposition 4.3 and Proposition 4.5, we have

$$\sum_{k=0}^{n-m} A_{n-m,k} \leq d^{2d} \sum_{k=0}^{\infty} \left(\frac{2}{k+1}\right)^{\beta-1}$$

$$A_{m,j} \leq d^{2d} \left(\frac{m+1}{(j+1)(m-j+1)}\right)^{\beta-1}$$

$$\leq d^{2d} \left(\frac{2}{j+1}\right)^{\beta-1}$$

$$B_{m,j} \leq A_{m,j} \leq d^{2d} \left(\frac{2}{j+1}\right)^{\beta-1}$$

for $0 \leq j \leq m$. Thus,

$$\sup_{t\in K} \frac{n^\beta[\theta(s')]^n}{\lambda(dn)!} |D_t^n F(u,v,x)|$$

$$\leq \frac{M}{\lambda C^2} \sum_{m=1}^{n} \left(\frac{1}{2}\right)^m A_{n-1,m-1} \left[\sum_{j=0}^{m} d^{4d} \left(\frac{2}{j+1}\right)^{2(\beta-1)}\right] \cdot \left[\sum_{k=0}^{n-m} d^{2d} \left(\frac{2}{k+1}\right)^{\beta-1}\right]$$

$$\leq \frac{M}{\lambda C^2} \left[\sum_{j=0}^{\infty} d^{4d} \left(\frac{2}{j+1}\right)^{2(\beta-1)}\right] \cdot \left[\sum_{k=0}^{\infty} d^{2d} \left(\frac{2}{k+1}\right)^{2(\beta-1)}\right] \cdot$$

$$\cdot \left[\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m d^{2d} \left(\frac{2}{m+1}\right)^{\beta-1}\right]$$

$$< \infty$$

for all $n \geq 1$. By (7.10), we have

$$\|F(u,v,x)\|_{s'} < \infty.$$

This completes the proof. ∎

The proof of the following lemma is elementary. We omit the proof.

Lemma 7.1. *Let γ,δ be two positive real numbers. Let $\{\zeta_k\}$ be a sequence satisfying*

$$\zeta_{k+1} \le \delta\zeta_k + \gamma\zeta_{k-1} \quad \text{for } k \ge 2$$
$$0 < \zeta_0 < \infty$$
$$0 < \zeta_1 < \infty.$$

Then the sequence $\{\zeta_k\}$ converges if and only if two real roots μ_1,μ_2 of the quadratic equation

$$x^2 - \delta x - \gamma = 0$$

have absolute values less than 1. Moreover,

$$\zeta_k \le c_1{\mu_1}^k + c_2{\mu_2}^k, \quad k = 0, 1, 2, \cdots$$

where c_1,c_2 are two constants determined by ζ_0 and ζ_1.

The following Theorem solves the Cauchy problem (7.9)-(7.10).

THEOREM 7.2. *Let the function $f(u, v, x)$ belong to Gevrey class 2 locally in its first two arguments, varying continuously with respect to x and satisfy $f(0,0,x) = 0$, $D_u f(0,0,x) = 0$ and $D_v f(0,0,x) = 0$ for all $x \in [0,2]$, i.e. there exist two constants $M > 0$, $b > 0$ such that for any integers $m = 0,1,2,\cdots$, $n = 0,1,2,\cdots$,*

$$|D_u^m D_v^n f(u,v,x)| \le M b^{m+n} \frac{2m!}{\tilde{m}^\beta} \frac{2n!}{\tilde{n}^\beta}.$$

Let $g(t)$ be a function of Gevrey class 2 in $t \ge T_0$ with support in $[T_0, T]$ where $T_0 > 0$ and $T > T_0$ are two finite numbers. Then there exists a constant $a > 0$ such that the Cauchy problem

$$u_t - u_{xx} = f(u, u_x, x) \quad \text{on } (0,2) \times [T_0, \infty) \tag{7.17}$$
$$u(0,t) = 0, u_x(0,t) = g(t) \quad \text{for } t \ge T_0 \tag{7.18}$$

has a solution $u(x,t)$ which is twice continuously differentiable with respect to x for $x < a$, infinitely differentiable with respect to t for $t \in [T_0, \infty)$, bounded for $x < a$, $t \in [T_0, \infty)$

and vanishes for $t \geq T$. Moreover, when $g(t)$ is small enough, the x-interval of existence will be greater than 1, i.e. $a > 1$.

PROOF: We convert the problem (7.17)-(7.18) to a first order system of differential equations by introducing the variables $u_1 = u$, $u_2 = u_x$ and $u_3 = u_t$. Then (7.13)-(7.14) can be rewritten as

$$\begin{aligned}
\frac{du_1}{dx}(x,\cdot) &= u_2(x,\cdot) \\
\frac{du_2}{dx}(x,\cdot) &= u_3(x,\cdot) - f(u_1(x,\cdot), u_2(x,\cdot), x) \\
\frac{du_3}{dx}(x,\cdot) &= \frac{\partial}{\partial t} u_2(x,\cdot)
\end{aligned}$$

with the Cauchy data

$$u_1(0,\cdot) = 0,\ u_2(0,\cdot) = g(\cdot),\ u_3(0,\cdot) = 0.$$

For $s \in [0,1]$, let $X_s = B_s(2,4)$, where $B_s(d,\beta)$ is defined in Section 4 with $K = [T_0, T+\epsilon]$ where ϵ is any finite number, $d = 2$, $\beta = 4$, λ is any fixed constant satisfying the assumption of Proposition 4.1, and the constants θ_0 and θ_1 satisfying $0 < \theta_0 < \theta_1 < \infty$ will be chosen so that the constant C in Proposition 4.2 is sufficiently small.

For simplification of notation, we let

$$F_1(u_2) = u_2 \tag{7.19}$$

$$F_2(u_1, u_2, u_3, x) = u_3 - f(u_1, u_2, x) \tag{7.20}$$

$$F_3(u_2) = \frac{\partial}{\partial t} u_2. \tag{7.21}$$

By substituting Proposition 7.1 for Proposition 4.5, all arguments in Theorem 5.1 for showing the existence of solutions apply and so we get the first part of the theorem. We are going to show that the x-interval of existence can be greater than 1 when $g(t)$ is sufficiently small by checking the proof of Theorem 2.1.

First for any constants $R_1 > 0$, $R_2 > R_{2,0}$, $R_3 > 0$ and for $u_i, \tilde{u}_i \in X_s$, $i = 1,2,3$,

$s \in [0,1]$ with $\|u_i\|_s < R_i$, $\|\tilde{u}_i\|_s < R_i$, we have for $0 \le s' < s \le 1$

$$\|F_1(u_2) - F_1(\tilde{u}_2)\|_{s'} \le \|u_2 - \tilde{u}_2\|_s \tag{7.22}$$

$$\begin{aligned}\|F_2(u_1,u_2,u_3,x) - F_2(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3,x)\|_{s'} &\le \|u_3 - \tilde{u}_3\|_s + \|f(u_1,u_2,x) - f(\tilde{u}_1,\tilde{u}_2,x)\|_s \\ &\le \|u_3 - \tilde{u}_3\|_s + N[\,\|u_1 - \tilde{u}_1\|_s + \|u_2 - \tilde{u}_2\|_s]\end{aligned} \tag{7.23}$$

$$\|F_3(u_2) - F_3(\tilde{u}_2)\|_{s'} \le \frac{C}{(s-s')^2}\|u_2 - \tilde{u}_2\|_s \tag{7.24}$$

where N is a constant depending on R_1 which will become sufficiently small when $R_1 \to 0$ by the assumption $D_u f(0,0,x) = 0$, $D_v f(0,0,x) = 0$; and C is the constant in Proposition 4.2 which can be chosen sufficiently small.

Now, let us go back to check the proof of Theorem 2.1. Since $u_1(0) = u_3(0) = 0$, by using (3.13) with $e_1 = 1$, $e_2 = 1$, $e_3 = 2$ and the inequalities (7.22)-(7.24), we get

$$\begin{aligned}\lambda_1^{(0)} &\le 4a_0 R_{2,0}\\ \lambda_2^{(0)} &\le N a_0 R_{2,0}\\ \lambda_3^{(0)} &\le 4C{a_0}^2 R_{2,0}\\ \lambda_1^{(k+1)} &\le 4a_0\lambda_2^{(k)}\\ \lambda_2^{(k+1)} &\le 4(Na_0\lambda_1^{(k)} + Na_0\lambda_2^{(k)} + \lambda_3^{(k)})\\ \lambda_3^{(k+1)} &\le 4C{a_0}^2\lambda_2^{(k)}\end{aligned}$$

and thus

$$\lambda_2^{(k+1)} \le \gamma\lambda_2^{(k-1)} + \delta\lambda_2^{(k)}$$

for $k \ge 0$, where $\gamma = 16(N+C){a_0}^2$ and $\delta = 4Na_0$. From Lemma 7.1, we know that the sequence $\{\lambda_2^{(k)}\}$ converges if and only if absolute values of both roots of quadratic equation

$$x^2 - \delta x - \gamma = 0 \tag{7.25}$$

are less than 1. Let μ_1 be the positive root and μ_2 be the negative root of equation (7.25). Then by computing μ_1, μ_2, we know that

$$\delta < 2, \gamma + \delta < 1, \gamma - \delta < 1 \text{ implies } 0 < \mu_1 < 1, -1 < \mu_2 < 0.$$

That is, if a_0 is chosen so that

$$4Na_0 < 2 \tag{7.26}$$

$$16(N+C){a_0}^2 + 4Na_0 < 1 \tag{7.27}$$

$$16(N+C){a_0}^2 - 4Na_0 < 1, \tag{7.28}$$

then the sequence $\{\lambda_2^{(k)}\}$ converges and so are $\{\lambda_1^{(k)}\}$ and $\{\lambda_3^{(k)}\}$. Especially,

$$\lambda_2^{(k)} \leq c_1\mu_1^k + c_2\mu_2^k \tag{7.29}$$

with

$$\begin{aligned} c_1 &= \frac{\gamma - 16{a_0}^2N^2 - 4Na_0\mu_2}{\sqrt{\delta^2+4\gamma}} R_{2,0} \\ &= \left(2N + \frac{4(N+C)}{\sqrt{N^2+N+C}} - \frac{6N^2}{\sqrt{N^2+N+C}}\right) a_0 R_{2,0} \\ &< \left(2N + 4\sqrt{N+C}\right) a_0 R_{2,0} \end{aligned} \tag{7.30}$$

and

$$\begin{aligned} c_2 &= \frac{4Na_0\mu_1 - \gamma - 16{a_0}^2N^2}{\sqrt{\delta^2+4\gamma}} R_{2,0} \\ &= \left(2N - \frac{4(N+C)}{\sqrt{N^2+N+C}} - \frac{2N^2}{\sqrt{N^2+N+C}}\right) a_0 R_{2,0} \\ &< 2Na_0R_{2,0}. \end{aligned} \tag{7.31}$$

We also need to choose a_0 so that (3.12) holds, that is, we need

$$\sum_{k=0}^{\infty} 2^{k+2}\lambda_1^{(k)} \leq \frac{R_1}{2} \tag{7.32}$$

$$\sum_{k=0}^{\infty} 2^{k+2}\lambda_2^{(k)} \leq \frac{R_2 - R_{2,0}}{2} \tag{7.33}$$

$$2{a_0}^{-1}\sum_{k=0}^{\infty} 2^{k+2}\lambda_3^{(k)} \leq \frac{R_3}{2}. \tag{7.34}$$

Since

$$\begin{aligned} \sum_{k=0}^{\infty} 2^{k+2}\lambda_2^{(k)} &\leq \sum_{k=0}^{\infty} 2^{k+2}\left[c_1\mu_1^k + c_2\mu_2^k\right] \\ &= \frac{4c_1}{1-2\mu_1} + \frac{4c_2}{1-2\mu_2}, \end{aligned} \tag{7.35}$$

if a_0 is chosen so that

$$2\mu_1 \le \frac{1}{2} \quad \text{and} \quad 2\mu_2 \ge -\frac{1}{2},$$

i.e.

$$4(N + \sqrt{N^2 + N + C})a_0 \le \frac{1}{2} \tag{7.36}$$

$$4(N - \sqrt{N^2 + N + C})a_0 \ge -\frac{1}{2}, \tag{7.37}$$

then by (7.30), (7.31) and (7.35),

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{k+2}\lambda_2^{(k)} &\le 8c_1 + \frac{8}{3}c_2 \\ &\le 32\left(\frac{2}{3}N + \sqrt{N+C}\right) a_0 R_{2,0},\end{aligned}$$

and thus if a_0 is also chosen so that

$$32\left(\frac{2}{3}N + \sqrt{N+C}\right) a_0 R_{2,0} < \frac{R_2 - R_{2,0}}{2}, \tag{7.38}$$

then (7.33) is satisfied. Now, we consider (7.32),

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{k+2}\lambda_1^{(k)} &= 4\lambda_1^{(0)} + \sum_{k=1}^{\infty} 2^{k+2}\lambda_1^{(k)} \\ &\le 16a_0R_{2,0} + \sum_{k=1}^{\infty} 2^{k+2}4a_0\lambda_2^{(k-1)} \\ &= 16a_0R_{2,0} + 32a_0\sum_{k=0}^{\infty} 2^k\lambda_2^{(k)} \\ &\le 16a_0R_{2,0} + 32a_0\sum_{k=0}^{\infty}\left[c_1(2\mu_1)^k + c_2(2\mu_2)^k\right] \\ &\le 16a_0R_{2,0} + 32a_0\left[\frac{c_1}{1-2\mu_1} + \frac{c_2}{1-2\mu_2}\right] \\ &\le 16a_0R_{2,0}\left[1 + 8(\frac{4}{3}N + \sqrt{N+C})a_0\right].\end{aligned}$$

Thus, if we require that

$$16a_0R_{2,0}\left[1 + 8a_0(\frac{4}{3}N + \sqrt{N+C})\right] \le \frac{R_1}{2}, \tag{7.39}$$

then (7.32) is satisfied. Similarly, to obtain (7.34), we require that

$$32Ca_0R_{2,0}\left[1+8Ca_0(\frac{4}{3}N+\sqrt{N+C})\right]\leq\frac{R_3}{2}. \tag{7.40}$$

Hence by putting all of these estimates together, a_0 should be chosen so that (7.26)-(7.28),(7.36)-(7.40) hold and so when the Cauchy data $g(t)$ is sufficiently small and the constants N and C are taken sufficiently small, a_0 can be chosen so that $a_0 > 2$ and thus the x-interval of existence $a = a_0/2 > 1$. This completes the proof. ∎

Remark. *In Theorem 7.1, we need analyticity of function f near $(0,0)$ to guarantee that $w_x(0,t)$ belongs to be Gevrey class 2 but in Theorem 7.2, we only need to assume that f is Gevrey class 2 in t to obtain existence for problem (7.17)-(7.18).*

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