The Core of a Game with a Continuum of Players and
Finite Coalitions: The Model and Some Results

by

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Abstract

In this paper we develop a new model of a cooperative game with a continuum of players. In our model, only finite coalitions - ones containing only finite numbers of players - are permitted to form. Outcomes of cooperative behavior are attainable by partitions of the players into finite coalitions; this is appropriate in view of our restrictions on coalition formation. Once feasible outcomes are properly defined, the core concept is standard - no permissible coalition can improve upon its outcome. We provide a sufficient condition for the nonemptiness of the core in the case where the players can be divided into a finite number of types. This result is applied to a market game and the nonemptiness of the core of the market game is stated under considerably weak conditions (but with finite types). In addition, it is illustrated that the framework applies to assignment games with a continuum of players.

1. Introduction

Models of games with a continuum of players have become widely used in game and economic theory. Nevertheless there are still subtle questions that arise in interpretation of these models. The questions which concern us here are the interpretation of the individual player, and the role of the individual player in cooperation. In this paper we provide a new model of a cooperative game with a continuum of players and obtain some results concerning the core which we hope will help us understand these issues.

Our model differs from those currently in the literature in that only subsets containing finite numbers of players are permitted to form coalitions. Consistent with our restriction on coalition formation, outcomes of cooperative behavior must be attainable by partitions of the players into finite coalitions. The partitions are required to be compatible with the distribution of players (described by a measure on the player set). Outcomes are then defined relative to feasible partitions. Once feasible outcomes are defined, the core concept is standard - no permissible (i.e., finite) coalition can improve upon an outcome in the core, called the f-core.

The motivation for our treatment of cooperative games with a continuum of players arises, as suggested above, in part from the interpretations of the existing models, initiated by Aumann (1964). One interpretation is that only portions of the total set of players are relevant; the individual player is some arbitrarily small set of positive measure. The second interpretation is that each point in the continuum represents an individual player, while the fact that the total set of players is given as a continuum is meant to approximate a situation with a large but finite player set.

The first interpretation comes from the analogy to mechanics, e.g., fluid mechanics, where a continuum is used to approximate a situation with a large but finite number of particles. In fluid mechanics this treatment is appropriate when it is not necessary to treat each particle as a distinct entity. In game theory and in micro-economics, virtually by definitions of the subjects, each player is a distinct entity, a separate decision making unit; thus the concept of the player must be precise. The second interpretation has the advantage that the integrity of the individual is maintained.

If we take the view that cooperation arises from the actions of individual players, then it seems reasonable that individual players should be able to be effective (i.e., non-negligible in its non-technical sense) members of coalitions. The first interpretation of the individual player as a set of "small" but positive measure allows the interpretation that individuals are non-negligible as members of coalitions (of positive measure) while the second does not. Therefore our view is that the standard core theory in continuum models fits the first interpretation but not the second. (See also the discussion in Aumann and Shapley (1974), 176-178).

In our model, each point in the continuum represents an individual player so the second interpretation fits. We view cooperation as arising from the actions of individuals and allow finite coalitions. In this fashion, we attempt to maintain a precise concept of the player while capturing some plausibility of the idea of the individual player as effective in facilitating cooperation.

Given that each point in the continuum of players is a distinct player, every coalition of positive measure contains a large number of players. If we take coalition formation costs into account, then the formation of such large coalitions seems unlikely. On this basis we restrict coalitions to be finite.

The preservation of the precise concept of the individual player and the restriction to finite coalitions allows us to view our model as the direct limit version of the Debreu-Scarfe (1963) and Shapley-Shubik (1966) models with the total set of players approximated by a continuum. Our model can also be viewed as a limit version of models of large games with nonempty approximate cores initiated by Wooders (1983) (see also Kaneko and Wooders (1982), Shubik and Wooders (1983), and Wooders and Zame (1984)).

The second interpretation of the individual player discussed above (and our approach) is consistent with approaches in non-cooperative game theory; thus we hope our model may permit a bridge between cooperative and non-cooperative theories in continuum models.

In the next section of the paper we develop our model and the concept of the f -core. Simple examples of assignment games are used to demonstrate several of the concepts. Section 3 consists of an application of our model to a market game. A proof of the nonemptiness of the f -core for a game with types is provided in Section 4 and is applied to a market game.

In a subsequent paper, Kaneko and Wooders (1984), we consider the nonemptiness of the f -core without the types assumption. The price we pay for removing this assumption is that we need uniformly bounded coalition sizes and strong comprehensiveness.

2. The Model of Continuum Games and the f-Core

2.1 The Player Set and Feasible Partitions

Let (N, B, μ) be a measure space, where N is a Borel subset of a complete separable metric space; B , the σ -algebra of all Borel subsets of N ; and μ , a nonatomic measure with $0 < \mu(N) < +\infty^*$. Each element in N is called a player and N is the player set. The measure μ represents the distribution of players. The σ -algebra B is necessary for measurability arguments but does not play any important game-theoretic role.

Let F be the set of all finite subsets of N . Each element S in F is called a finite coalition or simply a coalition. As motivated in Section 1, only finite subsets of players can form coalitions.

Remark 2.1. Since a singleton set is closed in N , every coalition is measurable.

Since only finite subsets of players are allowed to form coalitions, cooperative outcomes are attained by partitions of the player set into finite coalitions. Several conditions must be imposed to ensure that these partitions are compatible with the distribution of players described by the measure μ . For example, in a marriage (assignment) game, if the girls are the points in the interval $[0,1)$ and the boys, those in the interval $[1,3)$, then the set of coalitions $\{\{i, 1+2i\}: i \in [0,1)\}$ constitutes a partition of the set of players into coalitions. However, this partition is obviously not compatible with the distribution of players with respect to Lebesgue measure. Our conditions on feasibility will rule out this type of incompatibility.

* Under our assumptions, (N, B) is measure-theoretically isomorphic to $([0,1], C)$, where C is the σ -algebra of all Borel subsets of the interval $[0,1]$ (see Parthasarathy (1967), pp. 12-14).

A function ψ from a set A in $\mathcal{B}$ to a set B in $\mathcal{B}$ is called a measure-preserving isomorphism from A to B iff (i) ψ is a measure-theoretic isomorphism, i.e., ψ is 1 to 1, onto, and measurable in both directions, and (ii) $\mu(C) = \mu(\psi(C))$ for all $C \subset A$ with $C \in \mathcal{B}$. A partition p of N is feasible iff for any positive integer k

$$N_k^p \equiv \bigcup_{\substack{S \in p \\ |S|=k}} S \text{ is a measurable subset of } N; \text{ and}$$

each $N_k^p (k=1,2,\dots)$ has a partition $\{N_{kt}^p\}_{t=1}^k$, where each (2.1)

N_{kt}^p is measurable, with the following property:

there are measure-preserving isomorphisms $\psi_{k1}^p, \psi_{k2}^p, \dots, \psi_{kk}^p$

from N_{k1}^p to $N_{k1}^p, \dots, N_{kk}^p$, respectively, such that

$$\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\} \in p \text{ for all } i \in N_{k1}^p. *$$

Note that (2.1) implies that for any $S \in p$ with $|S| = k$, we have $S = \{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\}$ for some $i \in N_{k1}^p$.

Let Π be the set of feasible partitions.

Example 2.1

Consider the situation described above where the girls are the points in $[0,1)$, and the boys, those in $[1,3)$. The partition $\{\{i, 1+2i\}: i \in [0,1)\}$ violates (2.1). An example of a feasible partition is

$p = \{\{i, 2+i\}: i \in [0,1)\} \cup \{\{i\}: i \in [1,2)\}$. For this partition p ,

$$\text{we have } N_1^p = \bigcup_{\substack{S \in p \\ |S|=1}} S = [1,2), \quad N_2^p = \bigcup_{\substack{S \in p \\ |S|=2}} S = [0,1) \cup [2,3),$$

$N_{21}^p = [0,1)$ and $N_{22}^p = [2,3)$; then measure-preserving isomorphisms

satisfying (2.1) are given by $\psi_{11}^p(i) = i$, $\psi_{21}^p(i) = i$ and

$$\psi_{22}^p(i) = 2+i.$$

* $|S|$ denotes the number of players in the set S .

2.2 A Characteristic Function Game Without Side Payments

A characteristic function game V without side payments is a correspondence on F which assigns to each coalition $S \in F$ a subset $V(S)$ of R^S with the following properties:

$$V(S) \text{ is a nonempty, closed subset of } R^S \text{ for all } S \in F; \quad (2.2)$$

$$V(S) \times V(T) \subset V(S \cup T) \text{ for any } S, T \in F \text{ with } S \cap T = \emptyset; \quad (2.3)$$

$$\inf_{i \in N} \sup V(\{i\}) > -\infty; \quad (2.4)$$

$$\text{for any } S \in F, x \in V(S) \text{ and } y \in R^S \text{ with } y \leq x \text{ imply } y \in V(S); \quad (2.5)$$

$$\text{for any } S \in F, V(S) - \bigcup_{i \in S} [(\text{interior } V(\{i\})) \times R^{S-\{i\}}] \quad (2.6)$$

is nonempty and bounded.

These assumptions are all innocuous and we note only that (2.4) can always be obtained for a translation of a game, and that conditions (2.3) and (2.5) are called, respectively, super-additivity and comprehensiveness.

The game V will be required to be compatible with the measurability structure of the player set. Let p be an arbitrary feasible partition of N . For any positive integer k , we can regard $V(\{\psi_{k1}^p(i), \psi_{k2}^p(i), \dots, \psi_{kk}^p(i)\})$ as a correspondence from N_{k1}^p to R^k , i.e.,

$$V : N_{k1}^p \longrightarrow 2^{R^k}$$

$$i \longmapsto V(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\}) .$$

In this sense, we assume that

$$V(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\}) \text{ has an analytic graph, i.e., the set}$$

$$\{(i, x) \in N_{k1}^p \times R^k : x \in V(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\})\} \text{ is analytic in } \quad (2.7)$$

$$N_{k1}^p \times R^k .$$

This assumption is required for Lemma 2.1.

Remark 2.2. Every measurable set in the product σ -algebra $B(N_{k1}^p) \otimes B(R^k)$ is analytic, where $B(N_{k1}^p)$ and $B(R^k)$ are the σ -algebras of Borel subsets of N_{k1}^p and R^k , respectively. For a definition of analytic sets see Parthasarathy (1967, p. 15) or Meyer (1966), p. 34).

We next define the set of possible outcomes of the game. Some preliminary definitions are required.

Given a feasible partition p , define the set $H(p)$ by

$$H(p) = \{h \in L(N;R) : (h(j))_{j \in p(i)} \in V(p(i)) \text{ for almost all } i \in N\}, \quad (2.8)$$

where $L(N,R)$ is the set of measurable functions from N to R and $p(i)$ is the element of the partition p containing player i ; $H(p)$ is the outcome set relative to p . Let

$$H = \bigcup_{p \in \Pi} H(p). \quad (2.9)$$

Lemma 2.1. For any feasible partition p of N , $H(p) \neq \emptyset$.

Proof. Appendix

Let $\bar{R} = R \cup \{+\infty\}$. Define the outcome set H^* by

$$H^* = \{h \in L(N, \bar{R}) : \text{for some sequence } \{h^v\} \text{ in } H, \{h^v\} \text{ converges in measure to } h\}, \quad (2.10)$$

where " $\{h^v\}$ converges in measure to h " means that for any $\varepsilon > 0$,

$$\mu(\{i \in N : |h^v(i) - h(i)| > \varepsilon\}) \rightarrow 0 \text{ as } v \rightarrow \infty.$$

Note that $H(p) \subset H \subset H^*$ for all $p \in \Pi$.

Remark 2.3. In outcome space we allow $f(i) = +\infty$ but only on a null set since h^v (in (2.10)) cannot take on the value $+\infty$. Allowing $h \in H^*$ to be in $L(N, \bar{R})$, rather than $L(N, R)$, is simply for notational and expository convenience. See also Section 3.

Remark 2.4. If $\{h^v\}$ converges in measure to h , then $\{h^v\}$ has a subsequence which converges pointwise to h a.e. and, conversely, if $\{h^v\}$ converges pointwise to h a.e., then $\{h^v\}$ converges in measure. Therefore H^* can be defined by pointwise convergence, and in our proofs we use

whichever definition is convenient. Note that H^* is closed with respect to these convergences, i.e., if $\{h^v\}$ in H^* converges either pointwise or in measure to h , then $h \in H^*$.

Let h be a function in $L(N, \bar{R})$. We say that a coalition S in F can improve upon h iff for some $y \in V(S)$, $y_i > h(i)$ for all $i \in S$.

Now the f-core of the game V without side payments is defined to be the set

C_f :

$$C_f = \{h \in H^* : \text{no coalition in } F \text{ can improve upon } h\} . \quad (2.11)$$

An outcome h in the f-core C_f is stable in the sense that no coalition can improve upon h and is approximately feasible in the sense that $h \in H^*$. That is, the outcome h can be approximated by a sequence $\{h^v\}$ in H in the sense that for any $\varepsilon > 0$, the measure of players whose payoffs under h^v are not within ε from their payoffs under h can be made arbitrarily small by choice of v sufficiently large. In this sense, the f-core is the limit version of approximate cores, e.g., Shapley and Shubik [1966], Wooders [1983], Kaneko and Wooders [1982], Shubik and Wooders [1983] and Wooders and Zame [1984].

Example 2.2

We continue our marriage game example and describe the f-core of the game.

Suppose the marriage* of the i th girl and the j th boy yields a payoff of j utils to the girl and i utils to the boy (there is no transferability of utility) while remaining single yields a payoff of zero. A finite coalition can realize only those outcomes attainable by marriages between members of the coalition. An outcome in the f-core is given by f where

* Only marriages between boys and girls are allowed.

$$f(i) = \begin{cases} 2+i & \text{if } i \in [0,1) \\ 0 & \text{if } i \in [1,2) \\ i-2 & \text{if } i \in [2,3) \end{cases} ; \quad (2.12)$$

only the boys with high index numbers are married and, the higher one's index number, the higher the index number of one's partner. Figure 1 depicts the graph of f .

Figure 1

Formally, the characteristic function of the marriage game is defined as follows:

$$\begin{aligned} &\text{for any } i \in [0,3), v(\{i\}) = \{x_i : x_i \leq 0\} ; \\ &\text{for any } i \in [0,1) \text{ and } j \in [1,3) , \\ &\quad v(\{i,j\}) = \{(x_i, x_j) : x_i \leq j, x_j \leq i\} ; \\ &\text{for any } S \in F, \end{aligned} \quad (2.13)$$

$$v(S) = \bigcup_{p_S \in P(S)} \prod_{T \in p_S} v(T) ,$$

where $P(S) = \{p_S : p_S \text{ is a partition of } S \text{ such that } |T| = 1 \text{ or } (|T \cap [0,1)| = 1 \text{ and } |T \cap [1,3)| = 1) \text{ for all } T \in p_S\}$.

Then this function v satisfies (2.2)-(2.7), and the outcome f given by (2.13) is in the f -core of v . Furthermore the outcome f is the unique f -core outcome (up to sets of measure zero).

We note that the outcome f is, of course, in H^* and is actually exactly attainable by a feasible partition (the "a.e." qualification is not required).

2.3 A Characteristic Function Game with Side Payments

Games with side payments require a separate definition; however, the f -core is the same.

A game v with side payments is defined as a real-valued function on F with the following properties:

$$v(S \cup T) \geq v(S) + v(T) \quad \text{for any } S, T \in F \text{ with } S \cap T = \emptyset; \quad (2.14)$$

$$\inf_{i \in N} v(\{i\}) > -\infty; \text{ and} \quad (2.15)$$

for any feasible partition $p \in \Pi$ and any positive integer

$$k, v(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\}) \text{ is a measurable function from } N_{k1}^p \text{ to } R. \quad (2.16)$$

The f -core of a game v with side payments will be defined by representing the game v in terms of a game without side payments. That is, we define a game v_v by

$$v_v(S) = \{x \in R^S : \sum_{i \in S} x_i \leq v(S)\} \quad \text{for all } S \in F.$$

Then the game v_v satisfies (2.2)-(2.6) and

Lemma 2.2. $v_v(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\})$ has a measurable graph in $N_{k1}^p \times R^k$,
i.e., v_v satisfies (2.7).

Proof. Appendix.

From this Lemma it follows that a game with side payments can be treated as a special case of one without side payments. In particular, we can define the f-core of v as those outcomes in the f-core of v_v . We note that a necessary and sufficient condition for an outcome f of v_v to belong to the f-core of v_v is that

$$\sum_{i \in S} f(i) \geq v(S) \quad \text{for all } S \in F. \quad (2.17)$$

Example 2.3

Assume transferable utility and allow sidepayments among players in Example 2.2. Then the characteristic function v is

$$\begin{aligned} &\text{for all } i \in [0,3), v(\{i\}) = 0; \\ &\text{for all } i \in [0,1) \text{ and } j \in [1,3), \\ &\quad v(\{i,j\}) = i+j; \text{ and} \\ &\text{for all } S \in F, \end{aligned} \quad (2.18)$$

$$v(S) = \max_{P_S \in P(S)} \sum_{T \in P_S} v(T)$$

The outcome f given by (2.12) belongs to the f-core of this game also, because the only essential coalitions are boy-girl pairs, and because

$$\begin{aligned} f(i)+f(j) &= 2+i \geq i+j = v(\{i,j\}) \quad \text{for all } i \in [0,1), j \in [1,2) \\ f(i)+f(j) &= 2+i+j-2 = i+j = v(\{i,j\}) \quad \text{for all } i \in [0,1), j \in [2,3). \end{aligned}$$

In fact, f is again the unique outcome (up to sets of measure zero) in the f-core of the game v .

2.4 Some Lemmas on Feasible Partitions and the Outcome Space

Let A, B be sets in $\mathcal{B}$ with $\mu(A) = \mu(B) > 0$. We say that ψ is a measure-preserving isomorphism from A to B (mod 0) iff for some null sets A_0 and B_0 with $A_0 \subset A$ and $B_0 \subset B$, ψ is a measure-preserving isomorphism from $A - A_0$ to $B - B_0$.

Lemma 2.3. If $\mu(A) = \mu(B) > 0$, then there exists a measure-preserving isomorphism ψ (mod 0) from A to B .

Proof. Appendix.

This lemma ensures that we can construct feasible partitions. For example, let $\{N^1, N^2\}$ be a partition of N with $\mu(N^1) = \mu(N^2) = \frac{1}{2}\mu(N)$. Then, by Lemma 2.3, there is a measure-preserving isomorphism ψ from $N^1 - N_0^1$ to $N^2 - N_0^2$, where N_0^1 and N_0^2 are null subsets of N^1 and N^2 . We can then define a feasible partition p by:

$$p = \{\{i, \psi(i)\} : i \in N^1 - N_0^1\} \cup \{\{i\} : i \in N_0^1 \cup N_0^2\}.$$

Lemma 2.4. Let $h \in L(N, R)$. Assume that there is a partition $\{N_t\}_{t=1}^k$ of N and that there are measure-preserving isomorphisms $\psi_1, \dots, \psi_k$ (mod 0) from N_1 to $N_1, N_2, \dots, N_k$ such that

$$\begin{aligned} (h(\psi_1(i)), h(\psi_2(i)), \dots, h(\psi_k(i))) &\in V(\{\psi_1(i), \dots, \psi_k(i)\}) \\ \text{for all } i &\in \bigcap_{t=1}^k \text{Dom } \psi_t, \end{aligned} \tag{2.19}$$

where $\text{Dom } \psi_t$ is the domain of the function ψ_t . Then the function h belongs to H .

Proof. Appendix.

3. Market Games

3.1 The f-Core of a Market Game

To provide further motivation for our model of continuum games with finite coalitions, in this section we show that the f-core of a game derived from a market economy coincides with the "f-core of the economy", i.e. the f-core in allocation space. We consider a model of an exchange economy with m commodities where the player set (N, B, μ) is as defined in the preceding section. Each player has the same consumption set Ω , where

$$\Omega \text{ is closed in } R_+^m \text{ (endowed with the sup norm, } ||\cdot||); \text{ and} \quad (3.1)$$

$$\text{for any nonempty subset } S \subset \Omega, \text{ the vector } a^S, \text{ defined by its} \quad (3.2)$$

$$\text{coordinates } a_c^S = \inf \{a_c : a \in S\} \text{ (} c=1, \dots, m \text{), is in } \Omega. * \quad (3.2)$$

For simplicity, we assume that $0 \in \Omega$. For example, one choice for Ω is $\Omega = I_+^m = I_+ \times \dots \times I_+$ where I_+ is the set of all non-negative integers; all commodities are indivisible. Another choice is $\Omega = \{0, e^1, \dots, e^{m-1}\} \times R_+$, where e^c is the c th unit vector; this is the assignment market situation where a player wants at most one unit of one of the first $m-1$ indivisible commodities and the m th commodity is divisible.

Each player $i \in N$ has an initial endowment vector $\omega(i) \in \Omega$ and the function $\omega: N \rightarrow \Omega$ is measurable. Also, each player i has a utility function $U(i, \cdot)$ on Ω and we assume that $\inf_{i \in N} U(i, \omega(i)) > -\infty$.

Assumption A. The function $U: N \times \Omega \rightarrow R$ is measurable with respect to the product σ -algebra $B \otimes B(\Omega)$ **;

* This assumption will be used in the proof of Lemma 3.1.

** The σ -algebra of all Borel subsets of Ω is denoted by $B(\Omega)$.

Assumption B. For each $i \in N$, $U(i, \cdot)$ is continuous.

For the same purpose as the use of $\bar{R}$ rather than R in the preceding section (see Remark 2.3) and to simplify arguments, we extend Ω to

$$\bar{\Omega} = \Omega \cup \{(\infty, \dots, \infty)\} . \quad (3.3)$$

Then we assign $+\infty$ to $U(i, (\infty, \dots, \infty))$, i.e.,

$$U(i, (\infty, \dots, \infty)) = \infty \text{ for all } i \in N . \quad (3.4)$$

A function f in $L(N, \bar{\Omega})$ is called an allocation with respect to a partition p ($p \in \Pi$) iff

$$\sum_{j \in p(i)} f(j) \leq \sum_{j \in p(i)} \omega(j) \text{ a.e. in } N . \quad (3.5)$$

An allocation f with respect to p can be attained (a.e.) only by trading commodities within coalitions in p . Define the sets $A(p)$ ($p \in \Pi$), A and A^* by

$$A(p) = \{f \in L(N, \bar{\Omega}) : f \text{ is an allocation with respect to } p\} ; \quad (3.6)$$

$$A = \bigcup_{p \in \Pi} A(p) ; \quad (3.7)$$

$$A^* = \{f \in L(N, \bar{\Omega}) : \text{for some sequence } \{f^v\} \text{ in } A, \{f^v\} \text{ converges in measure to } f\} . \quad (3.8)$$

Note that $A(p) \subset A \subset A^*$ for all $p \in \Pi$ and it can be shown (with essentially the same proof as Lemma 2.1), that $A(p) \neq \emptyset$. We call a function f in A^* simply an allocation. Also, for any allocation f , we can find a pointwise convergent sequence $\{f^v\}$ in A where each f^v in the sequence can be attained a.e. by trades within finite coalitions.

The following lemma holds.

Lemma 3.1. $A^* = \{f \in L(N, \bar{\Omega}) : \int_N f(i) \leq \int_N \omega(i)\}$.

Proof. Appendix.

This lemma states that any function $f \in L(N, \bar{\Omega})$ with $\int_N f(i) \leq \int_N \omega(i)$ is an allocation, i.e., for some partition of the players into finite coalitions the allocation f is approximately attainable by trades within coalitions in the partition.

We say that a coalition S in F can improve upon a function f in $L(N, \bar{\Omega})$ iff there is a vector $(a^j)_{j \in S}$ such that

$$a^j \in \Omega \text{ for all } j \in S ; \quad (3.9)$$

$$\sum_{j \in S} a^j \leq \sum_{j \in S} \omega(j) ; \quad (3.10)$$

$$U(j, f(j)) < U(j, a^j) \text{ for all } j \in S . \quad (3.11)$$

The f-core of the market game (in allocation space) is defined to be

$$C_{fA} = \{f \in A^* : \text{no coalition in } F \text{ can improve upon } f\} . \quad (3.12)$$

This definition (3.12) of the f-core of a market game is obviously in the same spirit as the definition of the f-core of a game in characteristic function form. In the next section, we show that the f-core of the game in allocation space is equivalent to the f-core of the derived characteristic function game.

3.2 Characteristic Function Representation of the Market Game

Define the characteristic function V_M in the standard manner:
for any $S \in F$,

$$V_M(S) = \{x \in R^S : \text{for some } S\text{-allocation } (a^i)_{i \in S}, x_i \leq U(i, a^i) \text{ for all } i \in S\}, \quad (3.13)$$

where, by an S-allocation we mean a vector $(a^i)_{i \in S}$ satisfying (3.9) and (3.10). Then the sets $H_M(p)$, H_M and H_M^* are defined by (2.8), (2.9) and (2.10) and the f-core C_f of the game V_M , by (2.11).

Clearly, the game V_M satisfies (2.2)-(2.6).

Lemma 3.2. The game V_M satisfies condition (2.7).

Proof. Appendix.

Lemma 3.3. For any $p \in \Pi$, if $h \in H_M(p)$, then there is an $f \in A(p)$ such that $h(i) \leq U(i, f(i))$ a.e. in N .

Proof. Appendix.

We remark that the existence of both $h \in H_M(p)$ and $f \in A(p)$ is ensured by the Measurable Selection Theorem. However, since the spaces $H_M(p)$ and $A(p)$ are independently defined, Lemma 3.3 requires a proof.

The following theorem ensures that the general formulation of a game in characteristic function form given in Section 2 is adequate for the treatment of a market game. In particular, if we demonstrate nonemptiness of the f-core C_f of V_M , then the f-core C_{fA} in allocation space is nonempty; in the next section we show this nonemptiness.

Theorem 1 (i) For any h in the f-core C_f of the game V_M there is an allocation f in the f-core C_{fA} of the market game such that $U(i, f(i)) \geq h(i)$ for all $i \in N$.

(ii) For any f in the f -core C_{fA} of the market game, the utility representation h of f (i.e., h defined by $h(i) = U(i, f(i))$ for all $i \in N$) belongs to the f -core C_f of the game V_M .

Proof. Since the structure of "can improve upon" in the market game is fully described by the characteristic function V_M , it suffices to show that (i) an outcome h in C_f can be sustained by an allocation f in C_{fA} in the sense that $h(i) \leq U(i, f(i))$ for all $i \in N$, and (ii) the utility representation of an allocation in C_{fA} belongs to H_M^* .

(i) Suppose $h \in C_f$. Then there is a sequence $\{h^v\}$ in H_M such that $h^v(i) \rightarrow h(i)$ a.e. in N . For each v , there is a $p^v \in \Pi$ such that $h^v \in H_M(p^v)$. From Lemma 3.3, there is an allocation $f^v \in A(p^v)$ such that $h^v(i) \leq U(i, f^v(i))$ a.e. in N . From Lemma 3.1, f^v satisfies

$$\int_N f^v(i) \leq \int_N \omega(i) . \quad (3.14)$$

We can assume without loss of generality that $\int_N f^v(i)$ converges. Then we

can apply "Fatou's Lemma in m -dimensions" (Hildenbrand [1974, p. 69, Lemma 3]) to this sequence and state that there is an integrable function $\bar{f}$ such that

$$\bar{f}(i) \in Ls(f^v(i)) \text{ a.e. in } N ; \quad (3.15)$$

$$\int_N \bar{f}(i) \leq \liminf_N \int_N f^v(i) , \quad (3.16)$$

where $Ls(f^v(i))$ is the set of all cluster points of $\{f^v(i)\}$.

Hence $\int_N \bar{f}(i) \leq \int_N \omega(i)$, i.e., $\bar{f}$ is an allocation by Lemma 3.1. By (3.15),

there is a subsequence $\{f^{v\lambda}(i)\}$ for almost all i in N such that $f^{v\lambda}(i) \rightarrow \bar{f}(i)$ ($\lambda \rightarrow \infty$).

Since $U(i, f^v(i)) \geq h^v(i)$ and $h^v(i) \rightarrow h(i) \quad (v \rightarrow \infty)$, we have, by Assumption A, $h(i) \leq U(i, \bar{f}(i))$. That is, $h(i) \leq U(i, \bar{f}(i))$ a.e. in N . If $\bar{f}$ does not satisfy $h(i) \leq U(i, \bar{f}(i))$ for all $i \in N$, then we define the function f by

$$f(i) = \begin{cases} \bar{f}(i) & \text{if } U(i, \bar{f}(i)) \geq h(i) \\ (+\infty, \dots, +\infty) & \text{otherwise.} \end{cases}$$

This function f satisfies the condition required.

(ii) Let $f \in C_{fA}$. Then there is a sequence $\{f^v\}$ in A such that $f^v(i) \rightarrow f(i)$ a.e. in N . Since $f^v \in A$, there is a $p^v \in \Pi$ such that $f^v \in A(p^v)$. Then $(U(j, f^v(j)))_{j \in p^v(i)} \in V_M(p^v(i))$ for almost all i in N . Since $f^v(i) \rightarrow f(i)$ a.e. in N , $U(i, f^v(i)) \rightarrow U(i, f(i))$ a.e. in N by Assumption A. This means the limit function $h(\cdot) = U(\cdot, f(\cdot))$ belongs to H_M^* . Q.E.D.

3.3 The Relationships between the f-Core and the A-Core

We are now in a position to consider the relationships between the f -core concept and the core concept defined by Aumann [1964], which we will call the A-core. The A-core of the market game is

$$C_A = \{f \in L(N, \bar{\Omega}) : \int_N f(i) \leq \int_N \omega(i) \text{ and no subset of positive measure in } B \text{ can improve upon } f\},$$

where a subset S of positive measure is said to be able to improve upon f iff for some $g \in L(S, \bar{\Omega})$, $\int_S g(i) \leq \int_S \omega(i)$ and $U(i, g(i)) > U(i, f(i))$ for all $i \in S$. It is well-known that under some conditions, the A-core coincides with the set of competitive allocations (c.f., Aumann [1964], Hildenbrand [1974], Kahn and Yamazaki [1981]), where $f \in L(N, \bar{\Omega})$ is called a competitive allocation iff for some $p \in R_+^m$,

$$U(i, f(i)) = \max\{U(i, a) : a \in \Omega \text{ and } p \cdot a \leq p \cdot \omega(i)\} \text{ a.e. in } N ; \quad (3.18)$$

$$p \cdot f(i) \leq p \cdot \omega(i) \text{ a.e. in } N ; \text{ and} \quad (3.19)$$

$$\int_N f(i) \leq \int_N \omega(i) . \quad (3.20)$$

Let us assume

$$\Omega = \Omega_0 \times R_+^C, \text{ where } \Omega_0 \subset R_+^{m-|C|} \text{ and } C \subset M = \{1, \dots, m\}, \quad (3.21)$$

$C \neq \emptyset$; and

$$\text{for all } i \in N, U(i, a_{M-C}, a_C) \text{ is strictly monotone with respect to } a_C . \quad (3.22)$$

Under conditions (3.3) and (3.4), if a competitive allocation f exists then we can change f on a set of measure zero so that the resulting function, say f^* , is also a competitive allocation and has the property that

$$U(i, f^*(i)) \geq \max\{U(i, a) : a \in \Omega \text{ and } p \cdot a \leq p \cdot \omega(i)\} \text{ for all } i \in N . \quad (3.18')$$

It follows from (3.21) and (3.22) that

$$U(i, a) > U(i, f^*(i)) \Rightarrow p \cdot a > p \cdot \omega(i) .$$

Therefore if $S \in F$ can improve upon f^* then there is an S -allocation $(a^i)_{i \in S}$ such that $U(i, a^i) > U(i, f^*(i))$ for all $i \in S$, which implies that $p \cdot \sum_{i \in S} a^i > p \cdot \sum_{i \in S} \omega(i)$, a contradiction. We have now shown that the competitive allocation f^* belongs to the f -core C_{fA} . If the A -core coincides with the set of all competitive allocations, then

for any allocation f in the A -core, we can find another

$$\text{allocation } f^* \text{ in the } f\text{-core which is the same as } f \text{ a.e.} \quad (3.23)$$

Remark 3.1. Unless $(N, \mathcal{B}, \mu)$ is complete, it may not be possible that for every competitive allocation f , there is another allocation f^* , the same as f a.e., such that

$$U(i, f^*(i)) = \max \{U(i, a) : a \in \Omega \text{ and } p \cdot a \leq p \cdot \omega(i)\} \text{ for all } i \in N.$$

From Lemma 3.1, feasibility of outcomes is the same for Aumann's framework and for ours. In fact, Lemma 3.1 holds for any Borel subset of positive measure. It follows that Aumann's "can improve upon" can be viewed as the limit of "can improve upon" by partitions of the players into finite coalitions. Therefore Aumann's "can improve upon" concept is less restrictive than ours and it is natural that the A-core is contained in the f-core (up to null sets, of course).

The converse result, that the f-core is contained in the A-core, is easily demonstrated in some situations, for example, the Debreu-Scarfe [1963] model but with a continuum of players of each type and the same measure of players of each type. In more general settings, this converse is an open question of importance.

Our next example demonstrates that in other economic situations, the converse may not hold, even though for this example the f-core coincides with the set of competitive allocations.

Example 3.1. Let $N = [0, 5)$ be endowed with Lebesgue measure. The player set $[0, 5)$ is partitioned into three subsets: $[0, 1)$, the set of households; $[1, 3)$, the set of landlords of type 1 and $[3, 5)$, the set of landlords of type 2.

Each landlord has an apartment to rent, with a reservation rent

of 1 yen*. We imagine that the apartments of landlords of type 1 are located in a different area of the country from those of type 2. Landlord i 's consumption set is given by

$$\Omega_1 = \{0, e^1\} \times \mathbb{R}_+ \text{ if } i \in [1, 3) \text{ and } \Omega_2 = \{0, e^2\} \times \mathbb{R}_+ \text{ if } i \in [3, 5)$$

and his initial endowment, by

$$\omega(i) = (e^1, 0) \text{ if } i \in [1, 3) \text{ and } \omega(i) = (e^2, 0) \text{ if } i \in [3, 5) .^{**}$$

That is, landlord i owns an apartment and no money (a composite commodity).

Landlord i 's utility function $U(i, \cdot)$ is given as, for $i \in [1, 3)$

$$U(i, a^i) = \begin{cases} 1+m & \text{if } a^i = (e^1, m) \\ m & \text{if } a^i = (0, m) ; \end{cases}$$

and for $i \in [3, 5)$

$$U(i, a^i) = \begin{cases} 1+m & \text{if } a^i = (e^2, m) \\ m & \text{if } a^i = (0, m) . \end{cases}$$

Each household $i \in [0, 1)$ wants to rent at most one apartment from a landlord in either $[1, 3)$ or $[3, 5)$. The household's consumption set is given as $\Omega_h = \{0, e^1, e^2\} \times \mathbb{R}_+$, and its initial endowment is $\omega(i) = (0, 5)$. We allow households' utility functions to depend on external effects so households' utility functions are defined on the outcome space A^* , rather than on Ω_h . Let $U(i, f)$ ($i \in [0, 1)$, $f \in A^*$) be

* per minute. This is a reasonable rent in Tokyo.

**For simplicity, we have not defined the consumption sets as the same for all players; this could easily be done.

$$U(i, f) = \begin{cases} 5 + 5\mu(\{i \in [0, 1] : f_h(i) = e^1\}) + m & \text{if } f(i) = (e^1, m) \\ 2 + 10\mu(\{i \in [0, 1] : f_h(i) = e^2\}) + m & \text{if } f(i) = (e^2, m) \\ 0 + m & \text{if } f(i) = (0, m) \end{cases}$$

where $f(i) = (f_h(i), f_m(i))$ with $f_h(i) \in \{0, e^1, e^2\}$ and $f_m(i) \in R_+$. That is, household i 's utility comes from renting an apartment, the measure of players renting apartments in the same area, and the money remaining after paying the rent.

The allocations in the f -core of this game divides into three types:*

Type 1. All households rent apartments in $[1, 3)$ at 1 yen,

$$f(i) = \begin{cases} (e^1, 4) & \text{if } i \in [0, 1) \\ (e^2, 0) & \text{if } i \in [3, 5) \end{cases}$$

$$\mu(\{i \in [1, 3) : f(i) = (0, 1)\}) = \mu(\{i \in [1, 3) : f(i) = (e^1, 0)\}) = 1 ;$$

Type 2. All households rent apartments in $[3, 5)$ at 1 yen,

$$f(i) = \begin{cases} (e^2, 4) & \text{if } i \in [0, 1) \\ (e^1, 0) & \text{if } i \in [1, 3) \end{cases}$$

$$\mu(\{i \in [3, 5) : f(i) = (0, 1)\}) = \mu(\{i \in [3, 5) : f(i) = (e^2, 0)\}) = 1 ;$$

and

Type 3. Households rent apartments in both areas (at the same rent of 1 yen) and the distribution of households between areas is such that the utilities of households in different areas are equalized.

* For this example, "can improve upon" takes the allocations of other players as fixed. Since in our situation, the concept of a counter-coalition does not make sense, the above assumption is natural.

$$\mu(\{i \in [0,1]: f(i) = (e^1, 4)\}) = \frac{7}{15}$$

$$\mu(\{i \in [0,1]: f(i) = (e^2, 4)\}) = \frac{8}{15}$$

$$\mu(\{i \in [1,3]: f(i) = (0,1)\}) = \frac{7}{15}$$

$$\mu(\{i \in [1,3]: f(i) = (e^1, 0)\}) = \frac{23}{15}$$

$$\mu(\{i \in [3,5]: f(i) = (0,1)\}) = \frac{8}{15}$$

$$\mu(\{i \in [3,5]: f(i) = (e^2, 0)\}) = \frac{22}{15} \quad .$$

For allocations of Type 1, the utility of a household is 14, for Type 2, 16, and for Type 3, $11 \frac{1}{3}$. In all cases, rents are 1 yen because the measure of households is smaller than those of landlords of each type. Obviously, the outcomes of Types 1 and 3 are not Pareto-optimal nor are they in the A-core. The A-core consists only of the allocations of Type 2. However, the allocations of Types 1, 2 and 3 are all competitive allocations.

We remark that for the situation modelled in the above example the characteristic function defined in Section 2 does not adequately capture the structure of the economy nor determine the f-core of the economy. The definition of the characteristic function needs to be modified to reflect the externalities.

4. Nonemptiness of the f-Core in the Finite Type Case

We are now in a position to discuss the main theorem of this paper: if the player set N of a game can be partitioned into a finite number of types, then the game has a non-empty f-core under non-restrictive conditions. To state the theorem, some preliminary definitions are required.

Let (N, B, μ) be as described in Section 2.1 and let V be a game without side payments. Players i and j are called substitutes iff for any $S \in F$,

$$\begin{aligned} \text{if } i, j \notin S, \text{ then } x \in V(S \cup \{i\}) \Leftrightarrow x' \in V(S \cup \{j\}) \\ \text{where } x_\ell = x'_\ell \text{ for all } \ell \in S \text{ and } x_i = x'_j; \text{ and} \end{aligned} \quad (4.1)$$

$$\begin{aligned} \text{if } x \in V(S) \text{ and } i, j \in S, \text{ then } x' \in V(S), \text{ where} \\ x'_\ell = x_\ell \text{ for all } \ell \in S - \{i, j\} \text{ and } x'_j = x_i, x'_i = x_j; \end{aligned} \quad (4.2)$$

these conditions simply mean that the players i and j are completely identical with respect to the aspects described by the characteristic function V . The game V has the r-property with respect to $\{N_t\}_{t=1}^k$ iff

$$\begin{aligned} \{N_t\}_{t=1}^k \text{ is a partition of } N \text{ with } \mu(N_t) > 0 \text{ for all} \\ t=1, \dots, k, \text{ and all players in each } N_t (t=1, \dots, k) \text{ are} \\ \text{substitutes;} \end{aligned} \quad (4.3)$$

a game with the r-property is simply one with a finite number of types and a positive measure of players of each type. Assume that V has the

r-property with respect to $\{N_t\}_{t=1}^k$. For any $S \in F$, a payoff vector $y \in V(S)$ has the equal-treatment property (the ETP) iff

$$y_i = y_j \text{ for all } i, j \in N_t \cap S \text{ and } t=1, \dots, k. \quad (4.4)$$

The game V is per-capita bounded with respect to $\{N_t\}_{t=1}^k$ iff

there is a positive real number δ ($0 < \delta < 1$) and a K such that

$$\left. \begin{array}{l} S \in F \\ (1+\delta) \frac{\mu(N_t)}{\mu(N)} \geq \frac{|S \cap N_t|}{|S|} \geq (1-\delta) \frac{\mu(N_t)}{\mu(N)} \quad \forall t=1, \dots, k \\ x \in V(S) \text{ has the ETP} \end{array} \right\} \Rightarrow \left. \begin{array}{l} x_i < K \\ \text{for all } i \in S; * \end{array} \right. \quad (4.5)$$

that is, there is a constant K such that given any coalition S with approximately the same percentage of players of each type as N and any payoff x with the ETP in $V(S)$, we have $x_i < K$ for all $i \in S$.

Theorem 2 (Nonemptiness Theorem). Let V be a game without side payments. Assume V has the r-property and is per-capita bounded with respect to a partition $\{N_t\}_{t=1}^k$. Then the f-core of the game is nonempty.

The proof of Theorem 2 will be given in the next subsection.

The r-property is intuitively clear, but the per-capita boundedness assumption of Theorem 2 might need further explanation. If

$$\mu(N_1) = \dots = \mu(N_k) = \frac{\mu(N)}{k}, \text{ then (4.5) can be replaced by a weaker form:}$$

* If (4.5) is true for some δ , then it is also true for any δ^* ($0 < \delta^* < \delta$).

$$\left. \begin{array}{l}
 S \in F, \\
 |S \cap N_1| = \dots = |S \cap N_k|, \text{ and } \Rightarrow x_i < K \text{ for all } i \in S; \\
 x \in V(S) \text{ has the ETP}
 \end{array} \right\} \quad (4.5')$$

when the size of a coalition increases uniformly with respect to the percentage of agents of each type, per-capita payoffs are bounded. Even though condition (4.5') is in the same spirit as (4.5), it cannot be applied directly because the proportions of different types may vary.

Remark 4.1. As previously noted, our model provides a limit version of models of large games for which nonemptiness of approximate cores is obtained. As will be seen in the next subsection, our Theorem can be viewed as an extension of theorems on approximate cores in Wooders (1983) and Shubik and Wooders (1983).

The following Lemma states that (4.5) applies to market games with types and illustrates the non-restrictiveness of per-capita boundedness.

Lemma 4.1. Consider the market game described in Subsection 3.1.

Assume that there is a partition $\{N_t\}_{t=1}^k$ of N with $\mu(N_t) > 0$ ($t=1, \dots, k$) such that $\omega(i) = \omega(j)$ and $U(i, a) = U(j, a)$ for all $i, j \in N_t$ ($t=1, \dots, k$) and all $a \in \Omega$. Then the derived game V_M (defined by (3.13)) satisfies the r -property and per-capita boundedness with respect to $\{N_t\}_{t=1}^k$.

Proof. It is clear that V_M satisfies the r -property. Let us prove per-capita boundedness. Suppose the negation, i.e.,

$$\begin{aligned} & \forall \delta (0 < \delta < 1) \forall K \exists S \in F \\ & [(1+\delta) \frac{\mu(N_t)}{\mu(N)} \geq \frac{|S \cap N_t|}{|S|} \geq (1-\delta) \frac{\mu(N_t)}{\mu(N)}, t=1, \dots, k] \\ & \exists x \in V(S) [x \text{ has the ETP}] \exists i \in S: x_i \geq K. \end{aligned} \quad (4.6)$$

Let $\{K^v\}$ be an increasing sequence with $\lim_v K^v = +\infty$ and let $\delta (0 < \delta < 1)$ be given. Since the number of types is finite, we can assume without loss of generality that there is some t_0 ($1 \leq t_0 \leq k$) and sequences $\{S^v\}$ and $\{x^v\}$ such that for all v ,

$$\begin{aligned} & (1+\delta) \frac{\mu(N_{t_0})}{\mu(N)} \geq \frac{|S^v \cap N_{t_0}|}{|S^v|} \geq (1-\delta) \frac{\mu(N_{t_0})}{\mu(N)}, \\ & x^v \in V(S^v) \text{ has the ETP, and} \\ & x_i^v \geq K^v \text{ for all } i \in S^v \cap N_{t_0}. \end{aligned} \quad (4.7)$$

Since Ω is closed and $U(i, \cdot)$ ($i \in N_{t_0}$) is continuous on Ω , we can find a v_0 such that

$$U(i, a) \geq K^{v_0} \Rightarrow a > \frac{\mu(N)}{(1-\delta)\mu(N_{t_0})} \sum_{t=1}^k \omega^t, \quad (4.8)$$

where $\omega^t = \omega(i)$ for $i \in N_t$ ($t=1, \dots, k$). The condition $x^v \in V(S^v)$ implies that for some S^v -allocation $(a^{iv})_{i \in S^v}$, we have

$x_i^v \leq U(i, a^{iv})$ for all $i \in S^v$. Then it follows from (4.7) and

(4.8) that

$$\begin{aligned}
 \sum_{i \in S^{v_0}} a_i^{v_0} &> |S^{v_0} \cap N_{t_0}| \frac{\mu(N)}{(1-\delta)\mu(N_{t_0})} \sum_{t=1}^k \omega^t \\
 &\geq |S^{v_0}| \sum_{t=1}^k \omega^t \\
 &\geq \sum_{t=1}^k |S^{v_0} \cap N_t| \omega^t \\
 &= \sum_{i \in S^{v_0}} \omega(i) .
 \end{aligned}$$

This is a contradiction to the condition that $x^{v_0} \in v(S^{v_0})$.

Q.E.D.

From Theorems 1,2, and Lemma 4.1, we have

Corollary. Under the assumptions of Lemma 4.1, the f-core of the market game is non-empty.

Remark 4.2. Lemma 4.1 holds for any utility functions. However, we could transform the utility functions into bounded ones, and per-capita boundedness would be immediate. The substance of the corollary would be unchanged, since the f-core of the market game (in allocation space) does not depend on the particular choice of utility functions.

Remark 4.3. For nonemptiness of the f-core of a market game with types, we have essentially used only the existence of utility functions, which can be demonstrated under weak conditions (c.f. Debreu (1959, p. 56)).

In particular, no divisibility of commodities, no monotonicity, and no convexity are required.

4.2. Proof of Theorem 2

4.2.1. Without loss of generality, we can assume that $\sup V(\{i\}) = 0$ for all $i \in N$. First consider the case where $\mu(N_1), \dots, \mu(N_k)$ are all rational numbers. Then there are positive integers $p_1, q_1, \dots, p_k, q_k$ such that $\mu(N_t) = \frac{q_t}{p_t}$ for $t=1, \dots, k$. Put $m_t = p_1 \times \dots \times p_{t-1} \times q_t \times p_{t+1} \times \dots \times p_k$ for $t=1, \dots, k$. Then N_t can be partitioned into subsets $N_{t1}, \dots, N_{tm_t}$ ($t=1, \dots, k$) such that

$$\mu(N_{t\ell}) = \frac{1}{p_1 \times \dots \times p_k} \text{ for all } \ell = 1, \dots, m_t. \quad (4.9)$$

Let us construct a sequence $\{(A_r, V_r)\}_{r=1}^{\infty}$ of finite games as follows:

$$A_r = \bigcup_{t=1}^k A_{tr} \text{ and } A_{tr} = \{(t,1), \dots, (t, m_t)\} \quad (4.10)$$

for $t=1, \dots, k$;

$$\text{there is a 1 to 1 mapping } g : \bigcup_{t=1}^k \{(t,1), (t,2), \dots\} \rightarrow N \quad (4.11)$$

with $g(t,\ell) \in N_t$ for all $t=1, \dots, k$ and $\ell = 1, 2, \dots$;

and

$$V_r(S) = V(g(S)) \text{ for all } S \subseteq A_r \text{ and } r = 1, 2, \dots. \quad (4.12)$$

Then this sequence $\{(A_r, V_r)\}_{r=1}^{\infty}$ satisfies the conditions of the following lemma. In Section 4.3, both the terminology of the Lemma and the proof are described.

Lemma 4.2. (A generalization of Shubik and Wooders [1983, p.34 Theorem 2]).

Let $(m_1, \dots, m_k)$ be a vector of positive integers. Let $\{(A_r, V_r)\}$ be a sequence of (finite) replica games without side payments, where

$$A_r = \bigcup_{t=1}^k \{(t,1), \dots, (t, m_t)\}.$$

If the sequence satisfies per-capita boundedness, then some subsequence $\{(A_{r_v}, V_{r_v})\}_{v=1}^{\infty}$ of $\{(A_r, V_r)\}_{r=1}^{\infty}$ has a strong approximate core with the strong equal treatment property (the SETP) (4.15). That is, for any $\epsilon > 0$, there is a v_0 and an $x \in R^k$ with the following properties:

$$\prod_{t=1}^k x_t^{r_v m_t} = \prod_{t=1}^k [x_t^{r_v m_t}] \in V_{r_v}(A_{r_v}) \quad \text{for all } v \geq v_0; \quad (4.13)$$

$$\begin{aligned} &\text{for all } v \geq v_0 \text{ and for any } S \in A_{r_v} \text{ there does not exist} \\ &\text{a } y \in V_{r_v}(S) \text{ such that } y_i > x_t + \epsilon \text{ for all } i \in S \cap [t]_{r_v} \\ &\text{and } t=1, \dots, k; \text{ and} \end{aligned} \quad (4.14)$$

$$\begin{aligned} &\text{if } i \in [t]_{r_v} \text{ and } j \in [t']_{r_v} \text{ are substitutes, then} \\ &x_t = x_{t'}. \end{aligned} \quad (4.15)$$

Here $[t]_r$ denotes the set $\{(t,1), \dots, (t, m_t)\} (t=1, \dots, k)$, that is, the players of type t of the r th game.

Without loss of generality, we can assume that the sequence $\{(A_r, V_r)\}_{r=1}^{\infty}$ itself has a strong approximate core with the SETP.

Step 1. Let $\{\epsilon_v\}$ be a sequence of positive number with $\lim_{v \rightarrow \infty} \epsilon_v = 0$. From Lemma 4.2, we can choose an r_v for each ϵ_v such that there is an $x^{r_v} \in R^k$ satisfying (4.13), (4.14) and (4.15). It follows from

$\sup V(\{i\}) = 0$ and per-capita boundedness with respect to $\{N_t\}_{t=1}^k$ that $-\epsilon_v \leq x_t^{r_v} < K$ for $t=1, \dots, k$ and $v = 1, \dots$.

Then $\{x^{r_v}\}$ has a convergent subsequence. Again, without loss of generality, we can assume that $\{x^{r_v}\}$ itself converges to x^* . Note that $x^* \geq 0$.

Next define sequences $\{h^v\}$ in $L(N, R)$ and a function

$h^* \in L(N, R)$ by

$$h^v(i) = x_t^{r_v} \text{ if } i \in N_t \text{ and } t=1, \dots, k; \text{ and} \quad (4.16)$$

$$h^*(i) = x_t^* \text{ if } i \in N_t \text{ and } t=1, \dots, k. \quad (4.17)$$

Since $x^{r_v} \rightarrow x^*$, the sequence $\{h^v\}$ converges pointwise (in fact, uniformly) to h^* . Therefore if $h^v \in H$ for all v , then $h^* \in H^*$. Note that each h^v satisfies the SETP, i.e., if i and j are substitutes, then $h^v(i) = h^v(j)$, which implies that h^* also has the SETP.

Step 2. We now prove that $h^v \in H$ for all v . For $t=1, \dots, k$ and $\ell=1, \dots, m_t$, let $\{N_{t\ell}^1, \dots, N_{t\ell}^{r_v}\}$ be a partition of $N_{t\ell}$ such that

$$\mu(N_{t\ell}^s) = \frac{1}{r_v} (N_{t\ell}) = \frac{1}{r_v} \cdot \frac{1}{p_1 \times \dots \times p_k} \text{ for } s=1, \dots, r_v.$$

Then from Lemma 2.3, there is a measure-preserving isomorphism (mod 0), say

$\psi_{t\ell}^s$, from N_{11}^1 to $N_{t\ell}^s$ ($t=1, \dots, k$, $\ell=1, \dots, m_t$ and $s=1, \dots, r_v$).

Consider the coalition $T = \bigcup_{t=1}^k \bigcup_{\ell=1}^{m_t} \{\psi_{t\ell}^1(i), \dots, \psi_{t\ell}^{r_v}(i)\}$ for an

arbitrary $i \in \bigcap_t \bigcap_{\ell} \bigcap_s \text{Dom } \psi_{t\ell}^s$. This T has the same proportion of

players of each type as A_{r_v} , i.e.,

$$|T \cap N_t| = r_v m_t \quad \text{for } t=1, \dots, k.$$

Therefore, by (4.11), (4.12), (4.13) and (4.16), we have

$$(h^v(j))_{j \in T} \in V(T).$$

Since i is arbitrary and $\mu(\bigcap_t \bigcap_\ell \bigcap_s \text{Dom } \psi_{t\ell}^s) = \mu(N_{11}^1)$, it follows from Lemma 2.4 that $h^v \in H$.

Step 3. We now prove that there is no coalition $S \in F$ such that for some $y \in V(S)$, $y_i > h^*(i)$ for all $i \in S$. Suppose the contrary. Since $x^{r_v} \rightarrow x^*$ and $h^v(i) = x_t^{r_v}$ if $i \in N_t$ ($t=1, \dots, k$), there is a v_0 such that $\varepsilon_v < \min\{y_i - h^v(i) : i \in S\}$ for all $v \geq v_0$. That is, for all $v \geq v_0$,

$$\begin{aligned} y_i &> h^v(i) + \varepsilon_v = x_t^{r_v} + \varepsilon_v \quad \text{for all } i \in N_t \\ &\quad \text{and } t=1, \dots, k. \end{aligned} \tag{4.18}$$

Let S^* be a subset of A_{r_v} such that

$$|S^* \cap [t]_{r_v}| = |S \cap N_t| \quad \text{for all } t=1, \dots, k;$$

and let θ be a 1 to 1 mapping from S^* to S such that

$$\theta(S^* \cap [t]_{r_v}) = S \cap N_t \quad \text{for all } t=1, \dots, k.$$

(For sufficiently large v , we can find such an S^*). Then by the assumption of replication, $(y_i^*)_{i \in S^*} \equiv (y_{\theta(i)})_{i \in S^*}$ belongs to

$v_{r_v}(S^*)$, and by (4.16) and (4.18), $y_i^* > x_t^{r_v} + \varepsilon_v$ for all $i \in S^* \cap [t]_{r_v}$ and $t=1, \dots, k$. This is a contradiction to (4.14).

4.2.2. Let us consider the case where $\mu(N_1), \dots, \mu(N_k)$ are not necessarily rational numbers. Let $\{N_1^v\}, \{N_2^v\}, \dots, \{N_k^v\}$ be sequences such that

$$N_t^1 < N_t^2 < \dots \text{ for all } t=1, \dots, k; \quad (4.19)$$

$$\mu(N_t^v) \text{ is a rational number for all } t=1, \dots, k \text{ and } v=1, \dots; \quad (4.20)$$

$$\mu(N_t^v) \rightarrow \mu(N_t) \text{ (} v \rightarrow \infty \text{) for all } t=1, \dots, k; \text{ and} \quad (4.21)$$

$$\left(1 + \frac{\delta}{4}\right) \frac{\mu(N_t)}{\mu(N)} \geq \frac{\mu(N_t^v)}{\mu(N^v)} \geq \left(1 - \frac{\delta}{4}\right) \frac{\mu(N_t)}{\mu(N)} \quad (4.22)$$

for all $t=1, \dots, k$ and $v=1, 2, \dots$,

where $N^v = \bigcup_{t=1}^k N_t^v$ for all v . Consider the sequence of games

$\{(N^v, V^v)\}$, where V^v is the restriction of V to N^v for all $v=1, \dots$. For $\delta^v = \frac{\delta}{4}$, (N^v, V^v) satisfies per-capita boundedness

with respect to $\{N_t^v\}_{t=1}^k$. Indeed, if $S \in F$ and $S \subset N^v$ satisfies

$$\left(1 + \frac{\delta}{4}\right) \frac{\mu(N_t^v)}{\mu(N^v)} \geq \frac{|S \cap N_t^v|}{|S|} \geq \left(1 - \frac{\delta}{4}\right) \frac{\mu(N_t^v)}{\mu(N^v)},$$

then, by (4.22), we have

$$\begin{aligned}
(1+\delta) \frac{\mu(N_t)}{\mu(N)} &\geq (1+\frac{\delta}{4})^2 \frac{\mu(N_t)}{\mu(N)} \geq (1+\frac{\delta}{4}) \frac{\mu(N_t^v)}{\mu(N^v)} \\
&\geq \frac{|S \cap N_t^v|}{|S|} \geq (1-\frac{\delta}{4}) \frac{\mu(N_t^v)}{\mu(N^v)} \geq (1-\frac{\delta}{4})^2 \frac{\mu(N_t)}{\mu(N)} \\
&\geq (1-\delta) \frac{\mu(N_t)}{\mu(N)} .
\end{aligned}$$

This together with (4.5) implies that $x_i < K$ for any $x \in V^v(S) = V(S)$ with the ETP. Therefore the result of Subsection 4.2.1 can be applied to (N^v, V^v) and ensures the existence of a point h^v with the SETP in the f -core of the game (N^v, V^v) for all v . Since $h^v (v=1, \dots)$ has the SETP, we can define a vector x^v in R^k by

$$x_t^v = h^v(i) \text{ if } i \in N_t^v \text{ and } t=1, \dots, k .$$

Since $\sup V(\{i\}) = 0 \leq x_t^v \leq K$ for all $t=1, \dots, k$, the sequence $\{x^v\}$ has a convergent subsequence. We can assume without loss of generality that $\{x^v\}$ itself converges to x^* . Define h^* by

$$h^*(i) = x_t^* \text{ if } i \in N_t \text{ and } t=1, \dots, k .$$

Claim 1. No coalition in F can improve upon h^* .

On the contrary, suppose that for some $S \in F$ and some $y \in V(S)$ we have $y_i > h^*(i)$ for all $i \in S$. Then $x^v \rightarrow x^*$ implies that for some v , $y_i > h^v(i) = x_t^v$ if $i \in N_t^v$ ($t=1, \dots, k$). In this case we can find a coalition $S' \in F$ such that

$$|S' \cap N_t^v| = |S \cap N_t| \text{ for all } t=1, \dots, k .$$

Then S' can improve upon h^v in the game V^v , which is a contradiction.

Claim 2. $h^* \in H^*$.

Define $\bar{h}^v$ on N by

$$\bar{h}^v(i) = \begin{cases} h^v(i) & \text{if } i \in N^v \\ 0 & \text{otherwise} \end{cases}.$$

Then $\bar{h}^v \in H^*$ because $h^v \in H^{v*}$, where H^{v*} is the outcome space of (N^v, V^v) . It suffices to show that $\bar{h}^v(i) \rightarrow h^*(i)$ a.e. Since $\mu(\{i \in N: \bar{h}^v(i) = h^v(i)\}) \geq \mu(N^v)$, $\mu(N^v) \rightarrow \mu(N)$ and $h^v(i) \rightarrow h^*(i)$ for all $i \in \bigcup_{v=1}^{\infty} N^v$, we have $\bar{h}^v(i) \rightarrow h^*(i)$ a.e.

Q.E.D.

4.3. On Lemma 4.2

We first give the definitions required for Lemma 4.2. Let (A, V) be a finite player game without side payments and with properties (i) $V(S)$ is a closed nonempty subset of R^S for all $S \subset A$; (ii) $x \in V(S)$ and $y \in R^S$ with $x \geq y$ imply $y \in V(S)$ for all $S \subset A$; (iii) $V(S) \times V(T) \subset V(S \cup T)$ for all $S, T \subset A$ with $S \cap T = \emptyset$; and (iv) $V(S) - \bigcup_{i \in N} [(\text{interior } V(\{i\})) \times R^{S-\{i\}}]$ is nonempty

and bounded for all $S \subset A$. In this section, by a game, we mean a finite player game without sidepayments and conditions (i)-(iv).

Let $\{(A_r, V_r)\}_{r=1}^{\infty}$ be a sequence of games, where $A_r = \bigcup_{t=1}^k \{(t, 1), \dots, (t, r m_t)\}$ ($m = (m_1, \dots, m_k)$) is an arbitrary given vector

of positive integers). "Substitutes" are defined by (4.1) and (4.2). If all players in $\{(t,1), \dots, (t, m_t)\}$ ($t=1, \dots, k$) are substitutes, then the game (A_r, V_r) is called a replica game. A payoff vector $x \in V_r(S)$ is said to have the equal-treatment property (the ETP), iff $x_i = x_j$ for all $i, j \in S \cap [t]_r$ ($t=1, \dots, m$). The sequence $\{(A_r, V_r)\}_{r=1}^{\infty}$ is said to be per-capita bounded iff there exists a K such that for any r ,

$$\left[\begin{array}{l} S \subset A_r \\ \frac{|S \cap [t]_r|}{|S|} = m_t \text{ for all } t=1, \dots, k \\ x \in V_r(S) \text{ has the ETP} \end{array} \right] \Rightarrow x_i < K \text{ for all } i \in S.$$

In the case where $m = (1, \dots, 1)$, Lemma 4.2 without the SETP (4.15) is the same as Shubik and Wooders [1983, Theorem 2]. In their theorem, comprehensiveness of V_r was not assumed and their statement is slightly different from Lemma 4.2. However in the proof of their Theorem 1, they assumed comprehensiveness and demonstrated the conclusion of Lemma 4.2 with the ETP but not necessarily the SETP (Shubik and Wooders [1983, p. 44]). Their proof can be used without any substantive change in the case of $m = (m_1, \dots, m_k)$ arbitrary.

It is now easy to obtain the conclusion of Lemma 4.2. If some players of different types, say t and t' ($t < t'$) are substitutes, then we can treat all players of types t and t' as one type; we consider a new vector of numbers of players of each type,

$$m' = (m_1, \dots, m_t + m_{t'}, \dots, m_{t'-1}, m_{t'+1}, \dots, m_k)$$

and construct, by relabelling players, a sequence of replica games with respect to m' (in this sequence, all players of types t and t' are treated as the same type). By the above generalization of the Shubik-Wooders theorem, given $\varepsilon > 0$ we have an outcome with the ETP in the ε -core of a subsequence of the sequence of games. Then, again simply by relabelling, we can transform these outcomes into outcomes in a subsequence of the original games. These outcomes treat players of types t and t' identically. We repeat this procedure until we have the conclusion of Lemma 4.2.

Appendix

Proof of Lemma 2.1 Let k be any positive integer. The correspondence $V(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\})$ has an analytic graph by (2.7). Therefore the Measurable Selection Theorem (Hildenbrand (1974, p.54, Theorem 1)) can be applied to $V(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\})$ so there is a measurable selection $h^k(i) = (h_1^k(i), \dots, h_k^k(i)) : N_{k1} \rightarrow R^k$ such that $h^k(i) \in V(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\})$ a.e. in N_{k1}^p . We now have a countable number of selections $\{h^k\}_{k=1}^\infty$. Define a function $h : N \rightarrow R$ by

$$h(j) = h_t^k(i) \text{ if } j \in N_{kt}^p \text{ and } j = \psi_{kt}^p(i) \text{ for some } i \in N_{k1}^p \\ (t=1, \dots, k \text{ and } k=1, \dots).$$

Since any $i \in N$ belongs to N_k^p for some positive integer k , the function h is well-defined. Since

$$\{i \in N : h(i) > \alpha\} = \bigcup_{k=1}^\infty \{i \in N_k^p : \text{for some } j \in N_{k1}^p, i = \psi_{kt}^p(j) \\ \text{and } h_t^k(j) > \alpha\}$$

and since h^k and ψ_{kt}^p are measurable, the set $\{i \in N : h(i) > \alpha\}$ is a measurable subset of N . Q.E.D.

Proof of Lemma 2.2 Define $g : N_{k1}^p \times R^k \rightarrow R$ by

$$g(i, x) = \sum_{t=1}^k x_t - v(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\}).$$

Then g is a measurable function. Since the graph of $V_v(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\})$ can be represented as

$$\{(i, x) \in N_{k1}^p \times R^k : x \in V_v(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\})\} = \{(i, x) \in N_{k1}^p \times R^k : \\ g(i, x) \leq 0\},$$

the graph is measurable. Q.E.D.

Proof of Lemma 2.3 Let $([0, \mu(A)], \mathcal{B}([0, \mu(A)]), \lambda)$ be the measure space of Borel subsets of the closed interval $[0, \mu(A)]$ endowed with Lebesgue measure λ . Suppose that there exist measure-preserving isomorphisms ψ_1 and $\psi_2 \pmod{0}$ from $[0, \mu(A)] (= [0, \mu(B)])$ to A and B respectively. That is, there are null sets $L_A \subset [0, \mu(A)]$ (respectively, $L_B \subset [0, \mu(B)]$) and $A_0 \subset A$ ($B_0 \subset B$) such that ψ_1 (ψ_2) is a measure preserving isomorphism from $[0, \mu(A)] - L_A$ to $A - A_0$ (from $[0, \mu(B)] - L_B$ to $B - B_0$). Let $L = [0, \mu(A)] - L_A - L_B$. Then $\lambda(L) = \mu(A)$. Consider the restrictions of ψ_1 and ψ_2 to L . Since the composite function $\psi_2 \circ \psi_1^{-1}$ is a measure-preserving isomorphism from $\psi_1(L)$ to $\psi_2(L)$, the function $\psi_2 \circ \psi_1^{-1}$ is a measure-preserving

isomorphism $\pmod{0}$ from A to B . Therefore, it suffices to show the following theorem, where we assume without loss of generality that $\mu(A) = 1$.

Theorem. Let $(A, \mathcal{B}(A), \mu)$ be a measure space, where A is an uncountable Borel subset of a complete separable metric space, $\mathcal{B}(A)$, the σ -algebra of all Borel subsets of A , and μ , a nonatomic probability measure on $\mathcal{B}(A)$. Let $(I, \mathcal{B}, \lambda)$ be a measure space, where $I = [0, 1]$, $\mathcal{B}$, the σ -algebra of Borel subsets of I , and λ , Lebesgue measure. Then there is a measure-preserving isomorphism ψ from $A - A_1$ to $I - I_1$ for some $A_1 \subset A$ and $I_1 \subset I$ with $\mu(A_1) = \lambda(I_1) = 0$.

This theorem is stated in a slightly stronger form than Royden (1963, p.327, Theorem 9). However, it is sufficient to note that if one replaces

the section of Royden's proof dependent upon Royden [1963, p. 326, Theorem 8] with the obviously stronger analogue possible using Parthasarathy [1967, p. 15, Theorem 2.12], then the proof of the above form of the theorem is identical to Royden's. Q.E.D.

Proof of Lemma 2.4 We first dispose of the case where $k = 1$. In this case, $N_1 = N$ and $\psi_1(i) = i$ for almost all i in N . Set $p(i) = \{i\}$ for all i in N . Then $h(j)_{j \in p(i)} \in V(p(i))$ for almost all $i \in N$ so $h \in H$.

For the case where $k > 1$, let h' be the function on N defined by

$$h'(i) = \begin{cases} \max V(\{i\}) & \text{if } i \in N - \bigcup_{t=1}^k \psi_t \left(\bigcap_{\ell=1}^k \text{Dom } \psi_\ell \right) \\ h(i) & \text{otherwise.} \end{cases}$$

Then $h' \in L(N, R)$. Put

$$p = \{ \{ \psi_1(i), \dots, \psi_k(i) \} : i \in \bigcap_{t=1}^k \text{Dom } \psi_t \} \cup \{ \{ j \} : j \in N - \bigcup_{t=1}^k \psi_t \left(\bigcap_{\ell=1}^k \text{Dom } \psi_\ell \right) \}$$

Then $p \in \Pi$. If $p(i) = \{i\}$, then $h'(i) \in V(\{i\})$ by (2.2). Otherwise, there is a $j \in \bigcap_{t=1}^k \text{Dom } \psi_t$ such that $i = \psi_t(j)$ for some t , which together with (2.19) implies

$$(h'(\ell))_{\ell \in p(i)} = (h(\psi_1(j)), \dots, h(\psi_k(j))) \in V(\{\psi_1(j), \dots, \psi_k(j)\}) = V(p(i)).$$

Since $\psi_1, \psi_2, \dots, \psi_k$ are measure-preserving isomorphisms (mod 0) from N_1 to $N_1, \dots, N_k$, it holds that

$$\mu(N_1) = \mu(N_t) = \mu(\text{Dom } \psi_t) = \mu\left(\bigcap_{\ell=1}^k \text{Dom } \psi_\ell\right) = \mu\left(\psi_t \left(\bigcap_{\ell=1}^k \text{Dom } \psi_\ell\right)\right)$$

for all $t=1, \dots, k$ and

$$\begin{aligned} \mu(\{i \in N : p(i) = \{i\}\}) &= \mu(N) - \mu\left(\bigcup_{t=1}^k \psi_t \left(\bigcap_{\ell=1}^k \text{Dom } \psi_\ell\right)\right) \\ &= \mu(N) - \sum_{t=1}^k \mu(\text{Dom } \psi_t) = 0 \end{aligned}$$

Therefore $h' \in H$ and $\mu(\{i \in N : h'(i) \neq h(i)\}) \leq \mu(\{i \in N : \{i\} \in p\}) = 0$

imply $h \in H$.

Q.E.D.

Proof of Lemma 3.1

This lemma is critical to our results on market games and we believe will also be critical to future work. Since the later part of the proof (i.e., the proof that $\int_N f(i) \leq \int_N \omega(i) \Rightarrow f \in A^*$) is complex, we will provide a preliminary sketch of this part.

First we prove " $f \in A^* \Rightarrow \int_N f(i) \leq \int_N \omega(i)$ ". Since $f \in A^*$, from (3.8) and Remark 2.4, there is a sequence $\{f^v\}$ such that $f^v \in A$ for all v and f^v converges a.e. on N . For each v , there is a $p^v \in \Pi$ such that $f^v \in A(p^v)$. For each positive integer k , $\bigcup_{\substack{|S|=k \\ S \in p^v}} S \equiv N_k^v$ has a

partition $\{N_{k1}^v, \dots, N_{kk}^v\}$ with measure-preserving isomorphisms $\psi_{k1}^v, \dots, \psi_{kk}^v$ from N_{k1}^v to $N_{k1}^v, \dots, N_{kk}^v$, respectively, satisfying $\{S: S \in p^v \text{ and } |S| = k\} = \{\{\psi_{k1}^v(i), \dots, \psi_{kk}^v(i)\}: i \in N_{k1}^v\}$. Then it follows from (3.5) that for all v and k ,

$$\sum_{t=1}^k f^v(\psi_{kt}^v(i)) \leq \sum_{t=1}^k \omega(\psi_{kt}^v(i)) \text{ a.e. in } N_{k1}^v.$$

Therefore we have, for any v ,

$$\begin{aligned} \int_N f^v(i) &= \sum_{k=1}^{\infty} \sum_{t=1}^k \int_{N_{kt}^v} f^v(i) = \sum_{k=1}^{\infty} \int_{N_{k1}^v} \sum_{t=1}^k f^v(\psi_{kt}^v(i)) \\ &\leq \sum_{k=1}^{\infty} \int_{N_{k1}^v} \sum_{t=1}^k \omega(\psi_{kt}^v(i)) = \sum_{k=1}^{\infty} \sum_{t=1}^k \int_{N_{kt}^v} \omega(i) = \int_N \omega(i). \end{aligned}$$

Applying Fatou's lemma yields

$$\int_N f(i) \leq \liminf_{v \rightarrow \infty} \int_N f^v(i) \leq \int_N \omega(i).$$

We now prove that " $\int_N f(i) \leq \int_N \omega(i) \Rightarrow f \in A^*$ ".

As indicated earlier, we first provide a preliminary sketch of the proof.

Sketch

Suppose $\int_N f(i) \leq \int_N \omega(i)$. We first divide $R_+^m \cap \{x \in R^m : x_c \leq v, c=1, \dots, m\}$ into cubes of the same size $(\frac{1}{2^v})$. We then form a set of subsets $\{N_t^v\}_{t=1}^{\ell_v}$ such that all subsets have the same measures, and all players i in N_t^v have their allocation $f(i)$ and their endowments $\omega(i)$ in the same cubes. (This is carried out in Claim 1).

Now form the functions f^v and ω^v by taking $f^v(i)$ equal to the coordinate wise inf of f on the cube containing $f(i)$ and similarly, $\omega^v(i)$ equal to the coordinate-wise sup of ω on the cube containing $\omega(i)$. This has the consequence that for each t and all i and i' in N_t^v , we have $f^v(i) = f^v(i')$ and $\omega^v(i) = \omega^v(i')$.

It may be the case that $\int_N f^v(i)$ is too large for our purpose (because we ignore $\omega(i)$ when $\omega(i)$ is not contained in any of the cubes and because we make $\omega^v(i) \geq \omega(i)$ for all other values $\omega(i)$). Therefore, in Claim 2, we choose a subfamily of the subsets $\{N_t^v\}_{t=1}^{\ell_v}$ whose total measure tends to zero as $v \rightarrow \infty$. This subfamily is chosen so that if we redefine $f^v(i) = 0$ for all i in the union of members of the subfamily, then $\int_N f^v(i)$ is no longer too large. We carry this out and call the resulting function $\bar{f}^v$.

The next step is simple. The sequence $\{\bar{f}^v\}$ converges in measure to f and each member $\bar{f}^v$ of the sequence is attained by some feasible partition, where an element of the partition contains one member of N_t^v for each t (and players not in $\bigcup_{t=1}^{\ell_v} N_t^v$ are in singleton sets).

The Proof

Now to prove the lemma, without loss of generality we can assume that $\mu(N) = 1$. Let us suppose that $\int_N f_c(i) > 0$ for all $c = 1, \dots, m$.

Define

$$C^v(k) = \prod_{c=1}^m \left[\frac{k_c - 1}{2^v}, \frac{k_c}{2^v} \right) \text{ for } k = (k_1, \dots, k_m) \in \{1, 2, \dots, v2^v\}^m; \quad (A.1)$$

and

$$B^v = \{a \in R_+^m : a_c \geq v \text{ for some } c\}. \quad (v = 1, 2, \dots).$$

Claim 1 There is a sequence $\{N^v\}$ of subsets in B with the following properties:

Each N^v has a partition into some finite number of elements (A.2)

say $\{N_t^v\}_{t=1}^{\ell_v}$, with measure-preserving isomorphisms

$\psi_1^v, \dots, \psi_{\ell_v}^v$ from N_1^v to $N_1^v, \dots, N_{\ell_v}^v$, respectively, and

$$\mu(N_1^v) \rightarrow 0 \quad (v \rightarrow \infty);$$

$$\mu(N^v - N^{v+1}) = 0 \text{ for all } v \text{ and } \mu(N^v) \rightarrow \mu(N); \text{ and} \quad (A.3)$$

$$\text{for any } N_t^v, \text{ there are } k^1, k^2 \in \{1, \dots, v2^v\}^m \text{ such that} \quad (A.4)$$

$$\{f(i) : i \in N_t^v\} \subset C^v(k^1) \text{ and } \{\omega(i) : i \in N_t^v\} \subset C^v(k^2).$$

Note that of course $\mu(N_t^v) = \mu(N_1^v)$ for all t and the measure-preserving isomorphisms are "exact", i.e., not mod 0.

Proof of Claim 1. Define the sets $\tilde{M}^v(k^1, k^2)$ ($k^1, k^2 \in \{1, \dots, v2^v\}^m$) and M^v by

$$\tilde{M}^v(k^1, k^2) = \{i \in N: f(i) \in C^v(k^1) \text{ and } \omega(i) \in C^v(k^2)\}; \text{ and}$$

$$M^v = \{i \in N: f(i) \in B^v \text{ or } \omega(i) \in B^v\}.$$

Then $\bigcup_{k^1, k^2} \tilde{M}^v(k^1, k^2) \subset \bigcup_{k^1, k^2} \tilde{M}^{v+1}(k^1, k^2)$ and $\mu(\bigcup_{k^1, k^2} \tilde{M}^v(k^1, k^2)) =$

$\mu(N) - \mu(M^v) \rightarrow \mu(N)$. If $\mu(\tilde{M}^v(k^1, k^2)) > 0$, then we can choose a subset $M^v(k^1, k^2)$ of $\tilde{M}^v(k^1, k^2)$ such that

$$\mu(M^v(k^1, k^2)) \geq (1 - \frac{1}{2^v}) \mu(\tilde{M}^v(k^1, k^2)) ; \quad (\text{A.5})$$

$$\mu(M^v(k^1, k^2)) \text{ is a rational number.} \quad (\text{A.6})$$

Let $S_v = \{(k^1, k^2) \in \{1, \dots, v2^v\}^m: \mu(M^v(k^1, k^2)) > 0\}$. Without loss of generality, we can assume

$$\bigcup_{(k^1, k^2) \in S_v} M^v(k^1, k^2) \subset \bigcup_{(k^1, k^2) \in S_{v+1}} M^{v+1}(k^1, k^2) \text{ for all } v.$$

Let $\tilde{N}^v = \bigcup_{(k^1, k^2) \in S_v} \tilde{M}^v(k^1, k^2)$ ($v = 1, \dots$). Then $\tilde{N}^v \subset \tilde{N}^{v+1}$. It follows from

(A.5) that

$$\mu(\tilde{N}^v) = \mu(\bigcup_{(k^1, k^2) \in S_v} \tilde{M}^v(k^1, k^2)) \geq (1 - \frac{1}{2^v}) \mu(\bigcup_{k^1, k^2} \tilde{M}^v(k^1, k^2))$$

$$= (1 - \frac{1}{2^v}) [\mu(N) - \mu(M^v)] \rightarrow \mu(N) \quad (v \rightarrow \infty).$$

Since $\mu(M^v(k^1, k^2))$ is a rational number for all $(k^1, k^2) \in S_v$ by (A.6), we can find some subpartition, say $\{\tilde{N}_t^v\}_{t=1}^{l_v}$, of $\{\tilde{M}^v(k^1, k^2)\}_{(k^1, k^2) \in S_v}$ such that

$$\mu(\tilde{N}_1^v) = \dots = \mu(\tilde{N}_{l_v}^v) \leq \frac{1}{2^v}.$$

From Lemma 2.3, there are measure-preserving isomorphisms (mod 0)

$\psi_1^v, \dots, \psi_{\ell_v}^v$ from $\tilde{N}_1^v$ to $\tilde{N}_1^v, \dots, \tilde{N}_{\ell_v}^v$, respectively.

Let $N_t^v = \psi_t^v \left(\bigcap_{n=1}^{\ell_v} \text{dom } \psi_n^v \right)$ for $t = 1, \dots, \ell_v$ and let $N^v = \bigcup_{t=1}^{\ell_v} N_t^v$. Then

these sets $N_1^v, \dots, N_{\ell_v}^v$ and N^v have the same measures as those of $\tilde{N}_1^v, \dots, \tilde{N}_{\ell_v}^v$ and $\tilde{N}^v$, respectively. Therefore the claim holds. ■

Define the functions f^v and ω^v by their coordinates ($c=1, \dots, m$)

$$f_c^v(i) = \begin{cases} \inf \text{Proj}_c [C^v(k) \cap \Omega] & \text{if } f(i) \in C^v(k), \\ & k \in \{1, \dots, v2^v\}^m \text{ and } i \in N^v \\ 0 & \text{otherwise; and} \end{cases} \quad (\text{A.7})$$

$$\omega_c^v(i) = \begin{cases} \sup \text{Proj}_c [C^v(k) \cap \Omega] & \text{if } \omega(i) \in C^v(k), \\ & k \in \{1, \dots, v2^v\}^m \text{ and } i \in N^v \\ 0 & \text{otherwise} \end{cases} \quad (\text{A.8})$$

where $\text{Proj}_c [C^v(k) \cap \Omega] = \{x_c : x \in C^v(k) \cap \Omega\}$. Then $f^v(i) \in \Omega$ for all v and all $i \in N$ by (3.2) (while it may not be the case that $\omega^v(i) \in \Omega$). It follows from (A.3), (A.4) and (A.7) that

$$f^v(i) \leq f^{v+1}(i) \text{ for all } v \text{ and a.e. in } N; \quad (\text{A.9})$$

$$\int_N f^v(i) \leq \int_N f(i) \quad \text{for all } v. \quad (\text{A.10})$$

Since $f^v(i) \rightarrow f(i)$ a.e. in N by (A.3), (A.4), and (A.7), we have, by (A.9),

$$\int_N f^v(i) \rightarrow \int_N f(i). \quad (\text{A.11})$$

It follows from (A.11) and (A.8) that

$$\omega(i) \geq \omega^v(i) - \left(\frac{1}{2^v}, \dots, \frac{1}{2^v}\right) \quad \text{for all } v. \quad (\text{A.12})$$

Claim 2. There is a sequence $\{F^v\}$ of subsets in B such that for some v_0 ,

$$\text{for all } v \geq v_0, F^v = \bigcup_{t \in T_v} N_t^v \quad \text{for some } T_v \subset \{1, \dots, lv\}; \quad (\text{A.13})$$

$$\int_{F^v} f^v(i) \geq \int_{N - N^v} \omega(i) + \left(\frac{1}{2^v}, \dots, \frac{1}{2^v}\right) \quad \text{for all } v \geq v_0; \quad (\text{A.14})$$

$$\mu(F^v) \rightarrow 0 \quad (v \rightarrow \infty). \quad (\text{A.15})$$

Proof of Claim 2. Since $\int_{N-N^v} \omega(i) + (\frac{1}{2^v}, \dots, \frac{1}{2^v}) \rightarrow 0$ and

$\int_N f^v(i) \rightarrow \int_N f(i)$ by (A.3) and (A.11), we can find a v_0 such that

$$\int_{N^{v_0}} f^{v_0}(i) \geq \int_{N-N^{v_0}} \omega(i) + (\frac{1}{2^{v_0}}, \dots, \frac{1}{2^{v_0}}).$$

By (A.3), we have

$$\int_{N-N^v} \omega(i) + (\frac{1}{2^v}, \dots, \frac{1}{2^v}) \geq \int_{N-N^{v+1}} \omega(i) + (\frac{1}{2^{v+1}}, \dots, \frac{1}{2^{v+1}})$$

for all v .

These inequalities, together with (A.3) and (A.9), imply

$$\int_N f^v(i) \geq \int_{N^{v_0}} f^v(i) \geq \int_{N-N^v} \omega(i) + (\frac{1}{2^v}, \dots, \frac{1}{2^v}) \quad (\text{A.16})$$

for all $v \geq v_0$.

Let c be any element in $\{1, 2, \dots, m\}$. It follows from (A.16), (A.9), (A.2), and (A.4) that there is a sequence

$\{G_c^v\}_{v=v_0}^\infty$ such that for all $v \geq v_0$,

$$G_c^v = \bigcup_{t \in W_v} N_t^v \text{ for some } W_v \subset \{1, \dots, l_v\};$$

$$\int_{G_c^v} f_c^v(i) \geq \int_{N-N^v} \omega_c(i) + \frac{1}{2^v} :$$

$$\mu(G_c^v) \rightarrow 0 (v \rightarrow \infty).$$

Then let $F^v = \bigcup_{c=1}^m G_c^v$ for all $v \geq v_0$. The sequence $\{F^v\}$ satisfies

(A.13) - (A.15). This completes the proof of Claim 2. ■

We are now in a position to construct a sequence in $A = \bigcup_{p \in \Pi} A(p)$ which converges in measure to the function f . Define $\{\bar{f}^v\}$ by

$$\bar{f}^v(i) = \begin{cases} f^v(i) & \text{if } i \in N^v - F^v, v \geq v_0 \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A.17})$$

For any $\varepsilon > 0$, there is a $v_1 \geq v_0$ such that $\frac{1}{2^{v_1}} < \varepsilon$. Then it follows from (A.17) and (A.7) that for all $v \geq v_1$,

$$\mu(\{i \in N : \|\bar{f}^v(i) - f(i)\| > \varepsilon\}) \leq \mu(F^v \cup (N - N^v)), \quad (\text{A.18})$$

and from (A.3) and (A.15) that

$$\mu(F^v \cup (N - N^v)) \rightarrow 0 \quad (v \rightarrow \infty). \quad (\text{A.19})$$

That is, the sequence $\{\bar{f}^v\}$ converges in measure to f .

Next we have to show that for each $\bar{f}^v$, there is a feasible partition p^v of N such that $\bar{f}^v \in A(p^v)$. Define p^v by

$$p^v = \{ \{ \psi_1^v(i), \dots, \psi_{\ell_v}^v(i) \} : i \in N_1^v \} \cup \{ \{i\} : i \in N - N^v \} \quad (\text{A.20})$$

($v = 1, 2, \dots$)

(where the ψ_t^v 's are the isomorphisms given in Claim 1).

Then we have, for any $v \geq v_0$ and any $j \in N_1^v$,

$$\begin{aligned}
 \sum_{t=1}^{\ell v} \bar{f}^v(\psi_t^v(j)) &= \sum_{t=1}^{\ell v} f^v(\psi_t^v(j)) - \sum_{\psi_t^v(j) \in F^v} f^v(\psi_t^v(j)) \quad (\text{by (A.17)}) \\
 &= \frac{1}{\mu(N_1^v)} \left[\sum_{t=1}^{\ell v} \int_{N_1^v} f^v(\psi_t^v(i)) - \int_{F^v} f^v(i) \right] \quad (\text{by (A.13)}) \\
 &\leq \frac{1}{\mu(N_1^v)} \left[\int_{N^v} f^v(i) - \int_{N-N^v} \omega(i) - \left(\frac{1}{2^v}, \dots, \frac{1}{2^v}\right) \right] \quad (\text{by (A.14)}) \\
 &\leq \frac{1}{\mu(N_1^v)} \left[\int_N \omega(i) - \int_{N-N^v} \omega(i) - \left(\frac{1}{2^v}, \dots, \frac{1}{2^v}\right) \right] \\
 &\leq \frac{1}{\mu(N_1^v)} \left[\int_{N^v} \omega(i) - \left(\frac{1}{2^v}, \dots, \frac{1}{2^v}\right) \right] \\
 &\leq \frac{1}{\mu(N_1^v)} \int_{N^v} [\omega^v(i) - \left(\frac{1}{2^v}, \dots, \frac{1}{2^v}\right)] \quad (\text{by (A.8) and } \mu(N) = 1) \\
 &= \sum_{t=1}^{\ell v} [\omega^v(\psi_t^v(j)) - \left(\frac{1}{2^v}, \dots, \frac{1}{2^v}\right)] \leq \sum_{t=1}^{\ell v} \omega(\psi_t^v(j)) \quad (\text{by (A.12)}) .
 \end{aligned}$$

For all $j \in N-N^v$, (i.e., $p^v(j) = \{j\}$) it holds that $\bar{f}^v(j) = 0 \leq \omega(j)$. Therefore $\bar{f}^v \in A(P^v)$. By (A.19), $f \in A^*$.

Finally we remark on the case where $\int_N f_c(i) = 0$ for some $c \in \{1, 2, \dots, m\}$. Let $C = \{c : \int_N f_c(i) = 0\}$. If $C = \{1, \dots, m\}$, then clearly $f \in A^*$ (in fact, $f \in A(p)$ for all $p \in \Pi$). If $C \subset \{1, 2, \dots, m\}$ then we define $\bar{f}_c^v$ ($c \notin C$) in the same manner as above, and put, for $c \in C$,

$$\bar{f}_c^v(i) = 0 \text{ for all } i \in N.$$

Then $\bar{f}^v \in A$ for all v and $\bar{f}^v$ converges in measure to f .

This completes the proof.

Q.E.D.

Proof of Lemma 3.2 Let k be a positive integer. We will show that the graph $\{(i, x) \in N_{kl}^p \times R^k : x \in V_M(\{\psi_{kl}^p(i), \dots, \psi_{kk}^p(i)\})\}$ is analytic in $N_{kl}^p \times R^k$. Let $F(i)$ be the correspondence defined by

$$F(i) = \{(a, x) \in \Omega^k \times R^k : \sum_{t=1}^k \omega(\psi_{kt}^p(i)) \geq \sum_{t=1}^k a^t \text{ and}$$

$$U(\psi_{kt}^p(i), a^t) \geq x_t \text{ for all } t=1, \dots, k\}.$$

Then $F(i)$ has a measurable graph in $N_{kl}^p \times \Omega^k \times R^k$. Indeed, let us define functions on $N_{kl}^p \times \Omega^k \times R^k$ by

$$f_0(i, a, x) = \sum_{t=1}^k \omega(\psi_{kt}^p(i)) - \sum_{t=1}^k a^t; \text{ and}$$

$$f_t(i, a, x) = U(\psi_{kt}^p(i), a^t) - x_t \text{ for } t=1, \dots, k.$$

Then the graph of $F(i)$ can be written as

$$\{(i, a, x) \in N_{kl}^p \times \Omega^k \times R^k : f_t(i, a, x) \geq 0 \text{ for all } t=0, 1, \dots, k\}.$$

This implies that the graph is measurable in $N_{kl}^p \times \Omega^k \times R^k$.

The correspondence $V_M(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\})$ can be written as $V_M(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\}) = \text{Proj}_x F(i)$. Therefore it follows from Hildenbrand [1974, p. 59, Proposition 2] that $V_M(\{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\})$ has an analytic graph in $N_{k1}^p \times R^k$. Q.E.D.

Proof of Lemma 3.3 Let k be an arbitrary positive integer. By (2.1), any $S \in p$ with $|S| = k$ can be represented as $S = p(i) = \{\psi_{k1}^p(i), \dots, \psi_{kk}^p(i)\}$ for some $i \in N_{k1}^p$. Define $\Phi(i)$ on N_{k1}^p by

$$\Phi(i) = \{(a^1, \dots, a^k) \in \Omega^k : \sum_{t=1}^k \omega(\psi_{kt}^p(i)) \geq \sum_{t=1}^k a^t \text{ and}$$

$$U(\psi_{kt}^p(i), a^t) \geq h(\psi_{kt}^p(i)) \text{ for all } t=1, \dots, k\}.$$

It can be proved in the same way as illustrated in the proof of Lemma 3.2 that the correspondence $\Phi(i)$ has a measurable graph. The nonemptiness of $\Phi(i)$ follows from the assumption that $h \in H_M(p)$ and the definition of V_M . Therefore the Measurable Selection Theorem (Hildenbrand [1974, p. 54, Theorem 1]) can be applied to the correspondence and we conclude that there is a measurable function $(a_{k1}(\psi_{k1}^p(i)), \dots, a_{kk}(\psi_{kk}^p(i)))$ from N_{k1}^p to Ω^k such that $(a_{k1}(\psi_{k1}^p(i)), \dots, a_{kk}(\psi_{kk}^p(i))) \in \Phi(i)$ a.e. in N_{k1}^p . Since k is an arbitrary positive integer, we have a countable number of selections. Define a function $f: N \rightarrow \Omega$ by

$$f(j) = a_{kt}(\psi_{kt}^p(i)) \text{ if } j \in N_{kt}^p \text{ and } j = \psi_{kt}^p(i)$$

$$(k=1, 2, \dots).$$

Then this f belongs to $A(p)$, because f is measurable and for almost all $i \in N_k^p$ ($i = \psi_{kt}^p(i_o)$ for some $i_o \in N_{kl}^p$),

$$\begin{aligned} \sum_{j \in p(i)} f(j) &= \sum_{j \in p(i_o)} f(j) = \sum_{t=1}^k a_{kt}(\psi_{kt}^p(i_o)) \\ &\leq \sum_{t=1}^k \omega(\psi_{kt}^p(i)) = \sum_{j \in p(i)} \omega(j). \end{aligned}$$

It follows from the definition of Φ that $U(i, f(i)) \leq h(i)$

a.e. in N .

Q.E.D.

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