**TIME LIKE TRACE REGULARITY OF THE WAVE
EQUATION WITH A NONSMOOTH PRINCIPAL PART**

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Abstract

A trace regularity result is established for the multidimensional wave equation with nonsmooth variable coefficients in the principal part. It is shown that the time like trace of the solution can be as regular as the solution itself, provided that microlocal restrictions against the tangential oscillations of the coefficients.

Key words. trace regularity, microlocal Sobolev spaces, nonsmooth symbol classes, propagation of singularities, pseudo-differential cut-off.

AMS subject classification: 35L10, 35R25, 35S05

Abbreviated title: TRACE REGULARITY OF THE WAVE EQUATION

1 Introduction

A simplified model that governs many physical processes, such as acoustic and seismic wave propagation is the following wave equation

$$[\frac{1}{c^2(x,t)} \frac{d^2}{dt^2} - \nabla \sigma(x,t) \cdot \nabla] u(x,t) = f(x,t) .$$

where both coefficients $c(x,t)$ and $\sigma(x,t)$ are functions that may or may not be smooth. In this paper, we continue our study of trace regularity properties for the solution of the multidimensional wave equation with nonsmooth coefficients. The classical trace theorem in Sobolev spaces indicates that there will be a half derivative loss when a distribution is being restricted to a co-dimension one hypersurface. On the other hand, the standard method of energy estimates yields that for a strictly hyperbolic partial differential equation with smooth coefficients, the restriction (or trace) map to a co-dimension one space like hypersurface, mapping the solution to its trace, is from $H^s(\mathbb{R}^k)$ to $H^s(\mathbb{R}^{k-1})$ locally for any real s . It is our goal in this work to investigate the circumstances under which a time like trace of the solution is as regular as the solution itself. Obviously, even with smooth coefficients, the answer would be negative without any assumptions, which is essentially due to the illposedness nature of time like hyperbolic Cauchy problems, or the presence of grazing

rays. Moreover, the situation will be more complex when nonsmooth coefficients are present, where only limited initial regularity can be propagated.

In many applications, *e.g.* control and inverse problems, the situation where the coefficients (the medium) are time independent is of particular interest. Thus we shall throughout assume that the coefficients are time free in the model, *i.e.* $c(x, t) = c(x)$ and $\sigma(x, t) = \sigma(x)$, though some of the analysis may be extended to the time dependent case.

The following theorem is the main result of this paper. We show that with additional microlocal smoothness along the tangential directions in the coefficients, the time like trace can be as regular as the solution itself. Let us denote $d^2(x) = 1/c^2(x)$ and $\square_d = d^2(x)\partial_t^2 - \Delta$ and $\xi = (\xi', \xi_n)$ is the Fourier variable corresponding to $x = (x', x_n)$.

Theorem 1.1 *Suppose that $s > 3 + n/2$ and that u solves the problem*

$$[\square_d - \nabla\sigma(x) \cdot \nabla]u(x, t) = f(x, t), \quad (1.1)$$

$$u(x, t) \in H^l \cap H_{m\ell}^{l+1}(\gamma), \quad \text{near } \{t = 0\}, \quad (1.2)$$

with

$$\gamma = \{(x, t, \xi, \omega), (x, t) \in \Omega, (\xi, \omega) \in \mathbb{R}^{n+1}, |\xi'|^2 = d^2\omega^2 \text{ and } \xi_n = 0\},$$

for small $\epsilon_0 > 0$. Ω is a compact subset of $\{(x, t) \in \mathbb{R}^{n+1}, |t|, |x_n| \leq \epsilon_0\}$ and

$$\Gamma = \text{a conic neighborhood of } \gamma.$$

Assume that

$$(i) \ u \in H^{l-1} \cap H_{m\ell}^l(\Gamma), \ 1 \leq l \leq s;$$

$$(ii) \ d(x) \in H^s \cap H_{m\ell}^{l+1}(K) \text{ and } \nabla\sigma(x) \in H^{s-1} \cap H_{m\ell}^l(K), \ \Pi : \Gamma \subset T^*(\mathbb{R}^{n+1}) \rightarrow K \subset T^*(\mathbb{R}^n) \text{ is the projection map};$$

$$(iii) \ f \in H^{l-1} \cap H_{m\ell}^l(\Gamma).$$

Then

$$u|_{x_n=0} \in H_{loc}^l,$$

Trace regularity results are of great importance in diverse fields, such as control theory, boundary and initial boundary value problems of *P.D.E.s*, and in particular inverse problems. An interesting example arises from seismic imaging, where one wants to determine the mechanical properties of the earth through analyzing the seismogram measured on the surface of earth, *i.e.*, the solution on a time like surface. Clearly, modeling this process requires the most precise information about the time like trace.

In [1], we proved a trace theorem for general linear *P.D.E.s* with smooth variable coefficients, applying the Hörmander-Nirenberg pseudo-differential cutoff technique and the method of energy estimates. Our theorem indicates that the difficulty mentioned above may be resolved by imposing more smoothness against grazing ray directions. We then studied in [2] the wave equation with nonsmooth coefficients only at lower order terms. Compared to the general case in [1], a much simpler pseudo-differential cutoff was available, which

together with a result on propagation of singularities for strictly hyperbolic $\psi.d.o.$ equations with nonsmooth coefficients at lower order terms enabled us to prove a trace theorem.

The proof of Theorem 1.1 basically follows the framework developed in [2]. However, because of the presence of nonsmooth coefficients in the principal part, additional technical difficulties will necessarily take place. For example, the result on propagation of singularities which is essential in our proof becomes much harder to obtain. When the principal part has smooth or constant coefficients, as in [2], a generalized Rauch's lemma and a commutator lemma allowed us to extend a Beals-Reed theorem (the BRI, Theorem 1 in [6]) on propagation of singularities to a more suitable setting for linear problems. However, the same idea breaks down when the principal part becomes nonsmooth. To cure the difficulty, a refined approach was introduced by Beals and Reed in [7], where nonsmooth symbol classes and the corresponding calculus were developed. Their approach follows the general outline of Hörmander's proof of the theorem on propagation of singularities for $P.D.E.s$ with smooth coefficients [10], but of course the nonsmoothness of the principal part requires some very delicate commutator estimates. The key step was to establish a sharp Gårding's inequality which allowed them to reduce the microlocal problem to a local one. They then proved a theorem on propagation of singularities for $\psi.d.o$ equations with nonsmooth symbols (the BRII, Theorem 3.2 in [7]). In this paper, we prove an extended version of the Beals-Reed theorem (the BRII). Our theorem assures that under similar hypotheses some lower order microlocal regularity of the solution, for instance, H_{ml}^s -regularity (for $s \geq 0$), will also propagate along the null bicharacteristics. We also treat the case where coefficients may depend only on some of the variables, while we show that a better result can be expected. We emphasize that this is particularly important since the coefficients in our model are time independent!

We expect that the calculus of nonsmooth symbols developed in [7] and this work will find many other applications. Working on $P.D.E.s$ with nonsmooth coefficients, one would often encounter similar commutator estimates.

Recently, studies on trace regularity have received much attention. For solutions of the multidimensional wave equation with constant coefficients, Symes proved in [21], by using the method of geometric optics, that with finitely energy initial data compactly supported away from the boundary, the trace is as regular as the solution in the interior. Similar results were obtained independently by Lasiecka and Triggiani [15] based on an application of the classical Laplace-Fourier transform. They also established, see [16], a series of sharp global trace regularity results for second order hyperbolic equations with smooth coefficients and Dirichlet or Neumann boundary conditions. More recently, by using a microlocal cut-off technique, Lasiecka and Triggiani in [17] obtained an optimal tangential trace regularity theorem for the wave equation with smooth coefficients in a bounded domain. For a wave equation Kim [12] proved a trace regularity result by employing a microlocal method, which further allowed him to study the regularity of some boundary controls. Our approach differs from which mentioned above in several aspects: The data in this work need not have compact supports, and the coefficients in the model may be nonsmooth, furthermore our method relies heavily on various results on propagation of singularities. For more general geometry and functions, Joly studied in [14] some trace regularity results in the case of anisotropic Sobolev spaces in two dimension where the domain can be any arbitrary open set of $\mathbb{R}^2$.

The outline of the paper is as follows: In the next section, we introduce nonsmooth symbol classes which is more flexible than those defined in [7] in the sense that the coefficients may depend only on part of the variables. We then develop a calculus for this class of nonsmooth symbols which handles a more general class of functions with less regularity than in [7]. We establish a Gårding type inequality. Finally, we present a theorem on propagation of singularities. Section 3 is devoted to the proof of our main theorem. This is done by a combination of microlocal cut-off, the “fattening” lemma, the method of energy estimates, and the theorem on propagation of singularities.

Throughout, the reader is assumed to be familiar with the basic calculus of *Pseudodifferential Operators* (“*ψ.d.o.*”) as presented in [23] and [18]. A classical *ψ.d.o.* P of order m is denoted as $P \in OPS^m$ with its symbol $p \in S^m$. $ES(P)$ stands for *the essential support* of operator P . H^s is the standard L^2 -type Sobolev space and H_{loc}^s means a local Sobolev space. $\langle \xi \rangle$ means $(1 + |\xi|^2)^{1/2}$. The Fourier transform of a distribution u is expressed as $\hat{u}$. Usually, the constant from the Fourier Transform is assumed to be absorbed by the integral. For simplicity, C serves as a generalized positive constant the precise value of which is not needed. The characteristic function of a set Γ is represented by χ_Γ .

We conclude the introduction by a brief discussion about nonlinear wave equations. The approach of this work may be adopted nicely in the trace regularity study of a class of nonlinear wave equations. In fact, various results on propagation of singularities for semilinear and quasilinear wave equations have been developed by many people, in particular by Rauch [19], Bony [8], Beals and Reed [6], [7], Beals [3], [4], and Chemin [9]. But compared to the linear case, much higher overall smoothness of both the solution and coefficients would be required.

2 Propagation of microlocal regularity

We introduce some basic concepts, related material may be found in Beals and Reed [7]. General theoretical aspects as well as applications of nonsmooth microlocal analysis may be found in Rauch [19], Beals [5], and Taylor [24] and references therein.

Definition 2.1 *A distribution u is said to be in $H^s \cap H_{m\ell}^r(x_0, \xi_0)$ if there exist $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ with $\phi(x_0) \neq 0$ and a conic neighborhood $\gamma \subset \mathbb{R}^n \setminus \{0\}$ of ξ_0 such that*

$$\langle \xi \rangle^s (\phi u)^\wedge(\xi) \in L^2(\mathbb{R}^n) \text{ and } \langle \xi \rangle^r \chi_\gamma(\xi) (\phi u)^\wedge(\xi) \in L^2(\mathbb{R}^n).$$

Next, we define a nonsmooth symbol class, which follows [7] with one main difference, that is, the present symbol class includes symbols that depend only on part of the variables. An interesting special case deals with *P.D.E.s* with coefficients depending on part of the variables. For convenience, we denote $x = (x^1, x^2) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n-n_0}$, $1 \leq n_0 \leq n$. The frequency variables corresponding to $x = (x^1, x^2)$ are denoted by $\xi = (\xi^1, \xi^2) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n-n_0}$.

Definition 2.2 $S_{n_0}^{m;s,r}(K)$ is defined to be the collection of symbols $a(x^1, \xi)$, smooth in ξ , such that

$$a(x^1, \xi) / \langle \xi \rangle^m \in H^s \cap H_{m\ell}^r(K)$$

as a function of x^1 uniformly in ξ .

In other words, for each $(x_0^1, \xi_0) \in K$, there exist a function $\psi(x^1) \in C_0^\infty(\mathbb{R}^{n_0})$ with $\psi(x_0^1) \neq 0$ and a conic neighborhood γ of ξ_0 such that

$$\begin{aligned} \langle \zeta \rangle^s \mathcal{F}(\psi a)(\zeta, \xi) / \langle \xi \rangle^m &\in L^2(d\xi) \quad \text{and} \\ \langle \zeta \rangle^r \chi_\gamma(\zeta) \mathcal{F}(\psi a)(\zeta, \xi) / \langle \xi \rangle^m &\in L^2(d\xi) \end{aligned}$$

with norms independent of ζ .

In particular, when $n = n_0$, the symbol class becomes the one introduced in [7]. In that case, we denote

$$S^{m;s,r}(K) = S_n^{m;s,r}(K) .$$

More generally, dealing with operators with lower order terms, we also need the following definition.

Definition 2.3 For $k \geq 0$ an integer, $S_{st,n_0}^{m+k;s+k,r+k}(K)$ consists of symbols $a(x^1, \xi)$ of the form

$$a_{m+k}(x^1, \xi) + a_{m+k-1}(x^1, \xi) + \cdots + a_m(x^1, \xi)$$

where $a_m(x^1, \xi) \in S^{m;s,r}(K)$ and for $0 < j \leq k$

$$a_{m+j}(x^1, \xi) = \sum_l a_{j,l}(x^1) P_{m+j,l}(x^1, \xi) \in S^{m;s,r}(K)$$

with $a_{j,l}(x^1) \in H^{s+j} \cap H_{m\ell}^{r+j}(K)$ and $P_{m+j,l}(x^1, \xi) \in S_{1,0}^{m+j}$.

It is easy to observe that

$$S_{st,n_0}^{m+k;s+k,r+k}(K) \subset S_{n_0}^{m+k;s+k,r+k}(K) + \cdots + S_{n_0}^{m;s,r}(K) \subset S_{n_0}^{m+k;s,r}(K) .$$

We are now ready to state a linear theorem on propagation of singularities for $\psi.d.o.$ equations with nonsmooth coefficients. Again recall that $x = (x^1, x^2) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n-n_0}$, and $\xi = (\xi^1, \xi^2)$ is the corresponding frequency variable.

Theorem 2.1 Let $n_0/2 < s$, $0 \leq l \leq s, q$ and $q < l + s - n_0/2$ ($1 \leq n_0 \leq n$), and

$$a(x^1, \xi) \in S_{st,n_0}^{m+2;s+2,q+2}(K)$$

has real principal symbol a_{m+2} which is homogeneous of degree $m+2$ in ξ , and that γ is the null bicharacteristic through the characteristic point (x_0, ξ_0) . $K = \Pi\gamma$ with $\Pi : T^*(\mathbb{R}^n) \rightarrow T^*(\mathbb{R}^{n_0})$ the projection map. Assume that

- (i) $v \in H^{l+m} \cap H_{m\ell}^{q+m}(\gamma)$
- (ii) $v \in H^{l+m} \cap H_{m\ell}^{q+m+\epsilon}(x_0, \xi_0)$, for some $0 \leq \epsilon \leq 1$,
- (iii) $f \in H^l \cap H_{m\ell}^q(\gamma)$,

and that

$$a(x^1, D_x)v = f .$$

Then

$$v \in H_{m\ell}^{q+m+\epsilon}(\gamma) .$$

That is, if the solution v has improved microlocal regularity at a point on a null bicharacteristic, the improved regularity will stay for the rest of the null bicharacteristic.

Remarks on Theorem 2.1.

In the case that $l = s$ and $n_0 = n$, we speak of the original Beals-Reed theorem (the BRII). Compared to the BRII, Theorem 2.1 assures that weaker regularity of the solution may also be propagated along null bicharacteristics, and a better regularity result can be achieved when the coefficients depend only on some of the variables.

It seems that the regularity hypotheses on the coefficients and right-hand side cannot be improved much. However, the sharpness questions on this theorem and other theorems on propagation of singularities (the BRI, the BRII, and Theorem 2.1 in [2]) remain open. Needless to say, these questions are extremely important and worthwhile to pursue in the future.

The most precise information about the propagation of singularities may be obtained in the case of one space dimension, where the wave operator can be factored into products of two simple first order differential operators. Roughly speaking, the improved microlocal regularity is then propagated along null bicharacteristics with very few restriction on the order of smoothness. The best reference for one dimensional hyperbolic problems is Rauch and Reed [20].

2.1 About the proof of Theorem 2.1

Theorem 2.1 may be proved by following the general scheme of the proof of the BRII in [7] with some necessary modifications. The main ingredients of their proof are:

- The development of a calculus for the nonsmooth symbol class
- Construction of an appropriate microlocal cutoff b_0 by essentially following Bony's construction in [8]
- Proof of a generalized Gårding's inequality.

The key step was to develop the calculus of the nonsmooth symbol class, which led to a systematic way of handling commutators that involve either nonsmooth functions or nonsmooth symbols ($\psi.d.o.$ with nonsmooth coefficients). It was shown that the nonsmooth symbol class developed by Beals and Reed preserves important continuity properties of smooth $\psi.d.o.$.

We want to point out that when the differential operator of a differential equation has constant coefficients or with smooth principal part, the microlocal cutoff may be constructed in a much simpler fashion. In fact, following Hörmander's first proof of the theorem on propagation of singularities in [11], see also Nirenberg [18], there exists a $b_0 \in OPS^0$ such

that $[P_m, b_0]$ is of order $m - 2$ instead of $m - 1$, if P_m is the principal part of an m -th order partial differential operator with usual assumption on P_m . Having constructed this operator b_0 , one may reduce the microlocal regularity problem to a local one by acting b_0 to both sides of the *P.D.E.*.

For simplicity, we shall skip the formal proof of Theorem 2.1 with the understanding that the proof can be formally done by modifying the proof of the BRII ([7], pp 174-177). In order to do so, a calculus of general nonsmooth symbols must be developed, also a new Gårding's inequality must be established. Through this process, Theorem 2.2 will play a crucial role since it characterizes the fundamental Sobolev continuous properties of the class of nonsmooth symbols. The rest of this section is devoted to establish the necessary results for completing the proof of Theorem 2.1.

2.2 Sobolev continuity of nonsmooth symbols

We begin with some inequalities that will be used frequently.

Proposition 2.1 *Let*

$$C_g^2 = \sup_{\xi, \eta} \int |g(\zeta, \xi, \eta)|^2 d\zeta$$

and

$$C_K^2 = \sup_{\zeta, \eta} \int |K(\zeta, \xi, \eta)|^2 d\xi .$$

For $h \in L^2$, define

$$(Th)(\zeta, \eta) = \int K(\zeta, \xi, \eta) g(\zeta - \xi, \xi, \eta) h(\xi, \eta) d\xi .$$

Then

$$||Th|| \leq C_g C_K ||h|| .$$

Proof. Consider for any $f(\zeta, \eta) \in L^2$ and $||f|| \leq 1$,

$$\begin{aligned} |(Th, f)| &= \left| \int \int f(Th) d\zeta d\eta \right| \\ &= \left| \int \int \int K(\zeta, \xi, \eta) g(\zeta - \xi, \xi, \eta) f(\zeta, \eta) h(\xi, \eta) d\xi d\zeta d\eta \right| \end{aligned}$$

Interchanging the sequence of integrations and by Cauchy-Schwartz's inequality, we have

$$\begin{aligned} |(Th, f)| &\leq \int \int d\xi d\eta |h(\xi, \eta)| \left[\int d\zeta |K(\zeta, \xi, \eta) f(\zeta, \eta)|^2 \right]^{1/2} \cdot \left[\int d\zeta |g(\zeta - \xi, \xi, \eta)|^2 \right]^{1/2} \\ &\leq C_g ||h|| \cdot \left[\int \int \int d\xi d\eta d\zeta |K(\zeta, \xi, \eta) f(\zeta, \eta)|^2 \right]^{1/2} \\ &\leq C_g C_K ||h|| \cdot ||f|| \end{aligned}$$

hence the conclusion follows. □

Similarly, one can prove the following result.

Proposition 2.2 Denote $\zeta = (\zeta^1, \zeta^2)$ and $\eta = (\eta^1, \eta^2)$, where ζ^1 , η^1 and ξ are of the same dimension. Let

$$\begin{aligned} C_h^2 &= \sup_{\zeta} \int d\xi |h(\xi, \zeta)|^2 \\ C_g^2 &= \sup_{\zeta, \xi} \int d\eta |g(\eta, \zeta, \xi)|^2 \end{aligned}$$

and

$$C_G^2 = \sup_{\eta} \int \int d\xi d\zeta |G(\xi, \eta, \zeta)|^2.$$

For $v \in L^2$, define

$$T(v)(\eta) = \int \int d\xi d\zeta G(\xi, \eta, \zeta) h(\xi, \zeta) g(\eta^1 - \zeta^1 - \xi, \eta^2 - \zeta^2, \zeta, \xi) v(\zeta).$$

Then

$$||T(v)|| \leq C_h C_g C_G ||v||.$$

Proof. For any $f \in L^2(d\eta)$ with $||f|| \leq 1$, consider

$$\begin{aligned} |(T(v), f)| &\leq \int d\zeta |v(\zeta)| \int \int d\xi d\eta |h(\xi, \zeta) g(\eta^1 - \zeta^1 - \xi, \eta^2 - \zeta^2, \zeta, \xi)| |G(\xi, \eta, \zeta) f(\eta)| \\ &\leq \int d\zeta |v(\zeta)| \left[\int d\xi |h|^2 \int d\eta |g|^2 \right]^{1/2} \left[\int \int d\xi d\eta |G|^2 |f|^2 \right]^{1/2} \\ &\leq C_g C_h \int d\zeta |v(\zeta)| \left[\int \int d\xi d\eta |G|^2 |f|^2 \right]^{1/2} \\ &\leq C_g C_h C_G ||v|| ||f||. \end{aligned}$$

□

Proposition 2.3 Assume that K' is a closed cone which is strictly contained in an open cone K . If $\xi \in K'$, $\eta \in K^C$, where K^C is the complement of K , then

- (1) $|\xi - \eta| \geq C_1 |\xi|$, $C_1 > 0$;
- (2) if $|\xi| \geq C_0 > 0$, then $\langle \xi - \eta \rangle \geq C \langle \xi \rangle$.

The following theorem establishes continuity properties of the nonsmooth symbol class. Let $\Pi_1 : (x^1, x^2, \xi^1, \xi^2) \rightarrow (x^1, \xi^1, \xi^2)$.

Theorem 2.2 Let $n_0/2 < s$, $0 \leq l \leq s, q$, and $q < l + s - n_0/2$ ($1 \leq n_0 \leq n$). Suppose that $a(x^1, \xi) \in S_{n_0}^{m; s, q}(\gamma)$, $0 \leq m \leq s, l$, $\gamma = \Pi_1 \Gamma$. Then

$$a(x^1, D_x)$$

is a bounded operator from $H^l \cap H_{m\ell}^q(\Gamma)$ to $H^{l-m} \cap H_{m\ell}^{q-m}(\Gamma)$.

Remarks. We list some of the interesting special cases of Theorem 2.2.

- If $n_0 = n$ and $l = s$, then Theorem 2.2 becomes an earlier result of Beals and Reed (Theorem 1.3 in [7]). In addition, the case where $a(x, \xi) = a(x^1, \xi)$ gives rise to Rauch's Lemma [19].
- The case $a(x^1, \xi) = a(x^1)$ is the generalized Rauch's Lemma proved in [2] (Lemma 2.3).

Proof. *W.l.o.g.*, we assume that $s < q$, some obvious modification will yield the conclusion for $s \geq q$. For $u \in H^l \cap H_{m\ell}^q(\Gamma)$, we may assume u and $a(x^1, \xi)$ have compact supports in x near x_0 , and x^1 near x_0^1 respectively, where $(x_0, \xi_0) \in \Gamma$. Then

$$a(x^1, D_x)u(x) = \int \int e^{ix \cdot \xi} a(x^1, \xi) \hat{u}(\xi) d\xi$$

and

$$(au)^\wedge(\eta) = \int \hat{a}(\eta^1 - \xi^1, \xi^1, \eta^2) \hat{u}(\xi^1, \eta^2) d\xi^1.$$

Next since $a \in S_{n_0}^{m;s,r}(\gamma)$,

$$\hat{a}(\zeta, \xi) = f(\zeta, \xi) \langle \xi \rangle^m / \langle \zeta \rangle^s$$

where $\sup_\xi \int f^2(\zeta, \xi) d\zeta < \infty$.

Define $v(\xi) = \langle \xi \rangle^l \hat{u}(\xi) \in L^2$. Thus

$$\langle \eta \rangle^{l-m} (au)^\wedge(\eta) = \int K(\xi^1, \eta) f(\eta^1 - \xi^1, \xi^1, \eta^2) v(\xi^1, \eta^2) d\xi^1$$

with

$$K(\xi^1, \eta) = \langle \eta \rangle^{l-m} / \langle \eta^1 - \xi^1 \rangle^s \langle \xi^1, \eta^2 \rangle^{l-m}.$$

Therefore from Proposition 2.1, it suffices to show that

$$\sup_\eta \int |K(\xi^1, \eta)|^2 d\xi^1 < \infty$$

which follows immediately from Hölder's inequality and the assumption that $s > n_0/2$ and $l \geq m$.

Let θ be a conic neighborhood of ξ_0 such that $u \in H^l \cap H_{m\ell}^q(\theta)$, and $a(x^1, \xi) \in S_{n_0}^{m;s,r}(\sigma)$ where $\sigma = \Pi\theta$ the projection of θ onto the ξ^1 -space. Let $\theta' \subset \subset \theta$, be a strictly smaller conic neighborhood of ξ_0 and σ' is the projection of θ' . We must show that

$$\chi_{\theta'}(\eta) \langle \eta \rangle^{q-m} (au)^\wedge(\eta) \in L^2.$$

Write

$$\begin{aligned} \hat{a}(\zeta, \xi) &= \frac{\chi_\sigma(\zeta) f_1(\zeta, \xi) \langle \xi \rangle^m}{\langle \zeta \rangle^q} + \frac{\chi_{\sigma^c}(\zeta) f_2(\zeta, \xi) \langle \xi \rangle^m}{\langle \zeta \rangle^s} \\ \hat{u}(\xi) &= \frac{\chi_\theta(\xi) v_1(\xi)}{\langle \xi \rangle^q} + \frac{\chi_{\theta^c}(\xi) v_2(\xi)}{\langle \xi \rangle^l} \end{aligned}$$

where $f_i \in L^2(d\zeta)$ uniformly in ξ and $v_i \in L^2$. Then

$$\chi_{\theta'}(\eta) \langle \eta \rangle^{q-m} \hat{u} = \sum_{i,j=1,2} \int K_{ij}(\xi^1, \eta) f_i(\eta^1 - \xi^1, \xi^1, \eta^2) v_i(\xi^1, \eta^2) d\xi^1$$

where

$$\begin{aligned}
K_{11}(\xi^1, \eta) &= \frac{\chi_{\theta'}(\eta)\chi_{\sigma}(\eta^1 - \xi^1)\chi_{\theta}(\xi^1, \eta^2)\langle \xi^1, \eta^2 \rangle^m \langle \eta \rangle^{q-m}}{\langle \eta^1 - \xi^1 \rangle^q \langle \xi^1, \eta^2 \rangle^q} \\
K_{12}(\xi^1, \eta) &= \frac{\chi_{\theta'}(\eta)\chi_{\sigma}(\eta^1 - \xi^1)\chi_{\theta^C}(\xi^1, \eta^2)\langle \xi^1, \eta^2 \rangle^m \langle \eta \rangle^{q-m}}{\langle \eta^1 - \xi^1 \rangle^q \langle \xi^1, \eta^2 \rangle^l} \\
K_{21}(\xi^1, \eta) &= \frac{\chi_{\theta'}(\eta)\chi_{\sigma^C}(\eta^1 - \xi^1)\chi_{\theta}(\xi^1, \eta^2)\langle \xi^1, \eta^2 \rangle^m \langle \eta \rangle^{q-m}}{\langle \eta^1 - \xi^1 \rangle^s \langle \xi^1, \eta^2 \rangle^q} \\
K_{22}(\xi^1, \eta) &= \frac{\chi_{\theta'}(\eta)\chi_{\sigma^C}(\eta^1 - \xi^1)\chi_{\theta^C}(\xi^1, \eta^2)\langle \xi^1, \eta^2 \rangle^m \langle \eta \rangle^{q-m}}{\langle \eta^1 - \xi^1 \rangle^s \langle \xi^1, \eta^2 \rangle^l}.
\end{aligned}$$

Once again, by Proposition 2.1, it suffices to show that

$$\sup_{\eta} \int |K_{ij}(\xi^1, \eta)|^2 d\xi^1 < \infty.$$

We now estimate these kernels separately:

On support K_{11} , $\eta \in \theta'$, $\eta^1 - \xi^1 \in \sigma$, $(\xi^1, \eta^2) \in \theta$. Then

$$|K_{11}| \leq \frac{\langle \eta \rangle^{q-m}}{\langle \eta^1 - \xi^1 \rangle^q \langle \xi^1, \eta^2 \rangle^{q-m}} \in L^2(d\xi^1).$$

On support K_{12} , $\eta \in \theta'$, $\eta^1 - \xi^1 \in \sigma$, and $(\xi^1, \eta^2) \in \theta^C$. Then $\langle \eta^1 - \xi^1 \rangle \geq C\langle \eta \rangle$, hence

$$|K_{12}| \leq \frac{1}{\langle \eta^1 - \xi^1 \rangle^m \langle \xi^1, \eta^2 \rangle^{l-m}} \in L^2(d\xi^1)$$

since $l > n_0/2$.

On support K_{21} , $\eta \in \theta'$, $\eta^1 - \xi^1 \in \sigma^C$, and $(\xi^1, \eta^2) \in \theta$, therefore $\langle \xi^1, \eta^2 \rangle \geq C\langle \eta \rangle$. It follows that

$$|K_{21}| \leq \frac{C}{\langle \eta^1 - \xi^1 \rangle^s} \in L^2(d\xi^1).$$

On support K_{22} , $\eta \in \theta'$, $\eta^1 - \xi^1 \in \sigma^C$, and $(\xi^1, \eta^2) \in \theta^C$, therefore $\langle \xi^1, \eta^2 \rangle \geq C\langle \eta \rangle$. It follows that

$$|K_{22}| \leq \frac{C}{\langle \eta^1 - \xi^1 \rangle^{s+l-q}} \in L^2(d\xi^1)$$

because of the assumption $s + l - q > n_0/2$. □

2.3 Calculus of nonsmooth symbol class

With the presence of nonsmooth coefficients, one always needs to study various algebraic properties, such as products of two nonsmooth functions, the action of a nonsmooth symbol ($\psi.d.o.$) to a function, and commutators between smooth (or nonsmooth) $\psi.d.o.s$. To serve the proof of Theorem 2.1, we shall study compositions of an operator that has nonsmooth symbol with a smooth operator from left. The proof of Theorem 2.1 also requires a Gårding's inequality.

Definition 2.4 For $\Gamma \subset T^*(\mathbb{R}^n)$, $S_{bd}^{m;s,q,l,r}(\Gamma)$ is a collection of symbols that are in $S^{m;s,q}(\Gamma)$ and define bounded maps from

$$H^l \cap H_{m\ell}^r(\Gamma) \rightarrow H^{l-m} \cap H_{m\ell}^{r-m}(\Gamma) .$$

Lemma 2.1 Let $n_0/2 < s$, $0 \leq l \leq s$, $1 \leq n_0 \leq n$, $s \leq s_b$, $n/2 < s_b$, and $q < \min\{s + l - n_0/2, s_b + s - n/2\}$. Let $k \geq 0$ be an integer, and for $|\alpha| \leq k$

$$a(x^1, \xi) \in S_{n_0}^{m;s+k,q+k}(\gamma) , \quad \partial_\xi^\alpha b(x, \xi) \in S^{k-|\alpha|;s_b,q}(\Gamma) , \quad \text{and } \gamma = \Pi_1 \Gamma .$$

Then $b \circ a(x, \xi)$, the symbol of operator $b(x, D)a(x^1, D)$, satisfies

$$\{b \circ a(x, \xi) - \sum_{|\alpha| < k} \frac{1}{\alpha!} \partial_\xi^\alpha b D_{x_1}^\alpha a(x^1, \xi)\} \in S_{bd}^{m;s,q,l,q}(\Gamma) .$$

Proof.

$$(b \circ a)(x, D_x)u(x) = \int e^{ix \cdot \xi} b(x, \xi) (au)^\wedge(\xi) d\xi$$

it follows from

$$(au)^\wedge(\xi) = \int \hat{a}(\xi^1 - \zeta^1, \zeta^1, \xi^2) \hat{u}(\zeta^1, \xi^2) d\zeta^1$$

that

$$(b \circ a)(x, D_x)u(x) = \int \int e^{ix \cdot \xi} b(x, \xi) \hat{a}(\xi^1 - \zeta^1, \zeta^1, \xi^2) \hat{u}(\zeta^1, \xi^2) d\zeta^1 d\xi$$

and

$$(b \circ au)^\wedge(\eta) = \int \int \hat{b}(\eta - \xi, \xi) \hat{a}(\xi^1 - \zeta^1, \zeta^1, \xi^2) \hat{u}(\zeta^1, \xi^2) d\zeta^1 d\xi .$$

Substituting ξ^1 and ζ^2 by $\xi^1 + \zeta^1$ and ξ^2 , we have

$$(b \circ au)^\wedge(\eta) = \int \int \hat{b}(\eta^1 - \xi^1 - \zeta^1, \eta^2 - \zeta^2, \xi^1 + \zeta^1, \zeta^2) \hat{a}(\xi^1, \zeta^1, \zeta^2) \hat{u}(\zeta) d\zeta d\xi^1 .$$

On the other hand,

$$(\partial_\xi^\alpha b D_{x_1}^\alpha a)(x, D_x)u(x) = \int e^{ix \cdot \zeta} (\partial_\zeta^\alpha b D_{x_1}^\alpha a)(x, \zeta) \hat{u}(\zeta) d\zeta$$

and

$$\begin{aligned} (\partial_\xi^\alpha b D_{x_1}^\alpha au)^\wedge(\eta) &= \int \int e^{ix \cdot (\zeta - \eta)} (\partial_\zeta^\alpha b D_{x_1}^\alpha a)(x, \zeta) \hat{u}(\zeta) d\zeta dx \\ &= \int \int \partial_\zeta^\alpha \hat{b}(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2, \zeta) (\xi^1)^\alpha \hat{a}(\xi^1, \zeta) \hat{u}(\zeta) d\zeta d\xi^1 . \end{aligned}$$

Now, defining

$$r(x, D) = b \circ a(x, D) - \sum_{|\alpha| < k} \frac{1}{\alpha!} (\partial_\xi^\alpha b D_{x_1}^\alpha a)(x, D)$$

we then get by Taylor's theorem

$$(ru)\hat{(\eta)} =$$

$$\sum_{|\alpha|=k} C_\alpha \int \int \int_0^1 (1-t)^{k-1} (\partial_\zeta^\alpha \hat{b})(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2, \zeta^1 + t\xi^1, \zeta^2) dt \} (\xi^1)^\alpha \hat{a}(\xi^1, \zeta) \hat{u}(\zeta) d\xi^1 d\zeta \quad (2.1)$$

for the case $k = 0$, the term in braces should be replaced by $\hat{b}(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2, \zeta^1 + t\xi^1, \zeta^2)$.

Therefore

$$(ru)\hat{(\eta)} = \int \hat{r}(\eta - \zeta, \zeta) \hat{u}(\zeta) d\zeta ,$$

where

$$\begin{aligned} \hat{r}(\eta, \zeta) = & \int \sum_{|\alpha|=k} C_\alpha \int \int_0^1 (1-t)^{k-1} (\partial_\zeta^\alpha \hat{b})(\eta^1 - \xi^1, \eta^2, \zeta^1 + t\xi^1, \zeta^2) dt \} (D_{x^1}^\alpha a)(\xi^1, \zeta) d\xi^1 \quad k \neq 0 \\ & \int \hat{b}(\eta^1 - \xi^1, \eta^2, \xi^1 + \zeta^1, \zeta^2) \hat{a}(\xi^1, \zeta) d\xi^1 \quad k = 0 . \end{aligned}$$

Let $K' \subset\subset K$ be a small conic neighborhood of ξ_0 , $(x_0, \xi_0) \in \Gamma$, with K sufficiently small that estimates of Definition 2.2 hold for $\partial_\zeta^\alpha b$ on K and a on $\theta = \Pi K$ (Π the projection map from $\mathbb{R}^n$ to $\mathbb{R}^{n_0}$). In order to show

$$\{b \circ a(x, \xi) - \sum_{|\alpha| < k} \frac{1}{\alpha!} \partial_\xi^\alpha b D_{x^1}^\alpha a(x^1, \xi)\} \in S^{m;s,q}(\Gamma)$$

it suffices to prove

$$\langle \eta \rangle^s \frac{\hat{r}(\eta, \zeta)}{\langle \zeta \rangle^m} \in L^2(d\eta) \quad \text{and} \quad \langle \eta \rangle^q \frac{\chi_{K'}(\eta) \hat{r}(\eta, \zeta)}{\langle \zeta \rangle^m} \in L^2(d\eta)$$

This may be proved similarly as in the proof of Theorem 2.2 by treating ζ as a parameter, hence we shall omit it.

We next show that $r(x, D)$ is a bounded operator from $H^l \cap H_{m\ell}^q(\Gamma)$ to $H^{l-m} \cap H_{m\ell}^{q-m}(\Gamma)$.

Let us begin by proving the local part of the conclusion. W.l.o.g., only the case $k \neq 0$ is considered here. For $|\alpha| = k$, let

$$\langle \xi^1 \rangle^\alpha \hat{a}(\xi^1, \zeta) = \frac{h(\xi^1, \zeta) \langle \zeta \rangle^m}{\langle \xi^1 \rangle^s} \quad \text{and} \quad \hat{u}(\zeta) = v(\zeta) / \langle \zeta \rangle^l$$

where $h(\xi^1, \zeta) \in L^2(d\xi^1)$ uniformly in ζ and $v \in L^2$. The integral kernel in the expression (2.1) can be written

$$g(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2, \zeta, \xi^1) / \langle \eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2 \rangle^{s_b}$$

where $g(\eta, \zeta, \xi^1) \in L^2(d\eta)$ uniformly in ζ and ξ^1 .

Thus

$$\langle \eta \rangle^{l-m} (ru)\hat{(\eta)} = \sum_{|\alpha|=k} C_\alpha \int \int K(\xi^1, \eta, \zeta) g(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2, \zeta, \xi^1) h(\xi^1, \zeta) v(\zeta) d\xi^1 d\zeta$$

with

$$K(\xi^1, \eta, \zeta) = \frac{\langle \eta \rangle^{l-m}}{\langle \eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2 \rangle^{s_b} \langle \xi^1 \rangle^s \langle \zeta \rangle^{l-m}}.$$

From Proposition 2.2, we only need to show

$$K \in L^2(d\xi^1 d\zeta) \quad \text{uniformly in } \eta.$$

We separate it into several pieces:

If $|\eta|/3 \leq |\zeta|$, then

$$|K| \leq \frac{C}{\langle \eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2 \rangle^{s_b} \langle \xi^1 \rangle^s}$$

hence $K \in L^2(d\xi^1 d\zeta)$ since $s > n_0/2$ and $s_b > n/2$.

In the region where $|\zeta| < |\eta|/3 \leq |(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2)|$, then

$$\begin{aligned} |K| &\leq \frac{C}{\langle \xi^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2 \rangle^{s_b-l+m} \langle \xi^1 \rangle^s \langle \zeta \rangle^{l-m}} \\ &\leq \frac{C}{\langle \xi^1 \rangle^s \langle \zeta \rangle^{s_b}} \end{aligned}$$

In the region $|\zeta| < |\eta|/3$ and $|(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2)| < |\eta|/3$. Since

$$\eta = \zeta + (\eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2) + (\xi^1, 0)$$

we have

$$|\eta| \leq |\zeta| + |(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2)| + |\xi^1|$$

or $|\eta|/3 \leq |\xi^1|$. Thus

$$\begin{aligned} |K| &\leq \frac{C}{\langle \eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2 \rangle^{s_b} \langle \xi^1 \rangle^{s-l+m} \langle \zeta \rangle^{l-m}} \\ &\leq \frac{C}{\langle \eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2 \rangle^{s_b} \langle \zeta \rangle^s} \end{aligned}$$

i.e. $K \in L^2(d\xi^1 d\zeta)$.

We next deal with the microlocal part. Let $\theta' \subset \subset \theta$, then for $|\alpha| = k$, write

$$\langle \xi^1 \rangle^\alpha \hat{a}(\xi^1, \zeta) = \frac{\chi_{\theta'} h_1(\xi^1, \zeta) \langle \zeta \rangle^m}{\langle \xi' \rangle^q} + \frac{\chi_{\theta^c}(\xi^1) h_2(\xi^1, \zeta) \langle \zeta \rangle^m}{\langle \xi^1 \rangle^s}$$

where $h_i(\xi^1, \zeta) \in L^2(d\xi^1)$ uniformly in ζ . Let

$$\hat{u}(\zeta) = \frac{\chi_{K'}(\zeta) v_1(\zeta)}{\langle \zeta \rangle^q} + \frac{\chi_{K^c}(\zeta) v_2(\zeta)}{\langle \zeta \rangle^l}$$

with v_1 and $v_2 \in L^2$.

The integral kernel of (2.1) can be written as

$$\frac{\chi_{K'}(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \xi^2) g_1(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \xi^2, \zeta, \xi^1)}{\langle \eta^1 - \zeta^1 - \xi^1, \eta^2 - \xi^2 \rangle^r} + \frac{\chi_{K^c}(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \xi^2) g_2(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \xi^2, \zeta, \xi^1)}{\langle \eta^1 - \zeta^1 - \xi^1, \eta^2 - \xi^2 \rangle^{s_b}}$$

where $g_i(\eta, \zeta, \xi^1) \in L^2(d\eta)$ uniformly in ξ^1 and ζ .

Therefore, we have the following close form

$$\chi_{K'}(\eta) \langle \eta \rangle^{q-m} (ru)^\wedge(\eta) = \sum_{|\alpha|=k} C_\alpha \int \int \sum_{i,j,h=1,2} G_{i,j,h}(\xi^1, \eta, \zeta) g_i(\eta^1 - \zeta^1 - \xi^1, \eta^2 - \zeta^2, \zeta, \xi^1) h_j(\xi^1, \zeta) v_h(\zeta) d\xi^1 d\zeta$$

from Proposition 2.2, we only need to show that $G_{i,j,h} \in L^2(d\xi^1 d\zeta)$ uniformly in η . The fact that this is indeed so follows from the hypotheses and an analysis of eight different pieces of the kernel. The arguments are similar to the proof of Theorem 2.2 or the proof of the local part. For simplicity, we shall leave it to the interested reader. $\square$

A simple partition of unity argument and an application of Leibnitz's rule as in [7] (the proof of Corollary 1.6) give the following result.

Corollary 2.1 *Let $n_0/2 < s$, $0 \leq l \leq s$, $1 \leq n_0 \leq n$, $s \leq s_b$, $n/2 < s$, and $q < \min\{s + l - n_0/2, s_b + s - n/2\}$. Let $k \geq 0$ be an integer, and for $|\alpha| \leq k$*

$$a(x^1, \xi) \in S_{n_0}^{m+k; s+k, q+k}(\gamma), \quad \partial_\xi^\alpha b(x, \xi) \in S^{-|\alpha|; s_b, q}(\Gamma)$$

Then $b \circ a(x, \xi)$, the symbol of operator $b(x, D)a(x^1, D)$, satisfying

$$\{b \circ a(x, \xi) - \sum_{|\alpha| < k} \frac{1}{\alpha!} \partial_\xi^\alpha b D_{x^1}^\alpha a(x^1, \xi)\} \in S_{bd}^{m; s, q, l, q}(\Gamma).$$

The proof of Lemma 2.1 may be sharpened in a way similar to the proof of Lemma 1.12 in [7] to yield the following representation.

Lemma 2.2 *Let $n_0/2 < s$, $0 \leq l \leq s$, $1 \leq n_0 \leq n$, s , and $q \leq s + l - n_0/2$. Assume also that $k \geq 0$ is an integer and $\mu \geq 0$,*

$$a(x^1, \xi) \in S_{n_0}^{m; s+k, q+k}(\gamma), \quad \partial_\xi^\alpha b(x, \xi) \in S^{k+\mu}.$$

Then there exist symbols $r(x, \xi), \tilde{r}(x, \xi) \in S_{bd}^{m; s, q, l, q}(\Gamma)$ and $p(x, \xi), \tilde{p}(x, \xi) \in S_{1,0}^\mu$, such that

$$\{b \circ a(x, D) - \sum_{|\alpha| < k} \frac{1}{\alpha!} \partial_\xi^\alpha b D_x^\alpha a(x, D)\} = r(x, D)p(x, D) + \tilde{p}(x, D)\tilde{r}(x, D).$$

Accordingly, one can also prove a consequence of Lemma 2.2.

Corollary 2.2 *The conclusions of Lemma 2.2 hold if the assumptions on a and b are replaced with*

$$a(x^1, \xi) \in S_{n_0}^{m+k; s+k, q+k}(\gamma), \quad \partial_\xi^\alpha b(x, \xi) \in S^\mu.$$

Lemma 2.3 *Let $k \geq 1$, $n_0 < s$, $0 \leq l \leq s$, $q < s + l - n_0/2$, and*

$$a(x^1, \xi) \in S_{n_0}^{k; s+k, q+k}(\gamma)$$

then the adjoint of a is an operator with symbol

$$a^*(x, \xi) \in S_{n_0}^{k; s+k, r+k}(\gamma)$$

and

$$\{a^*(x, \xi) - \bar{a}(x, \xi)\} \in S_{bd}^{k-1; s+k-1, q+k-1, l, q}(\gamma).$$

The proof follows from the calculus of smooth $\psi.d.o.$ and Theorem 2.2 by observing that

$$a = \sum a_l(x^1) P_l(x^1, \xi), \quad P_{k,l} \in S^k$$

implies that

$$a^*(x, D) = \sum P_l^*(x, D) \bar{a}_l(x)$$

with $P_l^*(x, \xi) \in S^k$.

Finally, to complete the proof of Theorem 2.1, we need a Gårding type inequality. Let $\Pi : (x^1, x^2, \xi) \rightarrow (x^1, \xi)$ be a projection map.

Lemma 2.4 *Let $n_0/2 < s$, $0 \leq \epsilon \leq 1$, $x = (x^1, x^2) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n-n_0}$, and*

$$P(x, \xi) = a(x^1, \xi) P_0(x, \xi) + P_1(x, \xi) \geq 0 \tag{2.2}$$

where $a(x^1, \xi) \in S_{n_0}^{\epsilon, s+2\epsilon}(V)$, $P_0(x, \xi) \in S^0(U)$, and $P_1(x, \xi) \in S^\epsilon(U)$ for $V = \Pi U$. Then for all $u \in H_{comp}^0(U)$

$$\operatorname{Re} \langle P(x, D)u, u \rangle \geq -C \|u\|_{H^0(U)}^2.$$

Proof. The proof may be given by following the general outline of the proof of Lemma 3.1 in [7] with some necessary modifications. As in Taylor [23], let $b(D, x, D)$ be the Friedrichs symmetrization of P . Set $R(x, D) = b(D, x, D) - p(x, D)$. If one can show that

$$R(x, D) : H_{comp}^0 \rightarrow H_{loc}^0$$

is a bounded operator, then (2.2) would follow since $\langle b(D, x, D)u, u \rangle \geq 0$.

Let q be a smooth nonnegative even function, supported in $|\xi| \leq 1$ satisfying $\int q^2(\xi) d\xi = 1$, define

$$\begin{aligned} F(\xi, \zeta) &= \frac{1}{\langle \xi \rangle^{n/4}} q\left(\frac{\zeta - \xi}{\langle \xi \rangle^{1/2}}\right) \\ b(\eta, x, \xi) &= \int F(\eta, \zeta) P(x, \zeta) F(\xi, \zeta) d\zeta. \end{aligned}$$

Then

$$(b(D, x, D)u)^\wedge(\eta) = \int \hat{b}(\eta, \eta - \xi, \xi) \hat{u}(\xi) d\xi .$$

Thus

$$(R(x, D)u)^\wedge(\eta) = \int \hat{r}(\eta - \xi, \xi) \hat{u}(\xi) d\xi$$

where $\hat{R}(\eta, \xi) = \hat{b}(\eta + \xi, \eta, \xi) - \hat{P}(\eta, \xi)$.

If one can show that

$$\langle \eta^1 \rangle^s \langle \eta^2 \rangle^r |\hat{R}(\eta, \xi)| \leq C g(\eta, \xi) \quad (2.3)$$

with $g \in L^2(d\eta)$ uniformly in ξ for some $r > (n - n_0)/2$, then similar arguments as in the proof of Proposition 2.1 or Proposition 2.2 yield

$$Ru \in H^0 .$$

Observe that since $\int F^2(\xi, \zeta) d\zeta = 1$, we have

$$\hat{R} = \int F(\eta + \xi, \zeta) \hat{P}(\eta, \zeta) F(\xi, \zeta) d\zeta - \hat{P}(\eta, \xi) \quad (2.4)$$

$$\begin{aligned} &= \int F(\eta + \xi, \zeta) \{ \hat{P}(\eta, \zeta) - \hat{P}(\eta, \xi) \} F(\xi, \zeta) d\zeta \\ &\quad + \int \{ F(\eta + \xi, \zeta) - F(\xi, \zeta) \} \hat{P}(\eta, \xi) F(\xi, \zeta) d\zeta . \end{aligned} \quad (2.5)$$

On support of $F(\xi, \zeta)$, $|\zeta - \xi| \leq \langle \xi \rangle^{1/2}$, hence for large ξ one has $\zeta \approx \xi$. Then (2.4) and the hypotheses on a , P_0 , and P_1 imply that

$$|\hat{R}(\eta, \xi)| \leq \frac{\langle \xi \rangle^\epsilon}{\langle \eta^1 \rangle^{s+2\epsilon} \langle \eta^2 \rangle^r} g_0(\eta, \xi)$$

for any r with $g_0(\eta, \xi) \in L^2(d\eta)$ uniformly in ξ .

The estimate (2.3) can then be established by following the proof of Lemma 3.1 in [7]. $\square$

3 Proof of the main theorem

We now prove the main result of this paper, Theorem 1.1, a trace regularity theorem for the linear acoustic wave equation with nonsmooth coefficients in the principal part.

As we mentioned earlier, since the hypersurface $\{x_n = 0\}$ is a time-like surface, the method of energy estimates cannot be applied directly. Following [2], we alter the wave operator $\square$ by a microlocal cutoff technique so that $\{x_n = 0\}$ will become a space-like surface with respect to the new operator. More precisely, we shall construct a $\psi.d.o.$ equation which is strictly hyperbolic corresponding to the trace $\{x_n = 0\}$. Since the operator in our construction is differential in x_n , the standard method of energy estimates (for example in [13] or [23]) can then be employed to derive the basic estimate. Naturally, there will be a remainder term. We then show that the remainder term can be controlled by the microlocal hypotheses and Theorem 2.1.

Let $\tilde{\Pi}_2 : X \subset T^*(\mathbb{R}^k) \rightarrow Y \subset \mathbb{R}^k \times \mathbb{R}^{k_0}$ serve as a map for $k > k_0$,

$$\tilde{\Pi}_2(X) = \{(x, y, \xi) \in Y : (x, y, \xi, \eta) \in X\}.$$

Proof of Theorem 1.1. Let γ_0, γ_1 be two conic subsets of the set $\{(x, t, \xi', \omega) \in \mathbb{R}^{n+1} \times \mathbb{R}^n; (x, t) \in \Omega_0, (\xi', \omega) \in \mathbb{R}^n, d(x)|\omega| \geq |\xi'|\}$ where Ω_0 is a small compact neighborhood of Ω , and $\gamma_0^{cl} \subset \gamma_1^{int}$. Then, construct an operator $Q \in C^\infty(\mathbb{R}, OPS^0(\mathbb{R}^n))$, $q = q(x, t, \xi', \omega)$, whose symbol satisfies

- $ES(Q) \subset \gamma_1$ and $0 \leq q_0 \leq 1$;
- $q_0 = 1$ on $\gamma_0 \cap \{(x, t, \xi', \omega), |(\xi', \omega)| > 1\}$,

where $q_0(\xi', \omega)$ is the principal symbol of Q . Define another operator E as

$$E = Q\Box_{x',t}^d + (I - Q)\Delta_{x',t}^d,$$

where $\Box_{x',t}^d = d^2\partial_t^2 - \partial_{x'}^2$ and $\Delta_{x',t}^d = d^2\partial_t^2 + \partial_{x'}^2$.

Observe that the principal symbol of E

$$e_0 = q_0(d^2\omega^2 - |\xi'|^2) + (1 - q_0)(d^2\omega^2 + |\xi'|^2) \geq C(\omega^2 + |\xi'|^2),$$

for $|(\omega, \xi')| \geq \delta$, with some positive constants C, δ . Hence, E is an elliptic $\psi.d.o.$ of order two.

Let $\phi = \phi(x, t) \in C_0^\infty(\mathbb{R}^{n+1})$ with $\text{supp } \phi \subseteq \{|x_n| \leq \epsilon_0\}$. We then have a strictly symmetric hyperbolic problem

$$\begin{aligned} (-\partial_{x_n}^2 + E)\phi u &= \Box_d \phi u + (I - Q)(\Delta_{x',t}^d - \Box_{x',t}^d)\phi u \\ &= [\Box_d, \phi]u + \phi f + \phi \nabla \sigma \cdot \nabla u + 2(I - Q)\partial_{x'}^2 \phi u \\ &= d^2[\partial_t^2, \phi]u - [\Delta, \phi]u + \phi f + \phi \nabla \sigma \cdot \nabla u + 2(I - Q)\partial_{x'}^2 \phi u \end{aligned} \quad (3.1)$$

where $\Box_d = d^2\partial_t^2 - \Delta$.

Thus, we obtain a standard wave equation with x_n playing the role of “time”. This together with the fact that ϕ is compactly supported give us a symmetric hyperbolic Cauchy problem with zero Cauchy data. It follows from a hyperbolic energy estimate in Taylor [23] pp73-78,

$$\|(\phi u)|_{x_n=0}\|_l \leq C \|r.h.s. \text{ of (3.1)}\|_{l-1}$$

$$\leq C [\|\phi_1 u\|_l + \|\phi f\|_{l-1} + \|\phi \nabla \sigma \cdot \nabla u\|_{l-1} + \|(I - Q)\partial_{x'}^2 \phi u\|_{l-1, \Omega_1}], \quad (3.2)$$

where $\phi_1 \in C_0^\infty$, and $\phi_1 > 0$ on $\text{supp}(\phi)$, and $\text{supp}(\phi) \subset \subset \Omega_1$. Let $\tilde{\phi} \in C_0^\infty(\Omega_1)$ and $\tilde{\phi} = 1$ on Ω_2 , where $\text{supp}(\phi) \subset \subset \Omega_2 \subset \subset \Omega_1$. Then the above estimate leads to

$$\|(\phi u)|_{x_n=0}\|_l \leq C [\|\phi_1 u\|_l + \|\phi f\|_{l-1} + \|\phi \nabla \sigma \cdot \nabla u\|_{l-1} + \|\tilde{\phi}(I - Q)\partial_{x'}^2 \phi u\|_{l-1, \Omega_1}]$$

where we have used $\|(1 - \tilde{\phi})(I - Q)\partial_{x'}^2 \phi u\|_{l-1, \Omega_1} \leq C \|\phi u\|_r$ for any real r .

From the hypotheses, similar to the Proposition in Beals and Reed [6], one can show that $u \in H_{loc}^l$. Hence a generalized Schauder's lemma (Lemma 2.2 in [2]) yields

$$\phi \nabla \sigma \cdot \nabla u \in H^{l-1}.$$

Therefore, to complete the proof it suffices to show that

$$(I - Q) \partial_x^2 \phi u \in H_{loc}^{l-1}(\Omega_1),$$

which requires the use of Lemma 3.1 stated below. In order to apply Lemma 3.1, we choose $B = I - Q \in C^\infty(\mathbb{R}^1, OPS^0(\mathbb{R}^n))$ of order $m = 0$, $A = \square_d$ of order $m_0 = 2$, and $h = l - 2$ in the statement of Lemma 3.1.

Concerning the assumption (1) of Lemma 3.1, $Ell(A)$ defined as the microlocal elliptic region of $A = \square_d$ is easy to determine by knowing that the operator $\square_d$ is elliptic away from the set $\{d^2 \omega^2 = |\xi|^2\}$. To verify hypothesis (3), one only needs to look at

$$\square_d \partial_x^2 \phi u = [\square_d, \partial_x^2 \phi] u + \partial_x^2 \phi (\nabla \sigma \cdot \nabla u + f).$$

Then $\square_d \partial_x^2 \phi u \in H^{l-3}$ follows by some simple commutator arguments. Therefore, the only assumption that needs to be checked is

$$u \in H_{m\ell}^{l+1}([T^*(\mathbb{R}^{n+1}) \setminus Ell(\square_d)] \cap \tilde{\Pi}_2^{-1} ES(I - Q))$$

and this demands Theorem 2.1.

In the statement of Theorem 2.1 choose

$$(m, n_0, n, l, s, q, \epsilon) = (0, n, n + 1, l - 1, s - 2, l, 1)$$

then the microlocal hypotheses verify all the assumptions of Theorem 2.1. Notice that the main assumption, $s > 3 + n/2$, is required by the corresponding hypothesis (" $q < l + s - n_0/2$ ") in the statement of Theorem 2.1. Let γ_0 and γ_1 approach the set $\{(x, t, \xi, \omega) : (x, t) \in \Omega_0, d|\omega| \geq |\xi'|\}$. The set

$$[T^*(\mathbb{R}^{n+1}) \setminus Ell(\square)] \cap \tilde{\Pi}_2^{-1} ES(I - Q)$$

is contained in a small (conic) neighborhood of the Hamiltonian flow out of γ . Hence Theorem 2.1 and the microlocal initial hypotheses yield that

$$u \in H_{m\ell}^{l+1}([T^*(\mathbb{R}^{n+1}) \setminus Ell(\square)] \cap \tilde{\Pi}_2^{-1} ES(I - Q)).$$

We have completed the proof of Theorem 1.1. □

For the sake of completeness, we state a lemma given in [1].

Consider a smooth family of $\psi.d.o.$ $P \in C^\infty(\mathbb{R}^{k-k_0}, OPS^m(\mathbb{R}^{k_0}))$, i.e., for each $y \in \mathbb{R}^{k-k_0}$ with $k_0 < k$, $P(x, y, D_x) \in OPS^m(\mathbb{R}^{k_0})$. It is known, see [1] or [22], that P is not necessarily a $\psi.d.o.$ in $\mathbb{R}^k$. However, as shown in [1], a smooth family of $\psi.d.o.$ behaves like a $\psi.d.o.$.

Recall the map, for $k > k_0$,

$$\tilde{\Pi}_2(X) = \{(x, y, \xi) \in Y \subset \mathbb{R}^k \times \mathbb{R}^{k_0} : (x, y, \xi, \eta) \in X \subset T^*(\mathbb{R}^k)\}.$$

The normal bundle of a foliation $\mathbb{R}^k = \mathbb{R}^{k-k_0} \times \mathbb{R}^{k_0}$ is the set

$$\mathcal{N} = \{(x, y, \xi, \eta) \in \mathbb{R}^{k_0} \times \mathbb{R}^{k-k_0} \times \mathbb{R}^{k_0} \times \mathbb{R}^{k-k_0}, \xi = 0\}.$$

Lemma 3.1 (Fattening Lemma) *Let $B(x, y, D_x) \in C^\infty(\mathbb{R}^{k-k_0}, OPS^m(\mathbb{R}^{k_0}))$ and $A(x, y, D_x, D_y) \in OPS^{m_0}(\mathbb{R}^k)$, where $1 \leq k_0 \leq k$. Let*

$$\mathcal{N} = \{(x, \xi) \in \mathbb{R}^k \times \mathbb{R}^k, (\xi_1, \dots, \xi_{k_0}) = 0\}$$

be the normal bundle of $\mathbb{R}^{k_0} \times \mathbb{R}^{k-k_0}$. Also, assume that

- (1) *A is microlocal elliptic on a conic set $Ell(A)$, with $\mathcal{N} \subset\subset Ell(A)$;*
- (2) *$u \in H^h \cap H_{m\ell}^{h+1}([T^*(\mathbb{R}^k) \setminus Ell(A)] \cap \tilde{\Pi}_2^{-1}ES(B(\cdot, y, \cdot)))$;*
- (3) *$A\phi u \in H_{\text{loc}}^{h-m_0+1}(\mathbb{R}^k)$, where $\phi(x) \in C_0^\infty(\mathbb{R}^k)$.*

Then

$$B\phi u \in H_{\text{loc}}^{h-m+1}(\mathbb{R}^k),$$

in addition, if B is either a convolutional operator or its symbol has compact support in spatial variables,

$$B\phi u \in H^{h-m+1}(\mathbb{R}^k).$$

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References

- [1] G. Bao and W. Symes, A trace theorem for solutions of linear partial differential equations, *Math. Meth. in the Appl. Sci.* 14 (1991), 553-562.
- [2] G. Bao and W. Symes, Trace regularity for a second order hyperbolic equation with nonsmooth coefficients, *J. Math. Anal. & Appl.*, to appear.
- [3] M. Beals, Self-spreading and strength of singularities for solutions to semilinear wave equation, *Ann. of Math.* 118 (1983), 187-214.
- [4] M. Beals, Propagation of smoothness for nonlinear second order strictly hyperbolic equations, *Proc. of Symp. in Pure Math.* 43 (1985), 21-44.
- [5] M. Beals, *Propagation and Interaction of Singularities in Nonlinear Hyperbolic Problems*, Birkhäuser, Boston, 1989.
- [6] M. Beals and M. Reed, Propagation of singularities for hyperbolic pseudodifferential operators and applications to nonlinear problems, *Comm. Pure Appl. Math.* 35 (1982), 169-184.
- [7] M. Beals and M. Reed, Microlocal regularity theorems for nonsmooth pseudodifferential operators and applications to nonlinear problems, *Trans. Am. Math. Soc.* 285 (1984), 159-184.
- [8] J. -M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non-linéaires, *Ann. Scien. Ecole. Norm. Sup.* 14 (1981), 209-246.
- [9] J. Y. Chemin, Interaction contrôlée dans les équations aux dérivées partielles non linéaires, *C. R. Acad. Sci. Paris Sér. t Math.* 303 (1986), No. 10, 451-453.
- [10] L. Hörmander, On the existence and regularity of solutions of linear pseudo-differential equations, *Enseign. Math.* (2), 17 (1971), 121-133
- [11] L. Hörmander, Linear differential operators, *Actes Congr. Internat. Math.*, Vol.1 (Nice, 1970). Gauthier-Villars, Paris (1971), 121-133.
- [12] J. U. Kim, Trace regularity in the boundary control of a wave equation, preprint, 1992.
- [13] F. John, *Partial Differential Equations*, Springer-Verlag, New York, 1982.
- [14] P. Joly, Some trace theorems in anisotropic Sobolev spaces, *SIAM J. Math. Anal.*, Vol 23, No 3 (1992), 799-819
- [15] I. Lasiecka and R. Triggiani, Trace regularity of the solutions of the wave equation with homogeneous Neumann boundary conditions and compactly supported data, *J. Math. Anal. and Appl.*, 14 (1989), 49-71.

- [16] I. Lasiecka and R. Triggiani, Sharp regularity theory for second order hyperbolic equations of Neumann type, *Annali di Matem. Pure et Applic. (IV)* Vol. CLVII (1990).
- [17] I. Lasiecka and R. Triggiani, Uniform stabilization of the wave equation with Dirichlet-feedback control without geometric conditions, *Appl. Math. & Optimiz.*, to appear
- [18] L. Nirenberg, Lectures on Linear Partial Differential Equations, CBMS Regional Conf. Ser. in Math., No. 17, Amer. Math. Soc., Providence, R.I., 1973.
- [19] J. Rauch, Singularities of solutions to semilinear wave equations, *J. Math. Pures Appl.* 58 (1979), 299-308.
- [20] J. Rauch and M. Reed, Nonlinear microlocal analysis of semilinear hyperbolic systems in one space dimension, *Duke Math. J.*, 49 (1982), 379-475.
- [21] W. Symes, A trace theorem for solutions of the wave equation, and the remote determination of acoustic sources, *Math. Meth. in the Appl. Sci.* 5 (1983), 131-152.
- [22] M. Taylor, Reflection of singularities of solutions to systems of differential equations, *Comm. on Pure and Appl. Math.* 28 (1975), 457-475.
- [23] M. Taylor, *Pseudo-Differential Operators*, Princeton Univ. Press, Princeton, N.J., 1981.
- [24] M. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Birkhäuser, Boston, 1991.

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