**A NUMERICAL STUDY OF PARAMETER SENSITIVITIES
IN LAUDAU-GINZBURG MODELS OF PHASE TRANSITIONS
IN SHAPE MEMORY ALLOYS**

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A Numerical Study of Parameter Sensitivities in Landau-Ginzburg Models of Phase Transitions in Shape Memory Alloys

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ABSTRACT

In this paper we consider a one dimensional mathematical model for the dynamics of phase transitions in Shape Memory Alloys (SMA's). We employ a finite element scheme to investigate the dependence of solutions on the parameters that define the Landau-Devonshire potential.

Key words: Shape Memory Alloy, non-convex potential, hysteresis, conservation laws, finite-dimensional approximations, weak formulation.

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1. Introduction

In this paper we consider the system of partial differential equations

$$\rho u_{tt} - \beta \rho u_{xxt} - (\Psi_\epsilon)_x + (\Psi_{\epsilon_x})_{xx} = f \quad (1.1)$$

$$-\theta (\Psi_\theta)_t - k \theta_{xx} - \alpha k \theta_{xxt} - \beta \rho u_{xt}^2 = g \quad (1.2)$$

for $x \in \Omega \subset \mathbb{R}$ and $0 \leq t \leq T$. System (1.1)-(1.2) arises from the conservation laws of linear momentum and energy, respectively in a connected one-dimensional solid Ω . The variables involved are as follows: $u(x, t)$ = displacement, $\theta(x, t)$ = absolute temperature, ρ = mass density, k = coefficient of thermal conductivity, β = viscosity constant, α = thermal memory coefficient, $f(x, t)$ = distributed loads, $g(x, t)$ = density of heat sources, $\Psi(\epsilon, \epsilon_x, \theta)$ = specific Hemholtz free energy density and $\epsilon = u_x$ = linearized shear strain. For a complete analysis on the derivation of equations (1.1)-(1.2) see [21].

We consider materials for which the free energy Ψ is a nonconvex function of ϵ for certain values of θ and ϵ_x . This type of potential gives rise to a hysteretic behavior in the corresponding stress-strain laws. An interesting class of materials for which this happens are the so called “Shape Memory Alloys” (SMA’s). SMA’s are intermetallic materials (compounds of two ore more elements) that are able to sustain a residual deformation after the application of a sufficiently large stress, but they “remember” their original shape, to which they creep back, without the application of any external force, after they are heated above a certain critical temperature. The solid-solid (austenite $\leftrightarrow$ martensite) phase transitions that occur in these metals can be both stress and temperature-induced. Among the materials exhibiting this phenomena are AgCd, AuCd, CuAlNi, CuAuZn, NiAl, CuSn, NiTi, etc., (see [10] for a complete list). One important material, NiTi (also known as Nitinol), was discovered in 1962 and it exhibits a rather spectacular hysteretic behavior and strongly temperature dependent stress-strain laws (see [10]).

Since the early 1970’s several mathematical models to describe the dynamics of SMA’s have been proposed and studied ([1], [2], [3], [7], [8], [13], [14], [15], [16], [23]). Most of these

models, however, have several limitations (see [21]) and do not constitute practical tools for control design. Taking into account the most important properties of SMA's (e.g. hysteresis, strong temperature dependence and coupling, shape memory effects, etc.) and allowing for time-dependent external inputs, results in rather complicated mathematical systems like (1.1)-(1.2). There is a large amount of work in the area of existence and uniqueness of solutions, continuous dependence, etc., for such systems (see the review paper [21] and also [11], [12], [17], [18], [19], [20], [22]). In particular, the following theorem and its proof can be found in [21].

THEOREM 1.1. *Let $\beta > 0$, $\alpha = 0$, $\Omega = (0, 1)$ and the free energy Ψ given by the Landau-Ginzburg potential*

$$\Psi(\epsilon, \epsilon_x, \theta) = -C_v \theta \ln \left(\frac{\theta}{\theta_2} \right) + C_v \theta + C + \alpha_2 (\theta - \theta_1) \epsilon^2 - \alpha_4 \epsilon^4 + \alpha_6 \epsilon^6 + \gamma \epsilon_x^2 \quad (1.3)$$

where C is a constant, C_v is the specific heat, $\alpha_2, \alpha_4, \alpha_6, \gamma$ are positive constants and θ_1, θ_2 are critical temperatures. Consider the following initial and boundary conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad \text{for } x \in \Omega, \quad (1.4)$$

and for $0 \leq t \leq T$

$$\begin{aligned} u(0, t) = u_{xx}(0, t) = 0 = u(1, t) = u_{xx}(1, t), \\ \theta_x(0, t) = 0 \quad k\theta_x(1, t) = k_1 (\theta_\Gamma(t) - \theta(1, t)), \end{aligned} \quad (1.5)$$

where $\theta_\Gamma(t)$ is the temperature of the surrounding medium and k_1 is a positive constant. If the initial and boundary data is sufficiently restricted (see [21]), then there exists $t_1 > 0$ such that the initial boundary value problem (1.1)-(1.5) has a unique classical solution on the interval $[0, t_1)$.

The case $\alpha > 0$ seems to be physically irrelevant since the second law of thermodynamics is not satisfied in this case (see [21]). For this reason, we shall concentrate on the case $\alpha = 0$.

In order to test and validate these mathematical models, it is necessary to have “good” experimental data and practical computational algorithms for estimating the system parameters. Moreover, a simulation model can be used to guide the collection of experimental data

so that this data “enhances” the sensitivities of the solution with respect to an unknown parameter. In this paper we present a finite element scheme to numerically approximate the solutions of problem (1.1)-(1.2) for particular cases. The initial goal is to determine if an adequate choice of the potential Ψ and all the parameters involved, together with relevant initial and boundary conditions, can produce physically reasonable answers. If “reasonable” answers are produced, then an important second problem is to determine the model parameters that best fit experimental data. This is a particular type of inverse problem known as parameter estimation. The basic idea is to conduct an experiment (dynamic) to obtain data for a particular alloy and then to use a computational/optimization algorithm to estimate various parameters in the system (1.1)-(1.2) that best fit the observed data. The ultimate goal is to “automate” this process.

2. The Model Equations and Finite Dimensional Approximations

For the moment we consider the problem of developing approximations for the solutions of (1.1)-(1.2). With this in mind, let $\Omega = (0, 1)$, $\beta = 0$ and the free energy as in (1.3) with $\gamma = 0$, i.e.,

$$\Psi(\epsilon, \theta) = -C_v \theta \ln \left(\frac{\theta}{\theta_2} \right) + C_v \theta + C + \alpha_2 (\theta - \theta_1) \epsilon^2 - \alpha_4 \epsilon^4 + \alpha_6 \epsilon^6. \quad (2.1)$$

This type of potentials has been widely studied in the literature since the works of F. Falk ([5], [6], [9]). Experimental results have provided evidence that (2.1) captures much of the observed behavior of SMA's. Note that for θ close to θ_1 , $\Psi(\epsilon, \theta)$ is not a convex function of ϵ .

With Ψ as in (2.1), the system (1.1)-(1.2) becomes

$$(\Sigma) \begin{cases} \rho u_{tt} - \frac{\partial}{\partial x} (2\alpha_2 (\theta - \theta_1) u_x - 4\alpha_4 u_x^3 + 6\alpha_6 u_x^5) = f(x, t) \\ C_v \theta_t - 2\alpha_2 \theta u_x u_{xt} - k \theta_{xx} = g(x, t) \end{cases} \quad (2.2)$$

for $x \in \Omega = (0, 1)$ and $0 \leq t \leq T$. Temperature, displacement and velocity are given at time

$t = 0$ by

$$\theta(x, 0) = \theta_0(x), \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \quad (2.3)$$

A symmetric stress and the rate of thermal exchange are the boundary conditions:

$$\Psi_\epsilon(u_x, \theta) = \sigma_\Gamma(t), \quad k \frac{\partial \theta}{\partial \nu} = k_1(\theta_\Gamma(t) - \theta), \quad x = 0, 1 \text{ and } 0 \leq t \leq T, \quad (2.4)$$

where k_1 is a positive constant, $\theta_\Gamma(t)$ is the temperature of the surrounding medium, ν is the outward normal unit vector to $\partial\Omega$ and $\sigma_\Gamma(t)$ is an applied stress.

We note here that it is not known if the initial boundary value problem (2.2)-(2.4) is well-posed since we are considering here the case $\beta = \gamma = 0$ and slightly different boundary conditions than those in Theorem 1.1. However, as we shall see later, numerical experiments seem to suggest not only that the system is in fact well-posed, but also that physically reasonable solutions can be obtained for certain values of the model parameters. The main purpose here is to investigate this system to see if it produces physically reasonable answers and to set the stage for experimental and computational work needed to identify the model parameters.

We shall use a finite element model based on a weak formulation of (2.2)-(2.4) to investigate this system. In particular, consider the approximations of u and θ given by

$$u^N(x, t) = \sum_{i=-1}^{N+1} a_i^N(t) \phi_i^N(x), \quad \theta^N(x, t) = \sum_{i=-1}^{N+1} b_i^N(t) \phi_i^N(x), \quad (2.5)$$

where the ϕ_i^N 's are the standard cubic B-splines in $(0, 1)$, $a_i^N(t)$ and $b_i^N(t)$, $-1 \leq i \leq N+1$, are coefficients to be computed and $N+3$ is the order of the approximation.

The weak formulation of the first equation in (2.2) is given by

$$\begin{aligned} 0 &= \langle \rho u_{tt} - (\Psi_\epsilon)_x - f, v \rangle \\ &= \langle \rho u_{tt} - f, v \rangle + \langle \Psi_\epsilon, v_x \rangle - \Psi_\epsilon v \Big|_{x=0}^{x=1}, \end{aligned}$$

for all $v \in H^1(0, 1)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(0, 1)$. The corresponding finite dimensional approximating equation is given by (we suppress the index N whenever

the context is clear enough)

$$\rho \sum_{i=-1}^{N+1} \frac{d^2}{dt^2} a_i^N(t) \langle \phi_i, \phi_j \rangle = \langle f(\cdot, t), \phi_j \rangle + \sigma_\Gamma(t) \phi_j(x) \Big|_{x=0}^{x=1} - \langle \Psi_\epsilon(u_x^N, \theta^N), \phi_j' \rangle, \quad (2.6)$$

for $j = -1, 0, 1, 2, \dots, N+1$.

Similarly, the weak formulation of the second equation in (2.2) is

$$\begin{aligned} 0 &= \langle C_v \theta_t - 2\alpha_2 \theta u_x u_{xt} - k \theta_{xx} - g, v \rangle \\ &= \langle C_v \theta_t - 2\alpha_2 \theta u_x u_{xt} - g, v \rangle + k \langle \theta_x, v_x \rangle - k_1 \int_\Gamma (\theta_\Gamma - \theta) v dS, \end{aligned}$$

for all $v \in H^1(0, 1)$. Here $\Gamma = \partial\Omega = \{0, 1\}$ and we have used (2.4) in the integration by parts. The corresponding finite dimensional approximating equation is given by

$$\begin{aligned} C_v \sum_{i=-1}^{N+1} \frac{d}{dt} b_i^N(t) \langle \phi_i, \phi_j \rangle &= \langle g(\cdot, t), \phi_j \rangle - k \sum_{i=-1}^{N+1} b_i^N(t) \langle \phi_i', \phi_j' \rangle \\ &\quad + k_1 (\theta_\Gamma(0, t) - \theta^N(0, t)) \phi_j(0) \\ &\quad + k_1 (\theta_\Gamma(1, t) - \theta^N(1, t)) \phi_j(1) + 2\alpha_2 \langle \theta^N u_x^N u_{xt}^N, \phi_j \rangle. \end{aligned} \quad (2.7)$$

Since $\phi_i(0) = 0$ for $i = 2, 3, \dots, N+1$, $\phi_{-1}(0) = 1$, $\phi_0(0) = 4$ and $\phi_1(0) = 1$, it follows that

$$\theta^N(0, t) = b_{-1}^N(t) + 4b_0^N(t) + b_1^N(t), \quad (2.8)$$

and similarly,

$$\theta^N(1, t) = b_{N-1}^N(t) + 4b_N^N(t) + b_{N+1}^N(t). \quad (2.9)$$

Consequently, equation (2.7) can be written as

$$\begin{aligned} C_v \sum_{i=-1}^{N+1} \frac{d}{dt} b_i^N(t) \langle \phi_i, \phi_j \rangle &= \langle g(\cdot, t), \phi_j \rangle - k \sum_{i=-1}^{N+1} b_i^N(t) \langle \phi_i', \phi_j' \rangle \\ &\quad + k_1 (\theta_\Gamma(0, t) - b_{-1}^N(t) - 4b_0^N(t) - b_1^N(t)) \phi_j(0) \\ &\quad + k_1 (\theta_\Gamma(1, t) - b_{N-1}^N(t) - 4b_N^N(t) - b_{N+1}^N(t)) \phi_j(1) \\ &\quad + 2\alpha_2 \langle \theta^N u_x^N u_{xt}^N, \phi_j \rangle. \end{aligned} \quad (2.10)$$

In order to write equations (2.6) and (2.10) in matrix form, we define

$$A(t) \doteq [a_{-1}^N(t) \ a_0^N(t) \ a_1^N(t) \ \dots \ a_N^N(t) \ a_{N+1}^N(t)]^T, \quad (2.11a)$$

$$B(t) \doteq [b_{-1}^N(t) \ b_0^N(t) \ b_1^N(t) \ \dots \ b_N^N(t) \ b_{N+1}^N(t)]^T, \quad (2.11b)$$

$$M \doteq [\langle \phi_j, \phi_i \rangle]_{-1 \leq j, i \leq N+1} \quad (\text{the mass matrix}), \quad (2.11c)$$

$$S \doteq [\langle \phi'_j, \phi'_i \rangle]_{-1 \leq j, i \leq N+1} \quad (\text{the stiffness matrix}), \quad (2.11d)$$

$$F_1(t) \doteq [\langle f(\cdot, t), \phi_j \rangle]_{-1 \leq j \leq N+1}^T, \quad (2.11e)$$

$$F_2(t) \doteq [\sigma_\Gamma(t) \phi_j(x) \big|_{x=0}^{x=1}]_{-1 \leq j \leq N+1}^T, \quad (2.11f)$$

$$P(A, B) \doteq [-\langle \Psi_\epsilon(u_x^N, \theta^N), \phi'_j \rangle]_{-1 \leq j \leq N+1}^T, \quad (2.11g)$$

$$G_1(t) \doteq [\langle g(\cdot, t), \phi_j \rangle]_{-1 \leq j \leq N+1}^T, \quad (2.11h)$$

$$G_2(t) \doteq [k_1 \theta_\Gamma(0, t) \phi_j(0) + k_1 \theta_\Gamma(1, t) \phi_j(1)]_{-1 \leq j \leq N+1}^T, \quad (2.11i)$$

$$\tilde{S} \doteq k_1 \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{bmatrix}_{(N+3) \times (N+3)}, \quad \text{where} \quad T \doteq \begin{bmatrix} -1 & -4 & -1 \\ -4 & -16 & -4 \\ -1 & -4 & -1 \end{bmatrix}, \quad (2.11j)$$

and

$$Q(A, B) \doteq [2\alpha_2 \langle \theta^N u_x^N u_{xt}^N, \phi_j \rangle]_{-1 \leq j \leq N+1}^T. \quad (2.11k)$$

Here A , B , F_1 , F_2 , G_1 , G_2 , $P(A, B)$ and $Q(A, B)$ are all $(N + 3)$ -dimensional vectors. The matrices M , S and $\tilde{S}$ have dimension $(N + 3) \times (N + 3)$.

Using the notation introduced above, equations (2.6) and (2.10) can be written as the system

$$\begin{cases} \rho M A''(t) = F_1(t) + F_2(t) + P(A, B) \\ C_v M B'(t) = (-kS + \tilde{S})B(t) + G_1(t) + G_2(t) + Q(A, B). \end{cases} \quad (2.12)$$

By defining $C(t) \doteq A'(t)$ and $X(t) \doteq [A(t)^T \ C(t)^T \ B(t)^T]^T$, system (2.12) becomes

$$\begin{cases} A'(t) = C(t) \\ C'(t) = \rho^{-1} M^{-1} (F_1(t) + F_2(t)) + \rho^{-1} M^{-1} P(A(t), B(t)) \\ B'(t) = C_v^{-1} M^{-1} (-kS + \tilde{S})B(t) + C_v^{-1} M^{-1} (G_1(t) + G_2(t)) + C_v^{-1} M^{-1} Q(A, B). \end{cases} \quad (2.13)$$

We now define

$$\tilde{A}(t) \doteq \begin{bmatrix} 0 & I_{N+3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_v^{-1} M^{-1}(-kS + \tilde{S}) \end{bmatrix}_{3(N+3) \times 3(N+3)}, \quad (2.14)$$

where I_{N+3} is the identity matrix of order $N + 3$,

$$F(t) \doteq \begin{bmatrix} 0 \\ \rho^{-1} M^{-1} (F_1(t) + F_2(t)) \\ C_v^{-1} M^{-1} (G_1(t) + G_2(t)) \end{bmatrix}_{3(N+3) \times 1} \quad (2.15)$$

and

$$NL(X(t)) \doteq \begin{bmatrix} 0 \\ \rho^{-1} M^{-1} P(A(t), B(t)) \\ C_v^{-1} M^{-1} Q(A(t), B(t)) \end{bmatrix}_{3(N+3) \times 1}. \quad (2.16)$$

Finally, using (2.14)-(2.16), system (2.13) can be written in matrix form as

$$\frac{d}{dt}X(t) = \tilde{A}X(t) + NL(X(t)) + F(t) \quad (2.17)$$

Initial conditions for (2.17) are obtained by orthogonal projections. In particular,

$$\langle u_0(\cdot) - u^N(\cdot, 0), \phi_j(\cdot) \rangle = 0, \quad (2.18a)$$

$$\langle u_1(\cdot) - u_t^N(\cdot, 0), \phi_j(\cdot) \rangle = 0, \quad (2.18b)$$

$$\langle \theta_0(\cdot) - \theta^N(\cdot, 0), \phi_j(\cdot) \rangle = 0, \quad (2.18c)$$

for $-1 \leq j \leq N + 1$. Equations (2.18a-c) imply that

$$A(0) = M^{-1}U_0, \quad C(0) = M^{-1}U_1, \quad B(0) = M^{-1}\Theta_0, \quad (2.19)$$

where

$$U_0 \doteq [\langle u_0, \phi_{-1} \rangle \quad \langle u_0, \phi_0 \rangle \quad \dots \quad \langle u_0, \phi_{N+1} \rangle]^T, \quad (2.20a)$$

$$U_1 \doteq [\langle u_1, \phi_{-1} \rangle \quad \langle u_1, \phi_0 \rangle \quad \dots \quad \langle u_1, \phi_{N+1} \rangle]^T, \quad (2.20b)$$

$$\Theta_0 \doteq [\langle \theta_0, \phi_{-1} \rangle \quad \langle \theta_0, \phi_0 \rangle \quad \dots \quad \langle \theta_0, \phi_{N+1} \rangle]^T. \quad (2.20c)$$

By defining

$$X_0 \doteq [A(0)^T \quad C(0)^T \quad B(0)^T]^T, \quad (2.21)$$

the initial condition for (2.17) is given by

$$X(0) = X_0 \quad (2.22)$$

The resulting approximating finite dimensional system is given by

$$(\Sigma^N) : \begin{cases} \frac{d}{dt}X(t) = \tilde{A}X(t) + NL(X(t)) + F(t) \\ X(0) = X_0 \end{cases}, \quad (2.23)$$

for $0 \leq t \leq T$, where $\tilde{A}$, $F(t)$, $NL(X)$ and X_0 are as in (2.14), (2.15), (2.16) and (2.21), respectively.

3. Numerical Results

In order to determine the validity of this model and its possible use as an effective tool to predict the behavior of a given SMA under dynamical conditions, one must use physically meaningful values in the numerical approximations. However, some of the parameters in this model are not “physical” and therefore they need to be determined by parameter estimation techniques.

The numerical results presented below are based on data reported in [7] and [8] for the alloy $\text{Au}_{23}\text{Cu}_{30}\text{Zn}_{47}$: $\alpha_2 = 24 \text{ J cm}^{-3} \text{ K}^{-1}$, $\alpha_4 = 1.5 \times 10^5 \text{ J cm}^{-3}$, $\alpha_6 = 7.5 \times 10^6 \text{ J cm}^{-3}$, $\theta_1 = 208 \text{ K}$, $C_v = 2.9 \text{ J cm}^{-3} \text{ K}^{-1}$, $k = 1.9 \text{ W cm}^{-1} \text{ K}^{-1}$, $\rho = 11.1 \text{ g cm}^{-3}$. We employed a sixth order Runge–Kutta–Verner method to integrate (Σ^N) for this data. The first runs of the program were performed in an IBM 3090 with vectorization facilities and improved versions were run in a CRAY X-MP machine. We took $\theta_0(x) = 200 \text{ K}$, $u_0(x) = 0$, $u_1(x) = 0$, $\theta_\Gamma(t) = 200 \text{ K}$, $k_1 = 100$, $f = g = 0$, $T = 40 \times 10^{-3} \text{ sec.}$ and $\sigma_\Gamma(t)$ was defined to be a tensile load which was increased until it reached 250 MPa at time $t = 10^{-3} \text{ sec.}$, after which it was held constant at 250 MPa until $t = T$ (Figure 3.1).

Figure 3.2 shows the evolution of displacement (u), temperature (θ) and deformation (u_x), respectively, obtained with this data. Note that there seems to be a phase transition front moving towards the center of the wire, which reaches at approximately $t = 20 \times 10^{-3}$ sec. The temperature in the meantime rose by more than 40° K near the interfaces. We should note here that this type of numerical experiment provides valuable data and information for the design of laboratory experiments to collect dynamic data for parameter identification purposes. For instance, the temperature evolution depicted in Figure 3.2(B) provides important information about the possible location of sensors to detect stress-induced temperature increments. Figure 3.2(C) shows that the deformation exhibits sharp edges along the phase boundaries due mainly to the absence of viscosity effects.

We observed in Figure 3.2(B) that the temperature increased by more than 40° K due only to the applied stress. Experts in the field seem to agree on the fact that such an increment is too high for real materials. The coefficient that couples strain and temperature in our case is α_2 (see equation (2.1)). In the next experiment, we decreased α_2 by a factor of 10 and then by a factor of 100, keeping all the other parameters unchanged. When $\alpha_2 = 2.4$ (Figure 3.3), the highest temperature increment observed was only of about 4° K (Figure 3.3(B)). This lower temperature resulted now in larger deformations near the boundaries, in agreement with the fact that SMA's are more ductile at lower temperatures. When $\alpha_2 = 0.24$ (Figure 3.4), the highest temperature increment observed was less than 0.5° K (Figure 3.4(B)). Figure 3.5 shows a comparison of the different temperature evolutions obtained with the different values of α_2 . Assuming that all the other estimates in [7] and [8] are accurate, this numerical experiment suggests that the value $\alpha_2 = 24$ for this alloy is probably too high. Consequently, it would be important to conduct an experiment to collect data in this case, for use in a parameter estimation scheme.

In the next numerical experiment we studied the influence of k_1 and k . First, k_1 was increased by a factor of 5 (Figure 3.6) while keeping all the other parameters as those used in Figure 3.2. The influence of the external temperature (200° K) due to the larger value

of the boundary exchange coefficient $k_1 = 500$ results in much cooler boundaries (Figure 3.6(B)) and larger deformations near the endpoints of the wire (Figure 3.6(C)). When k was increased by a factor of 10 ($k = 19.0$, Figure 3.7), the thermal waves were damped much faster due to the larger heat conductivity coefficient (Figure 3.7(B)). Figures 3.8 and 3.9 show a comparison of the effects of changes in k and k_1 on temperature and deformation, respectively. Note that lower temperature regions near interfaces are always associated with larger deformations.

Finally, we studied the effects of changes in the coefficient α_4 . We first decreased α_4 by a factor of 10 ($\alpha_4 = 1.5 \times 10^4$, Figure 3.10). The greatest effects of this change were observed in the evolution of the temperature which is now damped throughout the whole wire, having an effect similar to that corresponding to increasing the heat conductivity. This is somewhat contrary to what we expected since α_4 is not associated with any thermal part of the free energy potential. A similar effect was observed when $\alpha_4 = 1.5 \times 10^3$ (Figure 3.11). Figures 3.12 and 3.13 show a comparison of the effects of this change in the coefficient α_4 on temperature and deformation respectively.

4. Conclusions

We have studied the influence of some of the model parameters on the evolution of displacement, temperature and deformation. The numerical experiments show that the Landau-Devonshire free energy (2.1) used in conjunction with the dynamic model previously described seem to provide interesting qualitative information about the dynamics of phase transitions in SMA's. The next step is to use dynamic experimental data to approximate the model parameters. This process is known as parameter identification and its implementation requires not only the design of adequate laboratory experiments but also the availability of efficient approximation schemes. This step will also determine whether this model can also

provide relevant quantitative information to be used as a tool for control design. We are already devoting efforts into this area of research.

Figure 3.1: Boundary stress input $\sigma_T(t)$

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Figure 3.2: Evolution of displacement (u), temperature (θ) and deformation (u_x) corresponding to the boundary stress input $\sigma_\Gamma(t)$ shown in Figure 3.1 and $\alpha_2 = 24$

$\alpha_2 = 2.4$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x, t) = 0$	$g(x, t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_r(t) = 200$

Figure 3.3: Evolution of displacement (u), temperature (θ) and deformation (u_x) corresponding to the boundary stress input $\sigma_r(t)$ shown in Figure 3.1 and $\alpha_2 = 2.4$

$\alpha_2 = 0.24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x, t) = 0$	$g(x, t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

Figure 3.4: Evolution of displacement (u), temperature (θ) and deformation (u_x) corresponding to the boundary stress input $\sigma_\Gamma(t)$ shown in Figure 3.1 and $\alpha_2 = 0.24$

$k = 1.9$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$N = 10$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$u_0(x) = 0$	$T = 40 \times 10^{-3}$	$f(x, t) = 0$	$g(x, t) = 0$
	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

Figure 3.5: Comparison of the temperature evolutions obtained with different values of α_2 and the boundary stress input $\sigma_\Gamma(t)$ shown in Figure 3.1

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 500$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x, t) = 0$	$g(x, t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

Figure 3.6: Evolution of displacement (u), temperature (θ) and deformation (u_x) corresponding to the boundary stress input $\sigma_\Gamma(t)$ shown in Figure 3.1 and $k_1 = 500$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 19.0$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x, t) = 0$	$g(x, t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

Figure 3.7: Evolution of displacement (u), temperature (θ) and deformation (u_x) corresponding to the boundary stress input $\sigma_\Gamma(t)$ shown in Figure 3.1 and $k = 19.0$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
		$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x, t) = 0$	$g(x, t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

Figure 3.8: Comparison of the temperature evolutions obtained with different values of k and k_1 and the boundary stress input $\sigma_\Gamma(t)$ shown in Figure 3.1

(A)

$$k_1 = 100, k = 1.9$$

(B)

$$k_1 = 100, k = 19.0$$

(C)

$$k_1 = 500, k = 1.9$$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^5$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$N = 10$	$T = 40 \times 10^{-3}$	$\rho = 11.1$	$C_v = 2.9$
$u_0(x) = 0$	$u_1(x) = 0$	$f(x, t) = 0$	$g(x, t) = 0$
		$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

Figure 3.9: Comparison of the deformation evolutions obtained with different values of k and k_1 and the boundary stress input $\sigma_\Gamma(t)$ shown in Figure 3.1

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^4$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x, t) = 0$	$g(x, t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

Figure 3.10: Evolution of displacement (u), temperature (θ) and deformation (u_x) corresponding to the boundary stress input $\sigma_\Gamma(t)$ shown in Figure 3.1 and $\alpha_4 = 1.5 \times 10^4$

$\alpha_2 = 24$	$\alpha_4 = 1.5 \times 10^3$	$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x, t) = 0$	$g(x, t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_T(t) = 200$

Figure 3.11: Evolution of displacement (u), temperature (θ) and deformation (u_x) corresponding to the boundary stress input $\sigma_T(t)$ shown in Figure 3.1 and $\alpha_4 = 1.5 \times 10^3$

(A)

(B)

(C)

$\alpha_2 = 24$		$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x, t) = 0$	$g(x, t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_r(t) = 200$

Figure 3.12: Comparison of the temperature evolutions obtained with different values of α_4 and the boundary stress input $\sigma_r(t)$ shown in Figure 3.1

(A)

$$\alpha_4 = 1.5 \times 10^5$$

(B)

$$\alpha_4 = 1.5 \times 10^4$$

(C)

$$\alpha_4 = 1.5 \times 10^3$$

$\alpha_2 = 24$		$\alpha_6 = 7.5 \times 10^6$	$\theta_1 = 208$
$k = 1.9$	$k_1 = 100$	$\rho = 11.1$	$C_v = 2.9$
$N = 10$	$T = 40 \times 10^{-3}$	$f(x, t) = 0$	$g(x, t) = 0$
$u_0(x) = 0$	$u_1(x) = 0$	$\theta_0(x) = 200$	$\theta_\Gamma(t) = 200$

Figure 3.13: Comparison of the deformation evolutions obtained with different values of α_4 and the boundary stress input $\sigma_\Gamma(t)$ shown in Figure 3.1

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