**BLOW-UP SURFACES FOR
NONLINEAR WAVE EQUATIONS, II**

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Abstract

In this second part, we prove that equation $\square u = e^u$ has solutions blowing up near a point of any analytic, space-like hypersurface in $\mathbf{R}^n$, without any additional condition; if $(\phi(x, t) = 0)$ is the equation of the surface, $u - \ln(2/\phi^2)$ is not necessarily analytic, and generally contains logarithmic terms. We then construct singular solutions of general semi-linear equations which blow-up on a non-characteristic surface, provided that the first term of an expansion of such solutions can be found. We finally list a few other simple nonlinear evolution equations to which our methods apply; in particular, formal solutions of soliton equations given by a number of authors can be shown to be convergent by this procedure.

1. INTRODUCTION.

We proved in part I of this paper that equation

$$\square u = e^u, \tag{1}$$

where $\square$ denotes the d'Alembertian in n space dimensions, has a singular solution of the form

$$u(x, t) = \ln(2/\phi^2) + v(x, t) \tag{2}$$

($x \in \mathbf{R}^n$, $t \in \mathbf{R}$, (x, t) close to (x_0, t_0)) where v has a holomorphic extension to a complex neighborhood of (x_0, t_0) , and $\phi(x_0, t_0) = 0$, $\phi(x, t) = t - \psi(x)$ with ψ analytic, if and only if the blow-up surface Σ defined by $t = \psi(x)$ is space-like and has zero scalar curvature for the metric induced on Σ by its embedding into Minkowski space. The strategy was to show that if we define w by

$$w(x, t) = [u - \ln(2/\phi^2) - u_0(x) - u_1(x)\phi(x, t)]/\phi(x, t)^2, \tag{3}$$

then w solves an equation of *Fuchsian type*; it was then proved that the initial-value problem for this equation, where the restriction of w to Σ is a prescribed holomorphic function, has a local analytic solution if and only if Σ is space-like and the curvature condition on Σ is satisfied; the solution is then unique. This procedure yields infinitely many solutions blowing up on Σ , depending on the choice of one arbitrary function on the surface.

We also recall that one consequence of this result is the *continuation* of the solutions of (1) of the form (2) beyond their blow-up surface. Indeed, if we agree on a continuation of the singular part $\ln(2/\phi^2)$, there is no choice for the

continuation of u , since its “regular part” v is analytic.¹ It seems convenient to assume that the equation holds in a generalized sense, which, however, is *not* the ordinary weak formulation, which does not apply directly.

In this second part, we obtain the following results:

1. We construct singular solutions for (1) near any blow-up surface which is merely assumed to be space-like and analytic. The representation (2) is replaced by

$$u(x, t) = \ln(2/\phi^2) + v(x, t, \phi \ln \phi), \quad (4)$$

where v is analytic in all its $(n + 2)$ arguments (§§2 and 3).

2. We prove that semi-linear systems for which the leading term of the expansion of a singular solution can be found, can be reduced to “generalized Fuchsian equations,” defined in §4, for which an existence theorem is sketched in §5.
3. We prove in §6 that our procedure applies to any equation of the form

$$\square u = P(u),$$

where P is a polynomial, and briefly describe the results for more general non-linearities. We apply the procedure to soliton equations, and obtain the convergence of the formal solutions generated by the method of Weiss, Tabor, and Carnevale (WTC).

¹There are two natural ways of continuing the function $\ln(2/t^2)$ for $t < 0$: either by the same expression, which still makes sense and is real, or by $\ln 2 - 2 \ln |t| \pm 2k\pi i$, k a non-zero integer.

The reader is referred to part I and to the references in §6 for the notation and a discussion of the literature.

2. LOGARITHMIC EXPANSIONS.

Sections 2 and 3 are devoted to the proof of the following result.

Theorem 1 *Let Σ be an analytic, space-like, hypersurface in Minkowski space. Fix a point $(x_0, t_0) \in \Sigma$. Then (1) has infinitely many solutions, depending on the choice of one arbitrary holomorphic function on Σ , defined near (x_0, t_0) and blowing up precisely on Σ . These solutions have the form*

$$u(x, t) = \ln(2/\phi^2) + v(x, t, \phi \ln \phi),$$

where v is analytic in all its $(n + 2)$ arguments, and $\phi = 0$ is an equation for Σ , with $\nabla \phi \neq 0$.

We reduce in this section Eq. (1) to the form (8) (see below), for which we solve the initial-value problem in §3.

Recall from Part I that if we define w by (3) off Σ , and v_0, v_1 by,

$$v_0 = \ln(1 - |D\psi|^2)$$

and

$$v_1 = -\frac{\Delta\psi}{(1 - |D\psi|^2)},$$

then w solves:

$$\begin{aligned}
& T^2 \left\{ (1 - |D\psi|^2)w_{TT} - \Delta'w + 2\psi^i \delta_i'^{''} w_{T; i'} + (\Delta\psi)w_T \right\} \\
& + T \left\{ 4(1 - |D\psi|^2)w_T + 4\psi^i \delta_i'^{''} w_{; i'} + 2w\Delta\psi - \Delta v_1 \right\} \\
& + \left\{ 2(1 - |D\psi|^2)w + v_1\Delta\psi + 2\psi^i \partial_i v_1 - \Delta v_0 \right\} \\
& \quad - \frac{2}{T}\Delta\psi + \frac{2}{T^2}(1 - |D\psi|^2) \\
& = \frac{2}{T^2}e^w \left\{ 1 + T(v_1 + Tw) + \frac{1}{2}T^2(v_1 + Tw)^2 \right. \\
& \quad \left. + \frac{1}{2}T^3(v_1 + Tw)^3 \int_0^1 (1 - \sigma)^2 \exp[\sigma T(v_1 + Tw)] d\sigma \right\}
\end{aligned} \tag{5}$$

where $T = t = \psi(x)$ and $X^i = x^i$ for $1 \leq i \leq n$.

Indeed, the terms in T^{-2} and T^{-1} in Eq. (38) of [3] vanish, and the equation takes the above form, after cancellation of two linear terms in w .

As in the first Part, we use the summation convention on repeated indices *one of which is in the upper position and the other in the lower position*. Thus, there is summation in $a^i b_i$ but not in $a_i b_i$. More generally, all indices are raised and lowered using the Kronecker delta. We let $\psi_i = \partial_i \psi$, $\psi_{ij} = \partial_{ij} \psi \dots$, and $\psi^i = \delta^{ik} \psi_k$, $\psi^{ij} = \delta^{ik} \delta^{jl} \psi_{kl} \dots$. In particular, $\sum_{1 \leq i \leq n} (\psi_i)^2 = \psi^i \psi_i$.

Let us set

$$Y = T \ln T,$$

and

$$w(T, X) = \lambda(X) \ln T + f(X, Y, T),$$

where

$$\lambda(X) = -\frac{2}{3}R(x)(1 - |D\psi|^2)^{-1},$$

R is the scalar curvature of Σ , given in terms of ψ in the Appendix of [3], and f is a new unknown.

We have, since $\partial_T Y = (1 + \ln T)$,

$$Tw(T, X) = \lambda Y + Tf(T, X, Y),$$

$$\begin{aligned} T\partial_T w &= \lambda(X) + Tf_T + T(1 + \ln T)f_Y \\ &= \lambda(X) + (T\partial_T + (T + Y)\partial_Y)f, \end{aligned}$$

and the equation for w becomes

$$\begin{aligned} &(1 - |D\psi|^2)(N(N + 3)f + 3\lambda) \\ &\quad T \left[(\Delta\psi)(\lambda + Nf) + 2\psi^i \delta_i^{i'} \partial_{i'}(Nf) - Y\Delta\lambda - \Delta'(Tf) \right] \\ &\quad + 4\psi^i \delta_i^{i'} \partial_{i'}(\lambda Y + Tf) - T\Delta v_1 \\ &\quad + v_1 \Delta\psi + \psi_i \partial_i v_1 - \Delta v_0 + 2(\Delta\psi)(\lambda Y + Tf) \\ &= (1 - |D\psi|^2) \{ (v_1 + \lambda Y + Tf)^2 \\ &\quad + T(v_1 + \lambda Y + Tf)^2 \int_0^1 (1 - \sigma)^2 \exp(T\sigma(v_1 + \lambda Y + Tf)) d\sigma \}, \end{aligned} \tag{6}$$

where

$$N = (T\partial_T + (T + Y)\partial_Y).$$

Observe that only Y occurs in this equation, not $\ln T$. We also note that if we let $T = 0$, we find, using the calculation of the constant term in Part I, that one must have

$$3\lambda(1 - |D\psi|^2) + 2R(x) = 0. \tag{7}$$

This justifies our choice of λ . With this choice, all terms except the first on the l.h.s. have either T or Y as a factor.

Eq. (6) is readily converted into a first-order system: let

$$z_1 = f;$$

$$z_2 = Nf;$$

$$z_{2+i} = T\partial_{i'}f \quad (1 \leq i' \leq n),$$

where $\partial_{i'} = \partial_{X^i}$. Thus, $z_2 = z_3 = \dots = 0$ for $T = Y = 0$.

We find

$$\begin{aligned} (1 - |D\psi|^2)(Nz_2 + 3z_2) &= -T\{\Delta\psi(\lambda + z_2) + 2\psi^i\delta_i^{i'}\partial_{i'}z_2 \\ &\quad - \delta^{ii'}\partial_{i'}(z_{2+i})\} + O(Y); \\ Nz_{2+i} &= (T\partial_T + (T + Y)\partial_Y)T\partial_{i'}f \\ &= T\partial_{i'}(Nf) + T\partial_{i'}f \\ &= T\partial_{i'}(z_2 + z_1). \end{aligned}$$

This system has the general form

$$Nz + Az = Th_1(X, Y, T, z, D_X z) + Yh_2(X, Y, T, z, D_X z), \quad (8)$$

where A is a constant matrix; in our case, A has eigenvalues 3 and 0, with respective multiplicities 1 and $n + 1$.

REMARK. The preceding considerations also apply to the problem

$$\square u = a_1 e^u + \sum_{j \geq 0} a_{-j}(x) e^{-ju}. \quad (9)$$

where $a_1(x)$ is bounded away from zero. Indeed, $\tilde{u} := u + \ln(a_1)$ solves then a similar equation with coefficients $\tilde{a}_j$ given by

$$\tilde{a}_1 = 1,$$

$$\tilde{a}_0 = a_0 + \square \ln(a_1),$$

and for $j \geq 1$,

$$\tilde{a}_{-j} = a_1^j a_{-j}.$$

Performing the reduction of this paragraph with

$$\lambda = -\frac{2}{3} \left(R - \frac{1}{2} [a_0 + \square \ln(a_1)] \right) (1 - |D\psi|^2)^{-1} \quad (10)$$

leads again to an equation of the form (8). The analysis applies also if terms of the form $\exp \{(j - 1/2)u\}$ with $j \neq 0$ are allowed in the nonlinearity, with minor modifications; this form includes in particular the Mikhailov-Fordy-Gibbons equation. More general fractional exponentials could presumably be handled by the techniques of §4 if desired.

3. SOLUTION OF SYSTEM (8).

We prove in this section that (8) has one solution if $z(0, 0, X)$ is prescribed. The relation between z and f implies that z_{2+i} and z_2 must vanish for $T = Y = 0$; there is therefore only one arbitrary scalar function in the data.

This will complete the proof of Theorem 1.

The argument is very similar in spirit to that of Part I, and we refer to it for references; the details are however somewhat different, as we will see. We therefore have included a complete proof.

Let us therefore consider the problem

$$(N + A)z = f(T, Y, z, Dz)$$

where $D = D_X$ and $f \equiv 0$ for $T = Y = 0$. We may assume, after redefining z , that we seek z such that $z(0) = 0$. Also, one may, by introducing new dependent variables, take f to be linear in Dz . We let

$$F[z] := f(T, Y, z, Dz).$$

Step 1. We first observe that

$$\begin{cases} (N + A) z(T, Y) = k(T, Y); \\ z(0, 0) = 0, \end{cases} \quad (11)$$

where f is analytic and vanishes for $T = Y = 0$ has a unique solution. Indeed, let

$$g(\sigma) = z(T\sigma, (T \ln \sigma + Y)\sigma)$$

for $0 < \sigma < 1$. We find

$$\frac{d}{d\sigma}(\sigma^A g(\sigma)) = \sigma^{A-1} k,$$

and since $g(\sigma)$ must be bounded as σ goes to zero, we have

$$\sigma^A g(\sigma) = \int_0^\sigma \tau^{A-1} k(\tau T, \tau(T \ln \tau + Y)) d\tau,$$

hence the desired solution must be given by

$$z(T, Y) = H[k] := \int_0^1 \sigma^{A-1} k(\sigma T, \sigma(T \ln \sigma + Y)) d\sigma. \quad (12)$$

That the integral converges follows from the fact that the contribution from k to the integral is $0(\sigma^{1-\epsilon})$ for any $\epsilon > 0$ since k vanishes at the origin.

Step 2. We define two norms. Assume f is analytic for $X \in \mathbf{C}^n$ and $d(X, \Omega) < 2s_0$ and $|u| < 2R$, for some positive constants s_0 and R , and an

open neighborhood Ω containing 0. We define, for $u = u(X)$,

$$\|u\|_s := \sup\{|u(X)| : d(x, \Omega) < s\}, \quad (13)$$

and, for $u = u(T, Y, X)$, and a a sufficiently small positive number,

$$|u|_a := \sup_{\substack{\delta := |T| + \theta|Y| < a(s_0 - s) \\ 0 \leq s < s_0}} \left\{ \delta^{-1} \|u\|_s(T, Y)(s_0 - s) \sqrt{1 - \frac{\delta}{a(s_0 - s)}} \right\}, \quad (14)$$

where $0 < \theta < 1$ is fixed. We wrote $\|u\|_s(T, Y)$ for the s -norm of $u(\cdot, T, Y)$.

We also let $\delta(\sigma) = \delta\sigma(1 - \theta \ln \sigma)$. Since

$$|\sigma T| + \theta\sigma|T \ln \sigma + Y| \leq \delta(\sigma)$$

for $|T| + \theta|Y| = \delta$ and $0 < \sigma < 1$, one finds that if $|u|_a < \infty$,

$$\|u\|_s(\sigma T, \sigma(T \ln \sigma + Y)) \leq \frac{\delta|u|_a}{s_0 - s} \left(1 - \frac{\delta(\sigma)}{a(s_0 - s)}\right)^{-1/2}.$$

This is, for us, how the norm $|u|_a$ will be used.

Step 3. We prove that one can estimate the s -norm of Hu in terms of the a -norm of u . From the definitions of our various norms, it follows that

$$\|Hu\|_s(T, Y) \leq \frac{\delta|u|_a}{s_0 - s} \int_0^1 \sigma^A (1 - \theta \ln \sigma) \left\{1 - \frac{\delta\sigma(1 - \theta \ln \sigma)}{s_0 - s}\right\}^{-1/2} d\sigma. \quad (15)$$

We may estimate σ^A by a constant. To go further, let us define

$$\rho = \frac{\sigma\delta(1 - \theta \ln \sigma)}{a(s_0 - s)}$$

so that

$$d\rho = \frac{\delta}{a(s_0 - s)} (1 - \theta \ln \sigma - \theta) d\sigma.$$

Since $-\ln \sigma \geq 0$, $-\theta \ln \sigma \geq -\theta(1 - \theta) \ln \sigma$ and

$$\frac{d\rho}{d\sigma} \geq \frac{\delta(1 - \theta \ln \sigma)}{a(s_0 - s)}(1 - \theta).$$

The integral in (15) now becomes, since ρ , like σ , varies from 0 to 1,

$$C \int_0^1 \frac{1 - \theta \ln \sigma}{\sqrt{1 - \rho}} \frac{d\sigma}{d\rho} d\rho$$

which is estimated using the bound on $\frac{d\rho}{d\sigma}$ and yields

$$\|Hu\|_s(T, Y) \leq \frac{C\delta|u|_a}{s_0 - s} \frac{a(s_0 - s)}{\delta(1 - \theta \ln \sigma)} \frac{d\rho}{\sqrt{1 - \rho}} = C_0 a|u|_a. \quad (16)$$

Step 4. Next, we observe that, since we have taken f to be linear in the derivatives,

$$\|F[u] - F[v]\|_{s'}(T, Y) \leq \frac{C\delta}{s - s'} \|u - v\|_s \quad (17)$$

for $0 < s' < s < s_0$, if $\|u\|_s < R$ and $\|v\|_s < R$.

Step 5. Let us now assume $|u|_a, |v|_a < R/(2C_0 a)$. Let $G[u] = F[Hu]$. We prove that

$$|G[u] - G[v]|_a \leq C_1 a|u - v|_a \quad (18)$$

for some constant C_1 . To this end, let $\sigma_j = j/n$, for $1 \leq j \leq n$, and

$$w_j = \int_0^{\sigma_j} \sigma^{A-1} u(\sigma T, \sigma(T \ln \sigma + Y)) d\sigma - \int_{\sigma_j}^1 \sigma^{A-1} v(\sigma T, \sigma(T \ln \sigma + Y)) d\sigma$$

and observe that

$$G[u] - G[v] = \sum_{j=1}^n F[w_j] - F[w_{j-1}]. \quad (19)$$

If we choose $s_j \in (s, s_0 - \delta(\sigma)/a)$ for every j , we find from Step 3 that

$$\|F[w_j] - F[w_{j-1}]\|_s \leq \frac{C\delta}{s_j - s} \|w_j - w_{j-1}\|_{s_j}.$$

On the other hand,

$$\|w_j - w_{j-1}\|_{s_j} = C \int_{\sigma_{j-1}}^{\sigma_j} |\sigma^{A-1}| \|u - v\|_{s_j}(\sigma T, \sigma(T \ln \sigma + Y)) d\sigma.$$

One checks directly that $\|w_j\|_{s_j} \leq R$ so that all formulae do make sense.

Let us fix s_j :

$$s_j = \min\{s(\sigma) : \sigma_{j-1} \leq \sigma \leq \sigma_j\},$$

where

$$s(\sigma) = \frac{1}{2} \left[s + s_0 - \frac{\sigma \delta}{a} (1 - \theta \ln \sigma) \right].$$

We then find, since $\sum_j s_j \chi_{[\sigma_{j-1}, \sigma_j]} \rightarrow s(\sigma)$ uniformly on $(0, 1)$, if $|T| + \theta|Y| < a(s_0 - s)$,

$$\|F[Hu] - F[Hv]\|_s \leq C\delta \int_0^1 \frac{\|u - v\|_{s(\sigma)}(\sigma T, \sigma(T \ln \sigma + Y))}{s(\sigma) - s} \frac{d\sigma}{\sigma}.$$

Therefore,

$$\|G[u] - G[v]\|_s(T, Y) \leq C\delta \int_0^1 \frac{\delta(\sigma) |u - v|_a}{\sigma(s(\sigma) - s)(s_0 - s(\sigma))} \left(1 - \frac{\delta(\sigma)}{a(s_0 - s(\sigma))}\right)^{-1/2} d\sigma.$$

Since

$$s(\sigma) - s = \frac{s_0 - s}{2} \left(1 - \frac{\delta(\sigma)}{a(s_0 - s)}\right)$$

and

$$s_0 - s(\sigma) = \frac{s_0 - s}{2} \left(1 + \frac{\delta(\sigma)}{a(s_0 - s)}\right)$$

we should let again $\rho = \frac{\delta(\sigma)}{a(s_0 - s)}$. Note also that $s_0 - s(\sigma) \leq s_0 - s$, so that

$$1 - \frac{\delta(\sigma)}{a(s_0 - s(\sigma))} \geq \frac{\delta(\sigma)}{a(s_0 - s)}.$$

Performing this change of variables, we find

$$\|G[u] - G[v]\|_s(T, Y) \leq C\delta \int_0^{\frac{\delta}{a(s_0 - s)}} \left(\frac{2|u - v|_a}{s_0 - s}\right)^2 (1 - \rho^2)^{-1} \frac{\delta(1 - \theta \ln \sigma)}{\sqrt{1 - \rho}} \frac{a(s_0 - s) d\rho}{(1 - \theta\delta(1 - \theta \ln \sigma))}$$

$$= \frac{4C\delta a}{s_0 - s} \frac{|u - v|_a}{(1 - \theta)} \int_0^{\frac{\delta}{a(s_0 - s)}} \frac{d\rho}{(1 - \rho)^{-3/2}} \leq C_1 \frac{\delta a |u - v|_a}{s_0 - s} \left(1 - \frac{\delta}{a(s_0 - s)}\right)^{-1/2}.$$

This is the desired estimate.

This ends the proof of Theorem 1.

4. GENERAL LOGARITHMIC EXPANSIONS.

The above reduction procedure can be applied in principle to equations and systems that are much more general than (1); examples are given in §6. However, in order to treat them, we need a few facts applicable to general classes of equations. Such results are obtained in the present section.

We consider a system of the form

$$tu_t + Au = f(t, x, u, u_x) \quad (20)$$

where A is independent of t , $x \in U \subset \mathbb{C}^n$, and f is analytic in its arguments. To simplify matters, the reader may assume A is also independent of x .

We assume that there is an exponent $\sigma = -p/q$, p and q being integers, $q > 0$, and an analytic vector $v_0(x)$, such that

$$t^\sigma f(t, x, v/t^\sigma, D_x(v/t^\sigma))$$

is holomorphic in v , Dv , x and $t^{1/q}$ near $x = 0$, $t = 0$, and vanishes for $v = v_0(x)$. We write $\sigma(A)$ for the spectrum of A . ($t = 0$) represents the blow-up surface. We therefore expect singular solutions of the form $t^\sigma(v_0(x) + t^{1/q}v_1(x) + \cdots)$; no assumption on the sign of σ is made.

Our results are as follows.

Theorem 2 *Assume that $\sigma(A) \subset \{\operatorname{Re} z \geq -m\}$, and assume $0 \in U$. Then there is an integer l such that (19) possesses a formal solution in powers of $z, z \ln z, \dots, z(\ln z)^l$, where $z = t^{1/q}$, with coefficients analytic in x . Such solutions involve a finite number of arbitrary analytic functions of x near 0.*

REMARK 1. If we require that no logarithms, or only a specified number of them, should occur in the solution, the arbitrary functions and f must satisfy certain conditions. This is precisely how the curvature condition for (1) arises. It may be appropriate to view such conditions as smoothness assumptions on the regular part of the solution.

REMARK 2. As we show on the examples, one can in practice be much more specific about the conditions and arbitrary functions involved, although this is often at the expense of very lengthy calculations. This theorem combined with the arguments of §6.1, enables one to check the existence of a formal solution without constructing it. The convergence of such formal solutions is dealt with in §5.

REMARK 3. The basic assumption, on the existence of σ expresses that singularities where the highest derivatives and the worst nonlinear terms balance each other are possible. It is verified on a remarkably large number of equations and systems of practical interest.

REMARK 4. Any system of the form

$$u_t = f(t, x, u, u_x)$$

is of the form considered here as can be seen by multiplying the equation by t . Thus, regular as well as singular solutions are encompassed by a single procedure. Blow-up on a surface ($t = \psi(x)$) can be reduced to the present situation by a change of coordinates.

Proof: Let

$$z = t^{1/q}$$

be a uniformizing parameter, and

$$v(z) = u(z^p)z^p.$$

It follows that

$$t\partial_t = \frac{z}{q}\partial_z;$$

the equation for u becomes

$$\frac{z}{q}v_z + (A - p/q)v = z^p f(z^q, x, vz^{-p}, v_x z^{-p}). \quad (21)$$

We write v_x for $D_x v$.

The right-hand side is holomorphic by assumption, and therefore has the form

$$f_0(x, v, v_x) + z f_1(z, x, v, v_x),$$

with f_0, f_1 analytic, and there is, also by assumption, a vector v_0 such that

$$(A - p/q)v_0 = f_0(x, v_0, v_{0x}).$$

Preliminary reduction. Let us introduce w such that

$$v = v_0 + zw,$$

so that w must solve

$$z \left\{ \frac{z}{q} w_z + \left(A - \frac{p-1}{q} \right) w \right\} = z f_1(z, x, u_0 + zw, u_{0x} + zw_x).$$

Since f_0 and f_1 are independent of w and w_x when $z = 0$, we have

$$f_0 + z f_1 = f_{10}(x) + z f_{11}(z, x, w, w_x).$$

Therefore,

$$\frac{z}{q} w_z + \left(A - \frac{p-1}{q} \right) w = f_{10}(x) + z f_{11}(z, x, w, w_x). \quad (22)$$

If $f_{10} \in \text{Ran}(A - \frac{p-1}{q})$,² (21) has the same form as problem (20) in the sense that the r.h.s. vanishes for $z = 0$, and we may repeat the above procedure. We therefore consider the case when

$$f_{10} \notin \text{Ran}(A - \frac{p-1}{q}) \quad (23)$$

for $x = 0$, and restrict our attention to a neighborhood of $x = 0$.

We prove next that one can recover a formal solution at the expense of allowing logarithmic terms in the expansions.

Logarithmic terms. We now introduce new variables $z_0 = z, z_1, \dots, z_l$ and consider the equation

$$Nw + Bw = (f_0 + z f_1)(x, z, u_0 + zw, (u_0 + zw)_x), \quad (24)$$

where $B = qA - (p-1)$ and

$$N := N_l := \sum_{k=0}^l (z_k + k z_{k-1}) \partial_k \quad (25)$$

²Ran denotes the range.

with $\partial_k = \partial_{z_k}$.

If $w(z, z_1, \dots, z_l, x)$ solves this equation, then

$$w(z, z \ln z, \dots, z(\ln z)^l, x)$$

solves (21). It therefore suffices for us to solve (23) in order to generate a solution of the desired form. We will refer to such an equation as a “generalized Fuchsian equation,” since it reduces to a Fuchsian equation for $l = 0$.

To prove the existence of the desired formal solution, we prove that if l is chosen large enough at the outset, one can find functions $\alpha_k(x)$ such that the equation for w follows if

$$w = \sum_{k \leq l} \alpha_k(x) \frac{z_k}{z_0} + \sum_{0 \leq j \leq l} z_j \lambda_j(x; Z),$$

where $Z = (z_0, \dots, z_l)$, and $(\lambda_0, \dots, \lambda_l)$ solves another generalized Fuchsian system where B has been replaced by a matrix C with

$$\sigma(C) = \sigma(B) + 1.$$

After a finite number of such steps, B will have been made invertible, and no more arbitrary functions, or logarithms, will be required. The result will be a formal solution of the form

$$w = \sum_{0 \leq k_0 \leq l} [\alpha_{k_0}(x) z_{k_0} / z_0 + \sum_{0 \leq j_0 \leq l} z_{j_0} [\sum_{0 \leq k_1 \leq l} \beta_{k_1}(x) z_{k_1} / z_0 + \dots]].$$

Since the solution we seek is $v(z_0, \dots, z_l, x) = u_0 + z_0 w$, we see that the denominators in w are cancelled and we obtain a formal solution of the required form.

Solution of $(N_l + B)w = f(x)$. After a linear change on the w 's, we may assume that A is in Jordan form. We need a Lemma:

Lemma. Let ρ denote the size of the largest Jordan block in B corresponding to the eigenvalue 0. Then for $l \geq \rho$ there is for every f , a solution of $(N_l + B)w = f$ such that $z_0 w$ is holomorphic in $z_0, \dots, z_l$. The solution is unique if B is invertible.

REMARK. Since we are only interested in the case where $z_k = t(\ln t)^k$, we find that if $f = 0$, then the equation reduces to $t f_t + B f = 0$. Assume, to fix ideas, that B is ρ -nilpotent.³ We find that the we can form the most general solution to our problem by adding to the one found below any vector of the form

$$\frac{1}{z_0} \left(\sum_{j=0}^{\rho-1} (-1)^j z_j \frac{B^j}{j!} \right) X$$

where $X(x)$ is a vector independent of Z .

Proof (Lemma): We may assume that B consists of a single $\rho \times \rho$ Jordan block; system to be solved then reads

$$N_l w_j + \mu w_j + w_{j+1} = f_j$$

for $1 \leq j \leq \rho - 1$ with

$$N_l w_\rho + \mu w_\rho = f_\rho.$$

This is readily solved if $\mu \neq 0$, since a solution independent of Z exists. If $\mu = 0$, we may find a particular solution, namely

$$w_j = \sum_{j=0}^{\rho-1} f_{\rho-j} \frac{(-1)^j}{(j+1)!} \frac{z_{j+1}}{z_0}.$$

This requires the variables $z_0, \dots, z_\rho$ to be available, hence that $l \geq \rho$. The homogeneous equation is treated directly, see the above Remark.

³This is the only case of relevance for the present problem.

This completes the proof of the Lemma.

Let us now consider the original problem.

Construction of the formal solution. Let ρ be the size of largest Jordan block in B for the eigenvalue 0. One can find, thanks to the above Lemma, functions $\alpha_0(x), \dots, \alpha_\rho(x)$ such that

$$(N_\rho + B)\left(\sum_{k \leq \rho} \alpha_k(x) \frac{z_k}{z_0}\right) = f(x, 0, u_0, u_{0x}). \quad (26)$$

We may now replace l by ρ if it is larger, and thereby *introduce new independent variables*.

Let us next introduce *new dependent variables* $\lambda_0, \dots, \lambda_r$ by

$$w = \sum_{k \leq \rho} \alpha_k(x) \frac{z_k}{z_0} + \sum_{0 \leq j \leq l} z_j \lambda_j(x; Z), \quad (27)$$

where $Z = (z_0, \dots, z_l)$. These new functions are *not* uniquely determined by w . The equation for w now gives us

$$(N_r + B)\left(\sum_j z_j \lambda_j\right) = \sum_j z_j f_j(x) + \sum_j z_j g_j(x, Z, z_0 \Lambda, z_0 \Lambda_x),$$

where $\Lambda = (\lambda_0, \dots, \lambda_l)$, and f_j, g_j are analytic in their arguments. Since

$$(N_l + B)\left(\sum_j z_j \lambda_j\right) = \sum_{j,k} z_j \left\{ B \lambda_j + \lambda_j + \sum_k [(z_k + k z_{k-1}) \partial_k \lambda_j + (j+1) \lambda_{j+1}] \right\},$$

it is sufficient to impose on Λ the relation

$$(N_l + \tilde{B} + 1 + J)\Lambda = g(x, Z, z_0 \Lambda, z_0 \Lambda_x), \quad (28)$$

where J is a block upper triangular matrix, and $\tilde{B}$ is a block diagonal matrix with the same eigenvalues as B .

Conclusion. We have by this process replaced the original equation (23) by another one of the same form, namely (27), but where the eigenvalues of B have been increased by one, and where the number of independent variables may have increased. Solutions of (27) with initial data of suitable form correspond to solutions of (23). After a finite number of such steps, B will become invertible and no more arbitrary functions or new variables are required. This produces, as explained above, a formal solution to the original problem containing logarithmic terms. No conditions are required here for the construction of this series, since any obstruction is removed by the introduction of a new variable, that is, by the introduction of a higher power of $\ln z_0$.

The next section proves the existence of a convergent series solution of the above form. The last section is devoted to examples.

5. GENERALIZED FUCHSIAN EQUATIONS.

We consider equations of the form

$$Nu + Au = f(x, Z, u, u_x) \quad (29)$$

with N, A as before. Let us write

$$N = \sum_j m_{ij} z_j \partial_i$$

and let M be the matrix with general term m_{ij} . If the right-hand side is $g(x, Z)$, independent of u or its derivatives, the solution is easily seen to be

$$u(x, Z) = Hg(x, Z) := \int_0^1 \sigma^{A-1} g(\sigma^M Z) d\sigma. \quad (30)$$

It follows that H is bounded using a variant of the a -norm defined by (14): We may mimic the arguments of §2 provided that we have a family of neighborhoods of the origin in Z space which are invariant under the characteristic flow of N . This is achieved as follows.

We observe that if M is a single $(\rho + 1) \times (\rho + 1)$ Jordan block, for the eigenvalue α , then the region

$$|z_0| + \theta|z_1| + \cdots + \theta^\rho|z_\rho| \leq \delta$$

is invariant provided that

$$0 < \theta < \min(1, \alpha/\rho).$$

In fact, if $\theta < (\alpha - \gamma)\rho$, then

$$\sigma^\alpha \sum_{j=1}^{\rho} (-\theta \ln \sigma)^j \leq \sigma^\gamma.$$

The proof of §3 is now rather similar, with N replaced by N_l throughout. The results of §3 correpond to the case when $\rho = 1$.

REMARK. The equations we consider are related to equations with several Fuchsian variables for which some existence results are available (see [1], [4] and more references therein); however the result we need does not seem to follow from the existing literature.

6. EXAMPLES.

This section is devoted to examples.

After proving briefly in §6.1 that any semilinear equation with rational nonlinearity, irrespective of its type, has solutions blowing up near any open portion of any surface such that the *first term* of an appropriate formal solution exists, we explain the procedure in some detail, in §6.2, on the case of

$$\square u = P(u),$$

where P is a polynomial. The assumption on the existence of the first term of a (non-zero) formal solution translates into the condition that the putative blow-up surface be space-like, if $\deg P$ is odd and the leading term of P has a positive coefficient (resp. time-like if this coefficient is negative), or that it be merely non-characteristic if $\deg P$ is even. We also write out the condition for the absence of logarithmic terms in the case $P(u) = u^2$, which has been extensively studied from other perspectives in the literature. Further remarks on exponential nonlinearities are also included.

We then turn in §6.3 to the justification of a number of formal expansions for other semilinear equations which are to be found in the literature, in particular those obtained in connection with the Painlevé test and its generalization, the “WTC method” which we described in [3]; those expansions correspond to solutions where logarithmic terms are absent: logarithmic terms were sometimes suspected, but were generally not studied in any detail since their appearance was interpreted as indicating that the equation under consideration was not integrable. For very special equations, conditions for the disappearance of logarithmic terms have been found. Our results prove in addition that

1. Formal singular solutions with logarithmic terms can always be found.

2. They converge and define solutions which are meromorphic, or exhibit branching, near the singularity surface.

Section 6.4 explains the consequences of these results for the problem of continuation of singular solutions after blow-up.

6.1. Semilinear systems. Any semilinear equation can be cast in the form of a system of Cauchy-Kowalewska type, therefore in Fuchsian form as well, by Rem. 4, §4: it suffices to consider the Cauchy problem with respect to a non-characteristic hypersurface. Moreover, if we insert $u = t^{-p/q}$, we see that the worst singularity coming from the linear terms will match the worst singularity in the nonlinear terms as soon as this is true in the original form of the equation. Therefore, it suffices to check for the possibility of a singular *leading term* to be sure that a convergent series solution can be constructed.

6.2. Nonlinear hyperbolic equations. For simplicity we restrict ourselves to two examples, and briefly comment on more general ones. Our first example is

$$\square u = \frac{2(m+1)}{(m-1)^2} u^m + \sum_{j=-\infty}^{m-1} a_j u^j, \quad (31)$$

where m is an integer greater than or equal to 2. Fix a hypersurface Σ with equation $t = \psi(x)$. Σ is assumed to be space-like if m is odd.

Theorem 3 *Equation (31) has singular solutions which, near the origin, blow up precisely on Σ . The rate of blow-up is $(t - \psi)^{-2/(m-1)}$ and the solution contains one arbitrary function. This solution is a convergent series in x , z and $z \ln z$ in general, where $z = (t - \psi)^{1/(m-1)}$.*

As usual, the logarithmic terms are absent whenever ψ satisfies an appropriate condition. This equation may be non-trivial, even in one space dimension.

This result is a simple consequence of the general ideas developed above; after introducing z instead of t as time variable, one lets $v = uz^2$. A Fuchsian equation for v follows. One then applies the general results of §§4 and 5. It suffices in fact to show the existence of a formal solution, since the convergence will follow.

REMARK 1. If m is even, Σ may be allowed to be time-like. This is because

$$u = z^{-2}(u_0(x) + Z.u_1(x) + \cdots), \text{ with } Z = (z, z \ln z), \quad (32)$$

where u_0^{m-1} is proportional to $(1 - |D\psi|^2)$; only for odd m does this force Σ to be space-like. It is conceivable, however, that blow-up surfaces be generated by smooth data on a *space-like* initial hypersurface cannot be time-like anywhere. Observe also that if $m = 2$, branching does not occur in the solution.

REMARK 2. The “rate of blow-up” in the theorem refers to the fact that as t tends to $\psi(x)$, $(t - \psi(x))^{2/(m-1)}u(x, t)$ tends to the an analytic function $u_0(x)$. The expansion of the previous remark gives of course a much more precise picture of the blow-up mechanism.

REMARK 3. The constant in front of u^m in (31) has been chosen for convenience in explicit calculations only. One can change it by scaling independent variables.

We include below the condition required in order that the logarithmic terms be absent, for the case $m = 2$, $a_j \equiv 0$. If $T = t - \psi$, and $u = (v_0 + Tv_1 + \cdots)/T^2$,

the condition reads

$$3(2\psi^i\partial_i + \Delta\psi - 4v_1)v_5 = \Delta v_4 + (v_3^2 + 2v_2v_4), \quad (33)$$

with

$$\begin{aligned} 10v_0v_1 &= -4\psi^i\partial_iv_0 - 2v_0\Delta\psi \\ 12v_0v_2 &= -\Delta v_0 - v_1\Delta\psi - 2\psi^i\partial_iv_1 - 6v_1^2 \\ 12v_0v_3 &= -\Delta v_1 - 12v_1v_2 \\ 10v_0v_4 &= -\Delta v_2 - v_3\Delta\psi + 2\psi^i\partial_iv_3 - 6(2v_1v_3 - v_2^2) \\ 6v_0v_5 &= -\Delta v_3 + 2v_4\Delta\psi - 4\psi^i\partial_iv_4 - 12(v_1v_4 + v_2v_3). \end{aligned}$$

One checks that any linear ψ satisfies this condition.

If this condition holds, v_6 is arbitrary and $(t - \psi(x))^2u(x, t)$ is analytic across Σ .

Despite its complexity, the case $m = 2$ is actually the simplest among polynomial nonlinearities. If $m > 2$, the condition for the absence of logarithms is found only after $v_0, \dots, v_{2m+1}$ have been computed.

REMARK 4. The above results extend with minor modifications to

$$\Delta u = P(u).$$

Replacing t by it in the hyperbolic results is permissible, since we are throughout dealing with holomorphic solutions; the condition that ψ be real when x is was imposed only in order to make the interpretation of the final results simpler.

6.3. The Painlevé test. We consider evolution equations of the form

$$u_t = f(u, D_x u, \dots, D_x^k u) \quad (34)$$

in one or two space dimensions. Weiss, Tabor and Carnevale (WTC) [6] have shown that for a number of equations integrable by Inverse Scattering, (the first terms of) an expansion of the form

$$\sum_{k \geq k_0} u_k \phi(x, t)^{k+\nu} \quad (35)$$

which formally solves the equation, can be found without any condition on ϕ other than the non-vanishing of ϕ_x .⁴ As pointed out in [3], these authors have focused on terminating series of this form which, in addition to providing closed form solutions, can often be related to the eigenvalue problem of which the evolution equation represents an iso-spectral deformation.

It follows from the results of §§4 and 5 that (i) such series can, as one may expect, be computed to all orders, and (ii) they *converge* in the vicinity of the set where ϕ vanishes. The special form of the equations enables one to avoid logarithmic terms.

We list a few of the simplest examples, without aiming at completeness: The *Korteweg-de Vries* equation

$$u_t + uu_x + \sigma u_{xxx} = 0$$

has $\nu = -2$, and u_4 and u_6 are arbitrary.

The *modified Korteweg-de Vries* equation

$$u_t - 3u^2 u_x + 2\sigma^2 u_{xxx} = 0$$

⁴This means that the singular surface defined by $\phi = 0$ is non-characteristic.

has $\nu = -1$ and u_3 and u_4 are arbitrary.

The *Kadomtsev-Petviashvili* equation

$$(u_t + uu_x + \delta u_{xxx})_x + u_{yy} = 0$$

has $\nu = -2$ and u_4, u_5 and u_6 are now arbitrary.

A few equations that are not known to be integrable by Inverse Scattering have been tested by this “generalized Painlevé test;” it is in particular known (Olver and McLeod [5], Clarkson, McLeod, Olver and Ramani [2]) that up to scaling, the only equations

$$u_{tt} - u_{xx} = f(u)$$

for which all such singular solutions are free of branching or essential singularities *in the whole complex plane* are linear combinations of $e^{\pm u}$ and $e^{\pm 2u}$. Indeed, even the equation for traveling wave solutions has otherwise movable critical points. One can further rule out all the equations that are not known to be integrable by Inverse Scattering since the other equations in this class do not have WTC expansions that are free from logarithms [2].

As we have shown above, all equations with

$$f(u) = e^u + g(e^{-u})e^{-u}$$

possess singular solutions which are meromorphic near their blow-up surface, and no logarithmic terms appear in one space dimension; this does not preclude the existence of more complicated singularities far from the blow-up set under consideration.

6.4. Continuation. Just as in the case of Eq. (1), the present results imply that all solutions constructed can be continued beyond their blow-up set, possibly after the introduction of a uniformizing variable. In general, branching cannot be avoided. It is due either to the presence of a fractional power of $(t - \psi(x))$ in the leading term, or to the appearance of logarithmic terms further down in the expansion. Due to this branching, the continuation is not uniquely determined, and one continuation might be preferable to another in a specific application.

In some special cases, branching may be avoided, and even the above ambiguity disappears. Thus, in the case of

$$\square u = 6u^2,$$

one finds that the solution is, when logarithmic terms are absent, actually meromorphic, and the continuation is uniquely determined. Similarly, for (1), we observe that e^u does not exhibit any branching.

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