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By

Caroline Fabre
Jean-Pierre Puel
and
Enrique Zuazua

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BY

CAROLINE FABRE

Université Paris XII-Val de Marne
U.F.R. Sciences, Laboratoire de Mathématiques
Av. du Général de Gaulle
94010 Creteil Cedex

and

Centre de Mathématiques Appliquées
Ecole Polytechnique
91128 PALAISEAU CEDEX, France

JEAN-PIERRE PUEL

Université de Versailles Saint-Quentin
Département de Mathématiques

and

Centre de Mathématiques Appliquées
Ecole Polytechnique
91128 PALAISEAU CEDEX, France

AND

ENRIQUE ZUAZUA*

Departamento de Matemática Aplicada
Universidad Complutense
28040 Madrid, Spain

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Abstract

We consider the semilinear heat equation $u_t - \Delta u + f(u) = 0$ in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, for $t > 0$ with Dirichlet boundary conditions $u = 0$ on $\partial\Omega \times (0, \infty)$. For $T > 0$ fixed we consider the map $S(T) : C_0(\Omega) \to C_0(\Omega)$ such that $S(T)u^0 = u(x, T)$ where u is the solution of this heat equation with initial data $u(x, 0) = u^0(x)$ and $C_0(\Omega)$ is the space of uniformly continuous functions on Ω that vanish on its boundary. When f is globally Lipschitz and for any $T > 0$ we prove that the range of $S(T)$ is dense in $C_0(\Omega)$. Our method of proof combines backward uniqueness results, a variational approach to the problem of the density of the range of the semigroup for linear heat equations with potentials and a fixed point technique. These methods are similar to those developed by the authors in an earlier paper in the study of the approximate controllability of semilinear heat equations.

key words and phrases: Semilinear heat equation, density of semigroup-map, approximate controllability.

1 Introduction and main results

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with boundary of class C^2 and f a globally Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$.

Consider the semilinear heat equation

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty) \\ u(x, 0) = u^0(x). \end{cases} \tag{1.1}$$

It is well known that, when $1 \leq p < \infty$, and $u^0 \in L^p(\Omega)$ equation (1.1) admits a unique solution

$$u \in C([0, \infty); L^p(\Omega)).$$

The same result holds in $C_0(\Omega)$ (the space of uniformly continuous functions on Ω that vanish on $\partial\Omega$ endowed with the norm of the supremum), i.e. if $u^0 \in C_0(\Omega)$ there exists a unique solution

$$u \in C([0, \infty); C_0(\Omega)).$$

For every $T > 0$ we define the map

$$[S(T)]u^0 = u(x, T) \tag{1.2}$$

where $u = u(x, t)$ is the solution of (1.1).

Our main result of this paper is as follows:

Theorem 1.1

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz. For every $T > 0$ the map

$$S(T) : C_0(\Omega) \to C_0(\Omega)$$

has dense range in $C_0(\Omega)$.

As an immediate Corollary we have the following result:

Corollary 1.1

For every $T > 0$ and $1 \leq p < \infty$ the range of the map $S(T) : L^p(\Omega) \to L^p(\Omega)$ is dense in $L^p(\Omega)$.

Remark 1.1

1) When f is linear, or more generally, in the framework of linear heat equations with a bounded potential $a = a(x,t)$,

$$u_t - \Delta u + a(x,t)u = 0 \tag{1.3}$$

the density of the range of the semigroup-map is a consequence of the following backward uniqueness result.

$$\text{(B.U.)} \qquad \left. \begin{cases} -\psi_t - \Delta\psi + a(x,t)\psi = 0 & \text{in} \quad \Omega \times (0,T) \\ \psi = 0 \quad \text{on } \partial\Omega \times (0,T) & \\ \psi(x,0) = 0 & \end{cases} \right\} \Rightarrow \psi \equiv 0$$

This backward-uniqueness result holds for every solution ψ with final data a bounded measure. Note that due to the smoothing effect of the heat equation the solution is smooth for every $0 \leq t < T$. Since the potential we consider is uniformly bounded the backward uniqueness property (B.U.) we need is classical and well known (see for instance [1], [5], [7] and [8]). In particular it is a direct consequence of Th. 1.8, p. 140 in S. Agmon and L. Nirenberg [1].

Let us briefly show how (B.U.) and a simple duality argument implies the density of the range of $S(T) : L^2(\Omega) \to L^2(\Omega)$ in $L^2(\Omega)$ where, now, $S(T)$ stands for the semigroup-map associated to (1.3). Suppose that $u^1 \in L^2(\Omega)$ is orthogonal to the range of $S(T)$ in $L^2(\Omega)$. We solve the system

$$\begin{cases} -\psi_t - \Delta\psi + a(x,t)\psi = 0 & \text{in} \quad \Omega \times (0,T) \\ \psi = 0 \quad \text{on} \quad \partial\Omega \times (0,T) & \\ \psi(T) = u^1 & \end{cases}$$

which has a unique solution $\psi \in C([0,T]; L^2(\Omega))$. Let u be a solution of

$$\begin{cases} u_t - \Delta u + a(x,t)u = 0 & \text{in} \quad \Omega \times (0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) & \\ u(0) = u^0 \in L^2(\Omega) & \end{cases}$$

for any $u^0 \in L^2(\Omega)$. Multiplying by u in the equation satisfied by ψ we get

$$\int_\Omega u(x,T)u^1 dx = \int_\Omega u^0\psi(x,0)dx, \quad \forall u^0 \in L^2(\Omega).$$

Since u^1 is orthogonal to $u(x,T)$ in $L^2(\Omega)$ we deduce that

$$\int_\Omega u^0\psi(x,0)dx = 0, \quad \forall u^0 \in L^2(\Omega)$$

but then $\psi(x,0) = 0$ in Ω and by (B.U.) $\psi \equiv 0$ and $u^1 = 0$.

2) Our proof of Theorem 1.1 is based on a fixed point technique. However, in order to apply this fixed point method we need some informations on how the range of $S(T)$ depends on the potential a that the duality argument above does not provide. Therefore, we will use a different approach to the problem of the density of the range of the semigroup for linear heat equations with potentials. This approach is variational and is inspired by our recent works ([3], [4]) on the approximate controllability of semilinear heat equations.

3) Our Theorem 1.1 is sharp in the sense that it fails for some nonlinearities that, at infinity, grow in a superlinear way.

Consider for instance

$$\begin{cases} u_t - \Delta u + u^3 = 0 & \text{in} \quad \Omega \times (0,\infty) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) = u^0(x). \end{cases} \tag{1.4}$$

It is easy to check that for every $T > 0$ there exists $C(T) > 0$ such that for every $u^0 \in L^2(\Omega)$ the solution of (1.4) satisfies

$$||u(\cdot,T)||_{L^2(\Omega)} \leq C(T).$$

Thus, the range of $S(T)$ is not dense in $L^2(\Omega)$.

On the other hand, the existence of an universal super-solution

$$w(t) = \frac{1}{\sqrt{2}} t^{-\frac{1}{2}}$$

implies that the range of $S(T)$ is not dense in any $L^p(\Omega)$ or in $C_0(\Omega)$.

In the case of non-linearities that grow superlinearly at infinity one may expect the density of the range of $S(T)$ in some ball of $C_0(\Omega)$ around the origin. However, our methods do not seem to provide such a result. □

The method of proof of Theorem 1.1 allows us to prove some variants of it. Let us mention one of them. Let us introduce the following set of initial data:

$$\mathcal{B} = \{\varphi \in L^\infty(\Omega) : \varphi(x) \in \lambda sgn(\phi(x)) \text{ for some } \lambda \in \mathbb{R} \text{ and some measurable function } \phi = \phi(x)\}. \tag{1.5}$$

By *sgn* we denote the multi-valued sign function:

$$sgn(s) = \begin{cases} 1 \text{ if } s > 0 \\ [-1,1] \text{ if } s = 0 \\ -1 \text{ if } s < 0. \end{cases}$$

We have the following result.

Theorem 1.2

If f is globally Lipschitz, for every $T > 0$, $S(T)\mathcal{B}$ is dense in $C_0(\Omega)$.

Remark 1.2

The initial data we use in the proof of Theorem 1.3 are in fact of the form

$$\varphi \in \lambda sgn(\phi)$$

where $\phi = \psi(x,0)$ with $\psi = \psi(x,t)$ solution of the heat equation

$$\begin{cases} -\psi_t - \Delta\psi + a(x,t)\psi = 0 \quad \text{in} \quad \Omega \times (0,T) \\ \psi(x,T) = \psi_0 \in M(\Omega) \\ \psi = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \end{cases}$$

for some potential $a \in L^\infty(\Omega \times (0,T))$. By $M(\Omega)$ we denote space of bounded measures in Ω (the dual of $C_0(\Omega)$).

In dimension $n = 1$ the results by S. Angenent [2] show that the zero set of $\psi(x,0)$ is of zero Lebesgue measure. Thus, in one space dimension, the density of the range of the semigroup map $S(T)$ holds with initial data of bang-bang form, i.e. $\varphi = \pm\lambda$ a.e. in Ω for some $\lambda \in \mathbb{R}$. □

The rest of this paper is organized as follows. In section 2 we prove Theorem 1.1 for linear equations of the form (1.3) by the variational method mentioned above. For the sake of completeness, first we give a direct proof of Corollary 1.1 in the linear framework. In section 3 we complete the proof of Theorem 1.1. Finally, in section 4 we prove Theorem 1.2.

In the sequel we will denote by Q the cylinder $\Omega \times (0,T)$ and by Σ the lateral boundary $\partial\Omega \times (0,T)$. The norm in $L^p(\Omega)$ will be denoted by $||\cdot||_p$ and the norm in $M(\Omega)$ by $|\cdot|$.

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2 The linear heat equation with a potential

Consider a potential $a = a(x,t) \in L^\infty(Q)$ with $T > 0$ fixed and the heat equation

$$\begin{cases} u_t - \Delta u + au = 0 \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(x,0) = u^0(x). \end{cases} \tag{2.1}$$

It is well known that if $u^0 \in L^p(\Omega)$ with $1 \le p < \infty$ then (2.1) has a unique solution

$$u \in C([0,T]; L^p(\Omega)).$$

On the other hand, if $u^0 \in C_0(\Omega)$ this solution belongs to $C([0,T]; C_0(\Omega))$.

We define the linear map

$$S(T)u^0 = u(x,T)$$

which is continuous from $L^p(\Omega)$ into $L^p(\Omega)$ when $1 \le p < \infty$ and from $C_0(\Omega)$ into $C_0(\Omega)$.

In this section we prove the following result

Proposition 2.1

(i) The range of $S(T): L^p(\Omega) \to L^p(\Omega)$ *is dense in* $L^p(\Omega)$ *when* $1 \leq p < \infty$.

(ii) The range of $S(T): C_0(\Omega) \to C_0(\Omega)$ *is dense in* $C_0(\Omega)$.

As we mentioned in the introduction this result can be easily proved by duality as a consequence of the backward uniqueness of solutions of the heat equation.

Our goal here is to give a proof based on a variational method that, in the sequel, will allow us to extend this density result to the semilinear equation (1.1).

We distinguish the cases $L^p(\Omega)$ with $1 < p < \infty$, $p = 1$ and $C_0(\Omega)$.

2. 1 The case $1 < p < \infty$

Let $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Consider the functional

$$J(\varphi^0) = \frac{1}{q}||\varphi(x,0)||_q^q + \alpha||\varphi^0||_q - \int_\Omega u^1\varphi^0 dx \tag{2.2}$$

where $u^1 \in L^p(\Omega)$ and $\alpha > 0$ are fixed such that $||u^1||_p > \alpha$ and $\varphi = \varphi(x,t)$ is the solution of

$$\begin{cases} -\psi_t - \Delta\varphi + a\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 \quad \text{on} \quad \Sigma \\ \varphi(T) = \varphi^0 \end{cases} \tag{2.3}$$

which belongs to $C([0,T]; L^q(\Omega))$.

We have the following result:

Proposition 2.2

The minimum of J *in* $L^q(\Omega)$ *is achieved at a unique* $\hat{\varphi}^0 \in L^q(\Omega)$. *If* $\hat{\varphi}$ *is the solution of (2.3) corresponding to this final data* ($\hat{\varphi}(T) = \hat{\varphi}^0$) *then the solution of*

$$\begin{cases} u_t - \Delta u + au = 0 & in \quad Q \\ u = 0 \quad on \quad \Sigma \\ u(x,0) = u^0(x) = |\hat{\varphi}(x,0)|^{q-2}\hat{\varphi}(x,0) & in \quad \Omega \end{cases} \tag{2.4}$$

satisfies

$$||u(T) - u^1||_p = \alpha. \tag{2.5}$$

Moreover, for $\alpha > 0$ *fixed,* $\hat{\varphi}^0$ *remains bounded in* $L^q(\Omega)$ *when* a *is uniformly bounded in* $L^\infty(Q)$ *and* u^1 *is relatively compact in* $L^p(\Omega)$.

In addition, if

$$
\left\{\begin{array}{lll} a_n \to a & in & L^\infty(Q) \text{ weak-}* \\ u_n^1 \to u^1 & in & L^p(\Omega) \end{array}\right. \tag{2.6}
$$

then

$$
\hat{\varphi}_n^0 \to \hat{\varphi}^0 \text{ in } L^q(\Omega) \text{ weak} \tag{2.7}
$$

where $\hat{\varphi}^0$ *is the minimizer associated to* a *and* u^1. *The corresponding initial values* $u_n^0 = |\hat{\varphi}_n(x,0)|^{q-2}\hat{\varphi}_n(x,0)$ *satisfy*

$$
u_n^0 \to u^0 = |\hat{\varphi}(x,0)|^{q-2}\hat{\varphi}(x,0) \text{ strongly in } L^\infty(\Omega). \tag{2.8}
$$

Remark 2.1

As a consequence of this Proposition we deduce that the map $S(T) : L^p(\Omega) \to L^p(\Omega)$ has dense range. □

Proof of Proposition 2.2

The functional J is continuous in $L^q(\Omega)$ and strictly convex. Moreover J is coercive. More precisely,

$$
\varliminf_{||\varphi^0||_q \to \infty} \frac{J(\varphi^0)}{||\varphi^0||_q} \geq \alpha. \tag{2.9}
$$

Thus, J achieves its minimum at a unique $\hat{\varphi}^0 \in L^q(\Omega)$.

Let us prove (2.9). Consider a sequence $(\varphi_n^0) \subset L^q(\Omega)$ such that $||\varphi_n^0||_q \to \infty$. Define

$$
I_n = \frac{J(\varphi_n^0)}{||\varphi_n^0||_q} = \frac{1}{q}||\varphi_n^0||_q^{q-1}||\tilde{\varphi}_n(x,0)||_q^q + \alpha - \int_\Omega u^1 \tilde{\varphi}_n^0 dx
$$

where $\tilde{\varphi}_n^0 = \varphi_n^0/||\varphi_n^0||_q$ and $\tilde{\varphi}_n = \varphi_n/||\varphi_n^0||_q$.

If

$$
\varliminf_{n \to \infty} ||\tilde{\varphi}_n(x,0)||_q > 0
$$

then (2.9) holds.

If this does not hold, then for a subsequence (that we still denote by the index n) we have

$$
||\tilde{\varphi}_n(x,0)||_q \to 0. \tag{2.10}
$$

By extracting subsequences (still denoted by the index n) we have

$$
\begin{array}{l} \tilde{\varphi}_n^0 \rightharpoonup \tilde{\varphi}^0 \quad \text{in } L^q(\Omega) \text{ weakly} \\ \tilde{\varphi}_n \rightharpoonup \tilde{\varphi} \text{ weakly in } L^q(Q) \end{array}
$$

where $\tilde{\varphi}$ is the solution of

$$
\left\{\begin{array}{l} -\tilde{\varphi}_t - \Delta\tilde{\varphi} + \tilde{\varphi} = 0 \quad \text{in } Q \\ \tilde{\varphi} = 0 \quad \text{on } \Sigma \\ \tilde{\varphi}(T) = \tilde{\varphi}^0. \end{array}\right.
$$

On the other hand (2.10) implies

$$\tilde{\varphi}(x,0) = 0$$

and thus, by backward uniqueness, $\tilde{\varphi} \equiv 0$ and $\tilde{\varphi}^0 \equiv 0$. This implies that

$$\tilde{\varphi}_n^0 \rightharpoonup 0 \text{ weakly in } L^q(\Omega)$$

and therefore

$$\lim_{n\to\infty} I_n \geq \lim_{n\to\infty} \alpha - \int_\Omega u^1 \tilde{\varphi}_n^0 dx = \alpha.$$

this concludes the proof of (2.9)

Let $\hat{\varphi}_0 \in L^q(\Omega)$ the minimum of J in $L^q(\Omega)$, since $||u^1||_p > \alpha$, $\hat{\varphi}^0 \neq 0$ and then the L^q-norm is differentiable. By calculating the first variation of J at $\hat{\varphi}^0$ we deduce that

$$\int_\Omega |\hat{\varphi}(x,0)|^{q-2}\hat{\varphi}(x,0)\theta(x,0)dx + \alpha||\hat{\varphi}^0||_q^{1-q}\int_\Omega |\hat{\varphi}^0|^{q-2}\hat{\varphi}^0\theta^0 dx - \int_\Omega u^1\theta^0 dx = 0 \tag{2.11}$$

for every solution $\theta = \theta(x,t)$ of

$$\begin{cases} -\theta_t - \Delta\theta + a\theta = 0 & \text{in} \quad Q \\ \theta = 0 \quad \text{on} \quad \Sigma \\ \theta(T) = \theta^0 \end{cases} \tag{2.12}$$

with $\theta^0 \in L^q(\Omega)$. Multiplying by θ in (2.4) and integrating over Q we deduce that

$$\int_\Omega |\hat{\varphi}(x,0)|^{q-2}\hat{\varphi}(x,0)\theta(x,0)dx = \int_\Omega u(T)\theta^0 dx$$

Combining (2.11) and (2.12) we obtain

$$u^1 - u(T) = \alpha||\hat{\varphi}^0||_q^{1-q}|\hat{\varphi}^0|^{q-2}\hat{\varphi}^0$$

and thus (2.5).

Let us prove now the uniform bound for the minimizers when a remains bounded in $L^\infty(Q)$ and u^1 in a compact set of $L^p(\Omega)$. We argue by contradiction. Let $\{a_n\}$ be a sequence of potentials and u_n^1 a sequence of final data such that

$$a_n \rightharpoonup a \text{ in } L^\infty(Q) \text{ weak-}*$$
$$u_n^1 \to u^1 \text{ in } L^p(\Omega) \text{ strongly}$$

and suppose that the corresponding minimizers $\{\hat{\varphi}_n^0\}$ satisfy

$$||\hat{\varphi}_n^0||_q \to \infty. \tag{2.13}$$

Let us denote by J_n the functional corresponding to the potential a_n and final data u_n^1. It is easy to check that, since $||u_n^1||_p > \alpha$,

$$J_n(\hat{\varphi}_n^0) \leq 0, \quad \forall n.$$

Thus, we will obtain a contradiction if we show that

$$\varliminf_{n\to\infty} \frac{J_n(\hat{\varphi}_n^0)}{||\hat{\varphi}_n^0||_q} \geq \alpha. \tag{2.14}$$

We proceed as in the proof of (2.6). Let us normalize the minimizers

$$\psi_n^0 = \hat{\varphi}_n^0 / ||\hat{\varphi}_n^0||_q$$

and let ψ_n be the solution of (2.6) with final data ψ_n^0. We have

$$I_n = \frac{J_n(\hat{\varphi}_n^0)}{||\hat{\varphi}_n^0||_q} = \frac{||\hat{\varphi}_n^0||_q^{q-1}}{q} ||\psi_n(x,0)||_q^q + \alpha - \int_\Omega u_n^1 \psi_n^0 dx.$$

Clearly, in view of (2.13), (2.14) holds if

$$\varliminf_{n\to\infty} ||\psi_n(x,0)||_q > 0.$$

If this does not hold, for a subsequence (still denoted by the index n) we have

$$||\psi_n(x,0)||_q \to 0. \tag{2.15}$$

Since ψ_n^0 and ψ_n are bounded in $L^q(\Omega)$ and $C([0,T]; L^q(\Omega))$ respectively, by extracting subsequences we have

$$\psi_n^0 \rightharpoonup \psi^0 \text{ weakly in } L^q(\Omega)$$
$$\psi_n \rightharpoonup \psi \text{ weakly in } L^q(Q).$$

But since the potentials weakly-$*$ converge in $L^\infty(Q)$ this does not suffice to pass to the limit in equation (2.3) satisfied by ψ_n with potential a_n.

However, ψ_n is also uniformly bounded in $X_q(0,T-\delta) = L^q(0,T-\delta; W_0^{1,q}(\Omega)) \cap W^{1,q}(0,T-\delta; L^q(\Omega))$ (see [6], p. 341) for every $\delta > 0$. The compactness of the embedding $X_q(0,T-\delta) \subset L^1(\Omega \times (0,T-\delta))$ allows us to pass to the limit in the equations and show that ψ satisfies:

$$\begin{cases} -\psi_t - \Delta\psi + a\psi = 0 & \text{in } Q \\ \psi = 0 & \text{on } \Sigma. \end{cases}$$

Let us now prove that ψ takes the final data ψ^0, i.e. $\psi(T) = \psi^0$. For every $\theta \in C^2(\overline{\Omega}) \cap C_0(\Omega)$ we have

$$\left| - \int_\Omega \psi_n^0 \theta dx + \int_\Omega \psi_n(t) \theta dx \right| = \left| \int_\Omega \int_t^T \psi_n \Delta\theta dx dt - \int_\Omega \int_t^T a_n \psi_n \theta dx dt \right|$$
$$\leq C(T-t)$$

for some $C > 0$ that does not depend on n. Passing to the limit as $n \to \infty$ we obtain

$$\left| - \int_\Omega \psi^0 \theta dx + \int_\Omega \psi(t) \theta dx \right| \leq C(T-t)$$

which shows that $\psi(T) = \psi^0$.

On the other hand, (2.15) implies that

$$\psi(x,0)=0$$

and therefore by backward uniqueness $\psi^0=0$. This shows that

$$\psi_n^0 \rightharpoonup 0 \text{ weakly in } L^q(\Omega)$$

and therefore

$$\varliminf_{n\to\infty} I_n \geq \varliminf_{n\to\infty}\left[\alpha-\int_\Omega u_n^1\psi_n^0 dx\right]=\alpha$$

which concludes the proof of (2.14).

Since $\hat\varphi_n^0$ is bounded in $L^q(\Omega)$ we may extract a subsequence (still denoted by the index n) such that

$$\hat\varphi_n^0 \rightharpoonup \psi^0 \quad \text{in} \quad L^q(\Omega) \text{ weakly .}$$

Let us see that $\psi^0=\hat\psi^0$ where $\hat\psi^0$ is the minimizer associated with a and u^1. It is sufficient to check that

$$J(\psi^0)\leq J(\varphi^0), \quad \forall\varphi^0\in L^q(\Omega). \tag{2.16}$$

By the weak lower semicontinuity of the L^q-norm and the strong convergence of $\{u_n^1\}$, we have

$$J(\psi^0)\leq \varliminf_{n\to\infty} J_n(\hat\psi_n^0)\leq \varliminf_{n\to\infty} J_n(\varphi^0), \quad \forall\varphi^0\in L^q(\Omega)$$

and on the other hand

$$\lim_{n\to\infty} J_n(\varphi^0)=J(\varphi^0), \quad \forall\varphi^0\in L^q(\Omega).$$

Thus (2.16) holds. Since the limit ψ^0 has been identified the whole sequence $\{\hat\varphi_n^0\}$ converges. This concludes the proof of (2.16).

Finally (2.8) is immediate from the regularizing effect of the heat equation since we have

$$\hat\varphi_n(\cdot,t)\rightharpoonup\hat\varphi(\cdot,t) \text{ in } L^\infty(\Omega)$$

for every $0\leq t<T$. □

2. 2 The case $p=1$

Consider the functional

$$J: L^\infty(\Omega)\to\mathbb{R}$$

defined as follows

$$J(\varphi^0)=\frac{1}{2}||\varphi(x,0)||_2^2+\alpha||\varphi^0||_\infty-\int_\Omega u^1\varphi^0 dx \tag{2.17}$$

for $u^1\in L^1(\Omega)$ and $\alpha>0$ fixed such that $||u^1||_1>\alpha$ and φ being the solution of (2.3) with initial data $\varphi^0\in L^\infty(\Omega)$. In this case $\varphi\in L^\infty(Q)$ and $\varphi\in C([0,T];C_0(\Omega))$. Thus J is well defined in $L^\infty(\Omega)$.

We have the following result.

Proposition 2.3

The minimum of J in $L^\infty(\Omega)$ is achieved at a unique $\hat{\varphi}^0 \in L^\infty(\Omega)$. If $\hat{\varphi}$ is the solution of (2.3) corresponding to this final data ($\hat{\varphi}(T) = \hat{\varphi}^0$) then, the solution u of

$$(2.18) \qquad \begin{cases} u_t - \Delta u + au = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u^0 = \hat{\varphi}(x,0) \end{cases}$$

satisfies

$$(2.19) \qquad ||u(T) - u^1||_1 \leq \alpha$$

Moreover, $\hat{\varphi}^0$ is uniformly bounded in $L^\infty(\Omega)$ when a is bounded in $L^\infty(Q)$ and u^1 remains in a compact set of $L^1(\Omega)$. In addition, if

$$(2.20) \qquad \begin{cases} a_n \rightharpoonup a \text{ weakly-}* \text{ in } L^\infty(Q) \\ u_n^1 \to u^1 \text{ strongly in } L^1(\Omega) \end{cases}$$

then

$$(2.21) \qquad \hat{\varphi}_n^0 \rightharpoonup \hat{\varphi}^0 \text{ weakly-}* \text{ in } L^\infty(\Omega)$$

where $\hat{\varphi}^0$ is the minimizer associated with the limit potential a and u^1 and the initial data $u_n^0 = \hat{\varphi}_n(x,0)$ satisfy

$$(2.22) \qquad u_n^0 \to u^0 = \hat{\varphi}(x,0) \text{ strongly in } L^\infty(\Omega).$$

Remark 2.2

This result implies that the range of $S(T) : L^1(\Omega) \to L^1(\Omega)$ is dense in $L^1(\Omega)$. □

Proof of Proposition 2.3

J is continuous in $L^\infty(\Omega)$ and convex. Moreover, as a consequence of the backward uniqueness, J is strictly convex. On the other hand, J is lower semicontinuous in the weak-$*$ topology of $L^\infty(\Omega)$.

The existence of a unique minimizer in $L^\infty(\Omega)$ is a direct consequence of the following coercivity property

$$(2.23) \qquad \underline{\lim}_{||\varphi^0||_\infty \to \infty} \frac{J(\varphi^0)}{||\varphi^0||_\infty} \geq \alpha.$$

The proof of (2.23) is analogous to that of (2.6). The minimizer $\hat{\varphi}^0$ is such that

$$\int_\Omega \hat{\varphi}(x,0)\theta(x,0)dx + \alpha[||\hat{\varphi}^0 + \theta^0||_\infty - ||\hat{\varphi}^0||_\infty] - \int_\Omega u^1\theta^0 dx \geq 0$$

for every $\theta^0 \in L^\infty(\Omega)$ where $\theta = \theta(x,t)$ is the solution of (2.12). Multiplying in (2.18) by θ we obtain

$$\int_\Omega \hat{\varphi}(x,0)\theta(x,0)dx = \int_\Omega u(T)\theta^0 dx.$$

Therefore

$$\int_\Omega (u(T) - u^1)\theta^0 dx \le \alpha[||\hat{\varphi}^0 + \theta^0||_\infty - ||\hat{\varphi}^0||_\infty] \le \alpha||\theta^0||_\infty, \quad \forall \theta^0 \in L^\infty(\Omega).$$

Thus

$$||u(T) - u^1||_1 \le \alpha.$$

The proof of the uniform bound on $\hat{\varphi}^0$ when a remains in a bounded set of $L^\infty(Q)$ and u^1 in a compact set of $L^1(\Omega)$ is similar to that of Proposition 2.2. The proof of (2.21) and (2.22) is also very similar. □

Remark 2.3

In this case we do not get $||u(T) - u^1||_1 = \alpha$ since the $L^\infty(\Omega)$-norm is not differentiable. □

2. 3 The $C_0(\Omega)$-case

This case requires to work in $M(\Omega)$, the space of finite measures. Let us recall that $M(\Omega)$ is the dual of $C_0(\Omega)$. The norm in $M(\Omega)$ is defined as follows:

$$|\mu| = \sup_{\substack{\varphi \in C_0(\Omega) \\ ||\varphi||_\infty \le 1}} |\langle \mu, \varphi \rangle|.$$

By $\langle \cdot, \cdot \rangle$ we denote the duality between $M(\Omega)$ and $C_0(\Omega)$.

Let us consider the functional

$$J : M(\Omega) \to \mathbf{R}$$

such that

$$J(\varphi^0) = \frac{1}{2}||\varphi(x,0)||_2^2 + \alpha|\varphi^0| - \langle \varphi^0, u^1 \rangle$$

for $u^1 \in C_0(\Omega)$ and $\alpha > 0$ fixed such that $||u^1||_\infty > \alpha$, where φ is the solution of (2.3). Let us recall that the solution of (2.3) with a measure as final data belongs to

$$\varphi \in L^\infty(0, T; L^1(\Omega)) \cap C([0, T); L^1(\Omega))$$

and takes the final data $\varphi^0 \in M(\Omega)$ in the following sense

$$\int_\Omega \varphi(x,t)\theta(x)dx \to \langle \varphi^0, \theta \rangle, \text{ as } t \nearrow T \text{ for every } \theta \in C_0(\Omega).$$

We have the following result:

Proposition 2.4

The minimum of J on $M(\Omega)$ is achieved at a unique $\hat{\varphi}^0 \in M(\Omega)$. If $\hat{\varphi}$ is the solution of (2.3) with this final data and u solves (2.18) then

$$\|u(T) - u^1\|_\infty \leq \alpha. \tag{2.24}$$

Moreover, $\hat{\varphi}^0$ remains bounded in $M(\Omega)$ when a is bounded in $L^\infty(Q)$ and u^1 remains in a relatively compact set of $C_0(\Omega)$. In addition, if

$$\begin{cases} a_n \rightharpoonup a & \text{weak-}* \text{ in } L^\infty(Q) \\ u_n^1 \to u^1 & \text{in } C_0(\Omega) \end{cases} \tag{2.25}$$

then

$$\hat{\varphi}_n^0 \rightharpoonup \varphi^0 \quad \text{weakly-}* \text{ in } M(\Omega) \tag{2.26}$$

and

$$u_n^0 = \hat{\varphi}_n(x,0) \to \quad u^0 = \hat{\varphi}(x,0) \text{ in } C_0(\Omega). \tag{2.27}$$

Remark 2.4

As a consequence of this result we obtain Proposition 2.1 in the $C_0(\Omega)$-case. □

Proof of Proposition 2.4

The functional J is strictly convex because of the backward-uniqueness property. On the other hand J is continuous in $M(\Omega)$.

The coercivity of J and the fact that it achieves its minimum in $M(\Omega)$ follow from the arguments of Proposition 2.2 and 2.3 but now one has to work with the weak-$*$ topology of $M(\Omega)$.

The uniform bound on the minimizers as well as (2.26) and (2.27) follow also by similar arguments. □

Remark 2.5

Note that, again, due to the non-differentiability of the $M(\Omega)$-norm we do not get $\|u(T) - u^1\|_\infty = \alpha$ but just $\|u(T) - u^1\|_\infty \leq \alpha$. □

3 The semilinear heat equation: Theorem 1.1

In this section we give the proof of Theorem 1.1. The same arguments allow to give a direct proof of Corollary 1.1.

Consider $u^1 \in C_0(\Omega)$ and $\alpha > 0$. Our goal is to find an initial data $u^0 \in C_0(\Omega)$ such that the solution u of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u^0 \end{cases} \tag{3.1}$$

satisfies

$$\|u(T) - u^1\|_\infty \le \alpha. \tag{3.2}$$

We consider first a nonlinearity f which is C^1 and globally Lipschitz and introduce the continuous and bounded function:

$$g(s) = \begin{cases} \dfrac{f(s) - f(0)}{s} & \text{if } s \neq 0 \\ f'(0) \quad \text{if } s = 0. \end{cases} \tag{3.3}$$

The general case where f is globally Lipschitz function (non-necessarily C^1) will be covered later on by a density argument.

For every $v \in L^1(Q)$ we consider the linearized equation

$$\begin{cases} u_t - \Delta u + g(v)u = -f(0) \quad \text{in } Q \\ u = 0 \quad \text{on } \Sigma \\ u(0) = u_0. \end{cases} \tag{3.4}$$

For each v, proceeding as in section 2.3, we will find an initial data $u_0 \in C_0(\Omega)$ (defined in a unique way) such that the solution u of (3.4) satisfies (3.2). Thus we will have a nonlinear map

$$K : L^1(Q) \to L^1(Q) \tag{3.5}$$

such that $Kv = u$ with u solution of (3.4) satisfying (3.2). The problem will be reduced to prove the existence of a fixed point for K.

Let us construct the map K.

We decompose the solution of (3.4) as follows

$$u = U + w \tag{3.6}$$

where U solves

$$\begin{cases} U_t - \Delta U + g(v)U = -f(0) \quad \text{in } Q \\ U = 0 \quad \text{on } \Sigma \\ U(0) = 0 \end{cases} \tag{3.7}$$

and w satisfies

$$\begin{cases} w_t - \Delta w + g(v)w = 0 \quad \text{in } Q \\ w = 0 \quad \text{on } \Sigma \\ w(0) = u^0. \end{cases} \tag{3.8}$$

Condition (3.2) is equivalent to

$$\|w(T) - (u^1 - U(T))\|_p \le \alpha. \tag{3.9}$$

As a consequence of Proposition 2.4 we know that given $u^1 \in C_0(\Omega)$ and $\alpha > 0$, for every $v \in L^1(Q)$ there exists $u^0 \in C_0(\Omega)$ (which of course depends on v) which is defined in a unique way by minimizing a suitable functional over $M(\Omega)$ such that the solution w of (3.8) satisfies (3.9). Moreover, since $g \in L^\infty(\mathbb{R})$, $g(v)$ is uniformly bounded in $L^\infty(Q)$ and due to the smoothing effect of the heat equation, $U(T)$ remains in a compact set of $C_0(\Omega)$. Thus, as a consequence of Proposition 2.4, u_0 belongs to a compact set of $C_0(\Omega)$ and therefore u remains in a compact set of $L^1(Q)$.

We have shown that K maps $L^1(Q)$ into a compact set of $L^1(Q)$. If we show that K is continuous, the existence of a fixed point of K will be a direct consequence of Schauder's Theorem.

Let us prove that $K : L^1(Q) \to L^1(Q)$ is continuous.

Consider a sequence $\{v_n\} \subset L^1(Q)$ such that

$$v_n \to v \quad \text{in } L^1(Q) \tag{3.10}$$

and let us prove that

$$u_n = Kv_n \to u = Kv \quad \text{in } L^1(Q). \tag{3.11}$$

Since g is continuous and uniformly bounded from (3.10) we deduce that

$$g(v_n) \rightharpoonup g(v) \text{ weakly-}* \text{ in } L^\infty(Q). \tag{3.12}$$

Therefore

$$U_n \to U \quad \text{in } L^1(Q). \tag{3.13}$$

Let us see that

$$w_n \to w \quad \text{in } L^1(Q). \tag{3.14}$$

In view of (3.12), the fact that $\{U_n(T)\}$ remains in a compact set of $C_0(\Omega)$ and Proposition 2.4 we deduce that the initial $\{u_n^0\}$ satisfy

$$u_n^0 \to u^0 \quad \text{in } C_0(\Omega)$$

and then (3.14) holds. From (3.13)-(3.14) we obtain (3.11).

Schauder's fixed-point Theorem ensures the existence of some $v \in L^1(Q)$ such that

$$u = Kv = v$$

but then $u = u(x,t)$ solves

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u^0 \end{cases}$$

and by construction

$$\|u(T) - u^1\|_\infty \leq \alpha.$$

This concludes the proof of Theorem 1.1 when $f \in C^1(\mathbb{R})$.

Let us consider now a globally Lipschitz function f and define the regularized sequence

$$f_n = f * \rho_n$$

where $\rho_n(s) = n\rho(ns)$ for some $\rho \in \mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \rho = 1$ and $\rho \geq 0$.

It is easy to check that $f_n \in C^1(\mathbb{R})$, $f_n \to f$ uniformly on compact intervals and that every f_n is globally Lipschitz with the same Lipschitz constant as f.

For $u^1 \in C_0(\Omega)$ and $\alpha > 0$ fixed we take, for each $n \in \mathbb{N}$, an initial data $u_n^0 \in C_0(\Omega)$ such that the solution u_n of

$$\begin{cases} u_t - \Delta u + f_n(u) = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u_n^0 \end{cases}$$

satisfies

$$\|u_n(T) - u^1\|_\infty \leq \alpha. \tag{3.15}$$

Since the nonlinearities f_n are uniformly globally Lipschitz and $f_n(0)$ is a bounded sequence, in view of Proposition 2.4 we deduce that $\{u_n^0\}$ is relatively compact in $C_0(\Omega)$. By extracting subsequences (that we denote by the index n) we deduce that

$$u_n^0 \to u^0 \text{ in } C_0(\Omega)$$
$$u_n \to u \text{ in } L^\infty(Q)$$

where u satisfies

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u^0 & \text{in } \Omega \end{cases}$$

and by (3.15),

$$\|u(T) - u^1\|_\infty \leq \alpha.$$

This concludes the proof of Theorem 1.1.

4 The semilinear heat equation: Theorem 1.3

This section is devoted to the proof of Theorem 1.2. Let us fix any $u^1 \in C_0(\Omega)$ and $\alpha > 0$. Our goal is to find some $u^0 \in \mathcal{B}$ such that the solution of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u^0 \end{cases} \tag{4.1}$$

satisfies

$$\|u(T) - u^1\|_\infty \leq \alpha. \tag{4.2}$$

We proceed in several steps.

Step 1. Suppose that $f \in C^1(\mathbb{R})$ and introduce the continuous and bounded function g as in (3.3). For every $v \in L^1(Q)$ consider the linearized problem

$$
\text{(4.3)} \qquad \begin{cases} u_t - \Delta u + g(v)u = -f(0) & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u^0. \end{cases}
$$

Consider any $q > 1$. We claim that there exists an initial data of the form

$$
\text{(4.4)} \qquad u^0(x) = \left(\int_\Omega |\hat{\varphi}(x,0)|^q \, dx \right) \quad |\hat{\varphi}(x,0)|^{q-2} \hat{\varphi}(x,0)
$$

so that (4.2) holds for the solution of (4.3). Moreover $\hat{\varphi} = \hat{\varphi}(x,t)$ is solution of

$$
\text{(4.5)} \qquad \begin{cases} -\varphi_t - \Delta \varphi + g(v)\varphi = 0 & \text{in } Q \\ \varphi = 0 & \text{on } \Sigma \\ \varphi(T) = \varphi^0 \end{cases}
$$

for some $\hat{\varphi}^0 \in M(\Omega)$ which is bounded in $M(\Omega)$ uniformly with respect to $v \in L^1(Q)$ and $q > 1$.

The proof of this claim is very similar to the proof of Proposition 2.4. For any $v \in L^1(Q)$ fixed we consider the functional

$$
J_q : M(\Omega) \to \mathbb{R}
$$

with

$$
J_q(\varphi^0) = \frac{1}{2q} \left(\int_\Omega |\varphi(x,0)|^q dx \right)^2 + \alpha |\varphi^0| - \langle \varphi^0, \widetilde{u^1} \rangle.
$$

where φ is solution of (4.5) and $\widetilde{u^1} = u^1 - U(T)$ where U solves (3.7). Minimizing J_q over $M(\Omega)$ we obtain $\hat{\varphi}^0$ such that the corresponding solution of (4.5) is so that (4.2) holds for the solution u of (4.3) with initial data (4.4). The arguments of Proposition 2.4 allow us to prove that $\hat{\varphi}^0$ is uniformly bounded in $M(\Omega)$ with respect to $v \in L^1(Q)$ and $q > 1$.

Step 2. For $u^1 \in C_0(\Omega)$, $\alpha > 0$ and $q > 1$ fixed we may proceed as in section 3 by applying Schauder's fixed point theorem and we deduce that there exists some $v \in L^1(Q)$ such that the solution u of (4.3)-(4.4) satisfies $u = v$. Thus u solves (4.1) for the initial data (4.4) and by construction satisfies also (4.2).

Step 3. Now, we have to pass to the limit as $q \to 1$. Since $\hat{\varphi}^0$ is uniformly bounded in $M(\Omega)$ and f is globally Lipschitz we deduce that $\hat{\varphi}(x,0)$ is relatively compact in $C_0(\Omega)$. By extracting a subsequence $q_n \downarrow 1$ we deduce that

$$
\begin{aligned}
&\hat{\varphi}_n^0 \rightharpoonup \hat{\varphi}^0 \quad \text{in } M(\Omega) \\
&\hat{\varphi}_n \rightharpoonup \hat{\varphi} \quad \text{in } L^1(Q) \\
&\hat{\varphi}_n(x,0) \to \hat{\varphi}(x,0) \quad \text{in } C_0(\Omega) \\
&u_n \to u \quad \text{in } L^1(Q)
\end{aligned}
$$

where $\hat{\varphi}_n^0$, $\hat{\varphi}_n$ and u_n denote respectively the minimizer of J_{q_n}, the corresponding solution of (4.5) with $v = u_n$ and the corresponding solution of (4.1).

It is easy to see that u solves (4.1) with the initial data

$$u^0 = \left(\int_\Omega |\hat{\varphi}(x,0)| dx \right) \psi$$

for some $\psi = \psi(x) \in sgn(\hat{\varphi}(x,0))$. By construction u satisfies (4.2). This concludes the proof of Theorem 1.3 when $f \in C^1(\mathbb{R})$.

Step 4. In the general case of a globally Lipschitz function the proof can be completed by approximation as in section 3. □

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