**OPTIMAL CONTROL OF A CONVECTIVE-DIFFUSIVE
FLUID PROBLEM**

By

Suzanne Lenhart

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SUZANNE LENHART*

Abstract. We consider optimal control of a parabolic differential equation, modeling one-dimensional fluid flow through a soil-packed tube in which a contaminant is initially distributed. A fluid is pumped through the tube to remove the contaminant. The convective velocity due to the fluid pumping is the control. The goal is to minimize a performance criterion which is a combination of the total contaminant at the final time and the cost of the control. The optimal control is characterized by an optimality system.

Key words. optimal control, partial differential equation, convective velocity, fluid flow

AMS(MOS) subject classifications. 49A22, 35K15, 76R05

1. Introduction. The increasing demand for groundwater as a drinking water resource requires the development of remediation methods. The damage caused by contaminants to the groundwater varies spatially due to the migration of the contaminants through the water and soil. Optimal control strategies for realistic contaminant transport models can expedite the remediation efforts. In an effort to understand optimal control of parabolic partial differential equations governing groundwater flow, we consider a simple example with one spatial dimension.

Consider saturated flow through an idealized tube packed with soil which contains some contaminant. The initial concentration of contaminant along the length of the tube is known. A fluid will be pumped through the tube (from left to right) to remove the contaminant. The contaminant concentration is represented by a function $u(x, t)$ with x denoting the length along the tube axis measured from flow entrance ($x = 0$) to its total length ($x = L$) and t denoting time. The contaminant concentration satisfies the solute transport equation [1],

$$u_t(x, t) = a u_{xx}(x, t) - Q(t)u_x(x, t) \quad \text{in} \quad D = (0, L) \times (0, T) \quad (1.1)$$

$$\text{with} \quad u(x, 0) = u_0(x), \quad 0 < x < L, \quad t = 0. \quad (1.2)$$

The function $Q(t)$ is the “average linear groundwater velocity” [1,8], which is convective velocity due to the fluid pumped into the tube. The coefficient a is the “dispersion-diffusion” coefficient, which reflects the effects of the diffusion of the solute in the water

*Mathematical Sciences Section, Engineering Physics and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831 and University of Tennessee, Mathematics Department, Knoxville, TN 37996

as well as the dispersive effects of the water flowing through the soil pores, and may be further defined as

$$a = a^* + \varepsilon Q$$

with a^* being the “diffusion coefficient” and ε being the “dispersity”. Since the effect of the ε dispersity term is small here in comparison with other effects [11], we assume

$$\varepsilon = 0 .$$

The boundary conditions are

$$u(0, t) = 0 \quad \text{for } t \in (0, T) , \ x = 0 \quad (1.3)$$

$$u_x(L, t) = 0 \quad \text{for } t \in (0, T) , \ x = L \quad (1.4)$$

which are interpreted as there is no contaminant at the face of the tube when the fluid enters and no dispersive flux at the face where it exists.

We take the convective velocity, $Q(t)$, as the control and the corresponding solution, $u(x, t) = u(Q)$, of (1.1)–(1.4) as the state variable. Our performance criterion (cost functional) is a combination of the final amount of contaminant and the cost of applying the control. We define our cost functional as

$$J(Q) = \alpha \int_0^L u(x, T) dx + \frac{\beta}{2} \int_0^T Q^2(t) dt .$$

The dependence on Q in $J(Q)$ is a convenient quadratic term measuring the cost of the control, other forms of dependence of Q in $J(Q)$ would also be reasonable. Our goal is to characterize the optimal control Q^* that minimizes our cost functional,

$$J(Q^*) = \min_{Q \in U_M} J(Q) .$$

We will justify later that this minimum can be achieved.

In section 2, we formulate our problem and define the solution space for (1.1). We then prove the existence of an optimal control. In section 3, we derive the optimality system, by differentiating the cost functional with respect to the control. The optimality system consists of the state equation (1.1)–(1.4) coupled with an adjoint equation, which has a backwards time orientation. The optimal control is characterized in terms of the unique solution of the optimality system.

For background on control of PDEs, see the fundamental book by Lions [6]. For techniques on control, similar to ones here, see Seidman and Zhou [9], Stojanovic [10], Lenhart and Bhat [3], and Lenhart and Wilson [4]. Recently Zeitouni and Opaluch [12,13]

have treated optimal control of the advection-diffusion equation, in which the control is the quantity of contaminating chemical applied as a source and the aquifer (spatial dimension) has one or two dimensions. Their objective is to maximize the total discounted value of profit over the time horizon under an upper constraint on the input contaminant. Our control problem has a different objective, and controlling a coefficient of a derivative terms, i.e. the convective velocity, is more difficult than controlling source terms.

2. Existence of optimal control. We now state our precise solution space for the state equation and then prove the existence of an optimal control.

Solutions of the state equation are weak solutions in

$$W = L^2(0, T; V)$$

where $V = H_{\{0\}}^1(0, L)$ and the subscript $\{0\}$ indicate the functions vanish at $x = 0$. See Lions [6] for explanation of the notation.

DEFINITION. A function $u \in W$ is a solution of

$$u_t = a u_{xx} - Q u_x \quad \text{in } D = (0, L) \times (0, T) \quad (2.1)$$

$$\begin{aligned} u(0, t) &= 0 && \text{on } (0, T), \quad x = 0 \\ u_x(L, t) &= 0 && \text{on } (0, T), \quad x = L \\ u(x, 0) &= u_0(x) && \text{on } (0, L), \quad t = 0. \end{aligned}$$

if

$$\int_0^T \langle u'(t), v(t) \rangle + \int_0^T \int_0^L (a u_x v_x + Q u_x v) = 0 \quad (2.2)$$

for all $v \in W$

where the bracket $\langle \cdot, \cdot \rangle$ in the integrand denotes the duality $V^* - V$.

Our control set for the convective velocity is

$$U_M = \{Q \in L^\infty(0, T) | 0 \leq Q(t) \leq M\}.$$

We assume

a is a positive constant

and

$$u_0 \in H^1(0, L).$$

For $Q \in U_M$, the existence of a unique solution $u = u(Q)$ in W satisfying (2.1) in the sense of (2.2) follows from standard results in [7].

THEOREM 2.1. *There exists an optimal control Q^* in U_M minimizing the cost function $J(Q)$.*

Proof. Choose a minimizing sequence $\{Q_n\}$, $n = 1, 2, \dots$, such that

$$\lim_{n \rightarrow \infty} J(Q_n) = \inf_{Q \in U_M} J(Q) .$$

Let $u^n = u(Q^n)$ be the corresponding concentrations, solutions to (2.1). To obtain the necessary apriori estimates on our minimizing sequence and associated concentrations, we need to change variables.

$$w = e^{-\lambda t} u$$

where λ is a positive constant to be chosen below.

Using the weak definition of solution,

$$\int_0^T \langle w_t^n, w^n \rangle + \iint_D (a w_x^n w_x^n + Q w_x^n w^n + \lambda (w^n)^2) = 0$$

Using $\frac{d}{dt} \left(\frac{(w)^2}{2} \right) = w w_t$, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^L (w^n)^2(x, T) + a \iint_D (w_x^n)^2 + \lambda \iint_D (w^n)^2 \\ &= \frac{1}{2} \int_0^L u_0^2 - \iint_D Q^n w_x^n w^n . \end{aligned}$$

Using Cauchy's inequality on the RHS, we obtain

$$\begin{aligned} \frac{1}{2} \int_0^L (w^n)^2(x, T) + a \iint_D (w_x^n)^2 + \lambda \iint_D (w^n)^2 &\leq \frac{1}{2} \int_0^L u_0^2 + \frac{a}{2} \iint_D (w_x^n)^2 \\ &+ \frac{M^2}{2a} \iint_D (w^n)^2 . \end{aligned}$$

Choose $\lambda = \frac{M^2}{a}$, we obtain

$$\frac{1}{2} \int_0^L (w^n)^2(x, T) + \frac{a}{2} \iint_D (w_x^n)^2 + \frac{M^2}{2a} \iint_D (w^n)^2 \leq \frac{1}{2} \int_0^L u_0^2 . \quad (2.3)$$

We conclude

$$\|w^n\|_W \text{ are uniformly bounded independent of } n ,$$

and thus

$$\|u^n\|_W \text{ are uniformly bounded independent of } n .$$

From these bounds, we can extract a subsequence with the following convergence properties:

$$\begin{aligned} u^n &\rightharpoonup u^* \text{ weakly in } W \\ Q^n &\rightharpoonup Q^* \text{ weakly in } L^2(0, T) , \end{aligned} \tag{2.4}$$

with $u^* \in W$. From these bounds and the PDE, we can extract a weakly convergent subsequence,

$$u_t^n \rightharpoonup u_t^* \text{ weakly in } L^2(0, T; V^*) . \tag{2.5}$$

We must show $u^* = u(Q^*)$, i.e. u^* is the concentration solution associated with Q^* . Consider the weak form of PDE satisfied by u^n and integrate by parts on the Q^n term, we obtain

$$\int_0^T \langle u_t^n, v \rangle + a \iint_D u_x^n v_x - \iint_D Q^n u^n v_x + \int_0^T Q^n u^n v(L, t) = 0 . \tag{2.6}$$

Using the compactness result of [5, Chapter 4, Proposition 4.2] coupled with (2.3) and (2.5), we obtain

$$\begin{aligned} \text{and} \quad u^n &\rightarrow u^* \text{ strongly in } L^2(\{L\} \times (0, T)) , \\ u^n &\rightarrow u^* \text{ strongly in } L^2(D) \end{aligned}$$

Thus we have sufficient convergence results to justify passing to the limit in (2.4) as $n \rightarrow \infty$ and we conclude $u^* = u(Q^*)$.

Now we verify that Q^* is an optimal control. Notice that estimate (2.3) gives a uniform bound on

$$\|u^n(x, T)\|_{L^2(0, L)} ,$$

and we can extract a weakly convergent subsequence from those traces on the top $t = T$. Using the lower semicontinuity of the cost function with respect to these weak convergences (basically applying Fatou's Lemma), we obtain

$$\begin{aligned} J(Q^*) &\leq \inf_n \int_0^L u^n(x, T) dx + \inf_n \frac{\beta}{2} \int_0^T (Q^n)^2(t) dt \\ &\leq \lim_{n \rightarrow \infty} J(Q^n) = \inf_Q J(Q) . \end{aligned}$$

Then Q^* is an optimal control. $\square$

3. Derivation of the optimality system. To obtain a characterization of the optimal control, we must derive the optimality system by differentiating the cost functional $J(Q)$ with respect to the control Q . Since the solution map $Q \rightarrow u(Q)$ is involved in $J(Q)$, we examine the differentiability of the solution map with respect to Q .

LEMMA 3.1. *The solution map*

$$Q \in U_M \rightarrow u = u(Q) \in W$$

is differentiable in the following sense:

$$\frac{u(Q + \varepsilon h) - u(Q)}{\varepsilon} \rightharpoonup \psi \quad \text{weakly in } W$$

as $\varepsilon \rightarrow 0$ where $Q + \varepsilon h \in U_M$, $h \in L^\infty(0, T)$. Moreover $\psi = \psi(Q; h)$ satisfies

$$\psi_t = a\psi_{xx} - Q\psi_x - hu_x \quad \text{in } D \tag{3.1}$$

$$\begin{aligned} \psi &= 0 \quad \text{when } t = 0 \\ \psi &= 0 \quad \text{when } x = 0 \\ \psi_0 &= 0 \quad \text{when } x = L, \end{aligned}$$

where $u = u(Q)$.

Proof. Once again to get the apriori estimates, we change variables,

$$w = e^{-\lambda t} u, \quad w^\varepsilon = e^{-\lambda t} u^\varepsilon$$

where $u = u(Q)$, $u^\varepsilon = u(Q + \varepsilon h)$. Considering the PDE solved by $w^\varepsilon - w$, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^L (w^\varepsilon - w)^2(x, T) + a \iint_D ((w^\varepsilon - w)_x)^2 + \lambda \iint_D (w^\varepsilon - w)^2 \\ &= - \iint_D Q(w^\varepsilon - w)_x (w^\varepsilon - w) - \varepsilon \iint_D h w_x^\varepsilon (w^\varepsilon - w). \end{aligned} \tag{3.2}$$

Noting that h is in L^∞ and $\|w_x^\varepsilon\|_{L^2}$ is uniformly bounded from an estimate like (2.3), we obtain

$$\frac{a}{2} \int_D ((w^\varepsilon - w)_x)^2 + \left(\lambda - \frac{M^2}{2a} - 1 \right) \iint_D (w^\varepsilon - w)^2 \leq K_1 \varepsilon^2$$

where K_1 depends on $\|h\|_\infty$ and $\|w_x^\varepsilon\|_{L^2}$. For λ chosen large, we have

$$\left\| \frac{w^\varepsilon - w}{\varepsilon} \right\|_W \leq K_2 ,$$

which carries over to

$$\left\| \frac{u^\varepsilon - u}{\varepsilon} \right\|_W \leq K_3$$

and the desired weak convergence to ψ is obtained. As in Lemma 2.1, we also have

$$\frac{u^\varepsilon - u}{\varepsilon} \rightarrow \psi \quad \text{strongly in } L^2(D) ;$$

we conclude ψ satisfies (3.1). $\square$

Remark 3.1. Estimate (3.2) gives the weak convergence of the traces on the top, $\frac{(u^\varepsilon - u)}{\varepsilon}(x, T)$, in $L^2(0, L)$ also.

We are ready to characterize the optimal control, by deriving the optimality system through differentiating $J(Q)$ with respect to Q at an optimal control.

THEOREM 3.1. *Given an optimal control Q in U_M , there exists a solution p in W to the adjoint problem,*

$$-p_t = a p_{xx} + Q p_x \quad \text{in } Q \tag{3.3}$$

$$\begin{aligned} p(x, T) &= 1 \quad \text{on } (0, L), \quad t = T \\ p(0, t) &= 0 \quad \text{on } (0, T), \quad x = 0 \\ a p_x(L, T) + Q(t) p(L, t) &= 0 \quad \text{on } (0, T), \quad x = L . \end{aligned}$$

Moreover,

$$Q(t) = \min \left\{ \frac{\alpha}{\beta} \left(\int_0^L p(x, t) u_x(x, t) dx \right)^+ , M \right\} \tag{3.4}$$

Remark 2.2. We note the weak formulation of (3.3) has the form:

$$- \int_0^T \langle p'(t), v(t) \rangle + \int_D (a p_x v_x - Q p_x v) + \int_0^T Q p v(L, t) = 0$$

Proof of Theorem 3.1. Suppose $Q(t)$ is an optimal control. Let $h \in L^\infty(0, T)$ such that $Q + \varepsilon h \in U_M$ for $\varepsilon > 0$. The derivative of $J(Q)$ with respect to q in the direction h satisfies.

$$\begin{aligned}
0 &\leq \lim_{\varepsilon \rightarrow 0} \frac{J(Q + \varepsilon h) - J(Q)}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \alpha \int_0^L \frac{(u^\varepsilon - u)}{\varepsilon}(x, T) dx + \frac{\beta}{2} \int_0^T \frac{((Q + \varepsilon h)^2 - Q^2)}{\varepsilon}(t) dt \\
&= \alpha \int_0^L \psi(x, T) dx + \beta \int_0^T Q h(t) dt
\end{aligned} \tag{3.5}$$

We used the weak convergence of $\frac{u^\varepsilon - u}{\varepsilon}$ at $t = T$ as noted in Remark 3.1.

Let p be the solution of the adjoint problem (3.3) (see [7]) and substitute the weak formulation with p into (3.5) to give

$$\begin{aligned}
0 &\leq \alpha \int_0^L p(x, T) \psi(x, T) dx + \beta \int_0^T Q h dt \\
&\quad + \alpha \iint_D (-p_t \psi + a p_x \psi_x - Q p_x \psi) + \alpha \int_0^T Q p \psi(L, t) dt .
\end{aligned}$$

Integration by parts yields

$$0 \leq \alpha \iint_D (p \psi_t + a p_x \psi_x + p Q \psi_x) + \beta \int_0^T Q h dt .$$

The RHS of (3.1) gives

$$\begin{aligned}
0 &\leq \alpha \iint_D (-h u_x p) + \beta \int_0^T Q h dt \\
&= \int_0^T h(t) \left(\beta Q(t) - \alpha \int_0^L u_x(x, t) p(x, t) dx \right) dt .
\end{aligned}$$

Using standard control arguments and bounds on our control set,

$$Q(t) = \min \left(M, \frac{\alpha}{\beta} \left(\int_0^L u_x(x, t) p(x, t) dx \right)^+ \right) . \quad \square$$

The next lemma enables us to simplify the above expression for $Q(t)$.

LEMMA 3.2. For M sufficiently large, any optimal control in U_M is strictly below that upper bound M , i.e. if $Q^* \in U_M$ is an optimal control, $Q^*(t) < M$.

Proof. Since $Q \equiv 0$ is an admissible control for every control set U_M ,

$$J(0) = \alpha \int_{\Omega} u^0(x, T) dx \geq J(Q^*) \geq \frac{\beta}{2} \int_0^T (Q^*(t))^2 dt ,$$

where u^0 satisfies

$$\begin{aligned} u_t^0 &= a u_{xx}^0 \text{ in } D \\ u^0 &= 0 \quad \text{when } x = 0 \\ u_x^0 &= 0 \quad \text{when } x = L \\ u^0 &= u_0 \quad \text{when } t = 0 . \end{aligned}$$

and $Q^* \in U_M$ is an optimal control. Thus $J(0)$ gives a bound on $\|Q^*\|_{L^2(0,T)}$ that is independent of M . Using the PDE satisfied by $u = u(Q^*)$, on an interval $(0, \tau)$ with $0 \leq \tau \leq T$,

$$\begin{aligned} \int_0^L a(u_x)^2(x, \tau) - \int_0^L a(u_x)^2(x, 0) &= \int_0^T \frac{d}{dt} \int_0^L a(u_x)^2 dx dt \\ &= 2 \int_0^{\tau} \int_0^L a u_x u_{xt} \\ &= -2 \int_0^{\tau} \int_0^L a u_{xx} u_t \\ &= - \int_0^{\tau} \int_0^L 2(u_t + Q^* u_x) u_t \\ &\leq - \int_0^{\tau} \int_0^L (u_t)^2 + \int_0^{\tau} \int_0^L (Q^*)^2 (u_x)^2 . \end{aligned}$$

Rearranging yields

$$a \int_0^L (u_x)^2(x, \tau) \leq a \int_0^L ((u_0)_x)^2 + \int_0^{\tau} (Q^*)^2(t) \int_0^L u_x^2(x, t) dx dt$$

and then Gronwall's Inequality implies,

$$a \int_0^L u_x^2(x, \tau) dx \leq C(\|Q^*\|_{L^2}^2),$$

which is bounded independent of M . Next, notice that p is bounded by the constant 1, which is a supersolution of the adjoint equation.

We now obtain our bound on Q^* , from the characterization (3.4),

$$Q^*(t) \leq \frac{\alpha}{\beta} \left(\int_0^L u_x^2(x, t) dx \right)^{1/2} \left(\int_0^L p^2(x, t) dx \right)^{1/2} \leq M_1$$

where M_1 is independent of M . $\square$

For $M > M_1$, our control representation (3.4), becomes

$$Q^*(t) = \frac{\alpha}{\beta} \left(\int_0^L u_x(x, t) p(x, t) dx \right)^+ . \quad (3.5)$$

We substitute this representation for Q^* into the state equation (2.1) and the corresponding adjoint problem (3.3), to obtain the following optimality system (OS):

$$\begin{aligned} (OS) \quad & u_t = au_{xx} - \frac{\alpha}{\beta} \left(\int_0^L u_x p(x, t) dx \right)^+ u_x && \text{in } D \\ & -p_t = ap_{xx} + \frac{\alpha}{\beta} \left(\int_0^L u_x p(x, t) dx \right)^+ p_x \end{aligned}$$

$$u = p = 0 \quad \text{when } x = 0$$

$$u_x = 0 \quad \text{when } x = L$$

$$ap_x + \frac{\alpha}{\beta} \left(\int_0^L u_x p(y, t) dy \right)^+ p = 0 \quad \text{when } x = L$$

$$u = u_0 \quad \text{when } t = 0$$

$$p = 1 \quad \text{when } t = T .$$

We know that solutions of (OS) exist by Theorems 2.1 and 3.1. We now prove solutions of (OS) are unique, which gives a characterization of the unique optimal control, given by (3.5), in terms of the unique solution of (OS).

THEOREM 3.2. *Solutions to the optimality system (OS) in $(W)^2$ are unique.*

Proof. Suppose (u, p) and $(\bar{u}, \bar{p})$ are two solutions to (OS). We change variables again,

$$w = e^{-\lambda t} u, \quad \bar{w} = e^{-\lambda t} \bar{u}, \quad q = e^{\lambda t} p, \quad \bar{q} = e^{\lambda t} \bar{p}$$

where λ is a positive constant to be chosen. Using the PDEs satisfied by $w, \bar{w}, q, \bar{q}$, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^L (w - \bar{w})^2(x, T) + \frac{1}{2} \int_0^L (q - \bar{q})^2(x, 0) + a \iint_D [((w - \bar{w})_x)^2 + ((q - \bar{q})_x)^2] \\ & + \lambda \iint_D [(w - \bar{w})^2 + (q - \bar{q})^2] + \frac{\alpha}{\beta} \int_0^T \left[\left(\int_0^L w_x q(y, t) dy \right)^+ q \right. \\ & \quad \left. - \left(\int_0^L \bar{w}_x \bar{q}(y, t) dy \right)^+ \bar{q} \right] (q - \bar{q})(L, t) dt \\ & = -\frac{\alpha}{\beta} \iint_D \left[\left(\int_0^L w_x q \right)^+ w_x - \left(\int_0^L \bar{w}_x \bar{q} \right)^+ \bar{w}_x \right] (w - \bar{w}) \\ & + \frac{\alpha}{\beta} \iint_D \left[\left(\int_0^L w_x q \right)^+ q_x - \left(\int_0^L \bar{w}_x \bar{q} \right)^+ \bar{q}_x \right] (q - \bar{q}) \end{aligned}$$

First estimate the boundary term on $L \times (0, T)$

$$\begin{aligned} & \frac{\alpha}{\beta} \int_0^T \left[\left(\int_0^L w_x q(y, t) dy \right)^+ q - \left(\int_0^L \bar{w}_x \bar{q}(y, t) dy \right)^+ \bar{q} \right] (q - \bar{q})(L, t) dt \\ & \leq \frac{\alpha}{\beta} \int_0^T \left(\int_0^L w_x q(y, t) dy \right)^+ (q - \bar{q})^2(L, t) dt \\ & + K_1 \left(\int_0^T \left(\int_0^L (w - \bar{w})_x q(y, t) dy + \int_0^L w_x (q - \bar{q})(y, t) dy \right)^2 (L, t) dt \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\int_0^T (q - \bar{q})^2(L, t) dt \right)^{1/2} \\
& \leq K_2 \iint_D (q - \bar{q})^2 + a/4 \iint_D [((w - \bar{w})_x)^2 + ((q - \bar{q})_x)^2]
\end{aligned}$$

where we used L^∞ bound on q and bounds on $\int_0^L w_x^2(x, t) dx$. (The bound on $\int_0^L w_x^2(x, t) dx$ can be obtained as in Lemma 3.2).

Second, estimate the interior first derivative terms,

$$\begin{aligned}
& \frac{\alpha}{\beta} \iint_D \left[\left(\int_0^L w_x q(y, t) dy \right)^+ q_x - \left(\int_0^L \bar{w}_x \bar{q}(y, t) dy \right)^+ \bar{q}_x \right] (q - \bar{q}) \\
& = \frac{\alpha}{\beta} \iint_D \left(\int_0^L w_x q(y, t) dy \right)^+ (q - \bar{q})_x (q - \bar{q}) \\
& \quad + \frac{\alpha}{\beta} \iint_D \bar{q}_x \left[\left(\int_0^L w_x q(y, t) dy \right)^+ - \left(\int_0^L \bar{w}_x \bar{q}(y, t) dy \right)^+ \right] (q - \bar{q}) \\
& \leq \frac{a}{4} \iint_D ((q - \bar{q})_x)^2 + K_3 \iint_D (q - \bar{q})^2 + \\
& \quad K_4 \left[\iint_D ((w - \bar{w})_x)^2 + \iint_D (q - \bar{q})^2 \right]^{1/2} \left[\int_0^T \left(\int_0^L q_x^2 \right) \left(\int_0^L (q - \bar{q})^2 \right) \right]^{1/2} \\
& \leq \frac{a}{4} \iint_D [((q - \bar{q})_x)^2 + ((w - \bar{w})_x)^2] + K_4 \iint_D (q - \bar{q})^2 .
\end{aligned}$$

Here we use a L^∞ bound on $\int_0^L (q_x)^2(x, t) dt$ which can be obtained from [2, Sections V.7, VI.5]. The other interior first derivative term can be obtained similarly.

Combining these estimates, we obtain

$$\frac{a}{4} \iint_D [((w - \bar{w})_x)^2 + ((q - \bar{q})_x)^2] + (\lambda - K) \iint_D [(w - \bar{w})^2 + (q - \bar{q})^2] \leq 0 .$$

For λ chosen larger than K , we obtain

$$w = \bar{w} \quad \text{and} \quad q = \bar{q} ,$$

and the desired uniqueness result. $\square$

This result gives a characterization of the unique optimal control in terms of the unique solution of the optimality system (OS).

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