**AXIALLY SYMMETRIC JET FLOWS ARISING FROM
HIGH SPEED FIBER COATING**

By

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Axially Symmetric Jet Flows Arising From High Speed Fiber Coating

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§0. Introduction.

In this paper we are concerned with a 3-dimensional axially symmetric flow which occurs in high speed coating of optical fibers. While the fiber is being drawn through a coater, it is immediately dressed with a protecting coating fluid. Figure 0.1 shows a generic coating.

Figure 0.1

The fluid, with flux Q , covers a portion of the fiber; two free surfaces are formed: the upper meniscus m_u and the lower meniscus m_l . There are basically three regimes of interest: the upper meniscus and its vicinity; the lower meniscus and its vicinity and the intermediate flow region with heat circulation and closed stream lines. For more discussion about the problem, we refer to [10] and the references given there.

The flow and heat transfer in the coater and the centering forces near the exit were considered by Abbott and Francis [1]. Here we shall consider a simplified model for the upper meniscus problem assuming that the coating fluid is inviscid and incompressible. This model can be formulated as a finite jet problem for the stream function u . Existence, uniqueness and asymptotic estimates of solutions will be established.

For other types of jet flows we refer to [3], [4], [6], [8], [11] and surveys [5] and [12].

This paper is organized as follows: In §1 we describe the simplified physical problem and state the main existence and uniqueness results for ideal fluids. In §2 we introduce a truncated variational problem and study its solution. Several properties of the free boundary of the truncated problem are established in §3. Existence and the uniqueness for the problem described in §1 are given in §4 and §5 respectively. In §6, we study the asymptotic behavior of the free boundary as its initial point approaches the axis of symmetry. Some extensions of previous results to some rotational flows are given in §7.

§1. The upper meniscus problem.

We take the axis of symmetry as the y -axis; it represents the optical fiber. Denote by G the union of the rectangle $\{0 < x < L, 0 < y < H\}$ and the three half-infinite strips

$$\begin{aligned} T_0 &= \{(x, y); \quad 0 < x < L_0, H \leq y < \infty\}, \\ T_1 &= \{(x, y); \quad 0 < x < L_1, -\infty < y \leq 0\}, \\ T_2 &= \{(x, y); \quad L \leq x < \infty, H_1 < y < H_0\}. \end{aligned}$$

Figure 1.1

We assume

$$(1.1) \quad 0 < L_0 < L, \quad 0 < L_1 < L \quad \text{and} \quad 0 < H_0 < H.$$

DEFINITION 1.1. The (axially symmetric) finite jet problem consists of finding a function u (the stream function), a curve Γ (the free boundary) and a positive number λ (the velocity on the free boundary) such that the following conditions are satisfied:

$$(1.2) \quad \Gamma \text{ is } C^1 \text{ and is given by} \\ \{(x, y); \quad y = f(x), 0 < x < L_0\}, \\ \text{where } y = f(x) \text{ is defined and continuous for } 0 \leq x \leq L_0,$$

$$(1.3) \quad f(L_0) = H, \quad f'(L_0-) = 0,$$

$$(1.4) \quad f(0) \text{ is finite and } f'(0+) = +\infty,$$

$$(1.5) \quad u \in C^0(\bar{G}), \quad 0 \leq u \leq Q,$$

$$(1.6) \quad \mathcal{L}u = u_{xx} + u_{yy} - \frac{1}{x}u_x = 0 \quad \text{in } \Omega \equiv \{u < Q\},$$

$$(1.7) \quad u = Q, \quad \frac{1}{x} \frac{\partial u}{\partial \nu} = \lambda \quad \text{on } \Gamma$$

and u satisfies the prescribed boundary conditions (see Figure 1.1),

$$(1.8) \quad u \text{ is in } C^1 \text{ in a } \bar{\Omega}\text{-neighborhood of } A,$$

$$(1.9) \quad \lim_{x \rightarrow 0+} \frac{1}{x^2} (Q - u(x, f(0))) = \frac{\lambda}{2}.$$

(1.3), (1.8) are called the smooth-fit conditions [8]. (1.4) implies that Γ is tangent to the y-axis at its endpoint. Finally, (1.9) roughly means that the speed of the flow is continuous at A' along a segment. We need this condition to establish uniqueness of solutions.

Some of the main results of this paper are the following:

THEOREM 1.2. *If $L_0 < 2(H - H_0)$, then for any $Q \geq 0$ there exists a unique solution of the finite jet problem.*

In the coating problem described in the introduction, L_0 is very small relative to $H - H_0$. Hence the assumption $L_0 < 2(H - H_0)$ is physically reasonable.

THEOREM 1.3. *If $L_0 < L_1$, then the free boundary Γ must lie below the line containing the point A and B .*

Remark 1.4. In general, the free boundary may contain points above $\{y = H\}$. In fact one can show that there exists a unique solution of the jet problem with Γ having points lying above the line $\{y = H\}$ if L_0 is large and $L - L_0$ is small.

§2. The truncated variational problem.

The variational approach of [2] and [3] will be used throughout the paper. However, first we need to truncate G by bounded domains D_μ . To do this, the following cases will be considered separately : i) $L_0 < L_1$ and ii) $L_0 \geq L_1$.

In the first case, the ray which goes from B to A will intersect the y-axis at a point C . We denote by γ the segment $\overline{AC}$. Let D denote the domain bounded by the y-axis, γ and $(\partial G) \cap \{y \leq H\}$. For any large μ , set (see Figure 2.1)

$$D_\mu = D \cap \{x < \mu\} \cap \{y > -\mu\},$$

$$E_\mu = D_\mu \cap \{L_0 < x < \mu\}.$$

In the second case, we simply set

Figure 2.1

$$D_\mu = G \cap \{x < \mu\} \cap \{-\mu < y < \mu\}.$$

Introduce the class K_μ of functions v in $H_{loc}^1(D_\mu)$ satisfying $0 \leq v \leq Q$ in D_μ and the boundary conditions described as in Figure 2.1.

Introduce the functional

$$J_{\lambda,\mu}(v) = \int_{D_\mu} (|\frac{\nabla v}{x}|^2 + \lambda^2 I_{\{v < Q\} \setminus E_\mu}) x dx dy$$

for $v \in K_\mu$, where I_M denotes the characteristic function of a set M .

Consider the truncated variational problem:

$$(2.1) \quad \begin{array}{ll} \text{Find} & u = u_{\lambda,\mu} \quad \text{such that} \\ & u \in K_\mu, \quad J_{\lambda,\mu}(u) = \min_{v \in K_\mu} J_{\lambda,\mu}(v). \end{array}$$

THEOREM 2.1. *There exists a unique solution u of problem 2.1 with $u_y \geq 0$.*

Proof. Define

$$u_0 = \begin{cases} Q & \text{for } y \geq H, \\ -\frac{Qx^2}{L_1^2(H+\mu)}(H-y) + Q & \text{for } y < H, \quad x \text{ is near } x = 0. \end{cases}$$

We extend the domain of u_0 into D_μ such that $u_0 \in K_\mu$. As easily verified, $J_{\lambda,\mu}(u_0) < \infty$. Hence $\min_{v \in K_\mu} J_{\lambda,\mu}(v)$ is finite, and the existence of a solution u then follows by using the lower semicontinuity of the functional $J_{\lambda,\mu}$ [8]. Since the minimizer u is a supersolution with respect to $\mathcal{L}$ (see [8]), we have, by the maximum principle,

$$(2.2) \quad u \geq -\frac{Q}{L_1^2}x^2 + Q \quad \text{in } D_\mu.$$

Notice that for fixed x , $u(x, \cdot)$ has its minimum at the bottom of D_μ and its maximum at the top of D_μ . We can rearrange u so that it becomes monotone increasing in the y -direction. Clearly, the rearrangement u^* of u is again in K_μ and by [8],

$$J_{\lambda,\mu}(u^*) \leq J_{\lambda,\mu}(u).$$

It follows that u^* is also a minimizer. For simplicity we denote u^* by u and thus obtain a solution u of (2.1) with $u_y \geq 0$.

To prove uniqueness let u_1, u_2 be two solutions with $\frac{\partial u_i}{\partial y} \geq 0$. We assume for the moment that $u_i, i = 1, 2$ are continuous in D_μ (see Lemma 2.3 below).

Extend u_i by Q to the region above D_μ and by $-\frac{Q}{L_1^2}x^2 + Q$ ($0 < x < L_1$), 0 ($x > L_1$) to the region below D_μ . For any small $\epsilon > 0$, define

$$(2.3) \quad u_1^\epsilon(x, y) = u_1(x, y + \epsilon),$$

$$(2.4) \quad v_1 = u_1^\epsilon \wedge u_2, \quad v_2 = u_1^\epsilon \vee u_2,$$

and

$$D_\mu^\epsilon = \{(x, y); \quad (x, y + \epsilon) \in D_\mu\}.$$

If we denote by K_μ^ϵ the set K_μ corresponding to D_μ^ϵ , then

$$(2.5) \quad v_1 \in K_\mu, \quad v_2 \in K_\mu^\epsilon.$$

Denote by $J_{\lambda,\mu}^\epsilon$ the functional $J_{\lambda,\mu}$ corresponding to D_μ^ϵ . As in [3] one can show that

$$(2.6) \quad J_{\lambda,\mu}^\epsilon(u_1^\epsilon) + J_{\lambda,\mu}(u_2) = J_{\lambda,\mu}(v_1) + J_{\lambda,\mu}^\epsilon(v_2).$$

Since u_1^ϵ and u_2 are minimizers in K_μ^ϵ and K_μ respectively, we deduce that

$$J_{\lambda,\mu}^\epsilon(u_1^\epsilon) = J_{\lambda,\mu}^\epsilon(v_2), \quad J_{\lambda,\mu}(u_2) = J_{\lambda,\mu}(v_1).$$

It follows that v_1 is a minimizer of $J_{\lambda,\mu}$ and thus a solution of

$$\mathcal{L}v_1 = 0 \quad \text{in} \quad \{v_1 < Q\}.$$

Notice that $\{u_2 < Q\} \subset \{v_1 < Q\}$, and $\{u_2 < Q\}$ is connected (since $\frac{\partial u_2}{\partial y} \geq 0$). By the maximum principle we deduce that

$$\text{either } u_1^\epsilon \wedge u_2 \equiv u_2 \quad \text{or} \quad u_1^\epsilon \wedge u_2 < u_2 \quad \text{in} \quad \{u_2 < Q\}.$$

On the other hand, $u_1^\epsilon > u_2$ at some points in $\{u_2 < Q\}$. It follows that

$$(2.7) \quad u_1^\epsilon \geq u_1^\epsilon \wedge u_2 \equiv u_2 \quad \text{in} \quad \{u_2 < Q\}.$$

Since $\frac{\partial u_1^\epsilon}{\partial y} \geq 0$ and $\frac{\partial u_2}{\partial y} \geq 0$, (2.7) actually holds throughout D_μ . Taking $\epsilon \rightarrow 0$, we obtain $u_1 \geq u_2$. By symmetry also $u_2 \geq u_1$ and thus $u_1 \equiv u_2$.

Theorem 2.1 implies that

$$\partial\{u_{\lambda,\mu} < Q\} = \{(x, y); \quad y = f_{\lambda,\mu}(x), \quad 0 < x < L_0\}$$

for some function $f_{\lambda,\mu}$, where $u = u_{\lambda,\mu}$ is the solution of (2.1) obtained in Theorem 2.1. Furthermore, by Lemma 10.4 of chapter 3 in [8], $u_{\lambda,\mu}$ and $f_{\lambda,\mu}$ depend on λ continuously.

LEMMA 2.2. *There exists a positive constant C , independent of μ , λ such that for any ball $B_r(X^0) \subset D_\mu$ with $X^0 = (x^0, y^0)$, $r < \frac{x^0}{2}$,*

$$\frac{1}{r} \int_{\partial B_r(X^0)} (Q - u) \geq C\lambda x^0$$

implies that $u < Q$ in $B_r(X^0)$, where $u = u_{\lambda,\mu}$ is any solution of (2.1).

The proof can be found in [8]. By using Lemma 2.2 and elliptic estimates we can prove the following “Bounded Gradient Lemma”:

LEMMA 2.3. For any $\delta > 0$, there exists a constant C_δ independent of μ such that

$$\sup_{D_\mu \setminus N_\delta} \frac{|\nabla u|}{x} \leq C_\delta,$$

where $u = u_{\lambda, \mu}$, N_δ is the union of balls of radius δ centered at corners of D_μ .

Proof. For points away from the y-axis, the proof can be found in [3]. It therefore suffices to consider the points near the y-axis.

Fix small $\delta > 0$. For any $X^0 = (x^0, y^0) \in \Omega_\delta = \{u < Q\} \setminus N_\delta$ with $x^0 \leq \frac{\delta}{2}$, we consider the case:

$$(2.8) \quad \begin{aligned} &\text{There is a largest ball } B_r(X^0) \subset \Omega_\delta \text{ with } 0 < r < \frac{x^0}{3} \\ &\text{such that } \partial B_r(X^0) \text{ contains a free-boundary point.} \end{aligned}$$

Then, applying Lemma 2.2, for all small $\epsilon > 0$,

$$\frac{1}{r + \epsilon} \int_{\partial B_{r+\epsilon}(X^0)} (Q - u) \leq C\lambda x^0.$$

Since u is a supsolution with respect to $\mathcal{L}$,

$$\rho_1^{\frac{3}{2}} \int_{\partial B_{\rho_1}} (Q - u) \leq \rho_2^{\frac{3}{2}} \int_{\partial B_{\rho_2}} (Q - u)$$

for any balls $B_{\rho_1}(X^0)$ and $B_{\rho_2}(X^0)$ with $\rho_1 \leq \rho_2 < \frac{x^0}{3}$ (see Appendix A for the proof). We hence deduce that

$$\frac{1}{r} \int_{\partial B_r(X^0)} (Q - u) \leq \frac{1}{r} \left(\frac{r + \epsilon}{r} \right)^{\frac{3}{2}} \int_{\partial B_{r+\epsilon}(X^0)} (Q - u) \leq \left(\frac{r + \epsilon}{r} \right)^{\frac{3}{2} + 1} C\lambda x^0.$$

Letting $\epsilon \rightarrow 0$, we get

$$(2.9) \quad \frac{1}{r} \int_{\partial B_r(X^0)} (Q - u) \leq C\lambda x^0.$$

Introduce

$$\tilde{u}(X) = \frac{Q - u(X^0 + rX)}{r}.$$

We can rewrite (2.9) in terms of $\tilde{u}$ as

$$(2.10) \quad \int_{\partial B_1(0)} \tilde{u} \leq C\lambda x^0.$$

$\tilde{u}$ also satisfies

$$(2.11) \quad \tilde{\mathcal{L}}\tilde{u} = \Delta\tilde{u} - \frac{r}{x^0 + rx} \tilde{u}_x = 0 \quad \text{in } B_1(0).$$

Since the bound of the coefficients of $\tilde{\mathcal{L}}$ is independent of r in $B_1(0)$, we have, by elliptic estimates,

$$|\nabla \tilde{u}(X)| \leq C \int_{\partial B_1} \tilde{u} \leq C \lambda x^0 \quad \text{for } X \in B_{\frac{1}{2}}(0)$$

and in particular,

$$(2.12) \quad |\nabla u(X^0)| = |\nabla \tilde{u}(0)| \leq C \lambda x^0.$$

So far we assumed that (2.8) holds. Now we consider the complementary case:

$$(2.13) \quad \text{There is a ball } B_{\frac{x^0}{3}}(X^0) \subset \Omega_\delta.$$

Set

$$\tilde{u}(X) = \frac{1}{r^2} (Q - u(X^0 + rX)), \quad r = \frac{x^0}{3}.$$

We still have

$$\tilde{\mathcal{L}}\tilde{u} = 0 \quad \text{in } B_1(0).$$

By (2.2),

$$0 \leq \tilde{u} \leq \frac{1}{r^2} \frac{Q}{L_1^2} (x^0 + rx)^2 \leq 16 \frac{Q}{L_1^2} \quad \text{in } B_1(0).$$

It follows by elliptic estimates that

$$|\nabla \tilde{u}(0,0)| \leq C,$$

i.e.

$$(2.14) \quad |\nabla u(X^0)| \leq Cr = C' x^0.$$

Combining (2.12) and (2.14) we have

$$(2.15) \quad \frac{|\nabla u|}{x} \leq C \quad \text{for all } 0 < x \leq \frac{\delta}{2}, \quad X \in D_\mu \setminus N_\delta$$

and Lemma 2.3 follows.

Lemma 2.3 implies the following “Nonoscillation Lemma” :

Let $W \subset D_\mu \cap \{u < Q\}$ be the set bounded by four curves:

$$\begin{aligned} l_1 : & \quad \{(x, y); \quad x_1 \leq x \leq x_1 + \epsilon_1, y = y_1\}, \\ l_2 : & \quad \{(x, y); \quad x_2 \leq x \leq x_2 + \epsilon_2, y = y_1 + h\} \quad h > 0 \end{aligned}$$

and

$$\gamma_1 : \quad \{(x^1(t), y^1(t)); \quad 0 \leq t \leq T\}, \quad \gamma_2 : \quad \{(x^2(t), y^2(t)); \quad 0 \leq t \leq T\},$$

where

$$x^1(t) < x^2(t) \quad \text{for } 0 < t < T.$$

We assume that $\bar{W}$ does not contain N_δ , the union of balls of radius δ centered at corners of D_μ .

LEMMA 2.4. Under the above assumptions, if $\gamma_i \subset \partial\{u < Q\} \cap \bar{D}_\mu$, $i = 1, 2$ (which implies that γ_i may be a part of the fixed boundary) and at least one of γ_i 's belongs to the free boundary, then

$$(2.16) \quad h \leq c_\delta(\epsilon_1 + \epsilon_2),$$

where c_δ is a constant independent of μ .

Proof. If we integrate the equation $\nabla(\frac{1}{x}\nabla u) = 0$ over W and use the divergence theorem, we get

$$\int_{\partial W} \frac{1}{x} \frac{\partial u}{\partial \nu} ds = 0,$$

i.e.

$$\int_{\gamma_1} \frac{1}{x} \frac{\partial u}{\partial \nu} ds + \int_{\gamma_2} \frac{1}{x} \frac{\partial u}{\partial \nu} ds = - \int_{l_1} \frac{1}{x} \frac{\partial u}{\partial \nu} ds - \int_{l_2} \frac{1}{x} \frac{\partial u}{\partial \nu} ds.$$

Notice that $\frac{\partial u}{\partial \nu} \geq 0$ on γ_i , $i = 1, 2$ and $\frac{1}{x} \frac{\partial u}{\partial \nu} = \lambda$ on γ_i if γ_i belongs to the free boundary; the left hand side of the equation above is $\geq \lambda h$. Since the right hand side is bounded by $c_\delta(\epsilon_1 + \epsilon_2)$ (by Lemma 2.3), (2.16) follows.

LEMMA 2.5. (Nondegeneracy Lemma) There exists a universal constant $c > 0$ such that for any ball $B_r(X^0)$ with center $X^0 = (x^0, y^0) \in D_\mu \setminus E_\mu$ and radius $r \leq \frac{x^0}{2}$, the following holds

$$\frac{1}{r} \int_{\partial B_r(X^0)} (Q - u) \leq c\lambda x^0$$

implies that

$$u \equiv Q \quad \text{in } B_{\frac{r}{8}}(X^0) \cap \{D_\mu \setminus E_\mu\},$$

where $u = u_{\lambda, \mu}$ and $Q - u$ is extended by zero into $B_r(X^0) \setminus D_\mu$.

The proof is given in [8]. Lemma 2.5 implies the following corollary:

COROLLARY 2.6. If $X^0 = (x^0, y^0)$ is a free boundary point in $D_\mu \setminus E_\mu$ and if $r \leq \frac{x^0}{2}$, then

$$\frac{1}{r} \int_{\partial B_r(X^0) \cap D_\mu} (Q - u) \geq c\lambda x^0;$$

in particular,

$$\sup_{\partial B_r(X^0) \cap D_\mu} (Q - u) \geq (c\lambda x^0)r.$$

§3. The free boundary $\Gamma_{\lambda, \mu}$ of the truncated solution $u_{\lambda, \mu}$.

Now we fix Q and denote $u_{\lambda, \mu}$, $\Gamma_{\lambda, \mu}$ simply by u , Γ .

Since the free boundary is analytic [2], [8] and $u_y \geq 0$, we can write the free boundary in the form $y = f_{\lambda, \mu}(x)$, or $f(x)$, where $f(x)$ is continuous on some open subset of $(0, L_0)$.

Proof. Clearly, it suffices to prove that Γ must stay strictly below γ . The idea of the proof bases on the proof of Lemma 3.1 in [6].

$$\Gamma_1 : \quad y = f(x) \quad \text{for} \quad 0 < x_0 < x < x_{\lambda, \mu} \leq L_0$$

Turn γ clockwise along Γ_1 a little so that we get another segment γ' which touches Γ_1 tangentially at N and intersects the y-axis at C' . See Figure 3.1.

Take a straight line l_s which contains γ' . Denote by M the farthest point on Γ'_1 to l_s , where Γ'_1 is a portion of Γ_1 which lies below l_s . Then translate l_s parallelly until it touches Γ_1 at M . Denote by E the intersection of l_s with $\partial D_\mu \setminus \{x = 0\}$. See Figure 3.1.

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Take $F \in \partial D_\mu \cap \bar{R}$, which is close to E . Denote by l_0 the distance from F to the y -axis. Denote by $\tilde{u}$ the solution of

$$(3.1) \quad \begin{aligned} \mathcal{L}\tilde{u} &= 0 & \text{in } R, \\ \tilde{u} &= Q & \text{on the } y\text{-axis, } \gamma', \widetilde{MN}, \text{ and } \overline{ME}, \\ \frac{\partial \tilde{u}}{\partial \nu} &= 0 & \text{on } \overline{EF}, \end{aligned}$$

and

$$(3.2) \quad \tilde{u} = -\frac{Q}{l_0^2}x^2 + Q \quad \text{on the remaining part of } \partial R.$$

The solution can be obtained by solving the same minimizing problem as (2.1) with D_μ replaced by R , λ by 0 and K_μ by an appropriate function class. By the maximum principle,

$$(3.3) \quad Q \geq \tilde{u} \geq -\frac{Q}{l_0^2}x^2 + Q \quad \text{in } R.$$

We claim that

$$(3.4) \quad \frac{\frac{\partial u}{\partial \nu}(M)}{\frac{\partial u}{\partial \nu}(N)} \geq \frac{\frac{\partial \tilde{u}}{\partial \nu}(M)}{\frac{\partial \tilde{u}}{\partial \nu}(N)},$$

where $\frac{\partial u}{\partial \nu}(P)$ ($\frac{\partial \tilde{u}}{\partial \nu}(P)$) denotes the outward normal derivative of u ($\tilde{u}$) at P .

Various versions of (3.4) have already been proved in [6] and [16], where they are called the “under-over theorem”. However, our present situation are quite different. We have to modify the proof as follows:

For any point $Z \in \widetilde{MN}$, introduce the function

$$(3.5) \quad \Omega_Z(X) = V(Z)(Q - \tilde{u}(X)) - \tilde{V}(Z)(Q - u(X)) \quad (X = (x, y)),$$

where

$$(3.6) \quad V(Z) = \frac{\partial u}{\partial \nu}(Z), \quad \tilde{V}(Z) = \frac{\partial \tilde{u}}{\partial \nu}(Z)$$

and the level curve:

$$(3.7) \quad \Delta_Z : \quad \Omega_Z(X) = 0$$

initiating at Z and going into the domain R_* where both u and $\tilde{u}$ are smaller than Q .

Notice that $\mathcal{L}\Omega_Z = 0$ in R_* and $\frac{\partial \Omega_Z}{\partial \nu}(X) = 0$ at $X = Z$. It follows by the maximum principle that Δ_Z does in fact initiate from Z into R_* . For the justification of extending Δ_Z , we refer to [6] and references therein.

Δ_Z cannot exit R_* on $\widetilde{MN}$ (otherwise $\Omega_Z \equiv 0$), the streamline $\{u = Q\}$ from N to the y-axis (where $\Omega_Z > 0$) and the segment $\overline{ME}$ (where $\Omega_Z < 0$).

Since Δ_Z , $V(Z)$ and $\tilde{V}(Z)$ depend continuously on Z , the subsets $\mathcal{M}$ and $\mathcal{N}$ of $\widetilde{MN}$ defined below are open:

$$\begin{aligned} Z \in \mathcal{M}, \quad \Delta_Z \text{ exits } R_* \text{ on the open segment } \overline{EF}; \\ Z \in \mathcal{N}, \quad \Delta_Z \text{ does not exit } R_* \text{ on the closure of } \overline{EF}. \end{aligned}$$

There are three possibilities; we consider them separately:

1) $\mathcal{M} = \widetilde{MN}$.

For any $Z \in \widetilde{MN}$, denote by G_Z the region bounded by Δ_Z , $\widetilde{MN}$, $\overline{ME}$ and $\overline{EF}$. Recalling $\frac{\partial u}{\partial \nu} \leq 0$ on $\overline{EF}$, we have

$$\frac{\partial}{\partial \nu} \Omega_Z = \tilde{V}(Z) \frac{\partial u}{\partial \nu} \leq 0 \quad \text{on } \overline{EF}, \quad \nu = \text{outward normal}.$$

Since also $\Omega_Z \leq 0$ on the remaining part of ∂G_Z , by the maximum principle,

$$\Omega_Z < 0 \quad \text{in } G_Z.$$

Hence

$$\frac{\partial}{\partial \nu} \Omega_Z > 0 \quad \text{at } M, \quad \nu = \text{outward normal},$$

i.e.

$$-V(Z) \frac{\partial \tilde{u}}{\partial \nu}(M) + \tilde{V}(Z) \frac{\partial u}{\partial \nu}(M) > 0.$$

Letting $Z \rightarrow N$, we obtain

$$\tilde{V}(N) \frac{\partial u}{\partial \nu}(M) \geq V(N) \frac{\partial \tilde{u}}{\partial \nu}(M)$$

and (3.4) follows.

2) $\mathcal{N} = \widetilde{MN}$.

Taking any $Z \in \widetilde{MN}$, denote by $\tilde{G}_Z$ the region bounded by Δ_Z , $\{y = -\mu\}$, the y-axis and the streamline $\{u = Q\}$ from N to the y-axis.

If Δ_Z terminates on the y-axis, then

$$(3.8) \quad \Omega_Z \geq 0 \quad \text{on } \partial \tilde{G}_Z.$$

If Δ_Z terminates on $\{y = -\mu\}$, then for some $X_0 = (x_0, -\mu)$, $0 < x_0 < l_0$,

$$\Omega_Z(X_0) = 0,$$

i.e.

$$(3.9) \quad V(Z) \frac{Q}{l_0^2} x_0^2 = \tilde{V}(Z) \frac{Q}{L_1^2} x_0^2,$$

where we used (3.2) and the boundary condition for u on $\{y = -\mu\}$. Hence for any point $X = (x, -\mu)$ with $0 < x < x_0$,

$$\Omega_Z(X) = V(Z) \frac{Q}{l_0^2} x^2 - \tilde{V}(Z) \frac{Q}{L_1^2} x^2 = 0 \quad \text{by (3.9).}$$

We still get (3.8). It follows that, by the maximum principle, $\Omega_Z > 0$ in $\tilde{G}_Z$ and

$$\frac{\partial \Omega_Z}{\partial \nu} < 0 \quad \text{at } N, \quad \nu = \text{outward normal},$$

i.e.

$$-V(Z) \frac{\partial \tilde{u}}{\partial \nu}(N) + \tilde{V}(Z) \frac{\partial u}{\partial \nu}(N) < 0.$$

Letting $Z \rightarrow M$, we obtain

$$\tilde{V}(M) \frac{\partial u}{\partial \nu}(N) \leq V(M) \frac{\partial \tilde{u}}{\partial \nu}(N)$$

and (3.4) follows again.

Finally, consider the case

$$\mathcal{M} \neq \emptyset, \quad \mathcal{N} \neq \emptyset.$$

Since $\mathcal{M}$ and $\mathcal{N}$ are open and disjoint, there exists a point $Z \in \widetilde{MN}$ such that Δ_Z terminates at F . Then as in cases 1), 2) we can easily verify that

$$\Omega_Z < 0 \quad \text{in } G_Z, \quad \text{and} \quad \Omega_Z > 0 \quad \text{in } \tilde{G}_Z.$$

Thus by the maximum principle,

$$\frac{\partial \Omega_Z}{\partial \nu}(M) > 0, \quad \frac{\partial \Omega_Z}{\partial \nu}(N) < 0,$$

i.e.

$$-V(Z) \frac{\partial \tilde{u}}{\partial \nu}(M) + \tilde{V}(Z) \frac{\partial u}{\partial \nu}(M) > 0$$

and

$$-V(Z) \frac{\partial \tilde{u}}{\partial \nu}(N) + \tilde{V}(Z) \frac{\partial u}{\partial \nu}(N) < 0.$$

Combining these two inequalities, we have

$$\frac{\frac{\partial u}{\partial \nu}(M)}{\frac{\partial \tilde{u}}{\partial \nu}(M)} > \frac{V(Z)}{\tilde{V}(Z)} > \frac{\frac{\partial u}{\partial \nu}(N)}{\frac{\partial \tilde{u}}{\partial \nu}(N)}$$

and (3.4) follows.

Having proven (3.4), we consider the similarity transformation

$$(3.10) \quad (x, y) \longmapsto (\rho x, \rho(y - y_0)) \quad (0 < \rho < 1)$$

which maps M to N , where $C'' = (0, y_0)$ is the intersection of the y -axis and the straight line passing through M, N .

Thus if

$$M = (x_M, y_M), \quad N = (x_N, y_N),$$

then

$$(3.11) \quad x_N = \rho x_M.$$

Denote by R_ρ the image of R under this transformation, and set

$$u_\rho(x, y) = \rho^2(Q - \tilde{u}(\frac{x}{\rho}, \frac{y}{\rho} + y_0)) \quad \text{for } (x, y) \in R_\rho.$$

Then u_ρ satisfies

$$\mathcal{L}u_\rho = 0 \quad \text{in } R_\rho.$$

Choose F so close to E that the segment $\overline{EF}$ is transformed by (3.10) to a segment lying outside R . We look at values of u_ρ and $Q - \tilde{u}$ on $\partial(R_\rho \cap R)$:

By (3.3) and the second equation in (3.1),

$$u_\rho \geq Q - \tilde{u} \quad \text{on } (\partial R) \cap \overline{R}_\rho.$$

On the remaining part $(\partial R_\rho) \cap R$ of $\partial(R_\rho \cap R)$,

$$\begin{aligned} u_\rho(x, y) &= \rho^2(Q - \tilde{u}(\frac{x}{\rho}, \frac{y}{\rho} + y_0)) \\ &= \rho^2 \frac{Q}{l_0^2} (\frac{x}{\rho})^2 \\ &= \frac{Q}{l_0^2} x^2 \geq Q - \tilde{u}(x, y), \end{aligned}$$

where we have used (3.3) in the last step. Thus

$$u_\rho \geq Q - \tilde{u} \quad \text{on } \partial(R_\rho \cap R).$$

By the maximum principle, we get

$$u_\rho > Q - \tilde{u} \quad \text{in } R_\rho \cap R.$$

Since $u_\rho = 0 = Q - \tilde{u}$ at N , we have, by applying the maximum principle,

$$\frac{\partial u_\rho}{\partial \nu} < \frac{\partial}{\partial \nu}(Q - \tilde{u}) \quad \text{at } N \quad \nu = \text{outward normal.}$$

Notice that the outward normal direction at N is the same as that at M . We have

$$\frac{\partial \tilde{u}}{\partial \nu}(N) < \rho \frac{\partial \tilde{u}}{\partial \nu}\left(\frac{x_N}{\rho}, \frac{y_N}{\rho} + y_0\right) = \rho \frac{\partial \tilde{u}}{\partial \nu}(M) \quad \text{by (3.11),}$$

i.e.

$$(3.12) \quad \frac{1}{\rho} < \frac{\frac{\partial \tilde{u}}{\partial \nu}(M)}{\frac{\partial \tilde{u}}{\partial \nu}(N)} \leq \frac{\frac{\partial u}{\partial \nu}(M)}{\frac{\partial u}{\partial \nu}(N)} \quad \text{by (3.4).}$$

Since, however, M, N lie on the free boundary,

$$\frac{\frac{\partial u}{\partial \nu}(M)}{\frac{\partial u}{\partial \nu}(N)} = \frac{x_M}{x_N} = \frac{1}{\rho} \quad \text{by (3.11),}$$

which contradicts (3.12).

Remark 3.2. In case $L_1 \leq L_0$, we can use the nonoscillation lemma (Lemma 2.4) to show that the free boundary $\Gamma_{\lambda, \mu}$ consists only of one piece provided μ is sufficiently large.

From now on we shall assume that $y = f_{\lambda, \mu}(x)$ is defined on $(0, x_{\lambda, \mu})$ with $0 < x_{\lambda, \mu} \leq L_0$. By the nonoscillation lemma, $f_{\lambda, \mu}(0+)$ and $f_{\lambda, \mu}(x_{\lambda, \mu}-)$ exist.

Denote by $h_{\lambda, \mu} = f_{\lambda, \mu}(0+)$, $A' = (0, h_{\lambda, \mu})$.

THEOREM 3.3. *If $h_{\lambda, \mu} > -\mu$, then the free boundary $\Gamma_{\lambda, \mu}$ is tangent at the y -axis. More precisely,*

$$(3.13) \quad f'_{\lambda, \mu}(0+) = +\infty.$$

Proof. For simplicity, denote $u_{\lambda, \mu}$, $\Gamma_{\lambda, \mu}$, $f_{\lambda, \mu}$ etc. by u , Γ , f respectively. We want to prove

$$\liminf_{x \rightarrow 0+} \frac{f(x) - f(0)}{x} = +\infty.$$

The proof consists of three steps:

Step 1. Suppose

$$(3.14) \quad \liminf_{x \rightarrow 0+} \frac{f(x) - f(0)}{x} = \alpha < +\infty.$$

By the nonoscillation lemma (Lemma 2.4), $\alpha > -\infty$. Choose $\gamma_n \rightarrow 0$ such that

$$\lim_{n \rightarrow \infty} \frac{f(\gamma_n) - f(0)}{\gamma_n} = \alpha.$$

We consider the blow-up sequence

$$u_n(X) = \frac{1}{\gamma_n^2}(Q - u(A' + \gamma_n X)).$$

By (2.2),

$$0 \leq u_n(X) \leq \frac{Q}{L_1^2} x^2.$$

It follows, by the bounded gradient lemma (Lemma 2.3) that for some subsequence

$$u_n \rightarrow u_0 \quad \text{uniformly in compact sets in } \overline{R}_+^2 = \{(x, y); x \geq 0\},$$

where, by the similar proof as for Lemma 3.6 in chapter 3 of [8], u_0 is an absolute minimizer for the functional

$$J(v) = \int_G (|\frac{\nabla v}{x}|^2 + \lambda^2 I_{\{v>0\}}) x dx dy$$

in any bounded domain G in R_+^2 with $v = u_0$ on ∂G .

Notice that the free boundary Γ_n of u_n can be written in the form

$$(3.15) \quad \Gamma_n = \{(x, y); \quad y = f_n(x) = \frac{f(\gamma_n x) - f(0)}{\gamma_n}; \quad 0 \leq x \leq \frac{x_{\lambda, \mu}}{\gamma_n}\}.$$

Choose $X_n^0 = (1, f_n(1)) \in \Gamma_n$. Then $X_n^0 \rightarrow X^0 = (1, \alpha)$ as $n \rightarrow \infty$. By the nondegeneracy lemma (Lemma 2.5) applied to u_n at point X_n^0 , we have

$$\int_{\partial B_{\frac{1}{2}}(X_n^0)} u_n \geq c\lambda > 0.$$

Letting $n \rightarrow \infty$, we get

$$\int_{\partial B_{\frac{1}{2}}(X^0)} u_0 \geq c\lambda,$$

which implies that $u_0 \not\equiv 0$.

Let

$$\Omega_0 = \{X; \quad u_0(X) > 0\} \neq \emptyset.$$

Then u_0 is a solution of the following free boundary problem:

$$(3.16a) \quad \mathcal{L}u_0 = 0 \quad \text{in } \Omega_0,$$

$$(3.16b) \quad u_0 = 0, \quad \frac{1}{x} \frac{\partial u_0}{\partial \nu} = -\lambda \quad \text{on } \Gamma_0,$$

$$(3.16c) \quad u_0(0, y) = 0,$$

and

$$(3.16d) \quad 0 \leq u_0 \leq \frac{Q}{L_1^2} x^2,$$

where $\Gamma_0 = (\partial\{X; u_0(X) > 0\}) \cap \Omega_0$ is an analytic curve [2].

Since $\frac{\partial u_n}{\partial y} \leq 0$, we have

$$(3.17) \quad \frac{\partial u_0}{\partial y} \leq 0.$$

It follows that the free boundary Γ_0 is an x-graph, i.e.

$$\Gamma_0 : \quad y = f_0(x)$$

for some function f_0 which is continuous in some open set in $\{x > 0\}$.

Since $\Gamma_n \rightarrow \Gamma_0$ ($n \rightarrow \infty$) in the Hausdorff distance (see Lemma 3.6 in chapter 3 of [8]), we have

$$f_0(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \text{at } x \text{ where } f_0(x) \text{ is finite,}$$

where f_n is defined in (3.15).

Hence

$$f_0(x) \geq \lim_{n \rightarrow \infty} f_n(x) \geq x \liminf_{n \rightarrow \infty} \frac{f(\gamma_n x) - f(0)}{\gamma_n x} \geq \alpha x, \quad \text{by (3.14),}$$

which implies that the free boundary Γ_0 lies above the ray

$$l_0 : \quad y = \alpha x, \quad x \geq 0.$$

Notice also that

$$\lim_{n \rightarrow \infty} f_n(1) = \alpha, \quad \text{and} \quad \lim_{n \rightarrow \infty} f_n(0) = 0.$$

Γ_0 , initiating at $(0,0)$, must touch l_0 at $(1, \alpha)$ (see Figure 3.2).

By the nonoscillation lemma (Lemma 2.4), $f_0(x)$ remains finite on $(0,1)$.

Step 2. We want to prove $\Gamma_0 \equiv l_0$ in $\{0 < x < 1\}$. Suppose that this is not true. We consider case $\alpha \leq 0$ and $\alpha > 0$ separately. Accordingly, we divide Step 2 into two substeps.

Substep 2.1. $\alpha \leq 0$.

Set $M' = (1, \alpha)$. We turn l_0 around M' a little such that the new ray $l'_0 : y = \tau x + y_0$, $x \geq 0$ satisfies

- $$(3.18) \quad \begin{aligned} (1) & \quad \tau < \alpha; \\ (2) & \quad l'_0 \text{ is tangent to } \Gamma_0 \text{ at } M = (x_M, y_M); \\ (3) & \quad l'_0 \text{ completely lies below } \Gamma_0 \text{ in } \{x > x_M\}; \\ (4) & \quad l'_0 \text{ intersects } \Gamma_0 \text{ at } N' = (x_{N'}, y_{N'}); \\ (5) & \quad l'_0 \cap \{x < x_{N'}\} \text{ lies above } \Gamma_0. \end{aligned}$$

(See Figure 3.2).

Denote by $N = (x_N, y_N) \in \Gamma_0$, the point on $\widetilde{MN'}$ which is farthest from l'_0 . Connect N

and E such that the segment $\overline{EN}$ is parallel to l'_0 . (See Figure 3.2)

Figure 3.2

For any large $\mu > 0$, denote by R_μ the region bounded by the y-axis, the segment $\overline{EN}$, the portion $\widetilde{MN}$ of Γ_0 , $\{y = -\mu\}$ and the part of the ray l'_0 starting from M .

Denote by F the intersection of l'_0 and $\{y = -\mu\}$, by F' a point on $\{y = -\mu\} \cap \partial R_\mu$ which is chosen close to F , and by T the part of $\{y = -\mu\} \cap \partial R_\mu$ from F' to the y-axis. (See Figure 3.2)

Denote by u_μ the solution of

$$(3.19a) \quad \mathcal{L}u_\mu = 0 \quad \text{in } R_\mu,$$

$$(3.19b) \quad u_\mu = 0 \quad \text{on } \partial R_\mu \setminus \{y = -\mu\},$$

$$(3.19c) \quad \frac{\partial u_\mu}{\partial \nu} = 0 \quad \text{on } \overline{FF'},$$

and

$$(3.19d) \quad u_\mu = \frac{Q}{L_1^2} x^2 \quad \text{on } T.$$

By the maximum principle,

$$(3.19e) \quad 0 < u_\mu < \frac{Q}{L_1^2} x^2 \quad \text{in } R_\mu.$$

We claim that

$$(3.20) \quad \lim_{\mu \rightarrow \infty} u_\mu = 0 \quad \text{uniformly in compact sets of } \overline{R_+^2}.$$

In fact, choose $\tau < \tau' < 0$. Then for any $\epsilon > 0$ we consider the auxiliary function

$$w_\epsilon(x, y) = \epsilon x^2 (\tau' x + y_E - y) \quad (E = (0, y_E)).$$

It is easy to check that

$$\mathcal{L}w_\epsilon = 3\epsilon\tau'x < 0 \quad \text{in } \{x > 0\},$$

and

$$w_\epsilon \geq 0 = u_\mu \quad \text{on } (\partial R_\mu) \setminus \{y = -\mu\}.$$

On $(\partial R_\mu) \cap \{y = -\mu\}$, we have $0 \leq x \leq \frac{-\mu - y_0}{\tau}$ so that

$$\begin{aligned} w_\epsilon &\geq \epsilon x^2 \left(\tau' \frac{-\mu - y_0}{\tau} + y_E + \mu \right) \\ &= \epsilon x^2 \left[\left(1 - \frac{\tau'}{\tau}\right) \mu - \frac{\tau'}{\tau} y_0 + y_E \right] \\ &> \frac{Q}{L_1^2} x^2 \geq u_\mu \quad \text{by (3.19e)} \end{aligned}$$

if μ is large enough.

It then follows by the maximum principle that

$$0 \leq u_\mu \leq w_\epsilon \quad \text{in } R_\mu$$

for all large $\mu > 0$. Letting $\mu \rightarrow \infty$ and then $\epsilon \rightarrow 0$, (3.20) follows.

By elliptic estimates we can choose μ so large that

$$(3.21) \quad |\nabla u_\mu(Z)| < |\nabla u_0(Z)| \quad \forall \quad Z \in \widetilde{MN},$$

where u_0 is the solution of (3.16) and (3.17).

In order to complete Substep 2.1, we prove another version of the under-over theorem in [16]:

$$(3.22) \quad \frac{|\nabla u_\mu(M)|}{|\nabla u_\mu(N)|} \leq \frac{|\nabla u_0(M)|}{|\nabla u_0(N)|}.$$

As in the proof of Theorem 3.1, for any point $Z \in \widetilde{MN}$, consider the function

$$(3.23) \quad \Omega_Z(X) = V(Z)u_\mu(X) - V_\mu(Z)u_0(X),$$

where

$$(3.24) \quad V(Z) = |\nabla u_0(Z)|, \quad V_\mu(Z) = |\nabla u_\mu(Z)|,$$

and the level curve

$$(3.25) \quad \Delta_Z : \quad \Omega_Z(X) = 0$$

initiating at Z and going into the domain $R_\star$ where both u_0 and u_μ are positive.

Δ_Z cannot exit R_* on $\widetilde{MN}$ (otherwise $\Omega_Z \equiv 0$); it also cannot exit the streamline $\widetilde{ON}$ (where $\Omega_Z > 0$) and the segment $\overline{MF}$ (where $\Omega_Z < 0$); finally Δ_Z cannot exit R_* on T , since by (3.16d), (3.19d) and (3.21), $\Omega_Z(X) > 0$ on T .

Define two open subsets $\mathcal{M}, \mathcal{N}$ of $\widetilde{MN}$ as follows:

$$\begin{aligned} Z \in \mathcal{M}, \quad & \Delta_Z \text{ exits } R_* \text{ on the segment } \overline{FF'}; \\ Z \in \mathcal{N}, \quad & \Delta_Z \text{ does not exit } R_* \text{ on } \overline{FF'}. \end{aligned}$$

From the consideration above, $\widetilde{MN} = \mathcal{M} \cup \mathcal{N}$. Since $\mathcal{M} \cap \mathcal{N} = \emptyset$ and $\widetilde{MN}$ is connect, we have

$$\text{either } \mathcal{M} = \widetilde{MN} \quad \text{or} \quad \mathcal{N} = \widetilde{MN}.$$

First we consider the case $\mathcal{M} = \widetilde{MN}$. For any $Z \in \widetilde{MN}$, denote by G_Z the region bounded by Δ_Z , $\widetilde{MN}$, $\overline{MF}$ and $\overline{FF'}$. By (3.17), (3.19c),

$$\frac{\partial \Omega_Z}{\partial \nu} = -V_\mu(Z) \frac{\partial u_0}{\partial \nu} \geq 0 \quad \text{on } \overline{FF'}, \quad \nu = \text{inward normal}.$$

Since $\Omega_Z \leq 0$ on the remaining part of ∂G_Z , by the maximum principle we have $\Omega_Z < 0$ in G_Z and

$$\frac{\partial \Omega_Z}{\partial \nu} < 0 \quad \text{at } M, \quad \nu = \text{inward normal}.$$

Taking $Z \rightarrow N$ we obtain

$$V(N) | \nabla u_\mu(M) | - V_\mu(N) | \nabla u_0(M) | \leq 0$$

and (3.22) follows.

If the complementary case $\mathcal{N} = \widetilde{MN}$ holds, then for any $Z \in \widetilde{MN}$, Δ_Z must terminate on the y-axis. Denote by $\tilde{G}_Z$ the domain bounded by $\widetilde{ON}$, $\widetilde{MN}$, Δ_Z and the y-axis. Then $\Omega_Z \geq 0$ on $\partial \tilde{G}_Z$. It follows by the maximum principle that $\Omega_Z > 0$ in $\tilde{G}_Z$ and

$$\frac{\partial \Omega_Z}{\partial \nu} > 0 \quad \text{at } N, \quad \nu = \text{inward normal}.$$

Taking $Z \rightarrow M$ we obtain

$$V(M) | \nabla u_\mu(N) | - V_\mu(M) | \nabla u_0(N) | \geq 0$$

and (3.22) follows.

Having proven (3.22), we consider the similarity transformation

$$(3.26) \quad (x, y) \mapsto (\rho x, \rho(y - y'_0)) \quad (0 < \rho < 1)$$

which maps M to N , where $O' = (0, y'_0)$ is the intersection of the y-axis and the straight line passing through M, N . Thus

$$(3.27) \quad \rho x_M = x_N.$$

Denote by R_μ^ρ the image of R_μ under this transformation, and set

$$(3.28) \quad u_\mu^\rho(x, y) = \rho^2 u_\mu\left(\frac{x}{\rho}, \frac{y}{\rho} + y'_0\right) \quad \text{for } (x, y) \in R_\mu^\rho.$$

Then u_μ^ρ satisfies

$$\mathcal{L}u_\mu^\rho = 0 \quad \text{in } R_\mu^\rho.$$

Choose F' so close to F that the segment $\overline{FF'}$ is transformed by (3.26) to a segment lying outside R_μ . Using (3.19b), (3.19d) and (3.19e), we can easily verify that

$$\begin{aligned} u_\mu^\rho &\geq u_\mu & \text{on } \partial(R_\mu \cap R_\mu^\rho), \\ u_\mu^\rho &= u_\mu = 0 & \text{at } N. \end{aligned}$$

By the maximum principle we get $u_\mu^\rho > u_\mu$ in $R_\mu \cap R_\mu^\rho$ and

$$\frac{\partial u_\mu^\rho}{\partial \nu}(N) > \frac{\partial u_\mu}{\partial \nu}(N), \quad \nu = \text{inward normal}$$

or

$$| \nabla u_\mu^\rho(N) | > | \nabla u_\mu(N) |.$$

Using (3.27) and (3.28), we get

$$\rho | \nabla u_\mu(M) | > | \nabla u_\mu(N) |.$$

Hence, by (3.22),

$$\frac{| \nabla u_0(M) |}{| \nabla u_0(N) |} > \frac{1}{\rho}.$$

Since, however, M and N lie on the free boundary,

$$\frac{| \nabla u_0(M) |}{| \nabla u_0(N) |} = \frac{x_M}{x_N} = \frac{1}{\rho} \quad \text{by (3.27)}$$

which is a contradiction.

Substep 2.2. $\alpha > 0$.

Choose $0 < \tau < \alpha$ and construct the ray l'_0 : $y = \tau x + y_0$ such that it satisfies the last four conditions of (3.18).

For any $\mu > 0$, we construct the domain R_μ in essentially the same way as in the case $\alpha \leq 0$ except that $\{y = -\mu\}$ no longer intersects l'_0 (see Figure 3.3).

Figure 3.3

Denote by u_μ the solution of

$$(3.29a) \quad \mathcal{L}u_\mu = 0 \quad \text{in } R_\mu$$

with the boundary conditions

$$(3.29b) \quad u_\mu = 0 \quad \text{on } (\partial R_\mu) \setminus \{y = -\mu\},$$

$$(3.29c) \quad u_\mu = \frac{Q}{L_1^2} x^2 \quad \text{on } \{y = -\mu\}.$$

By the maximum principle,

$$0 < u_\mu < \frac{Q}{L_1^2} x^2 \quad \text{in } R_\mu.$$

In order to continue with the proof, we need two auxiliary lemmas.

Introduce the polar coordinate system at $E = (0, y_E)$ such that $\{\theta = 0\}$ corresponds to $\{(0, y); y \geq y_E\}$. Consider the function

$$(3.30) \quad v(r, \theta) = r^{\beta+1} \sin^2 \theta P_{\beta-1}^{\frac{1}{2}+1}(\cos \theta), \quad \beta > 1$$

where $P_\mu^\lambda(z)$ are the ultraspherical functions or Gegenbauer functions (see reference [7], P.178, where the notation C_μ^λ is used) which satisfy the equation

$$(3.31) \quad (1 - z^2) \frac{d^2 w}{dz^2} - (2\lambda + 1)z \frac{dw}{dz} + \mu(\mu + 2\lambda)w = 0.$$

This implies that

$$\mathcal{L}v = 0.$$

LEMMA 3.4. For any $0 < \theta_0 < \pi$,

$$P_{\beta-1}^{\frac{1}{2}+1}(\cos \theta) > 0 \quad \text{if } 0 < \theta < \pi - \theta_0$$

provided $\beta > 1$ and $\beta - 1$ is sufficiently small.

The proof is given in Appendix B.

LEMMA 3.5. Let u_μ be the solution of (3.29). Then

$$\lim_{\mu \rightarrow \infty} u_\mu = 0 \quad \text{uniformly in compact sets in } \overline{R_+^2}.$$

Proof. Set

$$\theta_0 = \arctan \tau > 0.$$

Denote by Σ the sector

$$\Sigma : \quad \theta_0 < \theta < \pi.$$

Then $R_\mu \subset \Sigma$ for all $\mu > 0$. Set

$$(3.32) \quad w(r, \theta) = v(r, \pi - \theta),$$

where v is defined by (3.30). Choose $\beta > 1$ such that Lemma 3.4 holds. Then

$$\begin{aligned} \mathcal{L}w &= 0 & \text{in } \Sigma, \\ w &\geq 0 & \text{on } \partial\Sigma. \end{aligned}$$

Since $\beta > 1$, we also have

$$\frac{w}{x^2} \rightarrow \infty \quad \text{if } X = (x, y) \in \Sigma, \quad r = |X - E| \rightarrow \infty.$$

By the maximum principle we deduce that for any $\epsilon > 0$

$$u_\mu < \epsilon w \quad \text{if } X \in R_\mu$$

as long as μ is sufficiently large. It follows that

$$\lim_{\mu \rightarrow \infty} u_\mu = 0 \quad \text{uniformly in compact sets in } \overline{R_+^2}$$

and the proof is complete.

The completion of Substep 2.2.

By elliptic estimates and Lemma 3.5 we can choose μ so large that

$$(3.33) \quad |\nabla u_\mu(Z)| < |\nabla u_0(Z)|, \quad \forall Z \in \widetilde{MN}$$

where u_0 is the solution of (3.16) and (3.17). We shall show that (3.22) still holds at the present setting.

For any $Z \in \widetilde{MN}$, consider the function Ω_Z and the level curve Δ_Z which are defined by (3.23), (3.24) and (3.25).

By (3.16d), (3.29c) and (3.33), Δ_Z either exits $R_\star$ on the y-axis or stays in $R_\star$, where $R_\star$ is the domain where both u_0 and u_μ are positive. Define two open subsets $\mathcal{M}, \mathcal{N}$ of $\widetilde{MN}$ as follows:

$$\begin{aligned} Z \in \mathcal{M}; & \quad \Delta_Z \text{ stays in } R_\star; \\ Z \in \mathcal{N}; & \quad \Delta_Z \text{ exits } R_\star \text{ on the y-axis.} \end{aligned}$$

Since $\widetilde{MN} = \mathcal{M} \cup \mathcal{N}$, $\mathcal{M} \cap \mathcal{N} = \emptyset$, it follows that

$$\text{either } \mathcal{M} = \widetilde{MN} \quad \text{or} \quad \mathcal{N} = \widetilde{MN}.$$

We only prove (3.22) for the case $\mathcal{M} = \widetilde{MN}$, since the situation $\mathcal{N} = \widetilde{MN}$ can be considered similarly as we did in the case $\alpha \leq 0$.

For any $Z \in \mathcal{M} = \widetilde{MN}$, denote by G_Z the domain bounded by Δ_Z , $\widetilde{MN}$ and l'_0 . Consider the function

$$\Omega_Z^\epsilon(X) = \Omega_Z(X) - \epsilon w \quad \text{in } G_Z,$$

where w is defined by (3.32) and $\epsilon > 0$ is an arbitrary constant. Then

$$\begin{aligned} \mathcal{L}\Omega_Z^\epsilon &= 0 & \text{in } G_Z, \\ \Omega_Z^\epsilon &\leq 0 & \text{on } \partial G_Z, \end{aligned}$$

and

$$\lim_{|X| \rightarrow \infty} \Omega_Z^\epsilon < 0.$$

By the Phragmén-Lindelöf's Lemma [8], we have

$$\Omega_Z^\epsilon < 0 \quad \text{in } G_Z.$$

Letting $\epsilon \rightarrow 0$ we deduce, by the maximum principle, that $\Omega_Z < 0$ in G_Z and

$$\frac{\partial \Omega_Z}{\partial \nu} < 0 \quad \text{at } M, \quad \nu = \text{inward normal}.$$

Taking $Z \rightarrow N$ we obtain

$$V(N) \mid \nabla u_\mu(M) \mid - V_\mu(N) \mid \nabla u_0(M) \mid \leq 0$$

and (3.22) follows.

By virtue of (3.22), we can use the similarity transformation argument as in the case $\alpha \leq 0$ to derive a contradiction.

Henceforth, Γ_0 must coincide with l_0 in $(0,1)$.

Step 3. Consider another blow-up sequence

$$u_0^\gamma(X) = \frac{1}{\gamma^2} u_0(\gamma X)$$

for $\gamma \rightarrow 0$, where u_0 is the solution of (3.16). Then for some subsequence,

$$u_0^\gamma(X) \rightarrow u'_0(X) \quad \text{uniformly in compact sets in } \overline{R_+^2}$$

and $u'_0 \not\equiv 0$. Moreover,

$$\Gamma_0^\gamma \rightarrow l_0 \quad \text{in the Hausdorff distance,}$$

where Γ_0^γ is the free boundary of u_0^γ . By (3.16) we also have

$$\begin{aligned}\mathcal{L}u'_0 &= 0 && \text{in } \Sigma, \\ u'_0 &= 0, && \frac{1}{x} \frac{\partial u'_0}{\partial \nu} = -\lambda \quad \text{on } l_0, \\ u'_0(0, y) &= 0,\end{aligned}$$

and

$$0 \leq u'_0 \leq \frac{Q}{L_1^2} x^2.$$

where Σ is the sector bounded by the negative y-axis and the ray l_0 .

By the proof of Lemma 3.5, u'_0 is identically zero. This, however, contradicts $u'_0 \not\equiv 0$.

THEOREM 3.6. *If $h_{\lambda,\mu} > -\mu$, then $\Gamma_{\lambda,\mu} \cap \{x < L_1\}$ is star shaped with respect to $B = (L_1, 0)$. Consequently, the function*

$$\frac{f_{\lambda,\mu}(x)}{x - L_1} \quad \text{is monotone decreasing in } \{0 < x < \min\{L_1, x_{\lambda,\mu}\}\}$$

and

$$f'_{\lambda,\mu}(x) \geq \frac{f_{\lambda,\mu}(x)}{x - L_1} \quad \text{if } 0 < x < \min\{L_1, x_{\lambda,\mu}\},$$

where $x_{\lambda,\mu}$ is defined in Remark 3.2.

Proof. If the assertion is not true, then there exists a ray l_0 , initiating at B and intersecting $\Gamma_{\lambda,\mu} \cap \{x < L_1\}$ at least twice. We rotate l_0 counterclockwise with respect to the center B such that l_0 is tangent to $\Gamma_{\lambda,\mu}$ at M and the segment $\overline{MB}$ lies below $\Gamma_{\lambda,\mu}$. Since $f'_{\lambda,\mu}(0+) = +\infty$, we can find a segment $\overline{EN}$ which is parallel to $\overline{MB}$, tangent to $\Gamma_{\lambda,\mu}$ at N and lies above $\Gamma_{\lambda,\mu}$ (see Figure 3.4).

Figure 3.4

Proceeding as in the proof of Theorem 3.1, we can derive a contradiction by using the inequality (3.4).

Henceforth, the function

$$\frac{f_{\lambda,\mu}(x)}{x - L_1} \quad \text{is monotone decreasing in } (0, \min\{L_1, x_{\lambda,\mu}\})$$

and at points where $f_{\lambda,\mu}$ is differentiable,

$$f'_{\lambda,\mu}(x) \geq \frac{f_{\lambda,\mu}(x)}{x - L_1} \quad 0 < x < \min\{L_1, x_{\lambda,\mu}\}.$$

Since $y = f_{\lambda,\mu}(x)$ has analytic representation, this inequality actually holds for all $0 < x < \min\{L_1, x_{\lambda,\mu}\}$.

§4. The proof of Theorem 1.2.

LEMMA 4.1. *If $\lambda \geq \frac{QL_0}{L_1^2(H-H_0)}$, then the free boundary $\Gamma_{\lambda,\mu}$ must initiate on the half vertical line $\{x = L_0\} \cap \{y \leq H\}$.*

Proof. Suppose the assertion is not true. Recalling Theorem 3.3 and the smooth-fit theorem in [8], we can choose $H' \geq H$ such that $\Gamma_{\lambda,\mu}$ lies below the horizontal line $\{y = H'\}$ and touches it at $X_0 = (x_0, H')$ with $0 < x_0 < L_0$.

Consider the auxiliary function

$$w(x, y) = -\frac{Q}{L_1^2(H' - H_0)}x^2(H' - y) + Q.$$

By (2.2), we can easily verify that

$$\mathcal{L}w = 0 \quad \text{in } D_\mu \cap \{y \geq H_0\}$$

and

$$w \leq u_{\lambda,\mu} \quad \text{on } \partial\{D_\mu \cap \{y \geq H_0\}\}.$$

It follows by the maximum principle that $u_{\lambda,\mu} > w$ in $D_\mu \cap \{y > H_0\}$ and

$$\begin{aligned} \lambda = \frac{1}{x_0} \frac{\partial u_{\lambda,\mu}}{\partial \nu} &< \frac{1}{x_0} \frac{\partial w}{\partial \nu} = \frac{Qx_0}{L_1^2(H' - H_0)} \quad \text{at } X_0 \\ &< \frac{QL_0}{L_1^2(H - H_0)}, \end{aligned}$$

where ν is the outward normal. This contradicts the assumption of the lemma.

Set

$$(4.1) \quad \Sigma = \{\lambda; \quad \Gamma_{\lambda,\mu} \text{ does not initiate on } \{x = L_0\} \cap \{y \leq H\}\}.$$

Then $\Sigma \neq \emptyset$. Otherwise, by continuous dependence of $(u_{\lambda,\mu}, \Gamma_{\lambda,\mu})$ on λ (see Lemma 10.4 of chapter 3 in [8]), $u_{\lambda,\mu} \rightarrow u_0$, $\Gamma_{\lambda,\mu} \rightarrow \Gamma_0$ as $\lambda \rightarrow 0$, where $u_0 \not\equiv Q$ satisfies

$$\begin{aligned} \mathcal{L}u_0 &= 0 \quad \text{in } D_\mu, \\ u_0 &= Q, \quad \frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \neq \emptyset. \end{aligned}$$

Hence $u_0 \equiv Q$, which is a contradiction.

Set

$$(4.2) \quad \lambda_\mu = \sup_{\Sigma} \lambda.$$

By Lemma 4.1,

$$(4.3) \quad \lambda_\mu < \frac{QL_0}{L_1^2(H - H_0)}.$$

It follows, by continuous dependence of $(u_{\lambda,\mu}, \Gamma_{\lambda,\mu})$ on λ , that the pair

$$(u_\mu, \Gamma_\mu)$$

with $u_\mu = u_{\lambda_\mu,\mu}$, $\Gamma_\mu = \Gamma_{\lambda_\mu,\mu}$ solves the truncated problem in D_μ . In particular, Γ_μ initiates at the prescribed point $A = (L_0, H)$.

We now take a sequence $\mu = \mu_n \rightarrow \infty$ such that

$$\begin{aligned} \lambda_\mu &\rightarrow \lambda; \\ u_\mu &\rightarrow u \quad \text{uniformly in compact sets in } G; \\ f_\mu &\rightarrow f \quad \text{pointwise.} \end{aligned}$$

Then (u, Γ, λ) with

$$(4.4) \quad \Gamma : \quad y = f(x) \quad 0 \leq x_\lambda < x < L_0$$

satisfies all conditions in Definition 1.1 except possibly the conditions (1.4), (1.9). Furthermore, by Theorem 3.6,

$$\frac{f(x)}{x - L_1} \quad \text{is monotone decreasing in } x_\lambda < x < \min\{L_1, L_0\}$$

and

$$(4.5) \quad f'(x) \geq \frac{f(x)}{x - L_1}, \quad x_\lambda < x < \min\{L_1, L_0\}.$$

LEMMA 4.2. *If $0 < \lambda < \frac{2Q}{L_1^2}$, then the free boundary Γ must end on the y -axis at some finite point. More precisely, $x_\lambda = 0$ and $|f(0+)| < +\infty$, where x_λ is defined in (4.4).*

Proof. By the nonoscillation lemma (Lemma 2.4), $f(x_\lambda+) < +\infty$. Actually, in the case $L_0 < L_1$, the free boundary Γ lies below the segment γ (see Theorem 3.1). It remains to prove that $x_\lambda = 0$ and $f(0+) > -\infty$.

Suppose $x_\lambda > 0$. Then $f(x_\lambda+) = -\infty$. By (4.5),

$$f'(x) \geq \frac{f(x)}{x - L_1} > 0 \quad \text{for } f(x) < 0, \quad x_\lambda < x < \min\{L_1, L_0\}.$$

It follows that for $x_\lambda < x < x'_\lambda \leq L_0$, the free boundary Γ can be written in the form

$$(4.6) \quad x = g(y) \quad \text{on} \quad -\infty < y < 0$$

and

$$(4.7) \quad \begin{aligned} g(y) &\rightarrow x_\lambda, \\ 0 < g'(y) &= \frac{1}{f'(x)} \leq \frac{x - L_1}{f(x)} \rightarrow 0 \quad \text{as} \quad y \rightarrow -\infty. \end{aligned}$$

As in the proof of Theorem 9.3 of chapter 3 in [8], we consider a sequence of functions

$$u_n(x, y) = u(x, y + y_n)$$

for some $y_n \rightarrow -\infty$. Then for a subsequence

$$u_n \rightarrow v$$

uniformly in compact subsets of $\{0 < x < L_1\}$, where v is a solution of

$$(4.8) \quad \mathcal{L}v = 0 \quad \text{in} \quad x_\lambda < x < L_1,$$

$$(4.9) \quad v(L_1, y) = 0 \quad -\infty < y < \infty,$$

and

$$(4.10) \quad v = Q, \quad -\frac{1}{x} \frac{\partial v}{\partial x} = \lambda \quad \text{on} \quad x = x_\lambda, -\infty < y < \infty.$$

We conclude that, by uniqueness for the Cauchy problem,

$$v = -\frac{\lambda}{2}(x^2 - x_\lambda^2) + Q$$

and by (4.9),

$$\lambda = \frac{2Q}{L_1^2 - x_\lambda^2} > \frac{2Q}{L_1^2}$$

which contradicts the assumption of the lemma.

It remains to consider the case $x_\lambda = 0$ and $f(0+) = -\infty$. Notice that (4.6) and (4.7) still hold in the present situation. By the bounded gradient lemma, we can take a subsequence from u_n such that

$$u_n \rightarrow v$$

uniformly in compact subsets of $\{0 < x < L_1\}$ and

$$\nabla u_n \rightarrow \nabla v \quad \text{weakly in} \quad L_{loc}^2,$$

where v is a solution of

$$\begin{aligned} \mathcal{L}v &= 0 & \text{in } 0 < x < L_1, \quad -\infty < y < \infty, \\ v(0, y) &= Q, \quad v(L_1, y) = 0 & \text{for } -\infty < y < \infty. \end{aligned}$$

Since $0 < u_n < Q$, we also have $0 < v < Q$. By the Phragmén-Lindelöf's lemma [8],

$$(4.11) \quad v = -\frac{Q}{L_1^2}x^2 + Q \quad \text{in } 0 < x < L_1.$$

We shall prove below that

$$(4.12) \quad \lambda = \frac{2Q}{L_1^2}$$

Take any $\zeta \in C^\infty([0, L_1] \times [0, 1])$ with

$$\zeta = 0 \quad \text{on } \{y = 1\}, \{y = 0\} \quad \text{and } \{x = L_1\}.$$

Utilizing

$$\nabla u_n \rightarrow \nabla v \quad \text{weakly in } L^2([0, L_1] \times [0, 1]),$$

we get

$$\begin{aligned} \lambda \int_{\partial\{u_n < Q\}} \zeta &= \int_{\{u_n < Q\}} \frac{1}{x} \nabla \zeta \nabla u_n \rightarrow \int_{\{v < Q\}} \frac{1}{x} \nabla \zeta \nabla v \\ &= - \int_{\{v < Q\}} \frac{2Q}{L_1^2} \zeta_x \quad \text{by (4.11)} \\ &= \frac{2Q}{L_1^2} \int_0^1 \zeta(0, y) dy. \end{aligned}$$

On the other hand,

$$\lambda \int_{\partial\{u_n < Q\}} \zeta = \lambda \int_0^1 \zeta(g_n(y), y) \sqrt{1 + g_n'(y)^2} dy,$$

where $g_n(y) = g(y + y_n)$. By (4.7) and the Lebesgue dominated convergence theorem,

$$\lambda \int_0^1 \zeta(g_n(y), y) \sqrt{1 + g_n'(y)^2} dy \rightarrow \lambda \int_0^1 \zeta(0, y) dy \quad \text{as } n \rightarrow \infty.$$

It follows that $\lambda = \frac{2Q}{L_1^2}$, which still contradicts the assumption of the lemma. We have thus completed the proof of the lemma.

Combining this Lemma and (4.3), we conclude that if

$$L_0 < 2(H - H_0),$$

then $y = f(x)$ is defined on $(0, L_0)$ and $f(0+)$ is finite. By Theorem 3.3, $f'(0+) = +\infty$. Hence we obtain a solution of the finite jet problem once we proved the following theorem.

THEOREM 4.3. *The following limit exists:*

$$\lim_{x \rightarrow 0^+} \frac{1}{x^2} (Q - u(x, f(0))) = \frac{\lambda}{2}.$$

Proof. Set $v = Q - u$ and $A' = (0, f(0))$. We shall prove that

$$(4.13) \quad v(x, y) = \frac{\lambda}{2} x^2 + o(r^2) \quad r^2 = |X - A'|^2, \quad X \in \Omega = \{u < Q\}.$$

Suppose that (4.13) is not true. Then there exist sequences

$$X_n \in \Omega, \quad r_n = |X_n - A'| \rightarrow 0$$

such that

$$(4.14) \quad |v(X_n) - \frac{\lambda}{2} x_n^2| \geq c r_n^2 \quad \text{for some } c > 0.$$

Consider the blow-up sequence

$$u_n(X) = \frac{1}{r_n^2} (Q - u(A' + r_n X)).$$

By (2.2) and the bounded gradient lemma (Lemma 2.3), for a subsequence,

$$(4.15) \quad u_n(X) \rightarrow u_0 \quad \text{uniformly on compact sets of } \overline{R_+^2} = \{(x, y); \quad x \geq 0\}.$$

We claim that

$$(4.16) \quad u_0(x, y) \equiv \frac{\lambda}{2} x^2 \quad \text{in } R_+^2.$$

Indeed, since the free boundary Γ_n of u_n converges to the positive y-axis (see Theorem 3.3), u_0 is a solution of

$$\begin{aligned} \mathcal{L}u_0 &= 0 & \text{in } R_+^2, \\ u_0(0, y) &= 0 & \text{for } -\infty < y < \infty, \\ u_0 &\geq 0 & \text{in } R_+^2. \end{aligned}$$

by virtue of the uniqueness theorem in [14],

$$(4.17) \quad u_0(x, y) = x^2 w(x, y) \quad \text{in } R_+^2,$$

where w is an analytic function up to $\{x = 0\}$; it also even in x and satisfies

$$w_{xx} + w_{yy} + \frac{3}{x} w_x = 0 \quad \text{in } R_+^2.$$

Since $u_0 \geq 0$, we also have $w \geq 0$.

If we set $x = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}$, then

$$\Delta w = 0 \quad \text{and} \quad w \geq 0 \quad \text{in} \quad R^5.$$

It follows by the Liouville theorem that $w \equiv \text{const.} = a$. Thus by (4.17)

$$(4.18) \quad u_0(x, y) = ax^2 \quad \text{in} \quad R_+^2.$$

Using the argument involved in proving (4.12), we get $a = \frac{\lambda}{2}$ and (4.16) follows.

Going back to functions v and u_n , we note that

$$\frac{1}{r_n^2} v(X_n) = u_n(Y_n), \quad \text{where} \quad Y_n = \frac{X_n - A'}{r_n}, \quad |Y_n| = 1.$$

Since, for some subsequence,

$$Y_n \rightarrow Y = (x, y), \quad |Y| = 1,$$

it follows, by (4.15) and (4.16) that $u_n(Y_n) \rightarrow \frac{\lambda}{2}x^2$, i.e.

$$v(X_n) = r_n^2 \frac{\lambda}{2} x^2 (1 + o(1)).$$

Replacing x by $\frac{x_n}{r_n} + o(1)$, we obtain

$$v(X_n) = \frac{\lambda}{2} x_n^2 (1 + o(1)) = \frac{\lambda}{2} x_n^2 + o(r_n^2)$$

which contradicts (4.14).

By Theorem 3.3, for $\delta > 0$ small, the segment $\{0 < x < \delta, y = f(0)\}$ lies inside $\Omega = \{u < Q\}$. Substituting $X = (x, f(0))$ into (4.13) for x small, we get

$$\frac{1}{x^2} v(x, f(0)) = \frac{\lambda}{2} + o(1) \quad \text{as} \quad x \rightarrow 0+,$$

and the theorem follows.

§5. Uniqueness of solutions.

THEOREM 5.1. *The solution (u, Γ, λ) of the finite jet problem is uniquely determined by Q .*

Proof. Suppose that there are two solutions (u, Γ, λ) , $(\tilde{u}, \tilde{\Gamma}, \tilde{\lambda})$ corresponding to Q with $\tilde{\lambda} \geq \lambda$, $\tilde{u} \not\equiv u$. Denote by Ω , $\tilde{\Omega}$ the fluid region of u , $\tilde{u}$ respectively. Then

$$(5.1) \quad \tilde{\Omega} \not\subset \Omega.$$

Indeed, otherwise, by the maximum principle and the smooth-fit condition at A ,

$$\tilde{u} > u \quad \text{in } \tilde{\Omega}$$

and

$$\tilde{\lambda} = \frac{1}{x} \frac{\partial \tilde{u}}{\partial \nu} < \frac{1}{x} \frac{\partial u}{\partial \nu} = \lambda \quad \text{at } A = (L_0, H)$$

a contradiction.

For positive number $\tau > 0$, we introduce the translation

$$y \mapsto y + \tau.$$

Set

$$D^\tau = \{(x, y); (x, y - \tau) \in D\}$$

and

$$u^\tau(x, y) = u(x, y - \tau) \quad \text{in } D_\tau.$$

In virtue of (5.1), there exists a $\tau > 0$ such that Γ^τ lies above $\tilde{\Gamma}$ and the two free boundaries touch each other at $X_0 = (x_0, y_0)$ with $0 \leq x_0 < L_0$.

Suppose $x_0 > 0$. Since $u^\tau \leq \tilde{u}$ on $\partial\{\Omega^\tau \cap \tilde{\Omega}\}$, it follows, by the maximum principle that

$$(5.2) \quad u^\tau < \tilde{u} \quad \text{in } \Omega^\tau \cap \tilde{\Omega}$$

and

$$\tilde{\lambda} = \frac{1}{x_0} \frac{\partial \tilde{u}}{\partial \nu} < \frac{1}{x_0} \frac{\partial u^\tau}{\partial \nu} = \lambda \quad \text{at } X_0$$

which contradicts the assumption.

It remains to consider the case $x_0 = 0$. Notice that (5.2) still holds here; it then follows by the boundary point lemma in [17] that

$$(5.3) \quad \lim_{x \rightarrow 0+} \frac{1}{x} |\tilde{u}_x| < \lim_{x \rightarrow 0+} \frac{1}{x} |u_x^\tau| \quad \text{on } \{y < y_0\} \cap \{x = 0\}.$$

We choose a curve $T \subset \Omega^\tau \cap \tilde{\Omega}$, which connects a point on $\{y < y_0\} \cap \{x = 0\}$ and a point on $\tilde{\Gamma}$ at which $u^\tau < \tilde{u}$. Then for a sufficiently small $\epsilon > 0$, by (5.2), (5.3),

$$Q - \tilde{u} < (1 - \epsilon)(Q - u^\tau) \quad \text{on } T.$$

It follows, by the maximum principle, that

$$Q - \tilde{u} < (1 - \epsilon)(Q - u^\tau) \quad \text{in a neighborhood of } X_0.$$

Thus by Theorem 4.3,

$$\begin{aligned} \tilde{\lambda} &= 2 \lim_{x \rightarrow 0+} \frac{1}{x^2} (Q - \tilde{u}) \leq (1 - \epsilon) 2 \lim_{x \rightarrow 0+} \frac{1}{x^2} (Q - u^\tau) \quad \text{along } \{y = y_0\} \\ &= (1 - \epsilon) \lambda < \lambda. \end{aligned}$$

This still contradicts the assumption.

We thus conclude that $\tilde{\lambda} = \lambda$, $\tilde{u} = u$.

§6. Asymptotic estimate of Γ as $L_0 \rightarrow 0$.

We denote by (u, Γ, λ) the unique solution of the jet problem corresponding to Q .

THEOREM 6.1. *If Γ is given in the form*

$$\Gamma : \quad y = f(x) \quad 0 < x < L_0,$$

then there exists a negative function f_0 with $f_0(1) = 0$, monotone increasing in $(0, 1)$ such that

$$(6.1) \quad f(x) = L_0 f_0\left(\frac{x}{L_0}\right)(1 + o(1)) + H \quad \text{as } L_0 \rightarrow 0.$$

Proof. Set $\mathcal{O} = (0, H)$. Consider the blow-up sequence

$$(6.2) \quad u_{L_0}(X) = \frac{1}{\lambda L_0^2} (Q - u(\mathcal{O} + L_0 X)).$$

Denote by $\mathcal{R}$ the union of the fourth quadrant of R^2 and the half-strip $\{(x, y); 0 < x < 1, y \geq 0\}$. Then for any ball $B_\rho(0)$ we have, by the bounded gradient lemma,

$$(6.3) \quad |\nabla u_{L_0}| \leq C_\rho \quad \text{in } B_\rho(0) \quad \text{for all small } L_0$$

for some constant $C_\rho > 0$, independent of L_0 . It follows that for some sequence $L_0 \rightarrow 0$,

$$u_{L_0} \rightarrow u_0 \quad \text{uniformly in compact sets in } \mathcal{R}.$$

By the nondegeneracy lemma (Lemma 2.5), and the proof of Lemma 3.6 of chapter 3 in [8], $u_0 \not\equiv 0$ and u_0 is an absolute minimizer of the functional

$$J(v) = \int_G \left[\left| \frac{\nabla v}{x} \right|^2 + I_{\{v>0\} \setminus \{x>1\}} \right] x dx dy$$

for any bounded domain G in $\mathcal{R}$. Furthermore u_0 solves the free boundary problem

$$(6.4a) \quad \mathcal{L}u_0 = 0 \quad \text{in } \mathcal{R} \cap \{u_0 > 0\},$$

$$(6.4b) \quad u_0 = 0, \quad \frac{1}{x} \frac{\partial u_0}{\partial \nu} = -1 \quad \text{on } \Gamma_0 = \partial\{u_0 > 0\} \cap \mathcal{R},$$

and

$$(6.4c) \quad u_0(0, y) = 0 \quad \text{for } -\infty \leq y \leq \infty,$$

$$(6.4d) \quad u_0(x, 0) = 0 \quad \text{for } x > 1.$$

Moreover

$$(6.5) \quad \frac{\partial u_0}{\partial y} \leq 0 \quad \text{in } \mathcal{R}.$$

By [2], the free boundary Γ_0 is analytic, and (6.5) implies that Γ_0 is an x-graph, i.e.

$$(6.6) \quad \Gamma_0 : \quad y = f_0(x) \quad \text{in } I, \quad \text{an open set in } (0,1)$$

where f_0 is continuous in I . By the nonoscillation lemma (Lemma 2.4), Γ_0 consists of only one piece, i.e.

$$(6.7) \quad I = (x_0, 1) \quad \text{for some } x_0 \geq 0.$$

Since the free boundary Γ_{L_0} of u_{L_0} can be given in the form

$$\Gamma_{L_0} : \quad y = f_{L_0}(x) = \frac{1}{L_0}(f(L_0 x) - H) \quad \text{for } 0 < x < 1,$$

Lemma 3.6 of chapter 3 in [8] shows that

$$\Gamma_{L_0} \rightarrow \Gamma_0 \quad \text{in the Hausdorff distance as } L_0 \rightarrow 0$$

which implies that

$$(6.8) \quad f_{L_0}(x) \rightarrow f_0(x) \quad \text{in } (x_0, 1) \quad \text{for some } x_0 \geq 0.$$

To complete the proof of the theorem, it suffices to show that $x_0 = 0$ and f_0 is a monotone increasing negative function. The proof is given in the following three lemmas.

LEMMA 6.2. $x_0 = 0$ and $|f_0(0+)| < \infty$.

Proof. By the nonoscillation lemma, $f_0(x_0+) < \infty$. It remains to prove

$$x_0 = 0 \quad \text{and} \quad f_0(0+) > -\infty.$$

Suppose $x_0 \neq 0$. Then $x_0 > 0$ and $f_0(x_0+) = -\infty$. It follows that the free boundary $y = f_0(x)$ satisfies the “flatness condition” of [2] for each free boundary point (x_0, y_0) with $|y_0|$ large. Hence for some large enough $|y_0|$, $\Gamma_0 \cap \{y < y_0\}$ has the form

$$(6.9) \quad x = g(y)$$

and

$$(6.10) \quad \lim_{y \rightarrow -\infty} g'(y) = 0.$$

Recalling the proof of Lemma 4.2 (see also the proof of Theorem 9.3 of chapter 3 in [8]), we consider the functions

$$(6.11) \quad u_n(x, y) = u_0(x, y - n).$$

By the bounded gradient lemma, there exists a subsequence $u_{n'}$ of (6.11) such that

$$(6.12) \quad u_{n'} \rightarrow v \quad \text{uniformly in compact subsets of } \{0 < x < \infty\}$$

and v is a solution of

$$\begin{aligned} \mathcal{L}v &= 0 & \text{in } \{x > x_0\}, \\ v(x_0, y) &= 0 & -\infty < y < \infty, \\ \frac{1}{x} \frac{\partial v}{\partial x} &= 1 & \text{if } x = x_0, \quad -\infty < y < \infty. \end{aligned}$$

It follows, by uniqueness to the Cauchy problem, that

$$v = \frac{1}{2}(x^2 - x_0^2).$$

By (6.5) and (6.11), we have

$$u_0 \leq \frac{1}{2}(x^2 - x_0^2) \quad \text{in } \mathcal{R} \cap \{x > x_0\}.$$

Consequently, by the proof of Lemma 3.5, $u_0 \equiv 0$, contradicting the fact $u_0 \not\equiv 0$.

It remains to consider the case $x_0 = 0$ and $f_0(0+) = -\infty$. By (6.9), (6.10), (6.11), (6.12) and elliptic estimates, v is a solution of

$$\begin{aligned} \mathcal{L}v &= 0 & \text{in } R_+^2 = \{x > 0, -\infty < y < \infty\}, \\ v(0, y) &= 0 & \text{for } -\infty < y < \infty, \\ v &\geq 0 & \text{in } R_+^2. \end{aligned}$$

We have known by the proof of (4.18) that v has the form

$$v = ax^2 \quad \text{in } R_+^2,$$

where a is a constant. By (6.5) and (6.11), $u_0 \leq ax^2$ in $\mathcal{R}$. Then the proof of Lemma 3.5 implies

$$u_0 \equiv 0 \quad \text{in } \mathcal{R}$$

a contradiction.

LEMMA 6.3. *Let u_0 be the blow-up limit of (6.2). Then there exists a constant $a \geq 0$ such that*

$$u_0(x, y) + ax^2y \rightarrow 0$$

as $r^2 = x^2 + y^2 \rightarrow \infty$, $(x, y) \in \mathcal{R}_0 = \{(x, y); \quad x > 0, y < 0\}$.

Proof. Denote by Ω_0 the fluid region of u_0 . Then

$$\Omega_0 = \{(x, y); \quad 0 < x \leq 1, y < f_0(x)\} \cup \{(x, y); \quad x \geq 1, y < 0\}.$$

Because the end point $f(0+)$ of the free boundary Γ_0 is finite, $\partial\Omega_0$ contains an open subset of the y -axis (see Lemma 6.2). We can then use the uniqueness theorem in [14] to write u_0 in the form

$$(6.13) \quad u_0(x, y) = x^2 w(x, y) \quad \text{in } \Omega_0,$$

where w is analytic in $\Omega_0 \cup \{x = 0\}$; it is also even in x and satisfies

$$w_{xx} + w_{yy} + \frac{3}{x} w_x = 0 \quad \text{in } \Omega_0.$$

Set

$$\Omega = \text{int}\{(x_1, x_2, x_3, x_4, y) \in R^5; \quad ((x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}, y) \in \bar{\Omega}_0\}.$$

Then w is actually a solution of

$$\begin{aligned} \Delta w &= 0 & \text{in } \Omega, \\ w &= 0 & \text{on } \partial\Omega, \\ w &> 0, \quad w_y &\leq 0 & \text{in } \Omega. \end{aligned}$$

We claim that

$$(6.14) \quad \lim_{r \rightarrow \infty} (w(x, y) + ay) = 0, \quad r = \sqrt{x^2 + y^2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + y^2}$$

for some $a \geq 0$.

In fact, we extend w oddly into the region $\{y > 0\} \cap \{r > R_0\}$, where R_0 is chosen so large that the free boundary Γ_0 lies in the ball $\{r < R_0\}$. Then

$$\Delta w = 0 \quad \text{in } \{r > R_0\}$$

and

$$w_y \leq 0 \quad \text{in } \{r > R_0\}.$$

Since w_y is also harmonic and nonpositive in $\{r > R_0\}$, by Theorem 26 and related remarks on P.120-121 in [15],

$$\lim_{r \rightarrow \infty} w_y \quad \text{exists} \quad (\text{It possibly equals } -\infty).$$

We first show that this limit is not $-\infty$. To the contrary, suppose that $\lim_{r \rightarrow \infty} w_y = -\infty$. Choose $y_* < 0$ such that the free boundary Γ_0 lies above $\{y = y_*\}$. For large R , set

$$(6.15) \quad \Omega_R = \Omega \cap \{|x| < R, -R < y < \infty\}.$$

For any $\epsilon > 0$, consider the function

$$w_\epsilon = \epsilon w.$$

On $\Omega \cap \{|x| = R > 1\}$, we have, noticing that $y < 0$,

$$w_\epsilon(R, y) = \int_0^y \epsilon w_y(R, t) dt > - \int_0^y dt = -y > y_* - y$$

as long as R is large enough.

On $\Omega \cap \{y = -R\}$,

$$\frac{\partial}{\partial y}(w_\epsilon - (y_* - y)) = \epsilon w_y + 1 < 0 \quad \text{for sufficiently large } R.$$

Clearly,

$$w_\epsilon \geq y_* - y \quad \text{on the remaining part of } \partial\Omega_R.$$

By the maximum principle,

$$w_\epsilon \geq y_* - y \quad \text{in } \Omega_R \quad \text{for all large } R.$$

Letting $R \rightarrow \infty$ and then $\epsilon \rightarrow 0$, we have

$$y \geq y_* \quad \text{in } \Omega$$

which is impossible. Hence we have proved that $\lim_{r \rightarrow \infty} w_y$ is finite, i.e.

$$(6.16) \quad \lim_{r \rightarrow \infty} w_y = -a \quad 0 \leq a < \infty.$$

Choose a positive constant C such that

$$(6.17) \quad w \leq -ay + \frac{C}{r^3} \quad \text{on } \partial\Omega.$$

We want to prove that

$$(6.18) \quad w \leq -ay + \frac{C}{r^3} \quad \text{in } \Omega.$$

To do this, for any $\epsilon > 0$, consider

$$(6.19) \quad v_\epsilon = w + (a + \epsilon)y - \frac{C}{r^3} \quad \text{in } \Omega_R,$$

where Ω_R is defined by (6.15).

On $\Omega_R \cap \{|x| = R\}$, noticing that $y < 0$,

$$w(R, y) = \int_0^y w_y(R, t) dt \leq -(a + \frac{\epsilon}{2})y < -(a + \epsilon)y + \frac{C}{r^3}$$

provided R is large enough.

On $\Omega_R \cap \{y = -R\}$,

$$\frac{\partial v_\epsilon}{\partial \nu} = w_y + (a + \epsilon) + \frac{3Cy}{r^4} > 0$$

for R is sufficiently large.

Since $\Delta v_\epsilon = 0$ in $\Omega_R \setminus \{0\}$, by the maximum principle,

$$v_\epsilon = w + (a + \epsilon)y - \frac{C}{r^3} \leq 0 \quad \text{in } \Omega_R$$

for all large R . Letting $R \rightarrow \infty$ and then $\epsilon \rightarrow 0$, (6.18) follows.

Using the same argument we can prove that

$$(6.20) \quad -\frac{C}{r^3} \leq w + ay \quad \text{in } \Omega.$$

Combining (6.13), (6.18) and (6.20) we have

$$-\frac{Cx^2}{r^3} \leq u_0 + ax^2y \leq \frac{Cx^2}{r^3} \quad \text{in } \Omega_0.$$

Letting $r \rightarrow \infty$ the lemma follows.

LEMMA 6.4. f_0 is negative and monotone increasing in $(0, 1)$.

Proof. Set

$$\mathcal{R}_0 = \{(x, y); \quad x > 0, y < 0\}.$$

For any $\mu > 1$, denote by $\mathcal{R}_\mu$ the truncation of $\mathcal{R}_0$:

$$\mathcal{R}_\mu = \{(x, y); \quad 0 < x < \mu, -\mu < y < 0\}.$$

Introduce the functional

$$J_{\lambda, \mu}(v) = \int_{\mathcal{R}_\mu} (|\frac{\nabla v}{x}|^2 + \lambda^2 I_{\{v > 0\} \setminus \{x > 1\}}) x dx dy$$

for $v \in K_\mu = \{v \in H_{loc}^{1,2}(\mathcal{R}_\mu); \quad v = \phi \quad \text{on } \partial\mathcal{R}_\mu\}$, where $\phi = -x^2y$ satisfying $\mathcal{L}\phi = 0$.

Consider the minimizing problem:

$$(6.21) \quad \text{Find } u \in K_\mu \quad \text{with } J_{\lambda, \mu}(u) = \min_{K_\mu} J_{\lambda, \mu}(v).$$

By the same consideration as in §2, (6.21) has a solution u which is a subsolution with respect to the operator $\mathcal{L}$. Thus by the maximum principle,

$$(6.22) \quad u \leq -x^2y \quad \text{in } \mathcal{R}_\mu.$$

Hence we can rearrange u monotone increasingly in the x -direction and monotone decreasingly in the y -direction respectively such that two rearrangements u_1^* , u_2^* of u are still

minimizers of $J_{\lambda,\mu}$. Since a minimizer u with $u_x \geq 0$ or $u_y \leq 0$ is unique (see the proof of Theorem 2.1), it follows that $u_1^* = u_2^*$ and, consequently, (6.21) has a unique solution $u_{\lambda,\mu}$ with $\frac{\partial u_{\lambda,\mu}}{\partial x} \geq 0$ and $\frac{\partial u_{\lambda,\mu}}{\partial y} \leq 0$. Furthermore, the free boundary $\Gamma_{\lambda,\mu}$ of $u_{\lambda,\mu}$ can be written in the form

$$\Gamma_{\lambda,\mu} : \quad y = f_{\lambda,\mu}(x) \quad 0 < x < x_{\lambda,\mu}$$

for some monotone increasing negative function $f_{\lambda,\mu}$. By the continuous dependence of $(u_{\lambda,\mu}, \Gamma_{\lambda,\mu})$ on λ , there exists a unique λ_μ such that $\Gamma_\mu = \Gamma_{\lambda_\mu,\mu}$ initiates at $(0, 1)$. Set

$$u_\mu^* = \frac{1}{\lambda_\mu} u_{\lambda_\mu,\mu}.$$

Then

$$\begin{aligned} \mathcal{L}u_\mu^* &= 0 & \text{in } \mathcal{R}_\mu \cap \{u_\mu^* > 0\}, \\ u_\mu^* &= 0, \quad \frac{1}{x} \frac{\partial u_\mu^*}{\partial \nu} = -1 & \text{on } \Gamma_\mu, \end{aligned}$$

and

$$\frac{\partial u_\mu^*}{\partial x} \geq 0, \quad \frac{\partial u_\mu^*}{\partial y} \leq 0 \quad \text{in } \mathcal{R}_\mu.$$

By the bounded gradient lemma and elliptic estimates, for some $\mu = \mu_n \rightarrow \infty$,

$$u_\mu^* \rightarrow u^* \quad \text{uniformly in compact sets in } \mathcal{R}_0.$$

Denote by $\Gamma^* = \partial\{u^* > 0\} \cap \mathcal{R}_0$, then the proof of Lemma 6.2, 6.3 shows that (u^*, Γ^*) is a solution of the finite jet problem (6.4) and

$$(6.23) \quad u^* + bx^2y \rightarrow 0 \quad \text{as } r^2 = x^2 + y^2 \rightarrow \infty$$

for some $b \geq 0$.

Since $u_x^* \geq 0$, $u_y^* \leq 0$, Γ^* can be expressed in the form

$$(6.24) \quad \Gamma^* : \quad y = f^*(x) \quad \text{in } (0, 1)$$

for some monotone increasing negative function f^* in $(0, 1)$.

We claim that

$$(6.25) \quad f_0 \equiv f^*,$$

where f_0 is defined in (6.6) and (6.8).

Indeed, suppose $f_0 \not\equiv f^*$. We first consider the case $b \geq a$, $u^* \not\equiv u_0$. Denote by Ω_0 , Ω^* the fluid regions of u_0 , u^* respectively, then

$$(6.26) \quad \Omega_0 \not\subset \Omega^*.$$

Otherwise, by the Phragmén-Lindelöf's lemma, $u_0 < u^*$ in Ω_0 and this implies by the maximum principle that

$$-1 = \frac{1}{x} \frac{\partial u^*}{\partial \nu} < \frac{1}{x} \frac{\partial u_0}{\partial \nu} = -1 \quad \text{at } (1, 0), \quad \nu = \text{outward normal}$$

which is a contradiction.

For any positive number $\tau > 0$, we introduce the translation $y \mapsto y + \tau$, and set

$$\begin{aligned} \Omega_\tau^* &= \{(x, y); \quad (x, y - \tau) \in \Omega^*\}, \\ u_\tau^* &= u^*(x, y - \tau) \quad \text{in } \Omega_\tau^*. \end{aligned}$$

By (6.26), there exists a $\tau > 0$ such that Γ_τ^* lies above Γ_0 and they touch each other at $X_0 = (x_0, y_0)$ with $0 \leq x_0 < 1$.

Since

$$u_0 \leq u_\tau^* \quad \text{on } \partial\Omega_0$$

and

$$u_\tau^* \sim -bx^2(y - \tau) = -bx^2y + b\tau x^2 \geq u_0 \quad \text{as } r \rightarrow \infty, \quad (x, y) \in \Omega_0,$$

it follows by the Phragmén-Lindelöf's lemma that

$$u_0 < u_\tau^* \quad \text{in } \Omega_0.$$

We now apply the same argument as in the proof for Theorem 5.1 to get a contradiction.

The method above can also applies to the case $b < a$, $u^* \not\equiv u_0$. We thus conclude that $u_0 \equiv u^*$ and (6.25) follows.

§7. Flows with vorticity.

Due to the geometry of the coater and the high speed motion of the fiber, the coating fluid in general undergoes circulation. In this section we extend the previous model by including vorticity. Equation (1.6) is then replaced by

$$(7.1) \quad \mathcal{L}u = x^2 \zeta(u) \quad \text{in the fluid,}$$

where $\zeta(u)$ is a vorticity function (see [9]).

We consider the case

$$(7.2) \quad \zeta(u) = u.$$

Then the solution of (7.1), (7.2) in the vertical pipe of width L_1 with the boundary condition

$$u = Q \quad \text{on } x = 0; \quad u = 0 \quad \text{on } x = L_1$$

is given by the stream function:

$$(7.3) \quad \phi(x) = Q \cosh \frac{x^2}{2} - Q \coth \frac{L_1^2}{2} \sinh \frac{x^2}{2}$$

and the corresponding velocity vector is

$$\vec{v} = (0, Q \sinh \frac{x^2}{2} - Q \coth \frac{L_1^2}{2} \cosh \frac{x^2}{2}).$$

We can see that this flow moves downward and its speed decreases as x increases.

The example (7.3) describes the physical situation of the coating fluid near the fiber and hence somewhat justifies the assumption (7.2).

The mathematical definition of the problem is given in §1. In practice, L_0 and L_1 are very small relative to L , $H - H_0$. We assume that

$$(7.4) \quad \frac{1}{L_0} \sinh \frac{L_0^2}{2} < \min \left\{ \frac{1}{2} (1 - \sqrt{\frac{6}{8}}) \min \{L, \sqrt{2}\}, (H - H_0) \frac{\coth \frac{L_1^2}{2} - 1}{\coth \frac{L_1^2}{2}} \right\},$$

Since $L_0 \leq \frac{2}{L_0} \sinh \frac{L_0^2}{2}$ and $\frac{\coth \frac{L_1^2}{2} - 1}{\coth \frac{L_1^2}{2}} \sim 1$ as $L_1 \rightarrow 0$, (7.4) is not restrictive physically.

THEOREM 7.1. *If (7.4) holds, then for any $Q \geq 0$ there exists a unique solution of the finite jet problem.*

The proof is given at the end of the section.

As in §2, for any $\mu > \max\{L, H\}$ we introduce the truncated domains

$$\begin{aligned} D_\mu &= G \cap \{0 < x < \mu\} \cap \{-\mu < y < \mu\} \\ E_\mu &= D_\mu \cap \{L_0 < x < \mu\} \end{aligned}$$

and the functional

$$(7.5) \quad J_{\lambda, \mu}(v) = \int_{D_\mu} \left[\left| \frac{\nabla v}{x} \right|^2 + v^2 + \lambda^2 I_{\{v < Q\} \setminus E_\mu} \right] x dx dy, \quad \lambda > 0$$

over the admissible class

$$\begin{aligned} K_\mu &= \{v \in H^1(D_\mu); \quad v = Q \quad \text{on} \quad \{x = 0\}, \partial D_\mu \cap \{y \geq H_0\}, \\ &\quad v = 0 \quad \text{on} \quad \partial D_\mu \cap \{x \geq L_1\} \cap \{y \leq H_1\}, \\ &\quad v = \phi \quad \text{on} \quad \{0 < x < L_1\} \cap \{y = -\mu\}\}, \end{aligned}$$

where ϕ is defined by (7.3). Notice that the term v^2 in (7.5) represents the vorticity.

Consider the problem: find $u = u_{\lambda, \mu} \in K_\mu$ such that

$$(7.6) \quad J_{\lambda, \mu}(u) = \min_{K_\mu} J_{\lambda, \mu}(v).$$

By the variational principle in [9], any solution u of (7.6) belongs to $C^{0,1}(D_\mu)$ and satisfies

$$(7.7) \quad u \leq Q, \quad \mathcal{L}u \leq x^2 u \quad \text{in} \quad D_\mu;$$

$$(7.8) \quad \mathcal{L}u = x^2 u \quad \text{in} \quad \{u < Q\} \cap D_\mu.$$

Further, the free boundary $\Gamma_{\lambda, \mu} = \partial\{u < Q\} \cap D_\mu$ is an analytic curve. It is also worth mentioning that Lemma 2.2, Lemma 2.3 (the bounded gradient lemma) and Lemma 2.5 (the nondegeneracy lemma) hold for solutions of (7.6).

Analogously to Theorem 2.1, we have:

THEOREM 7.3. *There exists a unique solution $u = u_{\lambda,\mu}$ of (7.6) with $u_y \geq 0$.*

Proof. The existence of a minimizer u follows from the standard argument (see [9]). Since $u \geq \max\{\phi, 0\}$ on ∂D_μ , we get, by (7.7) and the maximum principle, that

$$(7.9) \quad u \geq \max\{\phi, 0\} \quad \text{in } D_\mu.$$

Therefore, the monotone increasing rearrangement u^* of u in the y -direction is again in K_μ . Since $J_{\lambda,\mu}(u^*) \leq J_{\lambda,\mu}(u)$, u^* is also a minimizer. For simplicity, we denote u^* by u . Thus $u_y \geq 0$. The uniqueness of such a u follows by the same argument used in the proof of Theorem 2.1.

Theorem 7.3 implies that there exists a function

$$y = f_{\lambda,\mu}(x), \quad 0 < x < L_0$$

such that if $0 < x < L_0$, $(x, y) \in D_\mu$, then

$$u(x, y) < Q \quad \text{if and only if} \quad y < f_{\lambda,\mu}(x)$$

and the free boundary $\Gamma_{\lambda,\mu}$ of u has the form

$$\Gamma_{\lambda,\mu} : \quad y = f_{\lambda,\mu}(x), \quad x \in I_\mu,$$

where I_μ is an open subset of $(0, L_0)$. We can write I_μ as a union of open intervals, i.e. $I_\mu = \cup_i (a_i, b_i)$, and define

$$(7.10) \quad x_{\lambda,\mu} = \sup_i b_i; \quad x_{\lambda,\mu} \leq L_0.$$

Actually $(x_{\lambda,\mu}, f_{\lambda,\mu}(x_{\lambda,\mu}-))$ is the point at which the free boundary $\Gamma_{\lambda,\mu}$ initiates. From now on we restrict the domain of $f_{\lambda,\mu}$ over $(0, x_{\lambda,\mu})$.

In order to study the free boundary, we need the following modified nonoscillation lemma.

LEMMA 7.4. *If, in addition to the assumptions of Lemma 2.4, W is contained in a rectangle $R = \{l < x < l + \epsilon, y_1 < y < y_1 + h\}$ and ϵ is sufficiently small, then*

$$(7.11) \quad h \leq c\epsilon,$$

where c is a constant independent of μ and ϵ .

Proof. Integrating the equation $\nabla(\frac{1}{x}\nabla u) = xu$ over W , we get

$$\int_{\partial W} \frac{1}{x} \frac{\partial u}{\partial \nu} = \iint_W xu.$$

Proceeding as the proof of Lemma 2.4, we find that

$$\begin{aligned} \lambda h &\leq c'\epsilon + \int_l^{l+\epsilon} \int_{y_1}^{y_1+h} xQ dy dx \\ &= c'\epsilon + Qh(\epsilon l + \frac{\epsilon^2}{2}). \end{aligned}$$

If $(\epsilon l + \frac{\epsilon^2}{2})Q < \frac{\lambda}{2}$, then

$$h \leq \frac{2c'}{\lambda}\epsilon = c\epsilon$$

and the lemma follows.

By Lemma 9.4 of chapter 3 in [8] and Lemma 7.4, $y = f_{\lambda,\mu}(x)$ is continuous on $[0, x_{\lambda,\mu}]$.

LEMMA 7.5. *There exists a constant $\lambda^* > 0$, independent of μ such that $\Gamma_{\lambda,\mu}$ initiates on $\{x = L_0\} \cap \{y < H\}$ for all $\lambda \geq \lambda^*$.*

Proof. Take a ball $B = B_r(X^0)$ with $X^0 = (L_0, y_0)$, $y_0 < H$ and $r = \frac{1}{2} \min\{L_0, y_0\}$. Set

$$\lambda^* = \frac{Q}{cL_0 \min\{L_0, y_0\}},$$

where c is the universal constant in Lemma 2.5. Then, for any $\lambda > \lambda^*$,

$$\frac{1}{r} \int_{B_r(X^0)} (Q - u) \leq \frac{Q}{r} \leq c\lambda^* L_0 \leq c\lambda x_0$$

which implies that $u \equiv Q$ in $B_{\frac{r}{2}}(X^0) \setminus E_\mu$ (see Lemma 2.5). Since $u_y \geq 0$ in D_μ , we get $f_{\lambda,\mu}(L_0) < y_0 < H$ which proves the lemma.

As in §4, we introduce the set

$$\Sigma = \{\lambda > 0; \quad \Gamma_{\lambda,\mu} \text{ does not initiate on } \{x = L_0, y \leq H\}\}.$$

Then $\Sigma \neq \emptyset$ (see the argument in §4). Set

$$\lambda_\mu = \sup_{\Sigma} \lambda.$$

By Lemma 7.5,

$$0 < \lambda_\mu \leq \lambda^*.$$

It follows that $\Gamma_\mu = \Gamma_{\lambda_\mu,\mu}$ initiates at $A = (L_0, H)$ and $u_\mu = u_{\lambda_\mu,\mu}$ is a solution of (7.6).

Hence for a sequence $\mu = \mu_n \rightarrow \infty$,

$$\begin{aligned} \lambda_\mu &\rightarrow \lambda, \\ u_\mu &\rightarrow u \quad \text{in } H_{loc}^1(G), \\ \Gamma_\mu &\rightarrow \tilde{\Gamma} \quad \text{in Hausdorff measure,} \end{aligned}$$

where $\tilde{\Gamma}$ is locally analytic and takes the form

$$y = f(x) \quad x \in I$$

for a continuous function f on I , where I is a subset of $[0, L_0]$ containing L_0 .

Denote by Γ the portion of the free boundary $\tilde{\Gamma}$ initiating from $A = (L_0, H)$. Then Γ can be written in the form

$$(7.12) \quad \Gamma : \quad y = f(x) \quad 0 \leq x_\lambda < x \leq L_0.$$

By the nonoscillation lemma (Lemma 7.4), $f(x_\lambda +)$ exists (it possibly be $+\infty$).

The triple (u, Γ, λ) forms a solution of the finite jet problem once we proved that $x_\lambda = 0$, $|f(0+)| < +\infty$ and $f'(0+) = +\infty$.

LEMMA 7.6. Suppose $L_0 \leq (1 - \sqrt{\frac{6}{8}}) \min\{L, \sqrt{2}\}$. Then $f(x_\lambda +) < +\infty$.

Proof. We assume the assertion is not true and derive a contradiction. Take a portion

$$\Gamma_1 : \quad y = f(x) \quad 0 \leq x_2 < x < x_1 \leq x_\lambda$$

of $\tilde{\Gamma}$ which is closest to Γ (Γ_1 may be the y-axis). Clearly, $f(x_1 -) = +\infty$.

By the nonoscillation lemma (Lemma 7.4), $x_\lambda - x_1 > 0$.

We shall prove the following inequality:

$$(7.13) \quad \lambda < \frac{Q}{4}(x_\lambda^2 - x_1^2).$$

To do this, consider the sequence

$$u_n(x, y) = u(x, y + n) \quad n = \text{natural numbers.}$$

By the bounded gradient lemma,

$$u_n \rightarrow u_0 \quad \text{uniformly in compact sets in } G.$$

Since Γ and Γ_1 satisfy the “flatness” condition for y large (see [9]), u_0 actually solves

$$(7.14a) \quad \mathcal{L}u_0 = x^2 u_0 \quad x_1 < x < x_\lambda, \quad -\infty < y < \infty;$$

$$(7.14b) \quad u_0(x_1, y) = Q, \quad u_0(x_\lambda, y) = Q, \quad -\infty < y < \infty;$$

$$(7.14c) \quad \frac{1}{x} \frac{\partial u_0}{\partial x} = \lambda \quad \text{at } x = x_\lambda, \quad -\infty < y < \infty.$$

The unique solution of this problem is given by

$$u_0 = Q \cosh \frac{x^2 - x_1^2}{2} + \frac{Q - Q \cosh \frac{x_\lambda^2 - x_1^2}{2}}{\sinh \frac{x_\lambda^2 - x_1^2}{2}} \sinh \frac{x^2 - x_1^2}{2}.$$

Hence, by (7.14c),

$$\lambda = Q \frac{\cosh \frac{x_\lambda^2 - x_1^2}{2} - 1}{\sinh \frac{x_\lambda^2 - x_1^2}{2}} < \frac{Q}{4}(x_\lambda^2 - x_1^2),$$

where we have used the inequality

$$\cosh \xi - 1 < \frac{1}{2} \xi \sinh \xi \quad \text{for } \xi \neq 0.$$

(7.13) gives an upper bound for λ , and we are going to estimate λ from below.

Denote by $X_* = (x_*, H_*)$ the lowest point on Γ . Clearly $x_\lambda < x_* \leq L_0$ and $-\infty < H_* \leq H$. Set

$$R = \{(x, y); \quad x_1 < x < L_*, H_* - h < y < H_*\}$$

for some $L_0 < L_* \leq L$ and $h > 0$. Then

$$\partial R \cap \tilde{\Gamma} = \partial R \cap \Gamma = \{X_*\}.$$

Consider the function

$$\psi = \sigma x(x^2 - x_1^2)(L_* - x)(e^{\beta y} - e^{\beta H_*})(e^{\beta y} - e^{\beta(H_* - h)}) + Q,$$

where $\sigma > 0$ and $\beta > 0$ are constant. We shall prove that, for an appropriate choice of σ and β , ψ is a supsolution in R .

It is easy to check that

$$\begin{aligned} \mathcal{L}\psi = & \sigma(-8x^2 + 3L_*x + \frac{x_1^2}{x}L_*)(e^{\beta y} - e^{\beta H_*})(e^{\beta y} - e^{\beta(H_* - h)}) \\ & + \sigma x(x^2 - x_1^2)(L_* - x)[\beta^2 e^{\beta y}(e^{\beta y} - e^{\beta(H_* - h)}) + 2\beta^2 e^{2\beta y} + \beta^2 e^{\beta y}(e^{\beta y} - e^{\beta H_*})]. \end{aligned}$$

Notice that for $(x, y) \in R$,

$$-8x^2 + 3L_*x + \frac{x_1^2}{x}L_* \geq -5x^2 \quad \text{and} \quad x(x^2 - x_1^2)(L_* - x) \leq x^3(L_* - x) \leq x^2 \frac{L_*^2}{4}.$$

We have

$$(7.15) \quad \mathcal{L}\psi \leq \sigma x^2 \{5(e^{\beta H_*} - e^{\beta(H_* - h)})^2 + \frac{L_*^2}{4}\beta^2 e^{\beta H_*}[(e^{\beta H_*} - e^{\beta(H_* - h)}) + 2e^{\beta H_*}]\}.$$

and

$$(7.16) \quad \psi \geq -\sigma L_*^2 \frac{L_*^2}{4}(e^{\beta H_*} - e^{\beta(H_* - h)})^2 + Q \quad \text{in } R,$$

We want to choose σ such that

$$(7.17) \quad \mathcal{L}\psi \leq x^2 \psi \quad \text{in } R.$$

Indeed, by (7.15) and (7.16), (7.17) holds provided

$$\begin{aligned} & \sigma \{5(e^{\beta H_*} - e^{\beta(H_* - h)})^2 + \frac{L_*^2}{4}\beta^2 e^{\beta H_*}[(e^{\beta H_*} - e^{\beta(H_* - h)}) + 2e^{\beta H_*}]\} \\ & = -\sigma \frac{L_*^4}{4}(e^{\beta H_*} - e^{\beta(H_* - h)})^2 + Q. \end{aligned}$$

By solving σ , we get

$$(7.18) \quad \begin{aligned} \sigma = Q \{ & 5(e^{\beta H_*} - e^{\beta(H_* - h)})^2 + \frac{L_*^2}{4}\beta^2 e^{\beta H_*} [(e^{\beta H_*} - e^{\beta(H_* - h)}) + 2e^{\beta H_*}] \\ & + \frac{L_*^4}{4}(e^{\beta H_*} - e^{\beta(H_* - h)})^2 \}^{-1}. \end{aligned}$$

Since ψ is a supersolution, $u < \psi$ in $R \cap G$ and

$$\begin{aligned} \lambda & \geq \frac{1}{x} \frac{\partial \psi}{\partial \nu} \quad \text{at } X_* \\ & = \frac{1}{x_*} \sigma x_* (x_*^2 - x_1^2) (L_* - x_*) \beta e^{\beta H_*} (e^{\beta H_*} - e^{\beta(H_* - h)}) \\ & \geq \sigma (x_\lambda^2 - x_1^2) (L_* - L_0) \beta e^{\beta H_*} (e^{\beta H_*} - e^{\beta(H_* - h)}) \quad \forall \beta > 0, \end{aligned}$$

where we have used the fact that $x_\lambda < x_* \leq L_0$. Letting $\beta \rightarrow 0$, noticing that $e^{\beta H_*} - e^{\beta(H_* - h)} \sim \beta h$, by (7.18) we get that

$$\lambda \geq \frac{(L_* - L_0)h}{(5 + \frac{L_*^4}{4})h^2 + \frac{L_*^2}{2}} (x_\lambda^2 - x_1^2) Q \geq \frac{Q}{4} (x_\lambda^2 - x_1^2)$$

provided

$$(7.19) \quad (L_* - L_0)h \geq \frac{L_*^2}{8} + \frac{1}{4}(5 + \frac{L_*^4}{4})h^2.$$

The last inequality has a positive solution h if and only if

$$(7.20) \quad (L_* - L_0)^2 - 4\frac{L_*^2}{8}\frac{1}{4}(5 + \frac{L_*^4}{4}) = (L_* - L_0)^2 - \frac{5}{8}L_*^2 - \frac{1}{32}L_*^6 \geq 0.$$

We choose $L_* = \min\{L, \sqrt{2}\}$. Then, by the assumption on L_0

$$(L_* - L_0)^2 - \frac{5}{8}L_*^2 - \frac{1}{32}L_*^6 \geq \frac{1}{8}L_*^2 - \frac{1}{32}L_*^6 = \frac{1}{32}L_*^2(4 - L_*^4) \geq 0.$$

Hence, $\lambda \geq \frac{Q}{4}(x_\lambda^2 - x_1^2)$, contradicting (7.13).

Lemma 7.6 implies that Γ lies under some horizontal line $\{y = H^*\}$ if $L_0 \leq (1 - \sqrt{\frac{6}{8}})\min\{L, \sqrt{2}\}$.

LEMMA 7.7. *Under the assumption of Lemma 7.6, if $0 < \lambda < Q \coth \frac{L^2}{2}$, then Γ must end on the y -axis at some finite point, i.e., $x_\lambda = 0$ and $|f(0+)| < +\infty$. Moreover, $f'(0+) = +\infty$.*

Proof. By Lemma 7.6, $f(x_\lambda+) < +\infty$. Now we only need to prove that $x_\lambda = 0$, $f(0+) > -\infty$ and $f'(0+) = +\infty$. This remaining part can be obtained by the same proof of that of Theorem 3.3 and Lemma 4.2, so we omit it here.

Proof of Theorem 7.1

The uniqueness of solutions follows similarly to the proof of Theorem 5.1. It remains to prove existence. By Definition 1.1 and Lemma 7.7, it suffices to prove that $\lambda < Q \coth \frac{L_1^2}{2}$.

Choose $H^* \geq H$ such that Γ lies below the horizontal line $\{y = H^*\}$ and touches it at $X_0 = (x_0, H^*)$ with $0 < x_0 \leq L_0$. Consider the auxiliary function

$$w(x, y) = Qe^{x(y-H^*)} + \frac{1}{H-H_0} Q \coth \frac{L_1^2}{2} (\sinh \frac{x^2}{2})(y-H^*)$$

in $G \cap \{H^* - H + H_0 \leq y \leq H^*\}$. We shall prove that w is a subsolution.

In fact,

$$\begin{aligned} \mathcal{L}w &= (y-H^*)^2 Qe^{x(y-H^*)} - \frac{1}{x} Q(y-H^*)e^{x(y-H^*)} \\ &\quad + Qx^2 e^{x(y-H^*)} + \frac{x^2}{H-H_0} Q \coth \frac{L_1^2}{2} (\sinh \frac{x^2}{2})(y-H^*) \\ &\geq x^2 w \quad \text{in } G \cap \{H^* - H + H_0 \leq y \leq H^*\} \end{aligned}$$

and

$$\begin{aligned} w(x, H^* - H + H_0) &= Qe^{x(H_0-H)} - Q \coth \frac{L_1^2}{2} \sinh \frac{x^2}{2} \\ &\leq Q \cosh \frac{x^2}{2} - Q \coth \frac{L_1^2}{2} \sinh \frac{x^2}{2} = \psi \leq u \quad \text{by (7.9).} \end{aligned}$$

Clearly, $w \leq u$ on the remaining part of $\partial\{G \cap \{H^* - H + H_0 \leq y \leq H^*\}\}$.

Having proved that w is a subsolution, we have that $w < u$ in $G \cap \{H^* - H + H_0 \leq y \leq H^*\}$ and

$$\begin{aligned} \lambda &\leq \frac{1}{x} \frac{\partial w}{\partial \nu} \quad \text{at } X_0 \\ &= Q + \frac{1}{H-H_0} Q \coth \frac{L_1^2}{2} \left(\frac{1}{x_0} \sinh \frac{x_0^2}{2} \right) \\ &\leq Q + \frac{1}{H-H_0} Q \coth \frac{L_1^2}{2} \left(\frac{1}{L_0} \sinh \frac{L_0^2}{2} \right) \\ &< Q \coth \frac{L_1^2}{2} \quad \text{by (7.4),} \end{aligned}$$

so that Theorem 7.1 follows.

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Appendix A.

PROPOSITION. Suppose $\mathcal{L}v \geq 0$ in $B_R(X^0)$ ($0 < R < \frac{x^0}{3}$) with $v \in H^1(B_R)$ and $v \geq 0$. Then for any two balls $B_{\rho_1}(X^0) \subset B_{\rho_2}(X^0) \subset B_R(X^0)$,

$$\rho_1^{\frac{3}{2}} \oint_{\partial B_{\rho_1}} v \leq \rho_2^{\frac{3}{2}} \oint_{\partial B_{\rho_2}} v.$$

Proof. We shall use formal calculation to carry out the proof. The rigorous verification can be given by using mollifiers.

Integrating $\mathcal{L}v \geq 0$ over an arbitrary ball $B_\rho(X^0) \subset B_R$, we get

$$(A1) \quad \int_{\partial B_\rho} \frac{\partial v}{\partial \rho} ds - \int_{B_\rho} \frac{1}{x} v_x \geq 0.$$

Since $\frac{1}{x} v_x = (\frac{1}{x} v)_x + \frac{1}{x^2} v$, we have

$$\begin{aligned} \rho \frac{d}{d\rho} \oint_{\partial B_\rho} v ds &= \frac{1}{2\pi} \int_{\partial B_\rho} \frac{\partial v}{\partial \rho} ds \geq \frac{1}{2\pi} \int_{B_\rho} \left(\frac{1}{x} v\right)_x \\ &= \frac{1}{2\pi} \int_{\partial B_\rho} \frac{1}{x} v \frac{x - x^0}{\rho} ds \\ &\geq -\frac{1}{2\pi\rho} \int_{\partial B_\rho} \frac{x^0}{x} v ds \end{aligned}$$

or

$$\rho \frac{d}{d\rho} \oint_{\partial B_\rho} v ds + \oint_{\partial B_\rho} \frac{x^0}{x} v ds \geq 0.$$

Notice that $\frac{x^0}{x} \leq \frac{x^0}{x^0 - \rho} \leq \frac{3}{2}$; it follows that

$$\rho \frac{d}{d\rho} \oint_{\partial B_\rho} v ds + \frac{3}{2} \oint_{\partial B_\rho} v ds \geq 0$$

or

$$\frac{d}{d\rho} (\rho^{\frac{3}{2}} \oint_{\partial B_\rho} v) \geq 0$$

and the proposition follows.

Appendix B. We give here the proof of Lemma 3.4.

From reference [7], P.178, formula (23),

$$(B1) \quad P_{\beta-1}^{\frac{1}{2}+1}(\cos \theta) = c_\beta \sin^{-2} \theta \int_0^\theta \cos((\beta + \frac{1}{2})\phi) (\cos \phi - \cos \theta)^{\frac{1}{2}} d\phi, \quad 0 < \theta < \pi$$

where c_β is a positive constant.

To prove the lemma, we need to estimate the integral

$$(B2) \quad I = \int_0^\theta \cos((\beta + \frac{1}{2})\phi)(\cos \phi - \cos \theta)^{\frac{1}{2}} d\phi \quad 0 < \theta < \pi.$$

Using the addition formulas for cosines, we can rewrite (B2) as

$$\begin{aligned} I &= \int_0^\theta (\cos \beta \phi \cos \frac{\phi}{2} - \sin \beta \phi \sin \frac{\phi}{2})(\cos \phi - \cos \theta)^{\frac{1}{2}} d\phi \\ &= \int_0^\theta \cos \beta \phi \cos \frac{\phi}{2} (\cos \phi - \cos \theta)^{\frac{1}{2}} d\phi - \int_0^\theta \sin \beta \phi \sin \frac{\phi}{2} (\cos \phi - \cos \theta)^{\frac{1}{2}} d\phi \\ (B3) \quad &= I_1 + I_2. \end{aligned}$$

Integrating by parts, we rewrite I_1 as

$$\begin{aligned} I_1 &= \frac{1}{\beta} \sin \beta \phi \cos \frac{\phi}{2} (\cos \phi - \cos \theta)^{\frac{1}{2}} \Big|_0^\theta - \frac{1}{\beta} \int_0^\theta \sin \beta \phi [\cos \frac{\phi}{2} (\cos \phi - \cos \theta)^{\frac{1}{2}}]' d\phi \\ &= \frac{1}{2\beta} \int_0^\theta \sin \beta \phi [\sin \frac{\phi}{2} (\cos \phi - \cos \theta)^{\frac{1}{2}} + \sin \phi \cos \frac{\phi}{2} (\cos \phi - \cos \theta)^{-\frac{1}{2}}] d\phi. \end{aligned}$$

Substituting this expression into (B3), we have

$$\begin{aligned} I &= \int_0^\theta \sin \beta \phi (\cos \phi - \cos \theta)^{-\frac{1}{2}} \left\{ \frac{1}{2\beta} [\sin \frac{\phi}{2} (\cos \phi - \cos \theta) + \sin \phi \cos \frac{\phi}{2}] \right. \\ &\quad \left. - \sin \frac{\phi}{2} (\cos \phi - \cos \theta) \right\} d\phi \end{aligned}$$

Set

$$\begin{aligned} f(\phi) &= \frac{1}{2\beta} [\sin \frac{\phi}{2} (\cos \phi - \cos \theta) + \sin \phi \cos \frac{\phi}{2}] - \sin \frac{\phi}{2} (\cos \phi - \cos \theta) \\ &= \frac{1}{2\beta} \sin \frac{\phi}{2} \cos \phi - \frac{1}{2\beta} \sin \frac{\phi}{2} \cos \theta + \frac{1}{2\beta} \sin \phi \cos \frac{\phi}{2} - \sin \frac{\phi}{2} \cos \phi + \sin \frac{\phi}{2} \cos \theta \\ (B4) \quad &= (\frac{1}{2\beta} - 1) \sin \frac{\phi}{2} \cos \phi + \frac{1}{2\beta} \sin \phi \cos \frac{\phi}{2} + (1 - \frac{1}{2\beta}) \sin \frac{\phi}{2} \cos \theta. \end{aligned}$$

If we can show that for all $\beta > 1$ with $\beta - 1$ being small,

$$f(\phi) > 0 \quad \text{for } 0 < \phi < \pi - \theta_0.$$

Then, for $\beta - 1 > 0$ so small that $\pi - \theta_0 < \frac{\pi}{\beta}$,

$$I = \int_0^\theta \sin \beta \phi (\cos \phi - \cos \theta)^{-\frac{1}{2}} f(\phi) d\phi > 0 \quad \text{for } 0 < \theta < \pi - \theta_0.$$

Lemma 3.4 then follows.

Indeed, since

$$(B5) \quad \sin \frac{\phi}{2} = \sin(\phi - \frac{\phi}{2}) = \sin \phi \cos \frac{\phi}{2} - \cos \phi \sin \frac{\phi}{2},$$

replacing $\sin \frac{\phi}{2} \cos \phi$ in (B4) by the other two terms in (B5), we have

$$\begin{aligned} f(\phi) &= (\frac{1}{2\beta} - 1)(\sin \phi \cos \frac{\phi}{2} - \sin \frac{\phi}{2}) + \frac{1}{2\beta} \sin \phi \cos \frac{\phi}{2} + (1 - \frac{1}{2\beta}) \sin \frac{\phi}{2} \cos \theta \\ &= (\frac{1}{\beta} - 1) \sin \phi \cos \frac{\phi}{2} + (1 - \frac{1}{2\beta}) \sin \frac{\phi}{2} + (1 - \frac{1}{2\beta}) \sin \frac{\phi}{2} \cos \theta \\ &= 2(\frac{1}{\beta} - 1) \sin \frac{\phi}{2} \cos^2 \frac{\phi}{2} + (1 - \frac{1}{2\beta}) \sin \frac{\phi}{2} + (1 - \frac{1}{2\beta}) \sin \frac{\phi}{2} \cos \theta \\ &= \sin \frac{\phi}{2} [2(\frac{1}{\beta} - 1) \cos^2 \frac{\phi}{2} + (1 - \frac{1}{2\beta}) + (1 - \frac{1}{2\beta}) \cos \theta]. \end{aligned}$$

Hence, if $0 < \theta < \pi - \theta_0$ for some $0 < \theta_0 < \pi$, then

$$f(\phi) \geq \sin \frac{\phi}{2} [-2(1 - \frac{1}{\beta}) + (1 - \frac{1}{2\beta})(1 + \cos(\pi - \theta_0))], \quad \text{for } 0 < \phi < \theta.$$

Notice that

$$\lim_{\beta \rightarrow 1} [-2(1 - \frac{1}{\beta}) + (1 - \frac{1}{2\beta})(1 + \cos(\pi - \theta_0))] = \frac{1}{2}(1 + \cos(\pi - \theta_0)) > 0.$$

We conclude that for any $\beta > 1$, $\beta - 1$ being small enough,

$$f(\phi) \geq \frac{1}{4}(1 + \cos(\pi - \theta_0)) \sin \frac{\phi}{2} > 0 \quad \text{for } 0 < \phi < \theta < \pi - \theta_0.$$

and Lemma 3.4 follows.

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