**A COMBINATORIAL PERTURBATION METHOD AND
ARNOLD'S WHISKERED TORI IN VORTEX DYNAMICS**

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A Combinatorial Perturbation Method and Arnold's Whiskered Tori in Vortex Dynamics

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Abstract

A combinatorial perturbation method for some n -body problems is presented. This method is used to reformulate a class of n -body problems in the form of near integrable hamiltonian systems with many degrees of freedom. The KAM-theory [4] is worked out for n -body problems defined on subgraphs G of the complete graph K_n . The main result in this paper is the existence of Arnold's whiskered tori [1] and diffusion for vortex lattice dynamics. Throughout the paper, combinatorial and graph-theoretic concepts play important roles, in particular the KAM-nondegeneracy and Melnikov transversability conditions are given in terms of binary trees $T(n)$. One of the main tools here is a combinatorial algorithm for symplectic transformations to Jacobi (relative) coordinates, which is based on binary trees $T(n)$.

I Introduction

In this paper, we will demonstrate the existence of Arnold's famous whiskered tori [1] in a class of hamiltonian n -body problems which includes vortex dynamics. The main tool is a new combinatorial perturbation method for hamiltonian n -body problems [cf. also [2], [3]]. This method will be used in conjunction with the Kolmogorov-Arnold-Moser (KAM)-theory [4], [5] for invariant tori, and the Melnikov-Arnold theory [6], [7] for transversal homoclinic intersections, i.e. intersecting whiskers.

In a 1964 paper [1], Arnold gave his well-known example of whiskered tori and the so-called Arnold Diffusion. Later, Holmes and Marsden [8] gave results in rigid body dynamics which exhibit these phenomena. All of these examples, including Arnold's, involve very special hamiltonians, chosen for the purpose of illustrating these phenomena without the burden of lengthy technical details. Arnold conjectured that whiskered tori are generic to many-body problems [1]. However, he cautioned that lengthy and even difficult calculations are involved.

Here we are concerned with a general method for constructing whiskered tori in a physically significant (i.e. naturally occurring) class of hamiltonian n -body problems. In particular, our main examples arise in vortex dynamics and numerical vortex methods [9], [10], [11]. In contrast to Ziglin [12] and Koiller [13] who demonstrated that chaotic behavior occurs in a system for four point vortices, we are primarily concerned with many body systems.

We will now outline the combinatorial perturbation method and give a plan of this paper. In any attempt to prove the existence of KAM-tori with or without whiskers, two main difficulties have to be overcome:

(A) formulation of a given n -body problem in perturbation form, i.e.

$$H = H_0 + H_1$$

where H_0 is *completely integrable* and,

(B) control of the size $|H_1|$ of the perturbation term.

The main ideas for overcoming the above difficulties are quite simple (in outline), and the combinatorial perturbation method consists of the following steps (cf. appendix A for graph-theoretic notions):

(1) Reformulate the hamiltonian $H(m_1, \dots, m_n, z_1, \dots, z_n)$ as a function on a graph G with n nodes.

(2) Transform symplectically to the so-called Jacobi variables, which are defined by another graph $T(n)$ (cf. [14], [15]); then the transformed hamiltonian (dropping primes) $H(m_1, \dots, m_n; \rho_1, \dots, \rho_{n-1})$ is treated as a function on the binary tree, $T(n)$.

(3) The natural partial order relation on $T(n)$ is now used to construct the

completely integrable term H_0 (which is thus defined combinatorially) and the perturbation term H_1 .

(4) The size of the perturbation term $|H_1|$ is controlled by focussing on a small part of phase-space, called the cone-set which is characterized by $T(n)$, and which correspond to clusters in the physical plane.

In section II, we will describe the class of n -body hamiltonians considered in this paper. In section III, we will formulate the combinatorial perturbation method in the context of KAM-theory. We will show in particular that the unperturbed term H_0 in step (3) above is not only completely - integrable but has the simple form of $(n - 1)$ - decoupled oscillators (not harmonic in general) (cf. Lemma 3.1). In addition, we will show that the KAM-nondegeneracy condition on $H_0(T(n))$ has a very simple form in terms of the graphs G and $T(n)$, and the “masses”, $\tilde{M} = (m_1, \dots, m_n)$. It turns out that for the complete graph $G = K_n$, the nondegeneracy condition is satisfied by all “masses” $\tilde{M}$ which satisfies the admissibility condition (3.1) for the symplectic transformation in step (2) above. Moreover, this condition holds for “almost all” masses $\tilde{M}$. One could say that the combinatorial perturbation method is optimal in the sense that it imposes minimal constraints on the “masses” of the hamiltonian n -body problems. The main results in this section, theorems 3.1, 3.2, are the existence of KAM-tori, for the whole class of hamiltonians (2.1) and for almost all “masses”.

In section IV, we will make the necessary modifications to the combinatorial perturbation method presented in section III, to prove the existence of whiskered tori and Arnold Diffu-

sion for a subclass of the hamiltonians (2.1), which govern the dynamics on certain periodic domains, e.g. lattice (cf. theorem 4.1). The proof is based on first, reformulating the n -body hamiltonian (4.1) in the form $H = H_0 + H_1$ where $H_0 = F + \sum G_s$, and $\sum G_s$ represents $(n - 2)$ - decoupled oscillators while F represents a “pendulum” with the usual heteroclinic manifold. Then a result of Graff and Moser [16] implies that the invariant tori of H_0 and their stable and unstable manifolds $W^\pm$ continue to the full hamiltonian for sufficiently small perturbations (cf. Lemma 4.2). Next, a higher-dimensional Melnikov integral [1], [8] shows that $W^\pm$ intersect transversally.

II Hamiltonian Functions on Graphs

In this section, we describe the class of hamiltonian functions that will be considered in this paper. Let $z_j \in C$, and $m_j \in R \setminus \{0\}$, for $j = 1, \dots, n$ represent the positions and “masses” (vorticities) of n particles. The following class, $\mathcal{H}$ of hamiltonian functions govern the motions of n particles in the physical plane: For each complete graph, K_n we consider the class $\mathcal{H}(G, \tilde{M})$ of Hamiltonians

$$(2.1) \quad H(m_1, \dots, m_n; z_1, \dots, z_n) = \sum_{(j,k) \in E(G)} m_j m_k \log [\xi(z_j - z_k)]$$

where $G \subseteq K_n$ is a subgraph of K_n , $\underline{M}$ is the mass-vector $(m_1, \dots, m_n)$, $E(G)$ is the edge set of G , and ξ is a complex analytic function such that $\xi(0) = 0$. The equations of motion for (2.1)

$$(2.2) \quad m_j \frac{\dot{z}_j}{z_j} = 2i \frac{\partial H}{\partial z_j} = i 2 \sum_{(j,k) \in E(G)} m_k \frac{\xi'(z_j - z_k)}{\xi(z_j - z_k)}$$

can be easily written in the standard hamiltonian form:

$$(2.3) \quad \dot{q}_j = \frac{\partial H}{\partial p_j} \quad , \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \quad , \quad j = 1, \dots, n$$

where

$$(2.4a) \quad H = \text{Re} [H(m_1, \dots, m_n; z_1, \dots, z_n)] ,$$

$$(2.4b) \quad q_j = \sqrt{|m_j|} x_j \quad , \quad p_j = \text{sgn}(m_j) \sqrt{|m_j|} y_j$$

$$(2.4c) \quad z_j = x_j + iy_j \quad , \quad j = 1, \dots, n$$

Example 1: One obtains the important examples of vortex dynamics [11], [17] when $\xi =$ identity and $G = K_n$ the complete graph with n nodes. The edge set $E(G)$ represents the connectivity in the n -body problem. When $G = K_n$, all the remaining particles are involved in the motion of each. Example 2: If $\xi(\cdot) = \sin \pi(\cdot)$, one obtains vortex lattice dynamics which arise, for instance in the point vortex discretization of a periodic vortex sheet and the von Karman vortex street [17], [18], [19], [20], [21]. This example will be studied in detail in section 4, where the existence of whiskered tori will be shown.

The hamiltonians (2.1) are thus functions on graphs G with n nodes where the nodes

$v_j = (n_j, z_j)$ represent symplectic variables, and the edges $(j, k) \in E(G)$ represents the terms in the sum (2.1). In general G may take the form of any subgraph of the complete graph K_n . This completes step (1) of the combinatorial perturbation method outlined in Section I. For an explicit example, consider the 5-body problem with $G = K_5$ and $\xi = \text{identity}$:

$$\begin{aligned}
(2.5) \quad H(m_1, \dots, m_5; z_1, \dots, z_5) = & m_1 m_2 \log(z_1 - z_2) + m_1 m_3 \log(z_1 - z_3) \\
& + m_1 m_4 \log(z_1 - z_4) + m_1 m_5 \log(z_1 - z_5) \\
& + m_2 m_3 \log(z_2 - z_3) + m_2 m_4 \log(z_2 - z_4) \\
& + m_2 m_5 \log(z_2 - z_5) + m_3 m_4 \log(z_3 - z_4) \\
& + m_3 m_5 \log(z_3 - z_5) + m_4 m_5 \log(z_4 - z_5).
\end{aligned}$$

III Combinatorial Perturbation

In this section, we formulate the combinatorial perturbation method for KAM-theory.

The main result here is:

Theorem 3.1: If the mass-vector $\tilde{M} = (m_1, \dots, m_n)$ in $\mathcal{H}(G, \tilde{M})$, (2.1) satisfy (a) the admissibility condition (3.1) of the symplectic transformation (B3), (B4) based on some binary tree $T(n)$, and (b) the KAM-nondegeneracy condition (3.11) for $T(n)$ and the subgraph $G_\bullet$ of K_n , then the hamiltonians $H(\tilde{M}; z_1, \dots, z_n)$ in subclass $\mathcal{H}(\mathcal{G}_\bullet, \tilde{\mathcal{M}})$, of $\mathcal{H}(G, \tilde{M})$ (2.1) admits

KAM-tori which form a positive measure set in phase-space.

In order to prove theorem 3.1, we will carry out steps (2)-(4) in the outline of the combinatorial perturbation method given in the introduction. For step (2), we will use a combinatorial algorithm that generates linear symplectic transformations to relative (Jacobi) variables from full binary trees, $T(n)$ (cf. [14], [15]). Since the details of this algorithm are available in [14], we will only summarize the pertinent parts in appendix B, and use the result here. Each leaf in the tree $T(n)$ corresponds to a particle in the n -body problem, i.e., a node in $G \subseteq K_n$.

From Lemma B1, we know that if the mass-vector $\tilde{M}$ satisfy the admissibility condition, (B6), i.e.,

$$(3.1) \quad A(T(n), \tilde{M}) \equiv \prod_{s=1}^{n-1} \left(\sum_{j \in X^+(s)} m_j \right) \left(\sum_{j \in X^-(s)} m_j \right) \neq 0$$

where $X^\pm(s)$ are defined in terms of $T(n)$, then the inverse (B8), (B9) of the transformation (B3, B4) for the pairs $(z_j - z_k)$ applied to (2-1) yields the new hamiltonian

$$(3.2a) \quad H'(\tilde{M}; \rho_1, \dots, \rho_{n-1}) = \sum_{(j,k) \in E_G(T(n))} m_j m_k \log \left[\xi \left(\sum_{\ell=1}^{n-1} C_\ell(j, k) \rho_\ell \right) \right]$$

which is now a function on the tree $T(n)$. In other words, the symplectic variables in (3.2) are $\{\rho_j\}$ which correspond to the internal nodes of $T(n)$ while the terms in the sum (3.2) correspond to the leaf-to-leaf paths, $(j, k) \in E_G(T(n))$ which connect leaf j to leaf k , where nodes j and k are adjacent in subgraph $G \subseteq K_n$ (cf. appendix B for this notation). In

appendix B, it is shown that the variables ρ_j correspond physically to the edges of a spatial full binary tree $T(n)$ whose leaves are given by the positions z_j of the n particles in the problem (cf. Figure 2) i.e., ρ_j are relative (Jacobi) variables, similar to those originally used in celestial mechanics [15].

We give an example of the above transformation which is based on the tree $T(5)$ in figure 1 and 2, using the 5-body hamiltonian, (2.5), with $\tilde{M} = (m_1, \dots, m_5)$ which is assumed to satisfy (3.1):

$$\begin{aligned}
H'(m_1, \dots, m_5; \rho_1, \dots, \rho_4) = & m_1 m_2 \log(C_1(1, 2)\rho_1) \\
& + m_1 m_3 \log(C_1(1, 3)\rho_1 + C_2(1, 3)\rho_2 + C_3(1, 3)\rho_3) \\
& + m_1 m_4 \log(C_1(1, 4)\rho_1 + C_2(1, 4)\rho_2 + C_3(1, 4)\rho_3) \\
& + m_1 m_5 \log(C_1(1, 5)\rho_1 + C_3(1, 5)\rho_3 + C_4(1, 5)\rho_4) \\
(3.2b) \quad & + m_2 m_3 \log(C_1(2, 3)\rho_1 + C_2(2, 3)\rho_2 + C_3(2, 3)\rho_3) \\
& + m_2 m_4 \log(C_1(2, 4)\rho_1 + C_2(2, 4)\rho_2 + C_3(2, 4)\rho_3) \\
& + m_2 m_5 \log(C_1(2, 5)\rho_1 + C_3(2, 5)\rho_3 + C_4(2, 5)\rho_4) \\
& + m_2 m_5 \log(C_1(2, 5)\rho_1 + C_3(2, 5)\rho_3 + C_4(2, 5)\rho_4) \\
& + m_3 m_4 \log(C_2(3, 4)\rho_2 + m_3 m_5 \log(C_2(3, 5)\rho_2 + C_3(3, 5)\rho_3 + C_4(3, 5)\rho_4) \\
& + m_4 m_5 \log(C_2(4, 5)\rho_2 + C_3(4, 5)\rho_3 + C_4(4, 5)\rho_4)
\end{aligned}$$

In this example, $G = K_5$ and thus, the set of leaf-to-leaf paths $E_G(T(5))$ is just $E(T(5))$, i.e.

all the leaf-to-leaf paths in $T(5)$.

The natural partial order relation for $T(n)$ (cf. appendix A) will play an important role in step (3) of this method. We will write $H(\tilde{M}; \rho_1, \dots, \rho_{n-1})$, (3.2) (dropping primes) in the form $H_0 + H_1$, by the following calculations:

$$(3.3) \quad H = \sum_{(j,k) \in E_G(T(n))} m_j m_k \log \left[\xi \left(\sum_{\ell=1}^{n-1} C_\ell(j, k) \rho_\ell \right) \right]$$

$$= \sum_{(j,k) \in E_G(T(n))} m_j m_k \log \left[\xi \left\{ C_s(j, k) \rho_s \left(1 + \sum_{\ell < s} \frac{C_\ell(j, k) \rho_\ell}{C_s(j, k) \rho_s} \right) \right\} \right]$$

where $<$ denotes “lower than” in the tree $T(n)$, and the index $s = N(j, k) \in \{1, \dots, n-1\}$ labels the node in the leaf-to-leaf path (j, k) , which is “nearest” to the root of $T(n)$. Continuing with the calculation, we use the partial order ($<$) on $T(n)$ to factor the hamiltonian obtaining:

$$(3.4a) \quad H = \sum_{(j,k) \in E_G(T(n))} m_j m_k \log \left[\{\} \cdot \left(\frac{\xi\{\}}{\{\}} \right) \right]$$

$$(3.4b) \quad = \sum_{(j,k) \in E_G(T(n))} m_j m_k \left(\log \{\} + \log \left(\frac{\xi\{\}}{\{\}} \right) \right)$$

$$(3.4c) = \sum_{(j,k) \in E_G(T(n))} m_j m_k \left(\log [C_s(j, k) \rho_s] + \log \left[1 + \frac{\sum_{\ell < s} C_\ell(j, k) \rho_\ell}{C_s(j, k) \rho_s} \right] \right) + \sum_{(j,k) \in E_G(T(n))} \log \left(\frac{\xi\{\}}{\{\}} \right)$$

where $\{\} \equiv C_s(j, k) \rho_s \left(1 + \sum_{\ell < s} \frac{C_\ell \rho_\ell}{C_s \rho_s} \right)$. We now make the assumption

$$(3.5) \quad |\{\}| \ll 1$$

which implies

$$(3.6) \quad \left(\frac{\xi\{\}}{\{\}} \right) \sim 1 + 0(\{\}).$$

In the last part of this calculation, we collect terms combinatorially using the partial order relation ($<$) again to obtain:

$$(3.7) \quad \begin{aligned} H = & \sum_{s=1}^{n-1} \left\{ \sum_{(j,k) \in E_s(G)} m_j m_k \right\} \log(C_s \rho_s) \\ & + \sum_{(j,k) \in E_G(T(n))} m_j m_k \left\{ \log \left[1 + \sum_{\ell < s} \frac{C_\ell(j,k) \rho_\ell}{C_s \rho_s} \right] + \log \left(\frac{\xi\{\}}{\{\}} \right) \right\} \\ = & H_0(\tilde{M}, G, T(n)) + H_1(\tilde{M}, G, T(n)) \end{aligned}$$

where $E_s(G)$ is the set of paths $(j, k) \in E_G(T(n))$ whose nearest node $N(j, k)$ to the root is node s , i.e., $E_G(T(n)) = \bigcup_{s=1}^{n-1} E_s(G)$. Note that the set $\{E_s, s = 1, \dots, n-1\}$ is a partition of the set $E(T(n))$ induced by the partial order ($\leq$) of the binary tree $T(n)$ when $G = K_n$. It follows from the Jacobi transformation (appendix B) that $C_s(j, k) \rho_s$ is independent of the pair (j, k) in E_s for $s = 1, \dots, n-1$; in other words, $C_s \rho_s$ is an invariant over the subset $E_s \subset E(T(n))$. By our construction (cf. (3.7)) we have the following results:

Lemma 3.1: $H_0(\tilde{M}, G, T(n))$ is the hamiltonian for $(n-1)$ decoupled nonlinear oscillators.

Lemma 3.1 implies that $H = H_0 + H_1$, (3.7) can be put in action-angle form in terms of I_j

and θ_j which satisfy the following equation

$$(3.8) \quad \rho_j = Q_j + iP_j = I_j e^{i\theta_j} \quad \text{for } j = 1, \dots, n-1.$$

We obtain, after taking the real part of $H(\tilde{M}, G, T(n))$ in (3.7):

$$(3.9a) \quad H_0(\tilde{M}, G, T(n)) = \sum_{s=1}^{n-1} \left\{ \sum_{(j,k) \in E_s(G)} m_j m_k \right\} \log(|C_s| I_s)$$

$$(3.9b) \quad H_1 = \sum_{(j,k) \in E_G(T(n))} m_j m_k \left[\log \left| 1 + \sum_{\ell < s} \frac{C_\ell(j, k) I_\ell e^{i\theta_\ell}}{C_s I_s e^{i\theta_s}} \right| + \log \left| \xi\{\}/\{\} \right| \right]$$

For our example in (2.5) and (3.2b), the hamiltonian H_0 (3.9a) takes the form given by:

$$\begin{aligned} H_0(m_1, \dots, m_5, K_n, T(5)) = & \\ & m_1 m_2 \log(|C_1| I_1) + m_3 m_4 \log(|C_2| I_2) \\ & + [m_1 m_3 + m_1 m_4 + m_2 m_3 + m_2 m_4] \log(|C_3| I_3) \\ & + [m_1 m_5 + m_2 m_5 + m_3 m_5 + m_4 m_5] \log(|C_4| I_4) \end{aligned}$$

In order to use Arnold's real analytic version of KAM-theory [4], we must extend (3.9) to complex action-angles. Step (4) of our method demonstrates that the perturbation H_1 in (3.9b) is small in the complex cone sets, N_ϵ^δ defined in the $(2n - 2)$ dimensional complex

phase-space of (3.9) in a hierarchical way by:

$$(3.10) \quad N_\varepsilon^\delta(T(n)) = \left\{ \begin{array}{l} \tilde{I} = (I_1, \dots, I_{n-1}) \in C^{n-1} \text{ such that } \operatorname{Re}(I_j) > 0, |I_j| < \varepsilon, \\ |I_\ell| < \sqrt{\varepsilon} |I_k| \text{ if } \ell < k; \text{ and } |I_m(\theta_j)| < \delta \end{array} \right\}$$

Note that the partial order relation $<$ of $T(n)$ plays a role in (3.10). The form of H_1 in (3.9b) and the fact that all the coefficients $C_\ell(j, k)$ are order 1 (cf. appendix B) yields a proof of the following results:

Lemma 3.2: In the cone set $N_\varepsilon^\delta(T(n))$, $\varepsilon \ll 1$, $\delta < \frac{1}{100}$, $|H_1(\tilde{M}, G, T(n))| \sim 0(\varepsilon)$ for all $\tilde{M}$ such that (3.1) holds, and for all subgraphs $G \subseteq K_n$.

Remark 3.1: Since the variables ρ_j correspond to the edges of $T(n)$, i.e. they represent relative distances between the particles, it is easily seen that the cone set N_ε^δ correspond to a tightly-bound clustered configuration of the n particles in the physical plane, (cf. Figure 3).

This completes the combinatorial perturbation procedure which puts a hamiltonian (2.1) in the form of a small perturbation of a completely integrable term $H_0, (\tilde{M}, G, T(n))$.

We can now establish that H_0 has the Twist property for certain mass-vectors $\tilde{M}$.

Lemma 3.3: If the mass-vector $\tilde{M}$ satisfy the combinatorial condition,

$$(3.11) \quad D(T(n), G, \tilde{M}) \equiv \prod_{s=1}^{n-1} \left\{ \sum_{(j,k) \in E_s(G)} m_j m_k \right\} \neq 0$$

for some graph $G \subseteq K_n$ and some full binary tree $T(n)$, then $H_0(\underline{M}, G, T(n))$, (3.9a) satisfies the KAM-nondegeneracy condition.

Proof: The KAM-nondegeneracy condition [4] is

$$(3.12) \quad 0 \neq \det \left[\partial^2 H_0 / \partial I_j \partial I_k \right] = \prod_{s=1}^{n-1} \sum_{(j,k) \in E_s(G)} m_j m_k \quad 1/I_s^2$$

because H_0 is decoupled. Since $I_s > 0$, (3.11) implies (3.12), which completes the proof. $\diamond$

Note that for given $G_* \subseteq K_n, T(n)$ and $\underline{M}, D(T(n)), G_*, \underline{M}$ (3.11) is invariant on the subclass $\mathcal{H}(G_*, \underline{M})$, (2.1), i.e. (3.11) does not depend on the complex analytic function ξ in (2.1).

Proof of Theorem 3.1

It is clear that we can work with the reduced hamiltonians (3.9) instead of the original ones (2.1). Suppose that $\underline{M}$ satisfies (3.1) for some tree $T(n)$, then Lemma B1 implies that there is a linear symplectic transformation which puts $H(\underline{M}, G)$, (2.1) in the form of $H(\underline{M}, G, T(n))$, (3.9). Clearly, $H(\underline{M}, G, T(n))$ is real-analytic in the actions, I_j and 2π - periodic in the angles, θ_j in the complex cone set $N_\varepsilon^\delta(T(n))$, (3.10). By Lemma 3.1, the only solutions of unperturbed term $H_0(\underline{M}, G, T(n))$ are $(n-1)$ copies of S^1 , i.e. $T^{(n-1)} = S^1 x \dots x S^1$. Moreover, by Lemma 3.3, $D(T(n), G, \underline{M}) \neq 0$, (3.11) implies that the KAM-nondegeneracy condition holds for each unperturbed hamiltonian, $H_0(\underline{M}, G, T(n))$, (3.9). Finally, by Lemma 3.2, the pertrubation term $H_1(\underline{M}, G, T(n))$, (3.9b) can be made arbitrarily small by choosing $\varepsilon < \varepsilon_0$. Since the KAM-bound depends on H_0 and the fixed complex domain $N_{\varepsilon_0}^\delta(T(n))$, the

real-analytic version of the KAM-Theorem [4] implies the result. $\diamond$

It remains to investigate what mass-vector $\tilde{M} \in [R \setminus \{0\}]^n$ are allowed under conditions (3.1) and (3.11). First we have the following result:

Proposition 3.1: If $G = K_n$ the complete graph, then (3.1) and (3.11) are equivalent for each full binary tree $T(n)$, i.e.

$$(3.13) \quad A(T(n), K_n, \tilde{M}) = D(T(n), K_n, \tilde{M})$$

Proof: We will show that for each full binary tree $T(n)$, and $s = 1, \dots, n-1$,

$$(3.14) \quad \left(\sum_{j \in \mathcal{X}^+(s)} m_j \right) \left(\sum_{k \in \mathcal{X}^-(s)} m_k \right) = \left\{ \sum_{(j,k) \in E_s(T(n))} m_j m_k \right\}$$

Since $\mathcal{X}^\pm(s)$ are the labels of leave that are lower than ($<$) node s and connected to it through the right (left) branch respectively, and E_s consists of paths, (j, k) connecting leaf j and leaf k , whose “highest” node is node s , we have

$$(3.15) \quad E_s = \mathcal{X}^+(s) \otimes \mathcal{X}^-(s), s = 1, \dots, n-1$$

where $\otimes$ denotes the direct product. Now (3.14) follows easily from (3.15) and this completes the proof. $\diamond$

Remark 3.2: Proposition 3.1 implies that the combinatorial perturbation method imposes

the minimum restriction on $\underline{M}$ when $G = K_n$. This is the case in vortex dynamics.

Remark 3.3: It is easily seen that both conditions (3.1) and (3.11) are satisfied if the mass-vector $\underline{M}$ has only one sign, i.e. $m_j m_k > 0$ for all $j \neq k$.

In the cases when G is a subgraph of K_n , and $\underline{M}$ has mixed signs, we have the following result:

Lemma 3.4: The set $\mathcal{Q}$ of mass-vectors $\underline{M}$ which satisfies (3.1) and (3.11) jointly for all full binary trees $T(n)$ and all subgraphs $G \subseteq K_n$ is open and has measure 1 in R^n .

Proof: By the same argument in the proof of Prop 3.1, $\underline{M} \in \mathcal{Q}$ satisfies the conditions for each $T(n)$ and $G \subseteq K_n$, and for $s = 1, \dots, n-1$,

$$(3.16) \quad \sum_{(j,k) \in E_s(G)} m_j m_k = \left(\sum_{j \in \mathcal{X}^+(s)} m_j \right) \left(\sum_{k \in \mathcal{X}^-(s)} m_k \right) \neq 0$$

Since there are a finite number (cf. Catalan numbers, appendix B) of full binary trees with n leave and a finite number of subgraphs G of the complete graph K_n with n nodes, the conditions (3.16) on $\underline{M} \in \mathcal{Q}$ translate into the geometric condition that $\mathcal{Q}$ is the complement in $[R \setminus \{0\}]^n$ of a finite number of hyperplanes defined by the sums $\sum m_j = 0$ appearing in (3.16). The result follows immediately from this geometric characterization of $\mathcal{Q}$. $\diamond$

An immediate corollary of Lemma 3.4, and theorem 3.1 is the result:

Theorem 3.2: On the measure one open set $\mathcal{Q}$ of mass-vectors $\underline{M}$, each subset $\mathcal{H}(G_*, \underline{M}) G_* \subseteq K_n$ of hamiltonians admit the Catalan number, C_n distinct sets of tori (with positive measure)

which correspond to C_n distinct types of clustering of n particles in the physical plane. Each type belongs to a full binary tree, $T(n)$.

IV Arnold's Whiskered tori in Vortex Dynamics

We will now prove the existence of whiskered tori and the transversal intersections of the whiskers (stable and unstable manifolds, $W^\pm$) of neighboring tori in lattice vortex dynamics. This latter phenomenon, also known as Arnold Diffusion [1], [22], is generic by Weinstein's theorem on the intersections of lagrangian submanifolds [23]. The specific vortex hamiltonian for a periodic domain is given by:

$$(4.1) \quad H(\tilde{M}, z_1, \dots, z_n, K_n) = \sum_{j \neq k} m_j m_k \log(\sin \pi(z_j - z_k))$$

By the following procedures, we will put (4.1) in the form $H = (F + \sum G_s) + H_1$ which is mentioned in the introduction:

Let us pick two distinguished vortices and without loss of generality, label them particles 1 and 2, i.e. nodes 1 and 2 in K_n . Now define the following disconnected subgraph $G \subset K_n$:

$$(4.2) \quad G = G_0 \cup \{(1, 2)\}$$

where $\{(1, 2)\}$ is the graph made up of nodes 1 and 2 and the edge (1,2) connecting them, while G_0 is the complete graph K_{n-2} on the remaining $(n - 2)$ nodes $j = 3, 4, \dots, n$.

The hamiltonian (4.1) becomes

$$(4.3) \quad H = m_1 m_2 \log(\sin \pi(z_1 - z_2)) + \sum_{(j,k) \in E(G_0)} m_j m_k \log(\sin \pi(z_j - z_k)) \\ + \sum_{(a,b') \in E(J)} m_a m_{b'} \log(\sin \pi(z_a - z_{b'}))$$

where $E(J)$, the edge set of the subgraph J

$$(4.4) \quad J = K_n \setminus G$$

consists of the edges (a, b') , with $a \in \{3, 4, \dots, n\}$ and $b' \in \{1, 2\}$.

Remark 4.1: In other words, the last term in (4.3) represents the coupling between the particles $j = 3, 4, \dots, n$ and the particles 1 and 2.

By transforming to the Jacobi variables for a suitable full binary tree $T(n)$ whose leaf 1 and 2 (corresponding to the distinguished vortices) are descendents of the same node (cf. Figure 1), we write H , (4.3) in the form of a function on $T(n)$:

$$(4.5) \quad H(M, \rho_1, \dots, \rho_n, G, K_n) = m_1 m_2 \log [\sin \pi(C_1(1, 2)\rho_1)] \\ + \sum_{(j,k) \in E_{G_0}(T(n))} m_j m_k \log \left[\sin \pi \left(\sum_{\ell=2}^{n-1} C_\ell(j, k)\rho_\ell \right) \right]$$

$$\begin{aligned}
& + \sum_{(a,b) \in E_J(T(n))} m_a m_b \log \left[\sin \pi \left(\sum_{\ell=1}^{n-1} C_\ell(a, b') \rho_\ell \right) \right] \\
& \equiv H^{(1)} + H^{(2)} + H^{(3)}
\end{aligned}$$

where $H^{(3)}$ is the coupling mentioned in Remark 4.1. Note that the complex Jacobi variable ρ_1 corresponds to the path (1,2) in $T(n)$, and it does not appear in the second term of (4.5).

Next, change variables $\rho_2, \dots, \rho_{n-1}$ to action-angles defined in (3.8) but leave ρ_1 in the form $\bar{Q} + i\bar{P}$ (dropping the subscript 1), where for $m_1 > 0, m_2 > 0$ (chosen for simplicity)

$$(4.6a) \quad \bar{Q} = K(x_1 - x_2)$$

$$(4.6b) \quad \bar{P} = K(y_1 - y_2)$$

$$(4.6c) \quad K = \sqrt{\frac{m_1 m_2}{m_1 + m_2}} = \frac{1}{C_1(1, 2)}$$

By taking the real part, following (2.5a), we put the first term $H^{(1)}$ in (4.5) in the form of the hamiltonian for a “generalized pendulum”

$$\begin{aligned}
(4.7) \quad F &= m_1 m_2 \log | \sin \pi(C_1(1, 2)\rho_1) | \\
&= \frac{m_1 m_2}{2} \log \left[\cosh \left(2\pi \frac{\bar{P}}{K} \right) - \cos \left(2\pi \frac{\bar{Q}}{K} \right) \right]
\end{aligned}$$

The hamiltonian's equations of F are:

$$(4.8a) \quad \dot{\bar{Q}} = \frac{\partial F}{\partial \bar{P}} = \frac{\pi m_1 m_2 \sinh 2\pi \frac{\bar{P}}{K}}{K \left[\cosh 2\pi \frac{\bar{P}}{K} - \cos 2\pi \frac{\bar{Q}}{K} \right]}$$

$$(4.8b) \quad \dot{\bar{P}} = \frac{-\partial F}{\partial \bar{Q}} = \frac{-\pi m_1 m_2 \sin 2\pi \frac{\bar{Q}}{K}}{K \left[\cosh 2\pi \frac{\bar{P}}{K} - \cos 2\pi \frac{\bar{Q}}{K} \right]}$$

Equations (4.8) are completely integrable and there is a heteroclinic orbit Γ on the zero “energy” manifold, defined by:

$$(4.9) \quad \begin{aligned} F &= \frac{m_1 m_2}{2} \log \left[\cosh \left(2\pi \frac{\bar{P}}{K} \right) - \cos \left(2\pi \frac{\bar{Q}}{K} \right) \right] \\ &= \frac{m_1 m_2}{2} \log 1 = 0 \end{aligned}$$

which connects the hyperbolic fixed points,

$$(4.10) \quad (\bar{Q}, \bar{P}) = \pm \left(\frac{K}{2}, 0 \right)$$

Although this heteroclinic orbit can be explicitly given in terms of Jacobi-elliptic functions [24], we do not discuss it here since this detail will not be needed later. Symmetry properties of this orbit play an important role.

The second term $H^{(2)}$ in (4.5) is put in the form of a perturbation of $(n - 2)$ -decoupled oscillators by the combinatorial method described in section III. Thus, following the calcu-

lations (3.3) - (3.9), we obtain

$$(4.11a) \quad H^{(2)} = \sum_{s=2}^{n-1} G_s + H_{11}$$

$$(4.11b) \quad G_s = \left[\sum_{(j,k) \in E_s(G_0)} m_j m_k \right] \log(I_s), \quad s = 2, \dots, n-1.$$

$$(4.11c) \quad H_{11} = \sum_{(j,k) \in E_{G_0}(T(n))} m_j m_k \log \left| 1 + \sum_{\ell < s(j,k)} \frac{C_\ell(j,k) I_\ell e^{i\theta_\ell}}{C_s I_s e^{i\theta_s}} \right| \\ + \sum_{(j,k) \in E_{G_0}(T(n))} m_j m_k \log \left| \frac{\sin \left(\sum_{\ell=2}^{n-1} C_\ell(j,k) I_\ell e^{i\theta_\ell} \right)}{\sum_{\ell=2}^{n-1} C_\ell(j,k) I_\ell e^{i\theta_\ell}} \right|$$

where $\sum_{s=2}^{n-1} G_s$ represents the decoupled oscillators. It is clear that the only solutions of the hamilton's equations of $\sum_{s=2}^{n-1} G_s$ are the tori

$$(4.12) \quad T^{(n-2)} = \underbrace{s^1 \times \dots \times s^1}_{n-2}$$

given by $I_\ell = r_\ell > 0$, $\ell = 2, \dots, n-1$ on which $\dot{\theta}_\ell = \Omega(r_\ell)t$. Collecting terms above we obtain the desired combinatorial perturbation form for the full hamiltonian (4.1) in the variables $I_2, \dots, I_{n-1}, \theta_2, \dots, \theta_{n-1}, \bar{Q}$ and $\bar{P}$:

$$(4.13a) \quad H = \left(F + \sum_{s=2}^{n-1} G_s \right) + H_1$$

$$(4.13b) \quad H_1 = H_{11} + H^{(3)}$$

$$(4.13c) \quad H^{(3)} = \sum_{(a,b') \in E_J(T(n))} m_a m_{b'} \log \left| \sin \pi \left[C_1(a, b')(\bar{Q} + i\bar{P}) + \sum_{\ell=2}^{n-1} C_\ell(j, k) I_\ell e^{i\theta_\ell} \right] \right|$$

Remark 4.2: Note that in this combinatorial perturbation form (4.13), the unperturbed (integrable) terms $F + \sum G_s$ are defined on the disconnected subgraph $G = G_0 + \{(1, 2)\}$, given by (4.2), although the original vortex hamiltonian (4.1) is defined on the complete graph K_n . The coupling term $H^{(3)}$ which is now the second term in the sum H_1 , is defined on the difference $J = K_n \setminus G$ given by (4.4). Moreover, the complete graph $G_0 = K_{n-2}$ on the nodes $3, 4, \dots, n$ appears in the “oscillator” term $\sum G_s$, while the graph $\{(1, 2)\}$ appears in the “pendulum” term F . Equations (4.1 - 4.13) complete steps 1-3 of the rough outline given in section I.

Next, we will show that the perturbation terms H_1 in (4.11) can be controlled under certain assumptions:

Lemma 4.1: With the actions $(I_2, \dots, I_{n-1})$ and angles in the complex cone set,

$$(4.14a) \quad N_\epsilon^\delta(T(n)) = \left\{ \begin{array}{l} I \equiv (I_2, \dots, I_{n-1}) \in C^{n-2}, \quad \theta \equiv (\theta_2, \dots, \theta_{n-1}) \in C^{n-2} \\ \text{such that} \\ |I_j| < \epsilon, \quad \text{Re}(I_j) > 0, \\ |I_\ell| < \sqrt{\epsilon} |I_k| \quad \text{if } \ell < k; \text{ and,} \\ |I_m(\theta_j)| < \delta \ll 1 \end{array} \right\}$$

and $(\bar{Q}, \bar{P})$ in a small complex neighborhood of the heteroclinic Γ

$$(4.14b) \quad H_\varepsilon^\delta = \{(\bar{Q}, \bar{P}) \in C^2 \mid F(\operatorname{Re}(\bar{Q}), \operatorname{Re}(\bar{P})) \sim 0(\varepsilon), |I_m(\bar{Q})|, |I_m(\bar{P})| < \delta\}$$

and under the conditions

$$(4.15a) \quad m_1 > 0, m_2 > 0, m_1, m_2 \sim 0(1)$$

$$(4.15b) \quad m_3, m_4, \dots, m_n \sim 0(\varepsilon^{1/2})$$

the perturbation term $H^{(3)} \sim 0(\varepsilon^{1/2})$ and $H_{11} \sim 0(\varepsilon^{3/2})$.

Proof: Under the conditions, (4.14 a,b), (4.15 a,b) with the parameter δ fixed at $\frac{1}{100}$, the terms

$$(4.16) \quad \left[C_1(a, b')(\bar{Q} + i\bar{P}) + \sum_{\ell=2}^{n-1} C_\ell(j, k) I_\ell e^{i\theta_\ell} \right], \text{ for each } (a, b') \in E_J(T(n))$$

are $0(\varepsilon)$ close to

$$(4.17) \quad \begin{cases} \frac{m_1}{m_1+m_2} \frac{(\bar{Q}+i\bar{P})}{K} & \text{if } b' = 2 \\ \frac{m_2}{m_1+m_2} \frac{(\bar{Q}+i\bar{P})}{K} & \text{if } b' = 1 \end{cases}$$

by virtue of the inverse transformation for the pairs $(z_a - z_b)$. Near the heteroclinic, (4.9), the terms (4.17) are $0(1)$ and bounded away from 0, and ± 1 . Thus the logarithms in $H^{(3)}$, (4.13c) are $0(1)$. Since the products $m_a m_b$, are the coefficients for these logarithmic terms, condition (4.15 a,b) implies that $H^{(3)} \sim 0(\varepsilon^{1/2})$. The remaining term H_{11} is $0(\varepsilon^{3/2})$ in the complex cone set, (4.14) by Lemma 3.2. $\diamond$

This completes step 4 of the modified combinatorial perturbation method.

Remark 4.3: Actions $I_2, \dots, I_{n-1}$ in the cone set (4.14) and $(\bar{Q}, \bar{P})$ in a small neighborhood of the heteroclinic (4.9), correspond physically to vortices 3, 4, ..., n clustered tightly and well-separated from the distinguished pair, e.g. vortices 1 and 2. The heteroclinic motion of this pair of vortices is depicted in figure(4), where the hyperbolic fixed points, (4.10) correspond to configurations $S^\pm$. Note that only the relative motions of the vortex lattice are described by the Jacobi variables. There may be additional rigid motions of the lattice with respect to the fluid; for example, when $m_1 \neq m_2$. The above physical interpretations are based on the correspondence between the variables ρ_j and the edges of a spatial tree $T(n)$, which is discussed in appendix B. Under the conditions (4.14) and (4.15 a,b) this correspondence guarantees that there are no collisions amongst the vortices. This is a necessary condition for the following steps of the Melnikov-Arnold method.

With the completion of the modified perturbation procedures, the hamiltonian, $H = (F + \sum G_s) + H_1$ given by (4.13) is now in the form suitable for a direct application of the vectorial Melnikov-Arnold Method [1], [6]. The unperturbed term $F + \sum_{s=2}^{n-1} G_s$ clearly has solution manifolds in the form of $(n-2)$ -dimensional tori with 1-dimensional heteroclinics Γ attached to them. The next result states that these solution manifolds, $T^{(n-2)} \times \Gamma$ continue under small perturbations:

Lemma 4.2: If the mass-vector $\tilde{M} = (m_1, m_2, m_3, \dots, m_n)$ satisfies the admissibility condition

(cf. (B6))

$$(I) \quad A(T(n), \tilde{M}) \equiv \prod_{s=1}^{n-1} \left(\sum_{j \in \mathcal{X}^+(s)} m_j \right) \left(\sum_{j \in \mathcal{X}^-(s)} m_j \right) \neq 0$$

and the modified nondegeneracy condition (cf. (3.11))

$$(II) \quad D(T(n), G_0, \tilde{M}) \equiv \prod_{s=2}^{n-1} \left\{ \sum_{(j,k) \in E_s(G_0)} m_j m_k \right\}$$

for the graph $G_0 = K_{n-2}$ defined in (4.2), and if moreover the mass-vector satisfies

$$(III) \quad m_3, m_4, \dots, m_n \sim 0(\varepsilon) ; m_1 > 0, m_2 > 0, m_1, m_2 \sim 0(1)$$

then for $\varepsilon \ll 1$ sufficiently small, a positive measure set of the invariant manifolds $T^{(n-2)} \times \Gamma$ of the unperturbed hamiltonian, $H_0(\tilde{M}, T(n), G) \equiv F + \sum_{s=2}^{n-1} G_s$ in the set $N_\varepsilon^\delta \times H_\varepsilon^\delta$, (4.14 a,b) continues under the perturbation H_1 , (4.13 b,c) to $(n-1)$ -dimensional manifolds, $T^{(n-2)} \times W^\pm$, where $W^\pm$ are the stable-unstable manifolds (or whiskers). The latter are Lagrangian submanifolds.

Proof: We will use a result of Graff and Moser [16], to show that the hyperbolic tori, $T^{(n-2)} \times \Gamma$ persists under sufficiently small perturbations. First, note that the full hamiltonian $H(\tilde{M}, T(n), G, K_n)$, (4.13) is real-analytic in the product of the complex cone set N_ε^δ and a small complex neighborhood of the heteroclinic manifold, (4.9); and 2π -periodic in the angles $\theta_2, \dots, \theta_{n-1}$ and $\bar{Q}$. Condition I guarantees that the linear symplectic transformation

which is used in (4.5) and based on the binary tree $T(n)$, is compatible with the mass-vector $\tilde{M}$. Condition II, by an argument similar to that in the proof of Lemma 3.3, implies that the “oscillator” hamiltonian $\sum_s G_s$ satisfies the KAM-nondegeneracy condition.

The $(n - 2)$ -dimensional manifolds given by (cf. (4.10)).

$$T^{(n-2)} \times \left(\pm \frac{K}{2}, 0 \right)$$

are clearly hyperbolic under the hamilton's equations of H_0 . All the conditions in the Graff-Moser theorem [16] are satisfied and thus the result follows if the perturbation H_1 is sufficiently small. Lemma 4.1 and condition III guarantees that this is true in the cone set N_ε^δ for sufficiently small ε . This completes the proof. $\diamond$

Following Holmes and Marsden [8], we compute the integrals

$$(4.18a) \quad \int_{-\infty}^{\infty} \{F, H_1\} dt$$

$$(4.18b) \quad \int_{-\infty}^{\infty} \{G_s, H_1\} dt \quad , \quad s = 3, \dots, n - 1$$

to leading order. The computations for the following result are carried out in appendix C.

Lemma 4.3:

$$(i) \quad \int_{-\infty}^{\infty} \{F, H_{11}\} dt = 0$$

$$(ii) \quad \int_{-\infty}^{\infty} \{G_s, H_{11}\} dt = 0$$

(iii) If the masses $\tilde{M}$ satisfies condition III in Lemma 4.2 and the actions

$I_2, \dots, I_{n-1}$ and $(\bar{Q}, \bar{P})$ are in $N_\epsilon^\delta \times H_\epsilon^\delta$, (4.14 a,b), and in addition

$$(4.19) \quad m_1 = m_2 = m$$

then, in (4.18 a,b) the leading order terms are:

$$(4.20a) \quad \int_{-\infty}^{\infty} \{F, H^{(3)}\} dt \sim 32\pi^4 m (C_R^2 I_R^2) \left(\frac{C_r I_r}{C_R I_R} \right) \left(\sum_{R,r,1 \in (a,2)} m_a m \right) \times M_1(\tilde{\theta}^0)$$

$$(4.20b) \quad M_1(\tilde{\theta}^0) \equiv \int_{-\infty}^{\infty} \left(\frac{\cosh 2\pi C_1 \bar{P} + \cos 2\pi C_1 \bar{Q}}{\cosh 2\pi C_1 \bar{P} - \cos 2\pi C_1 \bar{Q}} \right) \times \\ \times \sinh 2\pi C_1 \bar{P} \sin 2\pi C_1 \bar{Q} \sin(\Omega_R t - \Omega_r t) dt \left(\sin \theta_r^0 \cos \theta_R^0 - \sin \theta_R^0 \cos \theta_r^0 \right)$$

and

$$(4.21a) \quad \int_{-\infty}^{\infty} \{G_s, H^{(3)}\} dt \sim +4\pi^2 C_R I_R \times \\ \times \left(\sum_{(j,k) \in \mathbb{E}_s(G)} m_j m_k \right) \left(\sum_{R,s,1 \in (a,b)} m_a m_b C_s \right) M_s$$

$$(4.21b) \quad M_s(\tilde{\theta}^0) \\ \equiv \cdot \left[\int_{-\infty}^{\infty} \frac{\cosh 2\pi C_1 \bar{P} \cos \Omega_R t \cos \Omega_s t + \cos 2\pi C_1 \bar{Q} \sin \Omega_R t \sin \Omega_s t dt}{\cosh 2\pi C_1 \bar{P} - \cos 2\pi C_1 \bar{Q}} \right] \sin \theta_R^0 \cos \theta_s^0 \\ + \left[\int_{-\infty}^{\infty} \frac{\cosh 2\pi C_1 \bar{P} \sin \Omega_R t \sin \Omega_s t + \cos 2\pi C_1 \bar{Q} \sin \Omega_R t \sin \Omega_s t dt}{\cosh 2\pi C_1 \bar{P} - \cos 2\pi C_1 \bar{Q}} \right] \cos \theta_R^0 \sin \theta_s^0$$

Remark 4.4: In the proof of Lemma 4.3, the “oddness” of $\bar{Q}(t)$ and “evenness” of $\bar{P}(t)$ have been used. These properties are established easily from equations (4.8 a,b). Moreover the assumptions $m_1 = m_2$, (4.19) is not necessary but makes the calculations for Lemma 4.3, in particular (4.20), (4.21), shorter. The notation $R, r, 1 \in (a, b')$ and $R, s, 1 \in (a, b')$ in equations (4.20a) and (4.21a), means that the respective sums are over paths (a, b') , $b' = 1$ or 2 which contain the nodes $R, r, 1$ and $R, s, 1$ respectively. The labels R and r denote the root and its immediate descendent respectively in $T(n)$. For convenience the root is labelled $R = n - 1$. Note that the root R correspond to the largest $0(\epsilon)$ action, I_R (or edge ρ_R in the spatial tree $T(n)$). The $0(1)$ variables $(\bar{Q}, \bar{P})$ correspond to node 1 and thus the largest separation in $T(n)$. Note that C_1 in (4.20, 4.21) equals $\sqrt{2/m}$ under the assumption (4.19).

We will compute the Melnikov expressions (4.20b), (4.21b) along the unperturbed heteroclinic manifold, $T^{(n-2)} \times \Gamma$, i.e.

$$(4.22) \quad [\pm \bar{Q}(t), \pm \bar{P}(t), \Omega(r_2)t + \theta_2^0, \dots, \Omega(r_{n-1})t + \theta_{n-1}^0, r_2, \dots, r_{n-1}]$$

where the frequencies $\Omega(r_\ell)$ satisfy the Diophantine condition [4]. We look for a solution

$$(4.23) \quad \tilde{\theta}^0 \equiv (\theta_2^0, \dots, \theta_{n-1}^0)$$

to the Melnikov vector equation:

$$(4.24) \quad M_{\tilde{V}}(\tilde{\theta}^0) \equiv (M_1(\tilde{\theta}^0), M_2(\tilde{\theta}^0), \dots, M_{n-2}(\tilde{\theta}^0)) = (0, \dots, 0).$$

such that

$$(4.25) \quad \det \left[D_{\theta} M_{\sim} V(\theta^0) \right] \neq 0.$$

Lemma 4.4: (i) The Melnikov vector equation (4.24) is satisfied for

$$(4.26) \quad (\theta_2^0, \theta_3^0, \dots, \theta_{n-1}^0) = (0, 0, \dots, 0)$$

modulo integer multiples of π . (ii) The Melnikov nondegeneracy condition [8] is satisfied for $\theta_{\sim}^0$ given by (4.26) provided

$$(IV) \quad K_1 \equiv \sum_{R,r,1 \in (a,b)} m_a m_b \neq 0, \quad R = n-1, r = n-2$$

$$K_s \equiv \left(\sum_{(j,k) \in E_s(G_0)} m_j m_k \right) \left(\sum_{R,s,1 \in (a,b)} m_a m_b C_s(a, b') \right) \neq 0, s = 2, \dots, n-2$$

Proof: (i) From the expressions (4.20b, 4.21b), this follows immediately.

(ii) If we denote the integrals in (4.20b, 4.21b) by J_1 in M_1 , (4.20b) and $J_s^{1,2}$, $s = 2, \dots, n-2$ in M_s , (4.21b) where the superscript denotes the first and second integrals respectively, the

Jacobian matrix in (4.25) has the form:

$$(4.27) \quad D_{\theta} M_{\sim} V(\theta^0) = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 & J_1 & -J_1 \\ J_2^2 & 0 & \dots & \dots & \dots & 0 & -J_2^1 \\ 0 & J_3^2 & 0 & \dots & \dots & 0 & -J_3^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & J_{n-2}^2 & -J_{n-2}^1 \end{bmatrix}$$

where θ^0 is given by (4.26). The determinant has the form:

$$(4.28) \quad \det [D_{\theta} M_{\sim} V(\theta^0)] = \pm J_1 \left(\prod_{s=2}^{n-3} J_s^2 \right) (J_{n-2}^1 - J_{n-2}^2)$$

Since all the integrals $J_1, J_s^{1,2}, s = 2, \dots, n-2$ have nonzero values when evaluated along (4.22) and moreover, $J_{n-2}^1 \neq J_{n-2}^2$, we obtain the desired result provided conditions IV hold. These absolutely convergent integrals are evaluated numerically using suitable limits (cf. Robinson [25]). Note that the denominators in the integrands equal 1 $\diamond$.

Remark 4.5: It is well-known that the integrals, J_1 and $J_s^{1,2}, s = 2, \dots, n-2$ are absolutely convergent ones. Indeed the integrals (4.18 a,b) can be written in the form:

$$(4.29) \quad \int_{-\infty}^{\infty} \langle \tilde{n}(a), J_{\sim} \nabla H_1 \rangle dt$$

where $\tilde{n}(a)$ is the normal vector to the heteroclinic manifold, $T^{(n-2)} \times \Gamma$ at an arbitrary point $a = (\bar{Q}, \bar{P}, \theta_2, \dots, \theta_{n-1})$. Since $\tilde{n}(a)$ is defined by the gradient of the hamiltonian function,

H_0 the integrand in equation (4.29) decreases exponentially as $\underline{a}$ tends towards the torus $T^{(n-2)}$, i.e. the integrals (4.18 a,b) and therefore $J_1, J_1^{1,2}$ are absolutely convergent ones.

Collecting together Lemma 4.1 - 4.4, we obtain the proof of the main result in this paper:

Theorem 4.1: If the mass-vector $\underline{M} = (m_1, \dots, m_n)$ satisfy conditions I, II, and III on Lemma 4.2, for some full binary tree $T(n)$ (whose ~~nodes~~ ^{leave} 1,2 have the same parent) and moreover, the distinguished pair of vortices have equal “mass”, i.e. $m_1 = m_2$, then the vortex lattice hamiltonian (4.1) supports a positive measure set of whiskered KAM-tori in part of its phase-space, defined by $N_\epsilon^\delta(T(n)) \times H_\epsilon^\delta$, (4.14 a,b). Moreover, if $\underline{M}$ satisfy condition IV in Lemma 4.4 for the same tree $T(n)$, then the stable and unstable manifolds $W^\pm$ of these tori intersect transversally.

Remark 4.5: ⁶ The above theorem is proven under certain restrictions on the vorticities which are not the weakest ones possible, e.g. $m_1 = m_2$.

Conclusions

In view of the many results which connect the solutions of the n -body hamiltonians (2.1) to the 2-D Euler’s equations (cf. [26], [27], [28]), our main result on the existence of chaotic solutions in the form of Arnold Diffusion for the vortex lattice (4.1), indicates that chaotic behavior exists in 2-D ideal fluids. This phenomenon is sometimes called weak turbulence.

We expect that the combinatorial perturbation method can be applied to other n -body problems as well.

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Figure Captions

Figure 1 A full oriented binary tree, $T(5)$ with 5 leave (abstract).

Figure 2 Same tree $T(5)$ (spatial) with the Jacobi coordinates indicated.

Figure 3 The cone set in phase-space interpreted in terms of $T(5)$ and clustering in the physical plane.

Figure 4 Positions of vortices corresponding to the heteroclinic manifold $T^{(n-2)} \times \Gamma$.

Appendix A

A *complete graph*, K_n consists of N vertices, every pair of which, is linked by an *edge*, two vertices linked by an edge are *adjacent*. Two vertices, A and B in a graph, G are said to be *connected* if there is a *path* made up of the ordered set, $\{p, q, r, \dots s\}$ of vertices in G such that any two neighboring vertices in the set, are *adjacent*, and p is *adjacent* to A while s is *adjacent* to B . A *circuit* consists of at least two distinct *paths*, connecting the same pair of vertices in G . A *tree* is a graph without *circuits*. The *leaves* of a tree, T are the vertices of *degree*, 1 where the *degree* of a vertex, V is the number of edges incident at V . A *rooted tree* has a distinguished vertex, called the *root*. In a full *binary tree* each vertex other than the leaves has two *descendents*, where a *descendent* C of a vertex B is a vertex adjacent to B but further from the root than B . A vertex or node ℓ is lower than nodes s (denoted $\ell < s$) if the path from node ℓ to the root, R passes through node s .

A subgraph $G \subseteq K_n$ is a graph obtained from K_n by removing some of its edges and/or some of its nodes.

Appendix B

Since the results described here have appeared in print [14], [15], we will be very brief. In outline, the main result states that to every full binary tree $T(n)$ with n leave, there belongs a $2n \times 2n$ real symplectic matrix of the form

$$(B1) \quad \tilde{M}^n = \begin{bmatrix} \tilde{A} & \tilde{Q} \\ \tilde{Q} & \tilde{D} \end{bmatrix} \quad \tilde{A}^t \tilde{D} = \tilde{I}_n$$

which generates a linear canonical transformation to so-called Jacobi variables. An important property of this transformation is the fact that these variables are relative coordinates for the n -body problems in this paper.

Given a full binary tree $T(n)$ (cf. Figure 1) with n leave labelled by the integers $1, \dots, n$, (and without confusion) $n - 1$ internal nodes labelled by $1, \dots, n - 1$, define the subsets

$$\mathcal{X}^\pm(s) = \{j \in (1, \dots, n) \mid \text{leaf } j \text{ is below nodes } s, \text{ i.e. } j < s,$$

and connected to it via the right (left) branch}

$$(B2) \quad \mathcal{X}(s) = \mathcal{X}^+(s) \cup \mathcal{X}^-(s) \quad , \quad s = 1, \dots, n - 1$$

which are based on the partial order relation ($<$) for $T(n)$ (cf. Appendix A). The entries of

the $n \times n$ matrices $\tilde{A}(\tilde{M}, T(n))$, $\tilde{D}(\tilde{M}, T(n))$ in (B1) are now given by:

$$(B3) \quad A_{jk} = \left| \frac{M_j^+ M_j^-}{M_j} \right|^{1/2} \begin{cases} \text{sgn}(m_k) \sqrt{|m_k|} / M_j^+ & \text{if } k \in \mathcal{X}^+(j) \\ -\text{sgn}(m_k) \sqrt{|m_k|} / M_j^- & \text{if } k \in \mathcal{X}^-(j) \\ 0 & \text{if } k \notin \mathcal{X}(j) \end{cases}$$

$$j = 1, \dots, n-1$$

$$A_{nk} = \text{sgn}(m_k) \sqrt{|m_k|} / |M_{n-1}|^{1/2} \quad k = 1, \dots, n.$$

$$(B4) \quad D_{jk} = \text{sgn} \left(\frac{M_j^+ M_j^-}{M_j} \right) \left| \frac{M_j^+ M_j^-}{M_j} \right|^{1/2} \begin{cases} \sqrt{|m_k|} / M_j^+ & \text{if } k \in \mathcal{X}^+(j) \\ -\sqrt{|m_k|} / M_j^- & \text{if } k \in \mathcal{X}^-(j) \\ 0 & \text{if } k \notin \mathcal{X}(j) \end{cases}$$

$$j = 1, \dots, n-1$$

$$D_{nl} = \sqrt{|m_k|} \text{sgn}(M_{n-1}) / |M_{n-1}|^{1/2} \quad k = 1, \dots, n.$$

where

$$(B5) \quad M_j^\pm = \sum_{k \in \mathcal{X}^\pm(j)} m_k, \quad M_j = \sum_{k \in \mathcal{X}(j)} m_k$$

Lemma B1: For any labelled full binary tree $T(n)$ with n leave and mass-vector $\tilde{M} =$

$(m_1, \dots, m_n)$ which satisfies:

$$(B6) \quad M_j^\pm \neq 0, j = 1, \dots, n-1 \text{ and } M_{n-1} \neq 0$$

the matrices $A_{\approx}(M, T(n))$ and $D_{\approx}(M, T(n))$, (B3), (B4) satisfy the relation $A_{\approx}^t D_{\approx} = I_{\approx_n}$, i.e. $M_{\approx}^n$, (B1) is symplectic.

This symplectic matrix $M_{\approx}^n$ generates a linear canonical transformation to relative coordinates

$$(B7) \quad \rho_j = Q_j + iP_j \quad , \quad Q_{\approx} = Q_{\approx}(q) \quad , \quad P_{\approx} = P_{\approx}(p)$$

The $\rho_s, s = 1, \dots, n-1$ correspond to internal node s of the abstract tree $T(n)$ (figure 1) via the subsets $\mathcal{X}^\pm(s)$. On the other hand each complex variable ρ_s essentially represents (at least when $m_j > 0, j = 1, \dots, n$) the path connecting the left descendent to the right descendent of node s in the spatial tree, which is again denoted by $T(n)$, (cf. figure 2). The leave in this spatial tree represents the instantaneous positions of the particles, while each internal nodes is obtained by taking the weighted average of the positions of its two descendents (cf. [14], [15] for more details). In this way, the Jacobi variables $\rho_s, s = 1, \dots, n-1$ represent relative separations while ρ_n represent the “center of mass” of the system of particles.

Inverting the transformation (B7), we obtain the following formulae for the pairs

$$(B8) \quad z_j - z_k = \sum_{\ell=1}^{n-1} C_{\ell}(j, k) \rho_{\ell}$$

where the subscripts ℓ such that $C_{\ell}(j, k) \neq 0$ label the internal nodes ℓ in the leaf j to leaf k path (j, k) in $T(n)$; the coefficients $C_{\ell}(j, k)$ have the form:

$$(B9) \quad \frac{M_{\ell}^{\pm}}{M_{\ell}} \left| \frac{M_{\ell}}{M_{\ell}^+ M_{\ell}^-} \right|^{1/2}$$

For $m_1, \dots, m_n \sim 0(1)$ as in section 3, $C_{\ell}(j, k) \sim 0(1)$ over all leaf-to-leaf paths

$$(B10) \quad E(T(n)) = \{\text{leaf } j \text{ to leaf } k \text{ path } (j, k) \text{ in } T(n)\}$$

In section IV, $m_3, \dots, m_n \sim 0(\sqrt{\varepsilon})$ and $m_1, \dots, m_2 \sim 0(1)$ which implies that

$$C_1(1, 2), C_1(a, b') \sim 0(1) \text{ for } (a, b') \in E_{\overline{J}}(T(n)), J = K_n \setminus G$$

$$(B11) \quad C_{\ell}(j, k) \sim 0(\varepsilon^{-1/4}) \text{ for } (j, k) \in E_{G_0}(T(n)).$$

$$C_{\ell}(a, b') \sim 0(\varepsilon^{1/4}) \text{ for } (a, b') \in E_{\overline{J}}(T(n)) \text{ s.t. } a = 1, 2 \text{ and } b' < \ell.$$

where the sets G, G_0 and J are defined in (4.2) and (4.4).

The notation $E_G(T(n))$ (B10, B11) represents (after a slight abuse) the set of

paths (j, k) in $T(n)$ whose leaf j is adjacent to leaf k in the graph G , i.e. edge (j, k) is in the edge set $E(G)$. The notation $E_s^{(G)}$ represents the set of path (j, k) in $T(n)$ whose nearest node $N(j, k)$ to the root R is node s . The root will (without loss of generality) be labelled $n - 1$, i.e. $R = n - 1$, while the internal node 1's descendents will be labelled leaf 1 and 2. After applying (B8) to the original hamiltonians (2.1) in sections III and IV, it is easily seen that the transformed hamiltonians should be functions over the leaf-to-leaf paths $E_G(T(n))$.

The cone set N_s^δ in phase-space translates into tight clusters of particles defined on the spatial tree $T(n)$ (cf. Figure 3), with large $(O(\sqrt{\varepsilon}))$ differences in the lengths of its edges, depending on their positions with respect to the root R in terms of the partial order relation $(<)$.

Finally, the number of oriented unlabelled full binary trees $T(n)$ is given by the Catalan number [25]:

$$(B12) \quad C_{n-1} = \frac{1}{2n-1} \binom{2n-1}{n-1}$$

Appendix C

Proof of Lemma 4.3:

(i) Since H_{11} , (4.11c) does not depend on $(\bar{Q}, \bar{P})$, $\{F, H_{11}\} \equiv 0$.

(ii) Computing the *Poisson* bracket we obtain

$$\{G_s, H_{11}\} = \frac{-\partial G_s}{\partial I_s} \frac{\partial H_{11}}{\partial \theta_s}$$

which vanishes when integrated from $-\infty$ to ∞ because it consists of purely sinusoidal terms, with zero mean.

For the rest of the proof, we will need the fact that:

$$(C1) \quad \bar{Q}(t), \bar{P}(t) \text{ are odd and even functions, respectively.}$$

It is easy to see from the hamilton's equations, (4.8) that one of the pair $(\bar{Q}(t), \bar{P}(t))$ must be even and the other odd. Keeping the leading two terms in the expansion of the $\cosh$ in F , (4.7), we obtain the pendulum case of which F is a generalization with the same symmetries, i.e. $\cosh(2\pi C_1 \bar{P})$ and $(\bar{P})^2$ are both even. In the pendulum case, we know that $(\bar{Q}(t), \bar{P}(t))$ satisfies (C1) [8]. Thus we have the result (C1) for the generalized hamiltonian, F .

(iii) Under the assumptions in part (iii), we compute $\{F, H^{(3)}\}$ keeping only the

even functions of time since odd terms vanish upon integration from $-\infty$ to ∞ :

$$\begin{aligned}
\text{(C2)} \quad \{F, H^{(3)}\} &= \frac{\partial F}{\partial \bar{Q}} \frac{\partial H^{(3)}}{\partial \bar{P}} - \frac{\partial F}{\partial \bar{P}} \frac{\partial H^{(3)}}{\partial \bar{Q}} \\
&= 2\sqrt{2} \pi^2 m^{3/2} \sum_{(a,b')} \frac{m_a m_{b'} C_1(a, b')}{\cosh 2\pi (C_1 \bar{P} + \sum_\ell C_\ell I_\ell \sin \theta_\ell) - \cos 2\pi (C_1 \bar{Q} + \sum_\ell C_\ell I_\ell \cos \theta_\ell)} \times \\
&\quad \times \left[\begin{aligned} &\sinh \left(2\pi (C_1 \bar{P} + \sum_\ell C_\ell I_\ell \sin \theta_\ell) \right) \sin \left(2\pi C_1(1, 2) \bar{Q} \right) \\ &- \sin \left(2\pi (C_1 \bar{Q} + \sum_\ell C_\ell I_\ell \cos \theta_\ell) \right) \sinh \left(2\pi C_1(1, 2) \bar{P} \right) \end{aligned} \right] \\
&= 4\sqrt{2} \pi^2 m^{3/2} \sum_{(a,b')} \frac{m_a m_{b'} C_1(a, b')}{\cosh (2\pi C_1(a, b') \bar{P}) \cosh (2\pi \sum_\ell C_\ell I_\ell \sin \theta_\ell) - \cos (2\pi C_1 \bar{Q}) \cos (2\pi \sum_\ell C_\ell I_\ell \cos \theta_\ell)} \times \\
&\quad \times \left[\begin{aligned} &\sinh (2\pi C_1 \bar{P}) \cosh (2\pi \sum_\ell C_\ell I_\ell \sin \theta_\ell) \sin (2\pi C_1(a, 2) \bar{Q}) \cos (2\pi C_1(a, 2) \bar{Q}) \\ &- \sin (2\pi C_1 \bar{Q}) \cos (2\pi \sum_\ell C_\ell I_\ell \cos \theta_\ell) \sinh (2\pi C_1(a, 2) \bar{P}) \cosh (2\pi C_1(a, 2) \bar{P}) \end{aligned} \right]
\end{aligned}$$

In the last equality, we used the properties

$$\begin{aligned}
\text{(C3)} \quad C_1(a, 1) &= -C_1(a, 2) \quad \text{for each } a = 3, 4, \dots, n \\
C_1(a, 2) &= \frac{1}{2} C_1(1, 2) \quad \text{for each } a = 3, 4, \dots, n
\end{aligned}$$

of the coefficients in the inverse symplectic transformation (B9), which follows from the assumption, (4.19) that $m_1 = m_2$. Note that the sum $\sum_{(a,b')}$ is over all $(a,1), (a,2)$. Upon expanding the cosh and cos and after using the properties of the cone set:

$$\begin{aligned}
m_3, \dots, m_n &\sim 0 \left(\sqrt{\varepsilon} \right) \quad , \quad m_1 = m_2 = m \sim 0(1) \\
\text{(C4)} \quad I_2, \dots, I_{n-1} &\sim 0(\varepsilon)
\end{aligned}$$

$$I_\ell \setminus I_s \sim 0(\sqrt{\varepsilon}) \quad \text{for } \ell \text{ directly below } s \text{ in } T(n).$$

$$\bar{Q}, \bar{P} \sim 0(1)$$

we obtain the first 4 terms in powers of $\sqrt{\varepsilon}$:

$$\begin{aligned}
\text{(C5)} \quad & -4\sqrt{2}\pi^2 m^{3/2} \left\{ \begin{aligned} & \sin(\pi C_1(1,2)\bar{Q}) \sum m_a m_{\nu'} C_1(a,b') \sinh 2\pi C_1(a,b') \bar{P} \\ & + 2\pi^2 C_R^2 I_R^2 \sin(\pi C_1(1,2)\bar{Q}) \sin 2\Omega_R t \sin \theta_R^0 \sum m_a m_{\nu'} C_1(a,b') \sinh(2\pi C_1 \bar{P}) \\ & - 2\pi^2 C_R^2 I_R^2 \sin(\pi C_1 \bar{Q}) \sin 2\Omega_R t \sin \theta_R^0 \sum m_a m_{\nu'} C_1(a,b') \sinh(2\pi C_1 \bar{P}) \end{aligned} \right. \\
& + 16\sqrt{2}\pi^4 m^{3/2} (C_R^2 I_R^2) \left(\frac{C_r I_r}{C_R I_R} \right) \left(2 \sum_{R,r,1 \in (a,2)} m_a m_{C_1(a,2)} \right) \times \\
& \times \left[\left(\frac{\cosh(\pi C_1(1,2)\bar{P}) + \cos(\pi C_1 \bar{Q})}{\cosh(\pi C_1 \bar{P}) - \cos(\pi C_1 \bar{Q})} \right) \sinh(\pi C_1(1,2)\bar{P}) \sin(\pi C_1 \bar{Q}) \sin(\Omega_R t - \Omega_r t) \right] \\
& \times (\sin \theta_r^0 \cos \theta_R^0 - \sin \theta_R^0 \cos \theta_r^0)
\end{aligned}$$

where

$$\theta_\ell = \Omega_\ell t + \theta_\ell^0, \quad \Omega_\ell = \frac{\partial G_\ell}{\partial I_\ell}, \quad \ell = 2, \dots, n-1$$

and R, r denotes the root (node $n-1$) and its immediate descendent, node r ($r < R$) in the tree $T(n)$. The notation $(R, r, 1) \in (a, 2)$ in $2 \sum m_a m$ means that the sum is taken over all leaf-to-leaf paths $(a, 2)$ (from leaf $a \in \{3, 4, \dots, n\}$ to distinguished leaf 2) that contains the internal nodes 1, r and root R . The factor of 2 that appears in this sum is due to the fact that the original sum is over all paths $(a, b'), b' = 1, 2$.

The first term in (C5) vanishes on integrating from $-\infty$ to ∞ because it is odd; the second and third term in (C5) cancel. This leaves the fourth term which is the first nonzero term of $\int_{-\infty}^{\infty} \{F, H^{(3)}\} dt$, and has order $0(\varepsilon^{5/2})$ because (C4) implies (cf. (B11))

$$C_R I_R \sim 0(\varepsilon^{3/4}), \quad \frac{C_r I_r}{C_R I_R} \sim 0(\varepsilon^{1/2})$$

$$C_1(a, 2) = \frac{1}{2} C_1(1, 2) = \sqrt{\frac{2}{m}}$$

$$\sum_{R, r, 1 \in (a, 2)} m_a m \sim 0(\varepsilon^{1/2})$$

This yields (4.20 a,b) in Lemma 4.3.

The calculations for $\int_{-\infty}^{\infty} \{G_s, H^{(3)}\} dt$ are similar and yield the first nonzero term (4.21 a,b).

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