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By

C. Guillopé

and

J.C. Saut

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455-0436

Phone: 612/624-6066 Fax: 612/626-7370

URL: <http://www.ima.umn.edu>

EXISTENCE RESULTS FOR THE FLOW OF VISCOELASTIC FLUIDS WITH A DIFFERENTIAL CONSTITUTIVE LAW*

C. GUILLOPÉ⁽¹⁾ AND J.C. SAUT⁽¹⁾⁽²⁾

Introduction. This paper is mainly concerned with incompressible viscoelastic fluids which obey the constitutive law, of Oldroyd type ([15])

$$(0.1) \quad \underline{\tau} + \lambda_1 \frac{\mathcal{D}_a \underline{\tau}}{\mathcal{D}t} = 2\eta \left(\underline{D} + \lambda_2 \frac{\mathcal{D}_a \underline{D}}{\mathcal{D}t} \right),$$

where $\underline{\tau}$ is the (symmetric) extra-stress tensor (the total stress is given by $\underline{\sigma} = -p\underline{I} + \underline{\tau}$, where p is the hydrodynamic pressure); $\underline{D}$ is the rate of deformation tensor, $\underline{D}[\mathbf{u}] = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$, $\mathbf{u}$ being the velocity field; λ_1 is the relaxation time, λ_2 the retardation time, $0 \leq \lambda_2 < \lambda_1$; and η is the fluid viscosity. The symbol $\frac{\mathcal{D}_a}{\mathcal{D}t}$ denotes an objective (frame indifferent) derivative [1] [18]. More precisely,

$$(0.2) \quad \frac{\mathcal{D}_a \underline{\tau}}{\mathcal{D}t} = \left[\frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \right] \underline{\tau} + \underline{\tau} \underline{W} - \underline{W} \underline{\tau} - a(\underline{D} \underline{\tau} + \underline{\tau} \underline{D}),$$

where $-1 \leq a \leq 1$ and $\underline{W} = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^T)$ is the vorticity tensor.

The limit case $\lambda_1 > 0, \lambda_2 = 0$ corresponds to a purely elastic fluid, while the limit case $\lambda_1 = \lambda_2 = 0$ corresponds to a purely viscous (Newtonian) fluid. The constitutive law (0.1) may be used to describe solutions of polymers in a Newtonian solvent, or polymer melts with a fast relaxation mode. Note also that a nonzero retardation time is frequently introduced in the computation of flows of viscoelastic fluids (see e.g. [4] [6]) to prevent the development of short waves (Hadamard) instabilities [5] [10] [11]. We will consider only the case where $\lambda_2 > 0$.

The constitutive law (0.1) is coupled with the equations of conservation of momentum and of mass,

$$(0.3) \quad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla p = \nabla \cdot \underline{\tau} + \mathbf{f},$$

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⁽¹⁾Laboratoire d'Analyse Numérique, Bât. 425, C.N.R.S. and Université Paris-Sud 91405 Orsay Cedex

⁽²⁾Mathématiques, Université Paris-Val de Marne, 94010 Créteil Cedex.

$$(0.4) \quad \operatorname{div} \mathbf{u} = 0$$

in Ω (smooth open bounded set in $\mathbb{R}^n$, $n = 2, 3$). The velocity satisfies a non-slip boundary condition,

$$(0.5) \quad \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega = \Gamma$$

Moreover velocity and extra-stress are given at the initial time $t = 0$

$$(0.6) \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \underline{\underline{\tau}}|_{t=0} = \underline{\underline{\tau}}_0.$$

The data are $\mathbf{u}_0, \underline{\underline{\tau}}_0$, the (constant) density $\rho > 0$, and the external body forces $\mathbf{f}$. The unknowns are $\mathbf{u}, \underline{\underline{\tau}}$ and the pressure p .

Our goal is to study the initial value boundary problem (0.1)–(0.6), as well as the one associated with more general constitutive laws (see §1).

In the first paragraph we give a brief survey of classical constitutive laws of differential type to which our results apply. They include the models of Oldroyd, Giesekus, Leonov, Phan-Thien and Tanner, and Larson.

Then using Schauder's fixed point theorem we prove local existence of solutions of (0.1)–(0.6) for arbitrary data in some appropriate Sobolev space.

The third paragraph is devoted to *global* existence of solutions for sufficiently small data whose retardation time is not too small. This is deduced from uniform (in time) estimates on the local solution. An easy consequence is the exponential asymptotic stability of the null solution when $\mathbf{f} = 0$.

The last paragraph deals with stability of small solutions which leads to the existence of stable time-periodic (resp. steady) solutions of (0.1)–(0.6) when the forcing term is small and time-periodic (resp. steady). Finally we briefly discuss how to extend our results to models (such as the rubber-like model) which possess several relaxation times.

Part of our results were announced in a Note aux Comptes Rendus [8].

1. Constitutive laws of differential type. Many constitutive laws of differential type have been introduced to model the behavior of viscoelastic fluids. They are basically derived from molecular or continuum mechanics considerations. Most of them have the following form

$$(1.1) \quad \underline{\underline{\tau}} + \lambda_1 \frac{\mathcal{D}_a \underline{\underline{\tau}}}{\mathcal{D}t} + \underline{\underline{\beta}}(\underline{\underline{\tau}}, \underline{\underline{D}}) = 2\eta \left(\underline{\underline{D}} + \lambda_2 \frac{\mathcal{D}_a \underline{\underline{D}}}{\mathcal{D}t} \right)$$

where $\frac{\mathcal{D}_a}{\mathcal{D}t}$ is an objective time derivative of the type (0.2) and $\underline{\underline{\beta}}(\underline{\underline{\tau}}, \underline{\underline{D}})$ is a tensor valued nonlinear function of $\underline{\underline{\tau}}$ and $\underline{\underline{D}}$, constrained by certain restrictions due to objectivity. In fact, in all the examples we present here, $\underline{\underline{\beta}}$ is smooth and at least quadratic.

As in the Oldroyd model (0.1), λ_1 is a relaxation time and $\lambda_2, 0 \leq \lambda_2 < \lambda_1$ is a retardation time. For instance, the Oldroyd model has $\underline{\underline{\beta}} \equiv 0$. Other classical examples include the following.

(i) The (generalized) Larson's model [12] has $a = +1$ in (0.2) and

$$(1.2) \quad \underline{\underline{\beta}}(\underline{\underline{\tau}}, \underline{\underline{D}}) = 2Tr(\underline{\underline{\tau}} \underline{\underline{D}})\alpha(Tr \underline{\underline{\tau}})(\underline{\underline{\tau}} + \underline{\underline{I}})$$

where $\alpha(Tr \underline{\underline{\tau}})$ is a scalar function of $Tr \underline{\underline{\tau}}$.

(ii) The model of Phan-Thien and Tanner [16] has $a = +1$ in (0.2) and

$$(1.3) \quad \underline{\underline{\beta}}(\underline{\underline{\tau}}, \underline{\underline{D}}) = \alpha \underline{\underline{\tau}} Tr \underline{\underline{\tau}} \quad , \quad \alpha = \text{constant}.$$

It can be generalized (cf. [12]) to

$$(1.4) \quad \underline{\underline{\beta}}(\underline{\underline{\tau}}, \underline{\underline{D}}) = a \underline{\underline{\tau}}^2 + b \underline{\underline{\tau}},$$

where a and b are scalar functions of $Tr \underline{\underline{\tau}}$ and $\det \underline{\underline{\tau}}$.

(iii) The model of Giesekus [7] is a particular case of (1.4), where

$$\underline{\underline{\beta}}(\underline{\underline{\tau}}, \underline{\underline{D}}) = \alpha \underline{\underline{\tau}}^2 \quad , \quad \alpha = \text{constant}.$$

(iv) The Oldroyd 8 constants model [15]

$$(1.5) \quad \underline{\underline{\beta}}(\underline{\underline{\tau}}, \underline{\underline{D}}) = \mu_0(Tr \underline{\underline{\tau}})\underline{\underline{D}} + \nu_1 Tr(\underline{\underline{\tau}} \underline{\underline{D}})\underline{\underline{I}} + \mu_2 \underline{\underline{D}}^2 + \nu_2 Tr(\underline{\underline{D}}^2)\underline{\underline{I}},$$

where $\mu_0, \nu_1, \mu_2, \nu_2$ are constants.

We shall now consider the system (0.3), (1.1) and write it on a more tractable fashion.

We a priori decompose the extra-stress tensor as

$$(1.6) \quad \underline{\underline{\tau}} = \underline{\underline{\tau}}^s + \underline{\underline{\tau}}^p$$

where $\underline{\underline{\tau}}^s$ corresponds to the Newtonian part (the solvent) and $\underline{\underline{\tau}}^p$ to the purely elastic part (the polymer):

$$(1.7) \quad \begin{aligned} \underline{\underline{\tau}}^s &= 2\eta\lambda_2/\lambda_1 \underline{\underline{D}} \stackrel{\text{def}}{=} 2\eta_s \underline{\underline{D}} \quad , \\ \underline{\underline{\tau}}^p + \lambda_1 \frac{\mathcal{D}_a}{\mathcal{D}t} \underline{\underline{\tau}}^p + \tilde{\underline{\underline{\beta}}}(\underline{\underline{\tau}}^p, \underline{\underline{D}}) &= 2\eta(1 - \lambda_2/\lambda_1) \underline{\underline{D}} \stackrel{\text{def}}{=} 2\eta_p \underline{\underline{D}} \quad , \end{aligned}$$

where

$$(1.8) \quad \tilde{\beta}(\underline{\tau}^p, \underline{D}) \stackrel{\text{def}}{=} \beta(\underline{\tau}^p + \underline{\tau}^s, \underline{D}).$$

From now on we shall denote $\underline{\tau}^p = \underline{\tau}$ and rewrite (0.3), (1.1) in the form

$$(1.9) \quad \begin{cases} \rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla p = \eta_s \Delta \mathbf{u} + \nabla \cdot \underline{\tau} + \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \\ \underline{\tau} + \lambda_1 \frac{\mathcal{D}_a \underline{\tau}}{\mathcal{D}t} + \tilde{\beta}(\underline{\tau}, \underline{D}) = 2\eta_p \underline{D}, \end{cases}$$

where η_s and η_p are the “viscosities” of the solvent and of the polymer. Another reduction is obtained by using dimensionless variables. We define the Weissenberg number $We = \lambda_1 U/L$ and the Reynolds number $Re = \rho U L / \eta$. (U and L represent a typical velocity and a typical length of the flow.) Namely we set

$$\begin{aligned} \mathbf{x} &= \frac{\mathbf{x}^*}{L} & \mathbf{u} &= \frac{\mathbf{u}^*}{U}, & t &= t^* \frac{U}{L}, & \underline{\tau} &= \frac{\underline{\tau}^* L}{\eta U}, \\ p &= \frac{p^* L}{\eta U}, & \mathbf{f} &= \frac{\mathbf{f}^* L^2}{\eta U}, & \tilde{\beta} &= \frac{U}{L} \tilde{\beta}^* \end{aligned}$$

where stars are attached to dimensional variables. The dimensionless system (1.9) takes the form

$$(1.10) \quad \begin{cases} Re \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla p = (1 - \omega) \Delta \mathbf{u} + \nabla \cdot \underline{\tau} + \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \\ \underline{\tau} + We \frac{\mathcal{D}_a \underline{\tau}}{\mathcal{D}t} + \tilde{\beta}(\underline{\tau}, \underline{D}) = 2\omega \underline{D}, \end{cases}$$

where the retardation parameter ω , defined by

$$\omega = \eta_p / \eta = 1 - \lambda_2 / \lambda_1,$$

satisfies $0 < \omega < 1$.

Notation. Ω is a bounded domain in $\mathbf{R}^N$, $N = 2$ or 3 , with a regular boundary Γ . Throughout this paper we will use the following spaces: the Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, with norms $\| \cdot \|_{L^p}$; the Sobolev spaces $H^k(\Omega)$, $k = -1, 0, 1, \dots$ with norms $\| \cdot \|_k$ and inner product $((\cdot, \cdot))_k$; the Sobolev space $H_0^1(\Omega)$, and its dual $H^{-1}(\Omega)$; the vector spaces $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^k(\Omega)$ of vector valued or tensor valued functions with components in $L^2(\Omega)$ and $H^k(\Omega)$ respectively, their norms being denoted in the same way as above. To

simplify the notation we will also indicate the L^2 -norm by $|\cdot|$, and the Dirichlet norm, defined as the L^2 -norm of the gradient, by $\|\cdot\|$.

If I is an interval of $\mathbb{R}_+$, $\mathcal{C}(\bar{I}; \mathbf{H}^k)$ is the space of vector valued functions $\mathbf{u}(t, x)$ such that $\mathbf{u}(t, \cdot)$ belongs to $\mathbf{H}^k(\Omega)$ for all t in $\bar{I}$, and the function $t \rightarrow \mathbf{u}(t, \cdot)$ with values in $\mathbf{H}^k(\Omega)$ is continuous on $\bar{I}$. $\mathcal{C}_b(\bar{I}; \mathbf{H}^k)$ is the space of bounded continuous functions on I into $\mathbf{H}^k$. We will also use the function spaces $L^p(I; \mathbf{H}^k)$, $1 \leq p \leq \infty$, which consist of p -integrable functions with values in $\mathbf{H}^k$.

We now introduce the usual spaces of divergence-free vectors. H is the space of solenoidal L^2 vector functions whose normal trace on Γ is zero, equipped with the L^2 norm $|\cdot|$; we will use the orthogonal projection P of L^2 onto H ; V is the space of solenoidal $\mathbf{H}_0^1$ vector functions, equipped with the Dirichlet norm $\|\cdot\|$; $D(A)$ is the domain of the Stokes operator $A = -P\Delta$, $D(A) = V \cap \mathbf{H}^2(\Omega)$, equipped with the norm $|A\cdot|$, which actually is equivalent to the $\mathbf{H}^2$ norm $\|\cdot\|_2$.

In addition to the Stokes operator we will need the bilinear mapping B defined by

$$B(\mathbf{u}, \mathbf{v}) = P(\mathbf{u} \cdot \nabla) \mathbf{v},$$

for $\mathbf{u}$ and $\mathbf{v}$ regular enough. Also we will denote by $\underline{B}$ the bilinear tensor valued mapping defined by

$$\underline{B}(\mathbf{u}, \underline{\tau}) = (\mathbf{u} \cdot \nabla) \underline{\tau},$$

for $\mathbf{u}$ and $\underline{\tau}$ regular enough, and by $\underline{g}$ the bilinear tensor valued mapping defined by

$$\underline{g}(\underline{\tau}, \nabla \mathbf{u}) = \underline{\tau} \underline{W} - \underline{W} \underline{\tau} - a(\underline{D} \underline{\tau} + \underline{\tau} \underline{D}),$$

with $a \in [-1, 1]$. (For $a = 0$, $\underline{g}$ is the extra-term in (0.2) which makes the convective derivative of the symmetric tensor $\underline{\tau}$ frame indifferent).

Throughout this paper, we will prove our results only in the case $\tilde{\beta} = 0$. Actually these results can be easily extended to the models described in the Introduction. In fact all these models share the same linear part and differ only by high order nonlinear terms which are small when $\nabla \mathbf{u}$ and $\underline{\tau}$ are small.

We will therefore consider the problem

$$(1.11) \quad \mathbf{u}(\cdot, t) \in V, \quad \underline{\tau}(\cdot, t) \in \mathbf{H}^2(\Omega), \quad a.a.t,$$

$$(1.12) \quad Re(\mathbf{u}' + B(\mathbf{u}, \mathbf{u})) + (1 - \omega)A\mathbf{u} = P(\nabla \cdot \underline{\tau} + \mathbf{f}),$$

$$(1.13) \quad We(\underline{\tau}' + \underline{B}(\mathbf{u}, \underline{\tau}) + \underline{g}(\underline{\tau}, \nabla \mathbf{u})) + \underline{\tau} = 2\omega \underline{D}[\mathbf{u}],$$

$$(1.14) \quad \mathbf{u}(0) = \mathbf{u}_0 \quad , \quad \underline{\tau}(0) = \underline{\tau}_0,$$

where a prime denotes the partial derivative with respect to t .

The letters C_i , $i = 0, 1, \dots$, will denote different constants depending on the domain Ω , possibly on other data such as $\omega, Re, We, \underline{g}, \dots$, but certainly not on the unknowns $\mathbf{u}$ or $\underline{\tau}$.

The operator $\mathbf{B}$, as well as $\underline{B}$ and $\underline{g}$, has many continuity properties (cf e.g. R. Temam [20], or P. Constantin, C. Foias [3]). Among these, we will mainly use the ones given below, which are valid if $\mathbf{u}, \mathbf{v}$ and $\underline{\tau}$ are regular enough for the right hand sides of the inequalities to make sense:

$$(1.15) \quad \begin{aligned} |\mathbf{B}(\mathbf{u}, \mathbf{v})| &\leq C_0 |\mathbf{u}| \|\nabla \mathbf{v}\|_{L^\infty}, \\ &\leq C_0 |\mathbf{u}| \|\mathbf{v}\|_3, \\ &\leq C_0 \|\mathbf{u}\|_{L^\infty} \|\mathbf{v}\|_1. \end{aligned}$$

An obvious change of notation gives the corresponding estimates for $\underline{B}$ and $\underline{g}$.

Finally, $\mathbf{B}$ and $\underline{B}$ satisfy an orthogonality property; if $\mathbf{u} \in V$, then

$$(1.16) \quad (\mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{v}) = 0 \quad , \quad (\underline{B}(\mathbf{u}, \underline{\tau}), \underline{\tau}) = 0 \quad , \quad \forall \mathbf{v}, \forall \underline{\tau}.$$

In what follows, we will denote by $\partial_i (i = 1, 2, 3)$ the partial derivative with respect to x_i , and by $\nabla = (\partial_1, \partial_2, \partial_3)$ the gradient vector; by D^2 a second order derivative ∂_{ij}^2 with respect to x_i and x_j . We will also use the Einstein rule of repeated indices.

2. Local existence of regular solutions. In this section we implement a fixed point argument to show the existence of a regular solution to Problem (1.11)–(1.16) on a small time interval $(0, T^*)$. We prove also that this solution satisfies an energy inequality, which implies uniqueness in that class.

2.1. Existence of a local solution. We first study two linearized problems, one for the velocity $\mathbf{u}$, and the other for the symmetric tensor $\underline{\tau}$. The parameter ω will always satisfy: $\omega \in (0, 1)$.

Let T be a given positive real number, and let $Q_T = \Omega \times (0, T)$. We recall, without proof, some well known results for the time dependent Stokes problem, written as

$$(2.1) \quad \begin{cases} \mathbf{u}(\cdot) \in V, & \text{a.e. in } (0, T), \\ Re \mathbf{u}' + (1 - \omega) A \mathbf{u} = \mathbf{F}, & \text{a.e. in } (0, T), \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases}$$

where $\mathbf{F}$ is a given external body force.

LEMMA 3.1. Assume $\Gamma \in \mathcal{C}^2$, $\mathbf{F} \in L^2(Q_T)$, $\mathbf{u}_0 \in V$. Then the Stokes problem (2.1) admits a unique solution $\mathbf{u} \in L^2(0, T; D(A)) \cap \mathcal{C}([0, T]; V)$ such that $\mathbf{u}' \in L^2(Q_T)$ and $p \in L^2(0, T; H^1)$ (p is the associated hydrodynamic pressure). Moreover, there exists a constant $C_1 = C_1(Re, \omega, \Omega)$ such that

$$(2.2) \quad \begin{aligned} & \|\mathbf{u}\|_{L^2(0, T; D(A)) \cap L^\infty(0, T; V)}^2 + \|\mathbf{u}'\|_{L^2(Q_T)}^2 + \|p\|_{L^2(0, T; H^1)}^2 \\ & \leq C_1 \left(\|\mathbf{u}_0\|^2 + \|\mathbf{F}\|_{L^2(Q_T)}^2 \right). \end{aligned}$$

The proof of this result is given in Temam [20] (pp. 267–269.).

In an analogous way, we can prove that the solution is more regular, if the data are more regular. For instance we state, without proof again, the following lemma.

LEMMA 2.2. Assume $\partial\Omega \in \mathcal{C}^3$, $\mathbf{F} \in L^2(0, T; \mathbf{H}^1)$, $\mathbf{F}' \in L^2(0, T; \mathbf{H}^{-1})$, $\mathbf{u}_0 \in D(A)$. Then the unique solution of Problem (2.1) satisfies the following:

$$\begin{aligned} \mathbf{u} & \in L^2(0, T; \mathbf{H}^3) \cap \mathcal{C}([0, T]; D(A)), \\ \mathbf{u}' & \in L^2(0, T; V) \cap \mathcal{C}([0, T]; H), \\ p & \in L^2(0, T; H^2), \end{aligned}$$

and there exists a constant $C_2 = C_2(Re, \omega, \Omega)$ such that

$$(2.3) \quad \begin{aligned} & \|\mathbf{u}\|_{L^2(0, T; \mathbf{H}^3) \cap L^\infty(0, T; D(A))}^2 + \|\mathbf{u}'\|_{L^2(0, T; V) \cap L^\infty(0, T; H)}^2 + \|p\|_{L^2(0, T; H^2)}^2 \\ & \leq C_2 \left\{ |A\mathbf{u}_0|^2 + \|\mathbf{F}\|_{L^2(0, T; \mathbf{H}^1)}^2 + \|\mathbf{F}'\|_{L^2(0, T; \mathbf{H}^{-1})}^2 + |\mathbf{F}(0)|^2 \right\} \end{aligned}$$

The proof of this lemma uses estimates for $\mathbf{u}$, as well as for the time derivative $\mathbf{u}'$, which also satisfies a Stokes problem.

We turn now to the study of a linearized problem associated to the constitutive equation (1.13) for $\underline{\tau}$. We first show the existence and the uniqueness of a regular solution $\underline{\tau}$ to the following problem:

$$(2.4) \quad \begin{cases} We\{\underline{\tau}' + (\tilde{\mathbf{u}} \cdot \nabla)\underline{\tau} + \underline{g}(\underline{\tau}, \nabla\tilde{\mathbf{u}})\} + \underline{\tau} = 2\omega\underline{D}[\tilde{\mathbf{u}}] & \text{a.e. in } Q_T, \\ \underline{\tau}(0) = \underline{\tau}_0 & \text{a.e. in } \Omega, \end{cases}$$

where $\tilde{\mathbf{u}}$ is a given velocity field.

LEMMA 2.3. Assume $\partial\Omega \in \mathcal{C}^1$, $\tilde{\mathbf{u}} \in L^1(0, T; \mathbf{H}^3 \cap D(A))$, $\underline{\tau}_0 \in \mathbf{H}^2(\Omega)$. Then Problem (2.6) admits a unique solution $\underline{\tau}$ in $\mathcal{C}([0, T]; \mathbf{H}^2)$. Moreover, there exists a constant $C_0 = C_0(\Omega, \underline{g})$ such that

$$(2.5) \quad \|\underline{\tau}\|_{L^\infty(0, T; \mathbf{H}^2)} \leq \left(\|\underline{\tau}_0\|^2 + \frac{2\omega}{C_0 We} \right) \exp(C_0 \|\tilde{\mathbf{u}}\|_{L^1(0, T; \mathbf{H}^3)}).$$

If, in addition, $\tilde{u}$ belongs to $\mathcal{C}([0, T]; D(A))$, then τ' belongs to $\mathcal{C}([0, T]; H^1)$ and satisfies

$$(2.6) \quad \|\underline{\tau}'\|_{L^\infty(0, T; H^1)} \leq C_0 \left(\|\tilde{u}\|_{L^\infty(0, T; D(A))} + \frac{1}{C_0 We} \right) \left(\|\underline{\tau}_0\|_2 + \frac{2\omega}{C_0 We} \right) \exp(C_0 \|\tilde{u}\|_{L^1(0, T; H^3)}).$$

Proof. The existence of a unique solution to (2.4) follows from the application of the method of characteristics.

To obtain estimate (2.5) we write a differential inequality satisfied by $\|\underline{\tau}\|_2$. To this end, we multiply equation (2.4)₁ by $\underline{\tau}$ in $L^2(\Omega)$; we also apply the operator ∇ to (2.4)₁ and multiply the resulting equation by $\nabla \underline{\tau}$; finally we apply the operator D^2 to (2.4)₂ and multiply by $D^2 \underline{\tau}$ in $L^2(\Omega)$. Adding all those equations, and making use of the orthogonality property (1.16), we obtain:

$$(2.7) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} (We \|\underline{\tau}\|_2^2) + \|\underline{\tau}\|_2^2 = & 2\omega \left(\left(D[\tilde{u}], \underline{\tau} \right) \right)_2 \\ & - We \left\{ \left(\underline{B}(\nabla \tilde{u}, \underline{\tau}), \nabla \underline{\tau} \right) + 2(\underline{B}(\partial_i \tilde{u}, \partial_j \underline{\tau}), \partial_{ij} \underline{\tau}) \right. \\ & \left. + \left(\underline{B}(D^2 \tilde{u}, \underline{\tau}), D^2 \underline{\tau} \right) + \left(\left(\underline{g}(\underline{\tau}, \nabla \tilde{u}), \tilde{\tau} \right) \right)_2 \right\}. \end{aligned}$$

We now estimate the right hand side terms of (2.7) using the continuity properties (1.15), so that we deduce the following estimate:

$$(2.8) \quad \frac{1}{2} \frac{d}{dt} (We \|\underline{\tau}\|_2^2) + \|\underline{\tau}\|_2^2 \leq 2\omega \|\tilde{u}\|_3 \|\underline{\tau}\|_2 + C_0 We \|\tilde{u}\|_3 \|\underline{\tau}\|_2^2,$$

where C_0 depends only on Ω and $\underline{g}$. Inequality (2.8) implies, in particular, that

$$(2.9) \quad \frac{1}{2} \frac{d}{dt} \left(\|\underline{\tau}\|_2 + \frac{2\omega}{C_0 We} \right)^2 \leq C_0 \|\tilde{u}\|_3 \left(\|\underline{\tau}\|_2 + \frac{2\omega}{C_0 We} \right)^2$$

Integration of (2.9) over the time interval $(0, t) \subset (0, T)$ gives inequality (2.5). Therefore the function $\underline{\tau}$ belongs to $L^\infty(0, T; H^2)$.

To prove that actually $\underline{\tau}$ is continuous with values into $H^2(\Omega)$, we observe that the solution U of the following ordinary differential equation,

$$\begin{aligned} \frac{d}{dt} U(t, s; \mathbf{x}) &= \tilde{u}(t, U(t, s; \mathbf{x})) \quad \text{in } Q_T, \\ U(s, s; \mathbf{x}) &= \mathbf{x} \quad \text{in } \Omega, \end{aligned}$$

belongs to $\mathcal{C}([0, T] \times [0, T]; \mathbf{H}^3)$. Consequently, the representation formula

$$\begin{aligned} \underline{\tau}(t, \mathbf{x}) &= \underline{\tau}_0(\mathbf{U}(0, t; \mathbf{x})) \\ &+ \int_0^t \frac{1}{We} \left(2\mathbf{D}[\tilde{\mathbf{u}}] - \underline{\tau} - \underline{g}(\underline{\tau}, \nabla \tilde{\mathbf{u}})(s, \mathbf{U}(s, t; \mathbf{x})) \right) ds \end{aligned}$$

shows that $\underline{\tau}$ belong to $\mathcal{C}([0, T]; \mathbf{H}^2)$. (cf. e.g. J.-P. Bourguignon, H. Brezis [2] for a similar calculation.)

If now $\tilde{\mathbf{u}}$ belongs to $\mathcal{C}([0, T]; \mathbf{H}^2)$, then $\underline{\tau}'$ calculated with equation (2.4)₁,

$$\underline{\tau}' = \frac{1}{We} (2\omega \underline{D}[\tilde{\mathbf{u}}] - \underline{\tau}) - (\tilde{\mathbf{u}} \cdot \nabla) \underline{\tau} - \underline{g}(\underline{\tau}, \nabla \tilde{\mathbf{u}}),$$

belongs to $\mathcal{C}([0, T]; \mathbf{H}^1)$, and

$$(2.10) \quad \|\underline{\tau}'\|_2 \leq \frac{1}{We} (2\omega \|\tilde{\mathbf{u}}\|_2 + \|\underline{\tau}\|_1) + C_0 \|\tilde{\mathbf{u}}\|_2 \|\underline{\tau}\|_2,$$

which implies, in particular,

$$\|\underline{\tau}'\|_2 \leq C_0 \left(\|\tilde{\mathbf{u}}\|_2 + \frac{1}{C_0 We} \right) \left(\|\underline{\tau}\|_2 + \frac{2\omega}{C_0 We} \right).$$

Inequality (2.6) is then deduced from inequality (2.9).

We are now in a position to prove that Problem (1.11)–(1.14) admits at least one solution defined on a small time interval.

THEOREM 2.4. (*Local existence of a regular solution.*) Assume $\partial\Omega \in \mathcal{C}^3$, $f \in L^2_{loc}(\mathbf{R}_+; \mathbf{H}^1)$, $\mathbf{f}' \in L^2_{loc}(\mathbf{R}_+; \mathbf{H}^{-1})$, $\mathbf{u}_0 \in D(A)$, $\underline{\tau}_0 \in \mathbf{H}^2(\Omega)$.

Then there exist a $T^* > 0$, $\mathbf{u} \in L^2(0, T^*; \mathbf{H}^3) \cap \mathcal{C}([0, T]^*; D(A))$, with $\mathbf{u}' \in L^2(0, T^*; V) \cap \mathcal{C}([0, T^*]; H)$, $p \in L^2(0, T^*; \mathbf{H}^2)$, and $\underline{\tau} \in \mathcal{C}([0, T^*]; \mathbf{H}^2)$ such that $(\mathbf{u}, p, \underline{\tau})$ is a solution to Problem (1.11)–(1.14) in Q_{T^*} .

Proof. For $T > 0$, $B_1 > 0$ and $B_2 > 0$, we define the set

$$\begin{aligned} R_T = \Big\{ & (\tilde{\mathbf{u}}, \underline{\tau}), \tilde{\mathbf{u}} \in \mathcal{C}([0, T]; D(A)) \cap L^2(0, T; \mathbf{H}^3), \tilde{\mathbf{u}}' \in \mathcal{C}([0, T]; H) \cap L^2(0, T; V), \\ & \underline{\tau} \in L^\infty(0, T; \mathbf{H}^2), \underline{\tau}' \in L^\infty(0, T; \mathbf{H}^1), \tilde{\mathbf{u}}(0) = \mathbf{u}_0, \quad \underline{\tau}(0) = \underline{\tau}_0 \quad \text{in } \Omega, \\ & \|\tilde{\mathbf{u}}\|_{L^\infty(0, T; D(A)) \cap L^2(0, T; \mathbf{H}^3)}^2 + \|\tilde{\mathbf{u}}'\|_{L^\infty(0, T; H) \cap L^2(0, T; V)}^2 \leq B_1, \\ & \|\underline{\tau}\|_{L^\infty(0, T; \mathbf{H}^2)} \leq B_1, \quad \|\underline{\tau}'\|_{L^\infty(0, T; \mathbf{H}^1)} \leq B_2 \Big\}. \end{aligned}$$

(i) We first show that, if B_1 is large enough, then $R_T \neq \emptyset$, $\forall T > 0$. Let $\mathbf{u}^*$ be the solution of the following Stokes problem:

$$\begin{cases} \mathbf{u}^*(\cdot) \in V, & \text{a.e in } \mathbf{R}_+, \\ Re\mathbf{u}' + (1 - \omega)A\mathbf{u}^* = 0 & \text{a.e in } \mathbf{R}_+, \\ \mathbf{u}^*(0) = \mathbf{u}_0. \end{cases}$$

There exists a constant $C_3 = C_3(\omega, Re, \Omega)$ such that

$$\|\mathbf{u}^*\|_{L^\infty(0,T;D(A)) \cap L^2(0,T;\mathbf{H}^3)}^2 + \|\mathbf{u}'^*\|_{L^\infty(0,T;H) \cap L^2(0,T;V)}^2 \leq C_3 |A\mathbf{u}_0|^2.$$

Hence, if we choose

$$(2.11) \quad B_1 > \max \left\{ C_3 |A\mathbf{u}_0|^2, \|\underline{\tau}_0\|_2 \right\},$$

then $(\mathbf{u}^*, \underline{\tau}_0)$ belongs to R_T , for all $T > 0$. From now on, B_1 will satisfy (2.11).

(ii) We consider the mapping

$$\phi \begin{cases} R_T \longrightarrow X_T \stackrel{\text{def}}{=} \mathcal{C}([0, T]; V) \times \mathcal{C}([0, T]; \mathbf{H}^1) \\ (\tilde{\mathbf{u}}, \underline{\tau}) \longrightarrow (\mathbf{u}, \underline{\tau}) \end{cases}$$

where $\mathbf{u}$ and $\underline{\tau}$ are the unique solutions of (2.1) and (2.4) respectively, with $\mathbf{F} = -ReB(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + P\nabla \cdot \underline{\tau} + \mathbf{P}\mathbf{f}$. A fixed point of ϕ is clearly a solution of Problem (1.11)–(1.14).

We now prove the existence of T_* small enough so that $\phi(R_{T_*}) \subset R_{T_*}$. First of all, if $(\tilde{\mathbf{u}}, \underline{\tau})$ belongs to R_T , then there exists a constant $C_4 = C_4(Re, \Omega)$ such that

$$(2.12) \quad \begin{aligned} \|\mathbf{F}\|_{L^2(0,T;\mathbf{H}^1)}^2 &\leq C_4 B_1^2 T + 3\|\mathbf{f}\|_{L^2(0,T;\mathbf{H}^1)}^2, \\ |\mathbf{F}(0)|^2 &\leq C_4 |A\mathbf{u}_0|^4 + 3\|\tau_0\|_1^2 + 3|\mathbf{f}(0)|^2, \\ \|\mathbf{F}'\|_{L^2(0,T;\mathbf{H}^{-1})}^2 &\leq C_4 B_1^2 T + 3B_2^2 T + 3\|\mathbf{f}'\|_{L^2(0,T;\mathbf{H}^{-1})}^2. \end{aligned}$$

From Lemma 2.2 and 2.3, we then deduce that

$$(2.13) \quad \begin{aligned} &\|\mathbf{u}\|_{L^2(0,T;\mathbf{H}^3) \cap L^2(0,T,D(A))}^2 + \|\mathbf{u}'\|_{L^2(0,T;V) \cap L^\infty(0,T;H)}^2 \\ &\leq C_2 \left\{ |A\mathbf{u}_0|^2 + C_4 |A\mathbf{u}_0|^4 + 3|\mathbf{f}(0)|^2 + 3\|\mathbf{f}\|_{L^2(0,T;\mathbf{H}^1)}^2 + 3\|\mathbf{f}'\|_{L^2(0,T;\mathbf{H}^{-1})}^2 \right. \\ &\quad \left. + 3\|\underline{\tau}_0\|_1^2 + (2C_4 B_1^2 + 3B_2^2)T \right\}, \\ \|\underline{\tau}\|_{L^\infty(0,T;\mathbf{H}^2)} &\leq \left(\|\underline{\tau}_0\|_2 + \frac{2\omega}{C_0 We} \right) \exp(C_0 B_1^{1/2} T^{1/2}), \\ \|\underline{\tau}'\|_{L^\infty(0,T;\mathbf{H}^1)} &\leq \left(B_1^{1/2} + \frac{1}{C_0 We} \right) \left(\|\underline{\tau}_0\|_2 + \frac{2\omega}{C_0 We} \right) \exp(C_0 B_1^{1/2} T^{1/2}). \end{aligned}$$

Therefore, $\phi(R_T)$ is included into R_T if we choose T such that the right hand sides of inequalities (3.13) are bounded respectively by B_1, B_1 and B_2 . In fact, considering also relation (2.11), we see that it is sufficient to choose

$$(2.14) \quad \begin{aligned} B_1 &\geq \max \left\{ C_3 |A\mathbf{u}_0|^2, e^{\sqrt{2}} (\|\underline{\tau}_0\|_2 + \frac{2}{C_0 We}) \right\}, \\ 2C_2 \left(|A\mathbf{u}_0|^2 + C_4 |A\mathbf{u}_0|^4 + 3\|\underline{\tau}_0\|_1^2 + 3|\mathbf{f}(0)|^2 B_2 + 3\|\mathbf{f}\|_{L^2(0,T;\mathbf{H}^1)}^2 + 3\|\mathbf{f}'\|_{L^2(0,T;\mathbf{H}^{-1})}^2 \right) \\ B_2 &\geq e^{\sqrt{2}} \left(B_1^{1/2} + \frac{1}{C_0 We} \right) \left(\|\underline{\tau}_0\|_2 + \frac{2}{C_0 We} \right), \end{aligned}$$

and

$$(2.15) \quad T \leq \min \left(\frac{B_1}{2C_2(2C_4 B_1^2 + 3B_2^2)}, \frac{2}{C_0^2 B_1} \right).$$

Hence we have defined some constants B_1 and B_2 , depending on the data, and we have defined a time, say T_* , depending on B_1, B_2 and on the data, such that $\phi(R_{T_*}) \subset R_{T_*}$.

(iii) The conclusions of Theorem 2.4 follow from Schauder's fixed point theorem applied to the mapping ϕ on the convex set R_{T_*} . Indeed it is easily shown that R_{T_*} is closed in X_{T_*} . Moreover, by Ascoli's theorem, R_{T_*} is compact in X_{T_*} . Standard arguments prove that ϕ is continuous for the topology of $X_{T_*}^*$.

2.2. Uniqueness of the solution. In this section, we show that the solution obtained in Theorem 2.6 is the only one in the class of regular solutions. Precisely, the result reads as follows.

THEOREM 2.5. *Let $T > 0$. Problem (1.11)–(1.16) admits at most one solution $(\mathbf{u}, \underline{\tau})$ in the class $L^2(0, T; \mathbf{H}^3) \cap \mathcal{C}([0, T]; D(A)) \times \mathcal{C}([0, T]; \mathbf{H}^2)$. The pressure p is unique, up to an additive constant, in the space $L^2(0, T; H^2)$.*

Proof. As usual, we take the difference of two solutions $(\mathbf{u}^1, p^1, \underline{\tau}^1)$ and $(\mathbf{u}^2, p^2, \underline{\tau}^2)$ belonging to the class specified in the theorem. The vector function $\mathbf{u} = \mathbf{u}^1 - \mathbf{u}^2$ and the tensor function $\underline{\tau} = \underline{\tau}^1 - \underline{\tau}^2$ satisfy the following:

$$(2.16) \quad \begin{cases} Re \{ \mathbf{u}' + \underline{B}(\mathbf{u}^1, \mathbf{u}) + \mathbf{B}(\mathbf{u}, \mathbf{u}^2) \} + (1 - \omega) A \mathbf{u} = P \nabla \cdot \underline{\tau} & \text{in } (0, T), \\ We \left\{ \underline{\tau}' + \mathbf{B}(\mathbf{u}^1, \underline{\tau}) + \underline{B}(\mathbf{u}, \underline{\tau}^2) + \underline{g}(\underline{\tau}^1, \nabla \mathbf{u}) + \underline{g}(\underline{\tau}, \nabla \mathbf{u}^2) \right\} + \underline{\tau} = 2\omega \underline{D}[\mathbf{u}] & \text{in } Q_T, \end{cases}$$

together with zero initial conditions,

$$\mathbf{u}(0) = 0, \quad \underline{\tau}(0) = 0.$$

Taking the scalar product of (2.16) in $L^2(\Omega) \times L^2(\Omega)$ with $(\mathbf{u}, \underline{\tau}/2\omega)$, we obtain

$$(2.17) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(Re|\mathbf{u}|^2 + \frac{We}{2\omega} |\underline{\tau}|^2 \right) + (1-\omega)\|\mathbf{u}\|^2 + \frac{1}{2\omega} |\underline{\tau}|^2 \\ &= -Re(\mathbf{B}(\mathbf{u}, \mathbf{u}^2), \mathbf{u}) - \frac{We}{2\omega} (\underline{B}(\mathbf{u}, \underline{\tau}^1) + \underline{g}(\underline{\tau}^1, \nabla \mathbf{u}) + \underline{g}(\underline{\tau}, \nabla \mathbf{u}^2), \underline{\tau}). \end{aligned}$$

where we have used the orthogonality property (1.16). Estimates of the right hand side of (2.17) are deduced from the continuity properties (1.15) of $\mathbf{B}$, $\underline{B}$ and $\underline{g}$. We have

$$|B(\mathbf{u}, \mathbf{u}^2)| \leq C_0 \|\mathbf{u}^2\|_3 |\mathbf{u}|,$$

and, similarly,

$$|\underline{g}(\underline{\tau}, \nabla \mathbf{u}^2)| \leq C_0 \|\mathbf{u}^2\|_3 |\underline{\tau}|.$$

Moreover

$$|\underline{B}(\mathbf{u}, \underline{\tau}^1) + \underline{g}(\underline{\tau}^1, \nabla \mathbf{u})| \leq C_0 \|\underline{\tau}^1\|_2 \|\mathbf{u}\|.$$

From (2.17) we then deduce that solutions $(\mathbf{u}, \underline{\tau})$ of (2.16) satisfy the following energy inequality:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(Re|\mathbf{u}|^2 + \frac{We}{2\omega} |\underline{\tau}|^2 \right) + (1-\omega) \left(1 - \frac{\epsilon}{2(1-\omega)} \left(Re + \frac{We}{2} \right) \right) \|\mathbf{u}\|^2 + \frac{1}{2\omega} |\underline{\tau}|^2 \\ & \leq \phi_\epsilon \left(Re|\mathbf{u}|^2 + \frac{We}{2\omega} |\underline{\tau}|^2 \right), \end{aligned}$$

with $\phi_\epsilon = C_0 \|\mathbf{u}^2\|_3 + \frac{C_0^2}{\epsilon} \|\underline{\tau}^1\|_2^2$, $\epsilon > 0$. The function ϕ_ϵ is positive and, because of the class of solutions we consider, ϕ_ϵ belongs to $L^1(0, T)$. Therefore, choosing $\epsilon > 0$ small enough, we deduce from Gronwall's Lemma that $\mathbf{u} = 0$ and $\underline{\tau} = 0$ in Q_T , and that consequently the pressure associated to (2.16)₁ is constant in Q_T .

3. Global existence of regular solutions for small data. In this section, we show that the (unique) local solution obtained in Theorems 2.4 and 2.5 is actually defined on $\mathbb{R}_+$ if the data are small enough. To this end we derive some a priori bounds uniform in time, satisfied by that solution.

We also show that the problem linearized around the zero solution admits a unique global solution, which is uniformly bounded in time for any initial data.

3.1. Some a priori estimates. In this section, C_0 will denote different constants depending only on Ω , while C_i , $i = 1, 2, \dots$, will denote some constants depending on Ω , on the bilinear functionals $\mathbf{B}$ and $\underline{B}$, and possibly on the bilinear mapping $\underline{g}$.

Recall that the local solution $(\mathbf{u}, \underline{\tau})$ obtained in Theorem 2.4 satisfies

$$(3.1) \quad \begin{aligned} \mathbf{u} &\in L^2(0, T^*; H^3) \cap \mathcal{C}([0, T^*]; D(A)), \\ \underline{\tau} &\in \mathcal{C}([0, T^*]; H^2), \end{aligned}$$

together with equations (1.12) through (1.14).

Inequality (2.8) implies that there exists a constant C_1 such that

$$(3.2) \quad \frac{d}{dt} (We \|\underline{\tau}\|_2^2) + \frac{1}{2} \|\underline{\tau}\|_2^2 \leq 3\omega^2 \|\mathbf{u}\|_3^2 + C_1 \frac{We^2}{\omega^2} \|\underline{\tau}\|_2^4.$$

We estimate $\|\mathbf{u}\|_3$ via equation (1.12). There exist two constants C_0 and C_2 such that

$$(3.3) \quad \|\mathbf{u}\|_3^2 \leq \frac{C_0}{(1-\omega)^2} (Re^2 \|\mathbf{u}'\|^2 + \|\underline{\tau}\|_2^2 + \|\mathbf{f}\|_1^2 + C_2 Re^2 |Au|^4).$$

From equation (1.12) we also deduce an estimate for $|Au|$, which will be used later:

$$(3.4) \quad |Au|^2 \leq \frac{8}{(1-\omega)^2} \left\{ Re^2 \|\mathbf{u}'\|^2 + \|\underline{\tau}\|_1^2 + \|\mathbf{f}\|^2 + C_2 \frac{Re^6}{(1-\omega)^2} \|\mathbf{u}\|^6 \right\}.$$

Therefore, from equation (3.2) and (3.3), we deduce that $\|\underline{\tau}\|_2$ satisfies the following differential inequality:

$$(3.5) \quad \begin{aligned} \frac{d}{dt} (We \|\underline{\tau}\|_2^2) + \left(\frac{1}{2} - 3C_0 \left(\frac{\omega}{1-\omega} \right)^2 \right) \|\underline{\tau}\|_2^2 \\ \leq 3C_0 \left(\frac{\omega}{1-\omega} \right)^2 (Re^2 \|\mathbf{u}'\|^2 + \|\mathbf{f}\|_2^2 + C_2 Re^2 |Au|^4) + C_1 \frac{We^2}{\omega^2} \|\underline{\tau}\|_2^4. \end{aligned}$$

In what follows, inequality (3.5) is going to be used only for values of ω such that the coefficient of $\|\underline{\tau}\|_2^2$ in the left hand side of (3.5) is positive; we choose (for example) $\omega \leq \omega_0 = (1 + \sqrt{8C_0})^{-1}$ so that the coefficient of $\|\underline{\tau}\|_2^2$ is larger than $C_0 \omega^2 / (1-\omega)^2$.

We now derive differential inequalities for $\mathbf{u}$, $\mathbf{u}'$ and $\underline{\tau}'$. Taking the scalar product in H of (1.12) with Au , we obtain

$$\frac{1}{2} \frac{d}{dt} (Re \|\mathbf{u}\|^2) + (1-\omega) |Au|^2 = -Re((\mathbf{u} \cdot \nabla) \mathbf{u}, Au) + (\nabla \cdot \underline{\tau} + \mathbf{f}, Au),$$

which implies that there exists a constant C_3 such that

$$(3.6) \quad \frac{d}{dt} (Re \|\mathbf{u}\|^2) + (1-\omega) |Au|^2 \leq \frac{4}{1-\omega} \left(\|\underline{\tau}\|_1^2 + \|\mathbf{f}\|^2 + C_3 \frac{Re^4}{(1-\omega)^2} \|\mathbf{u}\|^6 \right).$$

We differentiate (1.12) with respect to t , and take the scalar product in H with u' . We also differentiate (1.13) with respect to t , and take the scalar product with $\underline{\tau}'/2\omega$. By addition of the resulting equations, the linear coupling terms cancel, and an energy estimate satisfied by u' and $\underline{\tau}'$ is obtained. Writing

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(Re|u'|^2 + \frac{We}{2\omega} |\underline{\tau}'|^2 \right) + (1-\omega)\|u'\|^2 + \frac{1}{2\omega} |\underline{\tau}'|^2 \\ &= -Re(B(u', u), u') - \frac{We}{2\omega} \left(B(u', \underline{\tau}) + g(\underline{\tau}, \nabla u') + g(\underline{\tau}', \nabla u), \underline{\tau} \right) \\ & \quad + (f', u') \end{aligned}$$

and estimating the quadratic terms as usual, we show that there exists a constant C_4 such that for all $\epsilon > 0$,

$$\begin{aligned} & \frac{d}{dt} \left(Re|u'|^2 + \frac{We}{2\omega} |\underline{\tau}'|^2 \right) + (1-\omega)\|u'\|^2 + \frac{1}{\omega} |\underline{\tau}'|^2 \\ & \leq \epsilon \|u\|_3^2 + \frac{2}{1-\omega} \|f'\|_{-1}^2 \\ (3.7) \quad & + C_4 \left\{ \frac{1}{\epsilon} \left(Re^2|u'|^4 + \frac{We^2}{\omega^2} |\underline{\tau}'|^4 \right) + \frac{We^2}{\omega^2(1-\omega)} \|\underline{\tau}\|_2^2 |\underline{\tau}'|^2 \right\}, \\ & \leq \epsilon \|u\|_3^2 + \frac{2}{1-\omega} \|f'\|_{-1}^2 \\ & + C_4 \left\{ \frac{1}{\epsilon} \left(Re^2|u'|^4 + \frac{We^2}{\omega^2} |\underline{\tau}'|^4 \right) + \epsilon \frac{We^2}{\omega^2(1-\omega)^2} \|\underline{\tau}\|_2^4 \right\}. \end{aligned}$$

We now multiply inequality (3.5) by $(1-\omega)^3/(6C_0Re^2\omega^2)$ and add inequality (3.7). This gives

$$\begin{aligned} & \frac{d}{dt} \left(Re|u'|^2 + \frac{We}{2\omega} |\underline{\tau}'|^2 + \frac{We}{Re^3} \frac{(1-\omega)^3}{3C_0\omega^2} \|\underline{\tau}\|_2^2 \right) \\ & + \frac{1-\omega}{2} \|u'\|^2 + \frac{1}{\omega} |\underline{\tau}'|^2 + \frac{1-\omega}{6Re^2} \|\underline{\tau}\|_2^2 \\ (3.8) \quad & \leq \epsilon \|u\|_3^2 + \frac{2}{1-\omega} \|f'\|_{-1}^2 + \frac{1-\omega}{2Re^2} \|f\|_1^2 + C_2 \frac{1-\omega}{2} |Au|^4 \\ & + C_5 \left\{ \frac{1}{\epsilon} \left(Re^2|u'|^4 + \frac{We^2}{\omega^2} |\underline{\tau}'|^4 \right) + \frac{We^2}{\omega^2} \left[\frac{\epsilon}{(1-\omega)^2} + \frac{(1-\omega)^3}{Re^2\omega^2} \right] \|\underline{\tau}\|_2^4 \right\}, \end{aligned}$$

where $C_5 = \text{Max}(C_4, C_1/6C_0)$. We make use of inequality (3.3) for estimating $\|u\|_3$ in

(3.8), and we choose $\epsilon = \frac{(1-\omega)^3}{8C_0Re^2}$. Inequality (3.8) now reads as follows:

$$\begin{aligned}
(3.9) \quad & \frac{d}{dt} \left(Re|\mathbf{u}'|^2 + \frac{We}{2\omega} |\underline{\tau}'|^2 + \frac{We}{Re^3} \frac{(1-\omega)^3}{3C_0\omega^2} \|\underline{\tau}\|_2^2 \right) \\
& + \frac{3}{8}(1-\omega)\|\mathbf{u}'\|^2 + \frac{1}{\omega} \|\underline{\tau}'\|^2 + \frac{1-\omega}{24Re^2} \|\underline{\tau}\|_2^2 \\
& \leq \frac{2}{1-\omega} \|\mathbf{f}'\|_{-1}^2 + \frac{5}{8} \frac{(1-\omega)}{Re^2} \|\mathbf{f}\|_1^2 + \frac{5C_2}{8}(1-\omega) |A\mathbf{u}|^4 \\
& + C_6 \left\{ \frac{Re^2}{(1-\omega)^3} \left(Re^2|\mathbf{u}'|^4 + \frac{We^2}{\omega^2} |\underline{\tau}'|^4 \right) + \frac{We^2(1-\omega)^3}{Re^2\omega^4} \|\underline{\tau}\|_2^4 \right\}
\end{aligned}$$

with $C_6 = \max \left(8C_0C_5, 1 + \frac{\omega^2}{8C_0(1-\omega_0)^2} \right)$.

Finally, we multiply inequality (3.6) by $(1-\omega)^2/192Re^2$ and add the resulting equation to (3.9); we also use inequality (3.4). It follows that there exists a constant C_7 such that

$$\begin{aligned}
(3.10) \quad & \frac{d}{dt} \left\{ \frac{(1-\omega)^2}{192Re^2} \|\mathbf{u}\|^2 + Re|\mathbf{u}'|^2 + \frac{We}{2\omega} |\underline{\tau}'|^2 + \frac{We}{Re^3} \frac{(1-\omega)^3}{3C_0\omega^2} \|\underline{\tau}\|_2^2 \right\} \\
& + \frac{3}{8}(1-\omega) \|\mathbf{u}'\|^2 + \frac{(1-\omega)^3}{192Re^2} |A\mathbf{u}|^2 + \frac{1}{\omega} |\underline{\tau}'|^2 + \frac{1-\omega}{48Re^2} \|\underline{\tau}\|_2^2 \\
& \leq \frac{2}{1-\omega} \|\mathbf{f}'\|_{-1}^2 + \frac{31}{48} \frac{1-\omega}{Re^2} \|\mathbf{f}\|_1^2 + \frac{160C_2}{(1-\omega)^3} |\mathbf{f}|^4 \\
& + C_7 \left\{ \frac{Re^2}{(1-\omega)^3} \left(Re^2|\mathbf{u}'|^4 + \frac{We^2}{\omega^2} |\underline{\tau}'|^4 \right) + \frac{1}{(1-\omega)^3} \|\underline{\tau}\|_1^4 + \frac{We^2}{Re^2} \frac{(1-\omega)^3}{\omega^4} \|\underline{\tau}\|_2^4 \right. \\
& \quad \left. + \frac{Re^2}{1-\omega} \|\mathbf{u}\|^6 + \frac{Re^8}{(1-\omega)^7} \|\mathbf{u}\|^{12} \right\},
\end{aligned}$$

for all ω , $0 < \omega \leq \omega_0 < 1$, ω_0 being a constant depending only on the domain Ω .

We are now in a position to prove that the local solution found in Theorem 2.4 exists for all time if the data are small enough.

3.2. Global existence of solutions for small data. Defining

$$\begin{aligned}
Y(t) = & \frac{(1-\omega)^2}{192Re^2} \|\mathbf{u}(t)\|^2 + Re \|\mathbf{u}'(t)\|^2 + \frac{We}{2\omega} |\underline{\tau}'(t)|^2 \\
& + \frac{We}{Re^3} \frac{(1-\omega)^3}{3C_0\omega^2} \|\underline{\tau}(t)\|_2^2,
\end{aligned}$$

we see that inequality (3.10) has the following form:

$$(3.11) \quad Y' + \lambda Y \leq \alpha(Y^2 + Y^3 + Y^6) + \beta,$$

where $\lambda > 0$, $\alpha > 0$ and $\beta > 0$ are some constants depending on the data. Then, under adequate hypotheses on $Y(0)$ and on the constant β , the function Y is bounded.

LEMMA 3.1. Let Y be a non-negative, absolutely continuous function satisfying inequality (3.11). Let M , $0 < M < M_0$, where M_0 is the unique solution of $M^5 + M^2 + M - \lambda/2\alpha = 0$.

If $Y(0) \leq M$, and $\beta \leq \lambda M/2$, then $Y(t)$ is bounded by M for all $t \geq 0$.

Proof. This result is proven by contradiction. Suppose that there exists a t such that $Y(t) > M$, and define $t^* = \inf\{t \in \mathbb{R}_+, Y(t) > M\}$. Then $Y(t^*) = M$ and $Y'(t^*) > 0$. But, from (3.11) and the hypothesis made on M and β , we deduce

$$Y'(t^*) \leq -\lambda Y(t^*) + \alpha(Y^2(t^*) + Y^3(t^*) + Y^6(t^*)) + \beta \leq 0$$

which contradicts the statements above.

Therefore $Y(t) \leq M$ for all $t \in \mathbb{R}_+$. $\square$

REMARK 3.2. Assume the hypotheses of Lemma 3.1. Then $Y(t) \leq M$, for all $t \geq 0$. This, together with inequality (3.11), implies

$$(3.12) \quad Y' + \lambda Y \leq \gamma Y^2 + \beta.$$

where $\gamma = \gamma(M) > 0$.

If $\beta = 0$ in (3.11) and in (3.12), then

$$(3.13) \quad Y(t) \leq \frac{Y(0) e^{-\lambda t}}{1 - \frac{\gamma}{\lambda} Y(0)}, \quad \forall t > 0,$$

for $Y(0) < \lambda/\gamma_0$.

Similarly there exists a continuous function $k : [0, \epsilon_1] \times [0, \epsilon_2] \subset [0, \lambda/2\gamma] \times [0, \lambda^2/4\gamma] \rightarrow \mathbb{R}_+$ such that $k(0, 0) = 0$ and

$$(3.14) \quad Y(t) \leq k(Y(0), \beta) \quad \forall t > 0. \quad \square$$

We now state the theorem of existence of a global solution.

THEOREM 3.3. Let $\partial\Omega \in \mathbb{C}^4$. There exists some $\omega_0, 0 < \omega_0 < 1$, depending on Ω , such that, if $0 < \omega < \omega_0$, and if $\mathbf{u}_0 \in D(A)$, $\underline{\tau}_0 \in H^2$, $\mathbf{f} \in L^\infty(\mathbb{R}_+; H^1)$, and $\mathbf{f}' \in L^\infty(\mathbb{R}_+; H^{-1})$ are small enough in their spaces, then Problem (1.11)–(1.14) admits a unique solution $(\mathbf{u}, \underline{\tau})$ defined for all times t , and satisfying:

$$\begin{aligned} \mathbf{u} &\in \mathcal{C}_b(\mathbb{R}_+; D(A)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^3), \\ \mathbf{u}' &\in \mathcal{C}_b(\mathbb{R}_+; H) \cap L^2_{\text{loc}}(\mathbb{R}_+; V), \\ \underline{\tau} &\in \mathcal{C}_b(\mathbb{R}_+; H^2), \quad \underline{\tau}' \in \mathcal{C}_b(\mathbb{R}_+; H^1). \end{aligned}$$

Proof.

(i) We have seen in Section 4.1 that a specific norm of the local solution obtained in Theorem 3.9 satisfies an inequality having the form (3.11), where α and λ depend on the domain Ω and on the parameters ω , We and Re , and where

$$\beta = \frac{2}{1-\omega} \|\mathbf{f}'\|_{L^\infty(\mathbf{R}_+; \mathbf{H}^{-1})}^2 + \frac{31}{48} \frac{1-\omega}{Re^2} \|\mathbf{f}\|_{L^\infty(\mathbf{R}_+; \mathbf{H}^1)}^2 + \frac{160 C_2}{(1-\omega)^3} \|\mathbf{f}\|_{L^\infty(\mathbf{R}_+; \mathbf{L}^2)}.$$

Lemma 3.1 shows that there exists a constant M_0 , depending on Ω, ω, We and Re such that if $Y(0) \leq M < M_0$ and $\beta \leq \lambda M/2$, then $Y(t)$ is bounded for all $t \in \mathbf{R}_+$. Note that $Y(0) \leq M$ if $\mathbf{u}_0, \underline{\tau}_0, \mathbf{f}$ are small in their respective spaces. Therefore, from the hypotheses of the Theorem we infer that

$$(3.15) \quad \begin{aligned} \mathbf{u} &\in L^\infty(\mathbf{R}_+; V) \cap L_{loc}^2(\mathbf{R}_+; D(A)), \\ \mathbf{u}' &\in L^\infty(\mathbf{R}_+; H) \cap L_{loc}^2(\mathbf{R}_+; V), \\ \underline{\tau} &\in L^\infty(\mathbf{R}_+; \mathbf{H}^2), \quad \underline{\tau}' \in L^\infty(\mathbf{R}_+; \mathbf{L}^2), \end{aligned}$$

if $\mathbf{f} \in L^\infty(\mathbf{R}_+; \mathbf{H}^2)$, $\mathbf{f}' \in L^\infty(\mathbf{R}_+; \mathbf{H}^{-1})$, and if the initial data satisfy $\mathbf{u}_0 \in V$, $\mathbf{u}'(0) \in V$, $\underline{\tau}_0 \in \mathbf{H}^2$, $\underline{\tau}'(0) \in \mathbf{L}^2$. These conditions are satisfied if $\mathbf{u}_0 \in D(A)$, $\mathbf{f}(0) \in \mathbf{L}^2$ and $\underline{\tau}_0 \in \mathbf{H}^2$.

(ii) Now, we deduce from inequality (3.4) that $\mathbf{u} \in L^\infty(\mathbf{R}_+; D(A))$, and from (3.3) that $\mathbf{u} \in L_{loc}^2(\mathbf{R}_+; \mathbf{H}^3)$. However, by interpolation, (3.15) implies that $\mathbf{u} \in \mathcal{C}_b(\mathbf{R}_+; D(A))$, and equation (1.12) implies that $\mathbf{u}' \in \mathcal{C}_b(\mathbf{R}_+; H)$.

The right hand side of equation (1.13), written as

$$(3.16) \quad We \left(\underline{\tau}' + (\mathbf{u} \cdot \nabla) \underline{\tau} \right) + \underline{\tau} = 2\omega \underline{D}[\mathbf{u}] - We \underline{g}(\underline{\tau}, \nabla \mathbf{u})$$

has its second member belonging to $L_{loc}^2(\mathbf{R}_+; \mathbf{H}^2)$, so that $\underline{\tau} \in \mathcal{C}_b(\mathbf{R}_+; \mathbf{H}^2)$. This together with (3.15) implies that the right hand side of (3.16) belongs to $\mathcal{C}_b(\mathbf{R}_+; \mathbf{H}^1)$ and that $\underline{\tau}' \in \mathcal{C}_b(\mathbf{R}_+; \mathbf{H}^1)$ because also $\underline{\tau}'(0) \in \mathbf{H}^1$.

This ends the proof of Theorem 3.3. $\square$

REMARK 3.4. (i) Condition $0 < \omega < \omega_0$ means that the fluid has a certain amount of Newtonian viscosity.

(ii) The conclusions of Theorem 3.3 are valid without any restriction on ω in the case $\Omega = \mathbf{R}^N$, or in the spatial periodic case. Actually, the restriction on ω arises in the treatment of the boundary terms contained in the linear coupled terms of equations (1.12) and (1.13). It is likely that using a special coordinate system near the boundary as in A. Valli ([21], [22]) would overcome this difficulty. (See also section 3.3 below, where uniform bounds on solutions of a linear problem are proved, and where an analysis of the coupling of those linear terms is proposed.)

(iii) More precise results can be obtained for one-dimensional flows, such as shearing or Poiseuille flows (see [9], where global existence of solutions is proved without any restriction on the size of the data).

The proof of Theorem 3.3 and Remark 3.2 yield immediately the following result.

COROLLARY 3.5. *Under the hypotheses of Theorem 3.3, we assume also $\mathbf{f} = 0$. Then the solution $(\mathbf{u}, \underline{\tau}) = (0, 0)$ is exponentially asymptotically stable in the H^2 norm.*

3.3. Boundedness and regularity of the solutions to a linearized problem. Let us consider the problem linearized around $(0, \underline{0})$:

$$(3.17) \quad Re\mathbf{u}' - (1 - \omega)\Delta\mathbf{u} + \nabla p = \nabla \cdot \underline{\tau} + \mathbf{F} \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$(3.18) \quad We\underline{\tau}' + \underline{\tau} = 2\omega D[u] + \underline{G} \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$(3.19) \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$(3.20) \quad \mathbf{u} = 0 \quad \text{on } \Gamma \times \mathbb{R}_+,$$

$$(3.21) \quad \mathbf{u}(0) = \mathbf{u}_0, \quad \underline{\tau}(0) = \underline{\tau}_0 \quad \text{in } \Omega,$$

where $\mathbf{F}$ is a given vector, and $\underline{G}$ is a given symmetric tensor.

We first show the existence of a unique solution.

LEMMA 3.6. *Let $\mathbf{F} \in L^\infty(\mathbb{R}_+; H^{-1})$, $\underline{G} \in L^\infty(\mathbb{R}_+; L^2)$, $\mathbf{u}_0 \in H$, $\underline{\tau}_0 \in L^2(\Omega)$. Then Problem (3.17)–(3.21) admits a unique solution $(\mathbf{u}, p, \underline{\tau})$ such that*

$$(3.22) \quad \begin{cases} \mathbf{u} \in L^2_{\text{loc}}(\mathbb{R}_+; V) \cap \mathcal{C}_b(\mathbb{R}_+; H), & p \in L^2_{\text{loc}}(\mathbb{R}_+; L^2), \\ \underline{\tau} \in \mathcal{C}_b(\mathbb{R}_+; L^2). \end{cases}$$

Proof. We approximate Problem (3.17)–(3.21) by a linear finite dimensional problem using any standard method. The a priori estimate needed for passing to the limit is the following obvious energy inequality:

$$(3.23) \quad \begin{aligned} & \frac{d}{dt} \left(Re|\mathbf{u}|^2 + \frac{We}{2\omega} |\underline{\tau}|^2 \right) + (1 - \omega)\|\mathbf{u}\|^2 + \frac{1}{2\omega} |\underline{\tau}|^2 \\ & \leq \frac{1}{1 - \omega} \|\mathbf{F}\|_{-1}^2 + \frac{1}{2\omega} |\underline{G}|^2. \end{aligned}$$

From (3.23), we deduce that Problem (3.17)–(3.21) possesses a solution satisfying

$$\mathbf{u} \in L^2_{\text{loc}}(\mathbb{R}_+; V) \cap \mathcal{C}(\mathbb{R}_+; H), \quad \mathbf{u}' \in L^2_{\text{loc}}(\mathbb{R}_+; V').$$

Integration of (3.18) on a finite interval $[0, t]$, $t > 0$, shows that actually $\underline{\tau}$ belongs to $\mathcal{C}(\mathbb{R}_+; L^2)$.

Finally, inequality (3.23) shows that $(\mathbf{u}, \underline{\tau})$ is bounded on $\mathbb{R}_+$. Uniqueness is also deduced from inequality (3.23). $\square$

Here we give a proof of the regularity of the solution to Problem (3.17)–(3.21). This proof will not extend to the nonlinear case where a convected derivated of the extra-stress tensor $\underline{\tau}$ is present.

PROPOSITION 3.7. Let $\mathbf{F} \in L^\infty(\mathbb{R}_+; \mathbf{H}^1)$, $\mathbf{F}' \in L^\infty(\mathbb{R}_+; \mathbf{H}^{-1})$, $\underline{G} \in L^\infty(\mathbb{R}_+; \mathbf{H}^2)$, $\underline{G}' \in L^\infty(\mathbb{R}_+; \mathbf{L}^2)$, $\mathbf{u}_0 \in D(A)$, $\underline{\tau}_0 \in \mathbf{H}^2$.

Then the (unique) solution to Problem (3.17)–(3.21) satisfies

$$(3.24) \quad \begin{cases} \mathbf{u} \in \mathcal{C}_b(\mathbb{R}_+; D(A)) \cap L^2_{\text{loc}}(\mathbb{R}_+; \mathbf{H}^3), \\ \mathbf{u}' \in \mathcal{C}_b(\mathbb{R}_+; H) \cap L^2_{\text{loc}}(\mathbb{R}_+; V), \\ \underline{\tau} \in \mathcal{C}_b(\mathbb{R}_+; \mathbf{H}^2), \quad \underline{\tau}' \in \mathcal{C}_b(\mathbb{R}_+; \mathbf{L}^2). \end{cases}$$

Proof. (i) We first write the energy inequality satisfied by $\mathbf{u}'$ and $\underline{\tau}'$. It reads,

$$(3.25) \quad \begin{aligned} & \frac{d}{dt} \left(Re|\mathbf{u}'|^2 + \frac{We}{2\omega} |\underline{\tau}'|^2 \right) + (1-\omega) \|\mathbf{u}'\|^2 + \frac{1}{2\omega} |\underline{\tau}'|^2 \\ & \leq \frac{1}{1-\omega} \|\mathbf{F}'\|_{-1}^2 + \frac{1}{2\omega} |\underline{G}'|^2 \end{aligned}$$

which shows that $\mathbf{u}' \in L^\infty(\mathbb{R}_+; H) \cap L^2_{\text{loc}}(\mathbb{R}_+; V)$ and $\underline{\tau}' \in L^\infty(\mathbb{R}_+; \mathbf{L}^2)$. Moreover, integration of (3.25) gives

$$(3.26) \quad \begin{aligned} & Re|\mathbf{u}'(t)|^2 + \frac{We}{2\omega} |\underline{\tau}'(t)|^2 + \left(1 - \omega - \frac{\epsilon}{\lambda_1}\right) \int_0^t e^{\alpha(s-t)} \|\mathbf{u}'(s)\|^2 ds \\ & \leq \left(Re|\mathbf{u}'_0|^2 + \frac{We}{2\omega} |\underline{\tau}'_0|^2 \right) e^{-\alpha t} + \frac{1}{\alpha} \left[\frac{1}{1-\omega} \|\mathbf{F}'\|_{L^\infty(\mathbb{R}_+; \mathbf{H}^{-1})}^2 + \frac{1}{2\omega} \|\underline{G}'\|_{L^\infty(\mathbb{R}_+; \mathbf{L}^2)}^2 \right] (1 - e^{\alpha t}), \\ & \quad \forall t > 0, \quad \forall \epsilon, \quad 0 < \epsilon \leq (1-\omega)\lambda_1, \end{aligned}$$

where λ_1 is the first eigenvalue of the Stokes operator, and $\alpha = \min \left(\frac{\epsilon}{Re}, \frac{1}{We} \right)$.

Equations (3.17) and (3.18) also show that $\mathbf{u}' \in \mathcal{C}_b(\mathbb{R}_+; H)$, $\underline{\tau}' \in \mathcal{C}_b(\mathbb{R}_+; \mathbf{L}^2)$, because $\mathbf{u}'(0) \in H$ and $\underline{\tau}'(0) \in \mathbf{L}^2$.

(ii) Next we eliminate $\Delta \mathbf{u}$ between equations (3.17) and (3.18), obtaining

$$Re\mathbf{u}' - \frac{1-\omega}{\omega} (We \nabla \cdot \underline{\tau}' + \nabla \cdot \underline{\tau} - \nabla \underline{G}) + \nabla p = \nabla \cdot \underline{\tau} + \mathbf{F},$$

and, thus, by projecting on H ,

$$(3.27) \quad (1-\omega)We P \nabla \cdot \underline{\tau}' + P \nabla \cdot \underline{\tau} = \omega Re \mathbf{u}' + P(-\omega \mathbf{F} + (1-\omega) \nabla \cdot \underline{G})$$

The right hand side of equation (3.27) belongs to $L^2_{\text{loc}}(\mathbb{R}_+; V)$, and $\underline{\tau}_0 \in \mathbf{H}^2$, so that $P \nabla \cdot \underline{\tau} \in \mathcal{C}(\mathbb{R}_+; V)$. Moreover, choosing ϵ small enough, precisely $\epsilon < \frac{1}{1-\omega} \frac{Re}{We}$, we may

write

$$(3.28) \quad \int_0^t e^{\frac{s-t}{(1-\omega)W\epsilon}} \|\mathbf{u}'(s)\| ds \leq \left(\int_0^t \|\mathbf{u}'(s)\|^2 e^{\alpha(s-t)} ds \right) \left(\int_0^t e^{(\frac{1}{(1-\omega)W\epsilon} - \alpha)(s-t)} ds \right)$$

and from inequality (3.26), it then follows that $P\nabla \cdot \underline{\underline{\tau}} \in \mathcal{C}_b(\mathbb{R}_+ V)$.

(iii) We now use equation (3.17), written as

$$(1 - \omega)A\mathbf{u} = P(\nabla \cdot \underline{\underline{\tau}} + \mathbf{F}) - R\epsilon \mathbf{u}',$$

where the right hand side belongs to $L^2_{\text{loc}}(\mathbb{R}_+; V) \cap \mathcal{C}_b(\mathbb{R}_+; H)$. The regularity theory for elliptic problems shows that $\mathbf{u} \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbf{H}^3) \cap \mathcal{C}_b(\mathbb{R}_+; D(A))$. Moreover, $\|\mathbf{u}\|_3$ satisfies an estimate

$$(3.29) \quad \int_0^t e^{\alpha(s-t)} \|\mathbf{u}(s)\|_3^2 ds \leq \text{constant},$$

for all $t > 0$, α being the same as in (3.26) and (3.28).

(iv) Finally, we see that the right hand side of equation (3.18) belongs to $L^2_{\text{loc}}(\mathbb{R}_+; \mathbf{H}^2)$, and satisfies an inequality of the type (3.29), so that $\underline{\underline{\tau}}$ belongs to $\mathcal{C}_b(\mathbb{R}_+; \mathbf{H}^2)$.

This proves the result. $\square$

REMARK 3.8. The result of Proposition 3.7 holds true if, instead of $\underline{\underline{G}} \in L^\infty(\mathbb{R}_+; \mathbf{H}^2)$, we assume $\underline{\underline{G}} \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbf{H}^2)$ together with an estimate of the form (3.29),

$$\int_0^t e^{\beta(s-t)} \|\underline{\underline{G}}(s)\|_2^2 ds \leq \text{constant},$$

for all t , $\beta > 0$. The parameter ϵ can be chosen small enough, so that inequalities (3.26), (3.28) and (3.29) hold true. However, this hypothesis on $\underline{\underline{G}}$ is too strong to be useful in the nonlinear case. $\square$

4. Periodic solutions. The main point in this section is to prove an L^2 stability result for small solutions of (1.11)–(1.14) and to deduce it from the existence of (small) periodic solutions. We shall follow the approach A. Valli used for the compressible Navier-Stokes equation ([21], [22]).

We therefore assume that $0 < \omega \leq \omega_0$ (see Theorem 3.3), and that $(\mathbf{u}^1, \underline{\tau}^1, p^1)$ and $(\mathbf{u}^2, \underline{\tau}^2, p^2)$ are global solutions of (1.11)–(1.14) corresponding to initial data $(\mathbf{u}_0^i, \underline{\tau}_0^i)$, $i = 1, 2$, and to the same body force $\mathbf{f}$. Let

$$\delta = \text{Max}_{i=1,2} \left(\|\mathbf{u}_0^i\|_2, \|\underline{\tau}_0^i\|_2, \|\mathbf{f}\|_{L^\infty(\mathbb{R}_+; \mathbf{H}^1)}, \|\mathbf{f}'\|_{L^\infty(\mathbb{R}_+; \mathbf{H}^{-1})} \right).$$

δ will be chosen to satisfy the smallness condition in Theorem 3.3. We set $\mathbf{u} = \mathbf{u}^1 - \mathbf{u}^2$, $\underline{\tau} = \underline{\tau}^1 - \underline{\tau}^2$, $p = p^1 - p^2$, and obtain the following equations for $(\mathbf{u}, \underline{\tau}, p)$:

$$(4.1) \quad Re \frac{\partial \mathbf{u}}{\partial t} - (1 - \omega) \Delta \mathbf{u} + \nabla p = \nabla \cdot \underline{\tau} + \mathbf{F},$$

$$(4.2) \quad We \frac{\partial \underline{\tau}}{\partial t} + \underline{\tau} = 2\omega \underline{D}[\mathbf{u}] + \underline{G},$$

with

$$(4.3) \quad \mathbf{F} = -(\mathbf{u}^1 \cdot \nabla) \mathbf{u}^1 + (\mathbf{u}^2 \cdot \nabla) \mathbf{u}^2,$$

$$(4.4) \quad \underline{G} = - \left\{ (\mathbf{u}^1 \cdot \nabla) \underline{\tau}^1 - (\mathbf{u}^2 \cdot \nabla) \underline{\tau}^2 + \underline{\tau}^1 \underline{W}^1 - \underline{W}^1 \underline{\tau}^1 \right. \\ \left. + a(\underline{D}^1 \underline{\tau}^1 + \underline{\tau}^1 \underline{D}^1) - \underline{\tau}^2 \underline{W}^2 + \underline{W}^2 \underline{\tau}^2 - a(\underline{D}^2 \underline{\tau}^2 + \underline{\tau}^2 \underline{D}^2) \right\},$$

where $\underline{D}^i = \underline{D}[\mathbf{u}^i]$, $\underline{W}^i = \underline{W}[\mathbf{u}^i]$ $i = 1, 2$. As in the proof of Theorem 2.5, we take the scalar product of (4.1) and (4.2) with $\omega \mathbf{u}, \tau$ respectively and add the resulting equations to get

$$(4.5) \quad \frac{d}{dt} \left(\frac{\omega Re}{2} |\mathbf{u}|^2 + \frac{We}{2} |\underline{\tau}|^2 \right) + \omega(1 - \omega) \|\mathbf{u}\|^2 + |\underline{\tau}|^2 \\ = \omega \int (\mathbf{u} \cdot \nabla) \mathbf{u}^1 \cdot \mathbf{u} dx + \int Tr(\underline{G} \underline{\tau}) dx$$

We have by using standard inequalities

$$(4.6) \quad \left| \int (\mathbf{u} \cdot \nabla) \mathbf{u}^1 \cdot \mathbf{u} dx \right| \leq \|\mathbf{u}\|_{L^6} \|\nabla \mathbf{u}^1\|_{L^3} \|\mathbf{u}\| \\ \leq C \|\mathbf{u}\|^2 \|\mathbf{u}^1\|_2 \\ \leq \frac{\omega(1 - \omega)}{4} \|\mathbf{u}\|^2$$

provided we have chosen δ small enough. The second integral in the right hand side of (4.5) contains several terms. First there is a term similar to (4.6), namely

$$(4.7) \quad \left| \int (\mathbf{u} \cdot \nabla) \underline{\tau}^2 \cdot \underline{\tau} dx \right| \leq \|\mathbf{u}\| \|\underline{\tau}^2\|_2 \|\underline{\tau}\| \\ \leq \frac{1}{4} |\underline{\tau}|^2 + \frac{\omega(1 - \omega)}{4} \|\mathbf{u}\|^2,$$

provided δ is small enough.

The other terms are of the form

$$\int (\phi(\underline{\tau}^1, \nabla \mathbf{u}^1) - \phi(\underline{\tau}^2, \nabla \mathbf{u}^2)) \cdot \underline{\tau} \, dx,$$

where ϕ is some bilinear operator. We estimate them as follows:

$$(4.8) \quad \left| \int \phi(\underline{\tau}, \nabla \mathbf{u}^1) \cdot \underline{\tau} \, dx \right| \leq C \|\mathbf{u}^1\|_3 |\underline{\tau}|^2$$

$$(4.9) \quad \left| \int \phi(\underline{\tau}^2, \nabla \mathbf{u}) \cdot \underline{\tau} \, dx \right| \leq C \|\underline{\tau}^2\|_{L^\infty} \|\mathbf{u}\| |\underline{\tau}| \\ \leq \frac{\omega(1-\omega)}{4} \|\mathbf{u}\|^2 + \frac{1}{4} |\underline{\tau}|^2$$

provided δ is small enough.

Finally, taking into account (4.5)–(4.9), we get after integrating in time

$$(4.10) \quad \frac{\omega Re}{2} |\mathbf{u}(t)|^2 + \frac{We}{2} |\underline{\tau}(t)|^2 + \int_0^t \left[\frac{\omega(1-\omega)}{2} \|\mathbf{u}\|^2 + \frac{1}{2} |\underline{\tau}|^2 \right] ds \\ \leq \frac{\omega Re}{2} |\mathbf{u}(0)|^2 + \frac{We}{2} |\underline{\tau}(0)|^2 \\ + C \int_0^t \|\mathbf{u}^1(s)\|_3 |\underline{\tau}(s)|^2 ds$$

But the uniform bounds on $\mathbf{u}^1$ and (3.3) immediately imply

$$\int_0^t \|\mathbf{u}^1(s)\|_3^2 ds \leq \eta + \eta t$$

where η is small when δ is small, and so

$$\int_0^t \|\mathbf{u}^1(s)\|_3 ds \leq t^{1/2} \left(\int_0^t \|\mathbf{u}^1(s)\|_3^2 ds \right)^{1/2} \\ \leq \eta^{1/2} (t + t^{1/2}).$$

We obtain finally by Gronwall's lemma

$$(4.11) \quad \frac{\omega Re}{2} |\mathbf{u}(t)|^2 + \frac{We}{2} |\underline{\tau}(t)|^2 \leq C(|\mathbf{u}(0)|^2 + |\underline{\tau}(0)|^2) \exp(\eta^{1/2} t).$$

We can repeat the same argument for $e^{\gamma t} \mathbf{u}(t)$ and $e^{\gamma t} \underline{\tau}(t)$ where γ is a small parameter, and by choosing η , hence δ , small enough with respect to γ , we get

$$(4.12) \quad |\mathbf{u}(t)|^2 + |\underline{\tau}(t)|^2 \leq C(|\mathbf{u}(0)|^2 + |\underline{\tau}(0)|^2) \exp(-\epsilon t),$$

where $\epsilon = 2\gamma - \eta^{1/2} > 0$.

We have therefore proved the following result.

THEOREM 4.1. *Let ω be as in Theorem 3.4. Let $\mathbf{u}_0^i \in \mathbf{H}^2(\Omega) \cap V$, $\underline{\tau}_0^i \in \mathbf{H}^2(\Omega)$, $i = 1, 2$, $\mathbf{f} \in L^\infty(\mathbf{R}_+; \mathbf{H}^1)$, $\mathbf{f}' \in L^\infty(\mathbf{R}_+; \mathbf{H}^{-1})$ be small enough. Then the corresponding solutions of (1.11)–(1.14) satisfy*

$$(4.13) \quad |\mathbf{u}^1(t) - \mathbf{u}^2(t)|^2 + |\underline{\tau}^1(t) - \underline{\tau}^2(t)|^2 \leq c(|\mathbf{u}_{10} - \mathbf{u}_{20}|^2 + |\underline{\tau}_{10} - \underline{\tau}_{20}|^2) e^{-\epsilon t}$$

for every $t \geq 0$, for some $\epsilon > 0$.

□

Having Theorem 4.1 in hand, the existence of small time periodic or steady solutions of (1.11)–(1.14) is straightforward. (See J. Serrin [19] for the incompressible Navier-Stokes equations or A. Valli [21], [22] for the compressible Navier-Stokes equations.) We just state the final results.

THEOREM 4.2. *Let ω be as in Theorem 3.3. Let $\mathbf{f} \in L^\infty(\mathbf{R}_+; \mathbf{H}^1(\Omega))$, $\mathbf{f}' \in L^\infty(\mathbf{R}_+; \mathbf{H}^{-1}(\Omega))$. Assume that $\mathbf{f}$ is T -periodic and that $\mathbf{f}, \mathbf{f}'$ are small enough. Then there exists a T -periodic solution $(\mathbf{u}, \underline{\tau})$ of (1.11)–(1.14) which satisfies the regularity properties stated in Theorem 3.3.*

Moreover, $(\mathbf{u}, \underline{\tau})$ is asymptotically stable and unique among all other small T -periodic solutions.

□

COROLLARY 4.3. *Let ω be as in Theorem 3.3. Let $\mathbf{f} \in \mathbf{H}^1(\Omega)$ be small enough. Then there exists a steady solution $(\mathbf{u}, \underline{\tau})$ of (1.11)–(1.14) which is in $\mathbf{H}^3 \times \mathbf{H}^2$. Moreover, $(\mathbf{u}, \underline{\tau})$ is asymptotically stable in L^2 and is unique among all other small steady solutions.*

REMARK 4.4. M. Renardy [17] has obtained by a direct method the existence of small steady solutions for all ω in the interval $[0, 1]$, but did not study their stability.

5. Several relaxation times. The models under study in the previous sections involve a single relaxation time (λ_1). For an integral model, this corresponds to a relaxation kernel of exponential type. More realistic models possess a more general kernel. Taking a sum of exponential leads, for the differential form of the model, to several relaxation times. For simplicity we only write the case of the Oldroyd models. In particular, the extra-stress tensor $\underline{\tau}$ is decomposed as

$$(5.1) \quad \underline{\tau} = \underline{\tau}_s + \underline{\tau}_p,$$

where

$$\underline{\tau}_p = \sum_{i=1}^N \underline{\tau}_i,$$

and

$$(5.2) \quad \begin{aligned} \underline{\tau}_i + \lambda_i \frac{\mathcal{D}_a \underline{\tau}_i}{\mathcal{D}t} &= 2 \eta_i \underline{D}, \quad \eta_i, \lambda_i > 0, \quad i = 1, \dots, N, \\ \underline{\tau}_s &= 2 \eta_s \underline{D}, \quad \eta_s \geq 0. \end{aligned}$$

We shall restrict ourselves to the case $\eta_s > 0$ (nonzero Newtonian contribution). The full equations of motion become (cf. (1.9))

$$(5.3) \quad \begin{cases} \rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla p - \eta_s \Delta \mathbf{u} = \sum_{i=1}^N \nabla \cdot \underline{\tau}_i + \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \\ \underline{\tau}_i + \lambda_i \frac{\mathcal{D}_a \underline{\tau}_i}{\mathcal{D}t} = 2 \eta_i \underline{D}, \end{cases}$$

or, in nondimensional form

$$(5.4) \quad \begin{cases} Re \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla p - (1 - \omega) \Delta \mathbf{u} = \nabla \cdot \underline{\tau}_i + \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \\ \underline{\tau}_i + W_i \frac{\mathcal{D}_a \underline{\tau}_i}{\mathcal{D}t} = 2 \omega_i \underline{D}, \end{cases}$$

where here $\omega = \frac{\eta_p}{\eta}$, $\omega_i = \frac{\eta_i}{\eta}$, $\eta = \eta_p + \eta_s$, $\eta_p = \sum_{i=1}^N \eta_i$, $w_i = \lambda_i \frac{U}{L}$, and $Re = \frac{\rho U L}{\eta}$.

All the results of Sections 2, 3 and 4 hold true for the system (5.4). They are proven by the very same methods.

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