**NEARLY OPTIMAL CONTROL OF QUEUES IN
HEAVY TRAFFIC WITH HETEROGENEOUS SERVERS**

By

K.M. Ramachandran

IMA Preprint Series # 442

August 1988

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455-0436

Phone: 612/624-6066 Fax: 612/626-7370

URL: <http://www.ima.umn.edu>

NEARLY OPTIMAL CONTROL OF QUEUES IN HEAVY TRAFFIC WITH HETEROGENEOUS SERVERS

K.M. Ramachandran*

Abstract. We analyze the approximately optimal control of queues in tandem with heterogeneous servers. The queues are assumed to be under heavy traffic. The controls are putting off/on various inputs and services. Discounted cost structure is considered. It is shown that the scaled controlled system converges weakly to controlled limit reflected diffusion. We also prove that the optimal policies for limit when adapted to the physical system is nearly optimal.

Keywords. Queueing networks, heterogeneous servers, heavy traffic approximation, weak convergence, nearly optimal control.

1. Introduction

A standard approach to analyze queues in heavy traffic is the following. Assuming that the arrival and service channels are independent, take the difference between the properly scaled arrivals and departures and use some form of central limit theorem and weak convergence to get a limit process. Then identify the limit process by specifying its moments. Usually, the limit is a Brownian motion with some drift. On the approximation of queues, we refer to the work of Reiman [14], Harrison [4], and Lemoine [11] for a nice exposure to the field. While these results are attractive, they do not specify how to manage a given system with certain specific limitations, such as finite buffers. Getting a "good" control policy to the queues in heavy traffic is important in order the limit analysis to be useful.

For special classes of control problems, such as queues are stable in the sense that the traffic intensity is less than unity, the optimal control of queues have been analyzed by Lin and Kumar [12], Crabill, Gross and Magasine [2], Stidham and Prabhu [15]. Assuming Markov structure of the problem, an essential role in these problems is played by dynamic programming equations, but when queues are unstable, as is the case in heavy traffic, and when the service and arrival rates are not mutually independent, we encounter a lot of problems. Due to the control effects, there will be impulses of standard type and also a non-standard problem of "multiple simultaneous" impulses. The weak limits are not simply just Brownian motions. In Kushner and Ramachandran [9], Ramachandran [13], the problem for tandem queues with single server at each service station was considered, optimal and nearly optimal controls were developed.

In the present paper, we consider an extension of the model developed by Lin and Kumar [12]. More realistic situations are taken into account. Consider the open queueing systems in tandem.

The buffer sizes are finite. Each service station P^i consists of two servers with different mean service rates which can depend on state. After being served at P^1 , the customer

* Department of Mathematics, University of South Florida, Tampa, FL 33620.

either leaves the system as a partially finished product or proceeds to the next station and joins the queue there. After being served at P^2 , the customer leaves the system. Here, we consider the control of this system with general arrival and departure rates and service rates are state and control dependent. The heavy traffic assumptions will be made precise later. The model considered in this work can be extended and generalized to the situation of more than two servers for each station.

An expression of a queue at each service station can be put as follows, For each $i = 1, 2$;

$$\begin{aligned} \text{Queue at } P^i = & (\text{external arrival to } P^i) \\ & + (\text{arrivals from other stations } P^j) + (\text{departures}) \\ & + (\text{increase of queue length due to shutting off services}) \\ & - (\text{decreases of queue length due to shutting off input}) \\ & + (\text{departures when the queue is empty}) \\ & - (\text{arrivals from other stations when they are empty}). \end{aligned}$$

We are now in a position to obtain a precise expression for the queues. Since we have to keep track of multiple impulses and the action times, the notations are not too nice. Nevertheless, it seems that this is not avoidable.

Let ϵ denote the traffic intensity. Let $\{\alpha^{i,\epsilon}(n)\}$ be the sequences of interarrival times of the customers from the exterior of the network to P^i , and let $\xi^{i,\epsilon}(n)$ be the indicator of the event that there was an arrival from exterior to P^i at time n . We shall use the following convention that the processor keeps processing even if the queue is empty, with the errors accounted for by an added reflection term. We denote P_j^i as the j th processor for the queue at station P^i . Let $\{\Delta_j^{i,\epsilon}(n)\}$ denote the sequences of service times for P_j^i with $\psi_j^{i,\epsilon}(n)$ being the indicator of the event that a service at P_j^i is completed at time n . Let $I^{i,j,\epsilon}(n)$ be the indicator function of the event that a completed service at P^i at time n is scheduled to be sent to P^j ($j=0$ denotes the exterior). Let $\{p_{ij}, i, j = 1, 2\}$ denote the probability that a completed service from P^i is to be routed to P^j , and $p_{i0} = 1 - \sum_{j=1}^2 p_{ij}$. Under the ϵ scaling, the buffer size at P^i is $\frac{B_i}{\epsilon}$, for some $B_i > 0$. For simplicity we will assume that $p_{21} = 0$ always.

We use the following control actions. The processor P_j^i can be shut off for a time, at a cost $k_j^i > 0$ and the exterior inputs to P^i can be shut off for a time, at a cost $k^{0,i} > 0$, all the costs to be paid at the moments of actions. We can shut off the link between P^i and P^j , by shunting out the partially finished product (customer) with a cost $c_{i,j} > 0$. There is a cost $q_{ij}\sqrt{\epsilon}$ per lost customer, $q_{ij} > 0$. If the buffer of P^i is full, then all the inputs to P^i must be turned off. There will be no change in the formulation if we allow all customers in P^i who have completed service are destined to return to P^i immediately.

Let $P_j^{i,\epsilon}(n)$ be the indicator of the event that P_j^i is working at time n , $P^{0,i,\epsilon}$ the indicator that the exterior input is not shut off at time n , and $C^{i,j,\epsilon}(n)$ be the indicator that the link connecting P^i to P^j is open at time n . The following definition is necessary to keep track of the precise impulse times. Let $N_j^{i,\epsilon}(n)$ (respectively $\tilde{N}_j^{i,\epsilon}(n)$) denote the

n th time that P_j^i is turned off (respectively, turned back on). Let $L^{i,j,\epsilon}(n)$ (resp., $\tilde{L}^{i,j,\epsilon}(n)$) denote the n th time that the link connecting P^i to P^j is shut off (resp., turned back on). Let $\nu_j^{i,\epsilon}(n) = \epsilon N_j^{i,\epsilon}(n)$ and $l^{i,j,\epsilon}(n) = \epsilon L^{i,j,\epsilon}(n)$, and similarly for $\tilde{\nu}_j^{i,\epsilon}(n)$, $\tilde{l}^{i,j,\epsilon}(n)$, the scaled times.

Let $X^{i,\epsilon}(n) = \sqrt{\epsilon}$ (number of customers in or waiting for service at P^i at time n). Set the interpolation $X^{i,\epsilon}(t) = X^{i,\epsilon}(t/\epsilon)$. Assume without loss of generality that all processors and links are working at time $t = 0$. Then $\tilde{\nu}_j^{i,\epsilon}(0) = \tilde{l}^{i,j,\epsilon}(n) = 0$ and $\tilde{\nu}_j^{i,\epsilon}(n) > \nu_j^{i,\epsilon}(n)$, $\tilde{l}^{i,j,\epsilon}(n) > l^{i,j,\epsilon}(n)$, for all $n > 0$.

To keep track of the flows in the system for purposes of the control problem and limit theorem, we specify separately the corrections to the flow due to empty queues and to the flow components due to the control actions. For notational simplicity, ϵ will be omitted in the terms in summands or integrands. The subscript c is for the phrase "combined" since we use it when there is a condition on the status of two controls simultaneously.

Define the arrival and departure processes as

$$A^{i,\epsilon}(t) = \sqrt{\epsilon} \sum_{n=1}^{t/\epsilon} \xi^{i,\epsilon}(n) \quad (1.1.a)$$

$$D^{i,j,\epsilon}(t) = \sqrt{\epsilon} \sum_{n=1}^{t/\epsilon} (\psi_1^i(n) + \psi_2^i(n)) I^{i,j}(n), \quad i \neq 0 \quad (1.1.b)$$

$Z^{i,j,\epsilon}$ defined below represents the lost output when the link from P^i to P^j is shunted to the exterior, for $i \neq j$ and $i, j \neq 0$.

$$\begin{aligned} Z^{i,j,\epsilon}(t) = & \sqrt{\epsilon} \sum_{n=1}^{t/\epsilon} \left(\psi_1^i(n) P_1^i(n) + \psi_2^i(n) P_2^i(n) \right) \\ & \cdot (1 - C^{i,j,\epsilon}(n)) I^{i,j}(n) I_{\{X^i(n) \neq 0\}} \end{aligned} \quad (1.1.c)$$

Define the reflection term $Y^{i,j,\epsilon}(\cdot)$ and the impulse control terms $U_k^{i,j,\epsilon}(\cdot)$, $k = 1, 2$ as follows.

$$Y^{i,j,\epsilon}(t) = \sqrt{\epsilon} \sum_{n=1}^{t/\epsilon} (\psi_1^i(n) P_1^i(n) + \psi_2^i(n) P_2^i(n)) I^{i,j}(n) I_{\{X^i(n)=0\}} \quad (1.1.d)$$

$$U_k^{i,j,\epsilon}(t) = \sqrt{\epsilon} \sum_{n=1}^{t/\epsilon} \psi_k^i(n) I^{i,j}(n) (1 - P_k^i(n)), \quad i \neq j, i \neq 0 \quad (1.1.e)$$

$$U^{0,i,\epsilon}(t) = \sqrt{\epsilon} \sum_{n=1}^{t/\epsilon} \xi^i(n) (1 - P^{0,i}(n)), \quad i \neq 0 \quad (1.1.f)$$

$$U_{k,c}^{j,i,\epsilon}(t) = \sqrt{\epsilon} \sum_{n=1}^{t/\epsilon} \psi_k^j(n) I^{j,i}(n) (1 - C^{j,i}(n) P_k^j(n)), \quad j \neq i, j \neq 0 \quad (1.1.g)$$

$$Y_c^{j,i,\epsilon}(t) = \sqrt{\epsilon} \sum_{n=1}^{t/\epsilon} (\psi_1^j(n)P_1^j(n) + \psi_2^j(n)P_2^j(n))C^{j,i}(n)I^{j,i}(n)I_{\{X^j(n)=0\}}, \quad j \neq i, j \neq 0 \quad (1.1.h)$$

Let

$$U^{1,\epsilon}(t) = U_1^{1,0,\epsilon}(t) + U_2^{1,0,\epsilon}(t) + U_1^{1,2,\epsilon}(t) + U_2^{1,2,\epsilon}(t) \quad (1.1.i)$$

$$U_c^{1,2,\epsilon}(t) = U_{1,c}^{1,2,\epsilon}(t) + U_{2,c}^{1,2,\epsilon}(t). \quad (1.1.j)$$

Then the representation for the queue length processes are

$$X^{1,\epsilon}(t) = A^{1,\epsilon}(t) - (D^{1,2,\epsilon} + D^{1,0,\epsilon}(t)) + (Y^{1,2,\epsilon}(t) + Y^{1,0,\epsilon}(t)) - U^{0,1,\epsilon}(t) + U^{1,\epsilon}(t) \quad (1.2.a)$$

$$X^{2,\epsilon}(t) = A^{2,\epsilon}(t) - D^{2,0,\epsilon}(t) + D^{1,2,\epsilon}(t) + Y^{2,0,\epsilon}(t) - Y_c^{1,2,\epsilon}(t) - U^{0,2,\epsilon}(t) - U_c^{1,2,\epsilon}(t). \quad (1.2.b)$$

Let $X^\epsilon(t) = (X^{1,\epsilon}(t), X^{2,\epsilon}(t))'$. Denote by π^ϵ or π the control policies which determine $\nu_j^{i,\epsilon}$, $\tilde{\nu}_j^{i,\epsilon}$ etc., and let E_x^π be the expectation given policy π and initial condition $X_0^\epsilon = x$. Let P denote the indicator functions $\{P^\alpha\}$ of the processors and links. Here the value function depends on the initial value of P . Then, for a bounded and continuous function $k(\cdot)$, and $\beta > 0$, the discounted cost is of the form

$$\begin{aligned} V^\epsilon(\pi, x, P) &= E_x^\pi \int_0^\infty e^{-\beta t} k(X^\epsilon(t)) dt \\ &+ E_x^\pi \sum_{i=1}^2 \sum_n \left(k_1^i e^{-\beta \nu_1^{i,\epsilon}(n)} + k_2^i e^{-\beta \nu_2^{i,\epsilon}(n)} \right) \\ &+ E_x^\pi \sum_n c_{1,2} e^{-\beta t^{1,2,\epsilon}(n)} \\ &+ E_x^\pi \int_0^\infty e^{-\beta t} (q_{1,2} dZ^{1,2,\epsilon}(t) + \sum_{i=1}^2 q_{0,i} dU^{0,i,\epsilon}(t)). \end{aligned} \quad (1.3)$$

Next, we will get the limit process through the weak convergence methods. We will also show that any optimal policy for the the limit process when adapted to the true system, gives us a δ -optimal policy. This result is extremely important because it is very difficult to obtain an optimal policy for the true system. There is also a possibility that the service for two customers at P_1^1 and P_2^1 are finished at the same time; they are both going to P^2 . However, since we assume that the service time at P_1^1 is different from that at P_2^1 , it is clear that given any $[0, t]$, there can be only finitely many such occurrences w.p.1. Henceforth, we will carry out our analysis in the complement of these points. Also, if the service time at P_1^1 is extremely small compared with that at P_2^1 , then in the limit, under heavy traffic conditions, the effect of P_1^1 will be negligible. Hence, without loss of generality, we can assume that shutting off any one of the servers will result in impulse in the limit, and shutting off both servers will only change the magnitude of the impulse.

2. Assumptions and Centering

For notational convenience, denote $\nu^\epsilon(n) = \{\nu_j^{i,\epsilon}(n), \tilde{\nu}_j^{i,\epsilon}(n), l^{i,j,\epsilon}(n), \tilde{l}^{i,j,\epsilon}(n)\}$ as a single sequence. $\{\tau^\epsilon(n)\}$ denote the sequence of event times indicated by all the elements of ν^ϵ in order. Define $R^{\alpha,\epsilon}(n) = 1, -1$ or 0 depending on whether or not the "control" with the same subscript was turned on or turned off or left unchanged at time $\tau^\epsilon(n)$. Let

$$R^\epsilon(n) = (R^{0,1,\epsilon}(n), R_1^{1,\epsilon}(n), R_2^{1,\epsilon}(n), R^{1,2,\epsilon}(n), R^{0,2,\epsilon}(n), R_1^{2,\epsilon}(n), R_2^{2,\epsilon}(n)).$$

The idea of introducing this term is that from $\{R^\epsilon(n), \tau^\epsilon(n)\}$, we can recover all control actions and their times. The corresponding impulses are denoted by $\delta U(n)$. The tripple $(\delta U(n), \tau(n), R(n))$ is said to be a control policy, which is admissible if the function

$$R(t) = \{X(0), \delta U(n)I_{\{\tau(n) \leq t\}}, R(n)I_{\{\tau(n) \leq t\}}, I_{\{\tau(n) \leq t\}}, n < \infty, X(t), Y(t)\}$$

is nonanticipative with respect to the limit process, which will be defined in the sequel. Let $E_{a,n}^{i,\epsilon}$ denote the expectation given the arrival, departure and control intervals and actions which ended by real time $S_{a,n}^{i,\epsilon}$ as well as the lengths of all other arrival and service intervals which started by but which might not have been completed by time $S_{a,n}^{i,\epsilon}$, where $S_{a,n}^{i,\epsilon} = \sum_{j=1}^n \alpha^{i,\epsilon}(j)$. Let $W_j^i(n)$ be the indicator function of a customer at n th time being served at the corresponding server P_j^i and let

$$S_{d,n}^{i,\epsilon} = \sum_{j=1}^n (\Delta_1^{i,\epsilon}(j)W_1^i(j) + \Delta_2^{i,\epsilon}(j)W_2^i(j)).$$

Similarly, we can define $E_{d,n}^{i,\epsilon}$ and the conditional variance $var_{a,n}^{i,\epsilon}, var_{d,n}^{i,\epsilon}$. Let $E_{a,n}^{i,\epsilon} \alpha^{i,\epsilon}(n+1) = \bar{\alpha}^{i,\epsilon}(n+1)$, $var_{a,n}^{i,\epsilon} \alpha^{i,\epsilon}(n+1) = (\sigma_a^{i,\epsilon}(n+1))^2$, $E_{d,n}^{i,\epsilon} \Delta_j^{i,\epsilon}(n+1) = \bar{\Delta}_j^{i,\epsilon}(n+1)$, $var_{d,n}^{i,\epsilon} \Delta_j^{i,\epsilon}(n+1) = (\sigma_{j,d}^{i,\epsilon}(n+1))^2$, and let $cov^{i,j,\epsilon}(\Delta_1^{i,\epsilon}(n+1), \Delta_2^{i,\epsilon}(n+1)) = 0$.

We make the following assumptions.

(A1) If $X^{2,\epsilon}(n) = 0$, then all inputs to P^2 are open. If $X^{1,\epsilon}(n) = 0$, then the inputs of P^1 is open. If some $X^{i,\epsilon}(n) = B_i$, for some $B_i > 0$, then all inputs to P^i are closed.

(A2) There are positive numbers $g_{a,i}, g_{d,i,j}, g_{d,i}$ and bounded and continuous functions $a^i(\cdot), d_j^i(\cdot), d^i(\cdot)$, such that

$$(\bar{\alpha}^{i,\epsilon}(n+1))^{-1} = g_{a,i} + \sqrt{\epsilon} a_i(n) + o(\sqrt{\epsilon})$$

$$(\bar{\Delta}_j^{i,\epsilon}(n+1))^{-1} = g_{d,i,j} + \sqrt{\epsilon} d_j^i(n) + o(\sqrt{\epsilon})$$

$$(\bar{\Delta}_1^{i,\epsilon}(n+1) + \bar{\Delta}_2^{i,\epsilon}(n+1))^{-1} = g_{d,i} + \sqrt{\epsilon} d^i(n) + o(\epsilon)$$

where $a_i(n) = a^i(X^\epsilon(S_a^{i,\epsilon}(n)))$, and $d_j^i(n) = d_j^i(X^\epsilon(S_d^{i,\epsilon}(n)))$, and $d^i(n) = d^i(X^\epsilon(S_d^{i,\epsilon}(n)))$.

(A3) The set

$$\{|\alpha^{i,\epsilon}(n)|^2, |\Delta_j^{i,\epsilon}(n)|^2; i, j, n \neq 0, \text{ small } \epsilon, \text{ all control actions}\}$$

is uniformly integrable.

(A4) *Heavy traffic assumption*

$$g_{a_1} = (1 - p_{11})g_{d,1}$$

$$p_{12}g_{d,1} + g_{a_2} = (1 - p_{22})g_{d,2}.$$

(A5) The routing variables $\{I^{i,j,\epsilon}(k); i, j, k\}$ are mutually independent, and independent of $\{\alpha^{i,\epsilon}(k), \Delta_j^{i,\epsilon}(k)\}$. Let $P\{I^{i,j,\epsilon} = 1\} = p_{ij}$.

(A6) There are continuous functions $\sigma_{a_i}(\cdot)$ and $\sigma_{d,i,j}(\cdot)$, such that

$$\sigma_a^{i,\epsilon}(n+1) = \sigma_{a_i}(X^\epsilon(S_a^{i,\epsilon}(n)) + \delta'_\epsilon)$$

$$\sigma_{j,d}^{i,\epsilon} = \sigma_{d_j}(X^\epsilon(S_d^{i,\epsilon}(n)) + \delta'')$$

where $\delta^\alpha \xrightarrow{\epsilon} 0$, uniformly in all the variables.

(A7) The uncontrolled $X(\cdot)$ has a unique solution (in the weak sense) for each initial condition.

Note that

$$(\bar{\Delta}_1^{i,\epsilon} + \bar{\Delta}_2^{i,\epsilon})^{-1} = (\bar{\Delta}_1^{i,\epsilon})^{-1}(\bar{\Delta}_2^{i,\epsilon})^{-1}((\bar{\Delta}_1^{i,\epsilon})^{-1} + (\bar{\Delta}_2^{i,\epsilon})^{-1})^{-1}$$

so in (A2), we get $g_{d,i} = \frac{g_{d,i,1}g_{d,i,2}}{g_{d,i,1} + g_{d,i,2}}$.

We can actually rewrite

$$\begin{aligned} Z^{1,2,\epsilon}(t) &= U_{1,c}^{1,2,\epsilon}(t) + U_{2,c}^{1,2,\epsilon}(t) - U_1^{1,2,\epsilon}(t) \\ &\quad - U_2^{1,2,\epsilon}(t) - \sum_1^\infty \int_{l^{1,2}(n) \cap t}^{\bar{l}^{1,2}(n) \cap t} dY^{1,2,\epsilon}(s). \end{aligned} \quad (2.1)$$

Let $\bar{S}_a^{i,\epsilon}(t) = \max\{\epsilon k; \epsilon S_{a,k}^{i,\epsilon}(t) \leq t\}$. This can be interpreted as the inverse of the interpolated arrival time function $\epsilon S_a^{i,\epsilon}(t/\epsilon)$. Define the centered processes

$$\begin{aligned} \tilde{A}^{i,\epsilon}(t) &= \sqrt{\epsilon} \sum_{k=1}^{t/\epsilon} \sum_{l=S_a^{i,\epsilon}(k)}^{S_a^{i,\epsilon}(k+1)-1} \left(\xi^i(l) - \frac{1}{\bar{\alpha}^i(k)} \right) \\ &= \sqrt{\epsilon} \sum_{k=1}^{t/\epsilon} \left(1 - \frac{\alpha^i(k)}{\bar{\alpha}^i(k)} \right) \end{aligned}$$

$$\begin{aligned}
\tilde{D}^{i,j,\epsilon}(t) &= \sqrt{\epsilon} \sum_{k=1}^{t/\epsilon} \sum_{l=S_d^{i,\epsilon}(k)}^{S_d^{i,\epsilon}(k+1)-1} \left((\psi_1^i(l) + \psi_2^i(l)) I^{i,j}(l) - \frac{p_{ij}}{\bar{\Delta}_1^i(k) + \bar{\Delta}_2^i(k)} \right) \\
&= \sqrt{\epsilon} \sum_{k=1}^{t/\epsilon} \left(I^{i,j}(S_d^{i,\epsilon}(k)) - p_{ij} \left(\frac{\Delta_1^i(k) + \Delta_2^i(k)}{\bar{\Delta}_1^i(k) + \bar{\Delta}_2^i(k)} \right) \right) \\
&= \sqrt{\epsilon} \sum_{k=1}^{t/\epsilon} \left(I^{i,j}(k) - p_{ij} \left(\frac{\Delta_1^i(k) + \Delta_2^i(k)}{\bar{\Delta}_1^i(k) + \bar{\Delta}_2^i(k)} \right) \right)
\end{aligned}$$

Last equality follows from (A5).

Write $A^{i,\epsilon}(\cdot)$ and $D^{i,j,\epsilon}(\cdot)$ in the following form

$$\begin{aligned}
A^{i,\epsilon}(t) &= \sqrt{\epsilon} \sum_{\epsilon k=1}^{\bar{S}_a^{i,\epsilon}(t)} \sum_{l=S_a^{i,\epsilon}(k)}^{S_a^{i,\epsilon}(k+1)-1} \left(\xi^i(l) - \frac{1}{\bar{\alpha}^i(k)} \right) \\
&\quad + \sqrt{\epsilon} \sum_{\epsilon k=1}^{\bar{S}_a^{i,\epsilon}(t)} \frac{\alpha^i(k)}{\bar{\alpha}^i(k)} = \tilde{A}^{i,\epsilon}(t) + \tilde{B}_a^{i,\epsilon}(t)
\end{aligned}$$

similarly

$$D^{i,j,\epsilon}(t) = \tilde{D}^{i,j,\epsilon}(t) + \tilde{B}_d^{i,j,\epsilon}(t)$$

where

$$\tilde{B}_d^{i,j,\epsilon}(t) = \sqrt{\epsilon} \sum_{\epsilon k=1}^{\bar{S}_d^{i,\epsilon}(t)} \left(\frac{\Delta_1^i(k) + \Delta_2^i(k)}{\bar{\Delta}_1^i(k) + \bar{\Delta}_2^i(k)} \right) p_{ij}.$$

Using (A2) in expression of queues, we write

$$\begin{aligned}
&\tilde{B}^{1,\epsilon}(t) - (\tilde{B}_d^{1,0}(t) + \tilde{B}_d^{1,2}(t)) \\
&= \sqrt{\epsilon} \sum_{\epsilon k=1}^{\bar{S}_a^{1,\epsilon}(t)} \alpha^1(k) \left(g_{a^1} + \sqrt{\epsilon} a_1(k) + o(\sqrt{\epsilon}) \right) \\
&\quad - \sqrt{\epsilon} \sum_{\epsilon k=1}^{\bar{S}_d^{1,\epsilon}(t)} (\Delta_1^1(k) + \Delta_2^1(k)) \left(g_{d,1} + \sqrt{\epsilon} d^1(k) + o(\sqrt{\epsilon}) \right) (1 - p_{11}). \tag{2.2}
\end{aligned}$$

Since

$$\sum_{\epsilon k=1}^{\bar{S}_a^{1,\epsilon}(t)} \alpha^1(k) = \frac{t}{\epsilon} \pmod{O(1)}$$

the principal term of the first sum is

$$g_{a^1} \frac{t}{\epsilon} \pmod{O(\sqrt{\epsilon})}$$

and the second term is

$$g_{d,1} \frac{t}{\epsilon} \pmod{O(\sqrt{\epsilon})}.$$

By (A4), $(1 - p_{11})g_{d,1} = g_{a^1}$. Hence,

$$\begin{aligned} & \sqrt{\epsilon} \sum_{\epsilon k=1}^{\bar{S}_a^{1,\epsilon}(t)} \alpha^1(k) g_{a,1} - \sqrt{\epsilon} \sum_{\epsilon k=1}^{\bar{S}_d^{1,\epsilon}} (\Delta^1(k) + \Delta^2(k)) g_{d,1} (1 - p_{11}) \\ &= g_{a^1} \frac{t}{\sqrt{\epsilon}} - (1 - p_{11}) g_{d,1} \frac{t}{\sqrt{\epsilon}} = 0. \end{aligned}$$

By using the definition of $\alpha^1(k)$, $a_1(k)$ and the fact that $X^\epsilon(k)$ changes by at most $O(\epsilon)$ per step,

$$\sqrt{\epsilon} \sum_{\epsilon k=1}^{\bar{S}_a^{1,\epsilon}(t)} \sqrt{\epsilon} \alpha^1(k) a_1(k) = \epsilon \sum_1^{\frac{t}{\epsilon}} a^1(X(k)) + \delta(\epsilon)$$

where $\delta(\epsilon) \xrightarrow{\epsilon} 0$. Similarly, for the second term we have $\epsilon \sum_1^{\frac{t}{\epsilon}} d^1(X(k)) + \delta(\epsilon)$.

(2.2) can be written as

$$\begin{aligned} & \epsilon \sum_1^{t/\epsilon} [a_1(X(k)) - (1 - p_{11}) d^1(X(k))] + \delta_\epsilon^1(t) \\ &= \int_0^t b^1(X^\epsilon(s)) ds + \delta_\epsilon^1(t) = B^{1,\epsilon}(t) + \delta_\epsilon^1(t). \end{aligned} \tag{2.3}$$

By similar calculation,

$$\begin{aligned} \tilde{B}_a^{2,\epsilon}(t) - \tilde{B}_d^{2,0,\epsilon}(t) + \tilde{B}_d^{1,2,\epsilon}(t) &= \int_0^t b^2(X^\epsilon(s)) ds + \delta_\epsilon^2(t) \\ &= B^{2,\epsilon}(t) + \delta_\epsilon^2(t) \end{aligned}$$

where $\delta_\epsilon^i \rightarrow 0$ uniformly in bounded intervals.

Define the processes

$$Y^{i,\epsilon}(t) = \sqrt{\epsilon} \sum_1^{t/\epsilon} (\psi_1^i(n) P_1^i(n) + \psi_2^i(n) P_2^i(n)) I_{\{X^i(n)=0\}} \tag{2.4}$$

and write

$$\bar{Y}_c^{1,2,\epsilon}(t) = \sum_{n=0}^{\infty} p_{12} \int_{\bar{l}^{1,2,\epsilon}(n)}^{l^{1,2,\epsilon}(n+1)} d\bar{Y}^{1,\epsilon}(s) \tag{2.5.a}$$

$$U_k^{1,2,\epsilon}(t) = \sum_{n=1}^{\infty} \int_{\nu_k^{1,\epsilon}(n) \cap t}^{\bar{\nu}_k^{1,\epsilon}(n) \cap t} dU_{k,c}^{1,2,\epsilon}(s) \tag{2.5.b}$$

Let $U^{1,j} = U_1^{1,j} + U_2^{1,j}$ and $U^1 = U^{1,1} + U^{1,2}$ and $U_c^{2,\epsilon} = U_1^{2,\epsilon} + U_2^{2,\epsilon}$. Then we have

Lemma 2:1. Under (A1)-(A6), the following hold.

$$Y^{1,j,\epsilon}(\cdot) - p_{12}Y^{1,\epsilon}(\cdot) \Rightarrow 0 \quad (2.6.a)$$

$$Y^{2,0,\epsilon}(\cdot) - (1 - p_{22})Y^{2,\epsilon}(\cdot) \Rightarrow 0 \quad (2.6.a')$$

$$Y_c^{1,2,\epsilon}(t) - \sum_{n=0}^{\infty} p_{12} \int_{\tilde{I}^{1,2,\epsilon}(n) \cap t}^{\tilde{I}^{1,2,\epsilon}(n+1) \cap t} dY^{1,\epsilon}(s) \Rightarrow 0 \quad (2.6.b)$$

$$Y_c^{1,2,\epsilon}(\cdot) - p_{12}Y^{1,\epsilon}(\cdot) \Rightarrow 0 \quad (2.6.c)$$

$$U^{1,j,\epsilon}(\cdot) - U^{1,\epsilon}(\cdot) \frac{p_{1j}}{(1 - p_{11})} \Rightarrow 0 \quad (2.6.d)$$

$$U_c^{1,2,\epsilon}(t) - \sqrt{\epsilon} p_{12} \sum_0^{t/\epsilon} (\psi_1^1(n) + \psi_2^1(n))(1 - (P_1^1(n) \vee P_2^1(n))P^{1,2}(n)) \Rightarrow 0 \quad (2.6.e)$$

$\tilde{I}^{i,j,\epsilon}(n) - I^{i,j,\epsilon}(n) \rightarrow 0$ and $\tilde{\nu}_k^{i,\epsilon}(n) - \nu_k^{i,\epsilon}(n) \rightarrow 0$ for each each n .

Proof: We will only show that

$$U^{1,0,\epsilon}(t) - U^{1,\epsilon}(t) \frac{p_{10}}{1 - p_{11}} \Rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

For other terms will follow similarly.

$$\begin{aligned} U^{1,0,\epsilon}(t) - U^{1,\epsilon}(t) \frac{p_{10}}{1 - p_{11}} &= U_1^{1,0,\epsilon}(t) + U_2^{1,0,\epsilon}(t) \\ &\quad - \frac{p_{10}}{1 - p_{11}} (U_1^{1,0,\epsilon}(t) + U_2^{1,0,\epsilon}(t) + U_1^{1,2,\epsilon}(t) + U_2^{1,2,\epsilon}(t)) \\ &= (1 - \frac{p_{10}}{1 - p_{11}}) (U_1^{1,0,\epsilon}(t) + U_2^{1,0,\epsilon}(t)) - \frac{p_{10}}{1 - p_{11}} (U_1^{1,2,\epsilon}(t) + U_2^{1,2,\epsilon}(t)) \\ &= \frac{p_{12}}{p_{10} + p_{12}} (U_1^{1,0,\epsilon}(t) + U_2^{1,0,\epsilon}(t)) - \frac{p_{12}}{p_{10} + p_{12}} (U_1^{1,2,\epsilon}(t) + U_2^{1,2,\epsilon}(t)) \\ &= \sqrt{\epsilon} \sum_{n=1}^{t/\epsilon} \{ \psi_1^1(n)(1 - P_1^1(n)) + \psi_2^1(n)(1 - P_2^1(n)) \} \frac{p_{12}I^{1,0} - p_{10}I^{1,2}}{p_{10} + p_{12}} \\ &= \tilde{U}^{1,0,\epsilon}(t). \end{aligned}$$

Then $\tilde{U}^{1,0,\epsilon}(t)$ is a martingale whose variance is bounded by $c^\epsilon(t) \rightarrow 0$ as $\epsilon \rightarrow 0$ or else $EU^{0,1,\epsilon}(t)$ will diverge by the way of filling the buffer and hence forcing $P^{0,1}$ to be shut off once or more and as a result the cost will go to infinity. Hence the proof. Q.E.D.

We can rewrite (1.2) as follows.

$$\begin{aligned} X^{1,\epsilon}(t) &= \tilde{A}^{1,\epsilon}(t) - (\tilde{D}^{1,0,\epsilon}(t) + \tilde{D}^{1,2,\epsilon}(t)) + B^{1,\epsilon}(t) \\ &\quad + (Y^{1,0,\epsilon}(t) + Y^{1,2,\epsilon}(t)) - U^{0,1,\epsilon}(t) + U^{1,\epsilon}(t) + \rho^{1,\epsilon}(t) \\ &= W^{1,\epsilon}(t) + B^{1,\epsilon}(t) + (1 - p_{11})Y^{1,\epsilon}(t) - U^{0,1,\epsilon}(t) + U^{1,\epsilon}(t) + \rho^{1,\epsilon}(t) \quad (2.7.a) \end{aligned}$$

$$\begin{aligned}
X^{2,\epsilon}(t) &= \tilde{A}^{2,\epsilon}(t) - \tilde{D}^{2,0,\epsilon}(t) + \tilde{D}^{1,2,\epsilon}(t) + B^{2,\epsilon}(t) + Y^{2,0,\epsilon}(t) \\
&\quad - Y_c^{1,2,\epsilon}(t) - U^{0,2,\epsilon}(t) - U_c^{1,2,\epsilon}(t) + \rho^{2,\epsilon}(t) \\
&= W^{2,\epsilon}(t) + B^{2,\epsilon}(t) + (1 - p_{22})Y^{2,\epsilon}(t) - p_{12}Y^{1,\epsilon}(t) \\
&\quad - U^{0,2,\epsilon}(t) - U_c^{1,2,\epsilon}(t) + \hat{\rho}^{2,\epsilon}(t)
\end{aligned} \tag{2.7.b}$$

where $\rho^{i,\epsilon}(\cdot)$ and $\hat{\rho}^{i,\epsilon}$ are "small error" processes and

$$\begin{aligned}
V^\epsilon(\pi, x, P) &= E_x^\pi \int_0^\infty e^{-\beta t} k(X^\epsilon(t)) dt \\
&\quad + E_x^\pi \sum_{i=1}^2 \sum_n (k_1^i e^{-\beta \nu_1^{i,\epsilon}(n)} + k_2^i e^{-\beta \nu_2^{i,\epsilon}(n)}) \\
&\quad + E_x^\pi \sum_n c_{1,2} e^{-\beta l^{1,2,\epsilon}(n)} \\
&\quad + E_x^\pi \int_0^\infty e^{-\beta t} [q_{1,2} d(U_c^{1,2,\epsilon}(t) - U^{1,2,\epsilon}(t)) \\
&\quad + \sum_{i=1}^2 q_{0,i} dU^{0,i,\epsilon}(t)] + \hat{\rho}^{3,\epsilon}(t)
\end{aligned} \tag{2.8}$$

We will show later that for any sequence of controls π^ϵ with $\sup_{t \leq T} V^\epsilon(\pi^\epsilon, x, P) < \infty$, $\sup |\rho^{i,\epsilon}(t)| \rightarrow 0$ in distribution for any $T < \infty$ and so is $\sup_{t \leq T} |\hat{\rho}^{i,\epsilon}(t)| \rightarrow 0$.

By impulsive nature of "control" part of the cost (2.8), on any bounded time interval, there are only a finite number (w.p.1) of subintervals on which the controls are active. By definitions, the reflection terms $Y^{i,j,\epsilon}(\cdot)$ cannot increase on these control intervals. The next result tells us that we can obtain the "reflection" terms as continuous functions of the "non-control" data. The main reason is that $Y^{1,\epsilon}(\cdot)$ can only increase when both $P^{0,1}$ and P^1 are working. Also, $Y^{2,\epsilon}(\cdot)$ can increase only when all of P^1 , $C^{1,2}$ and $P^{0,2}$ are on. Hence, one can use the set up of Reiman [14].

Lemma 2.2. There is a unique continuous function $F(\cdot)$ such that

$$\begin{aligned}
&(Y^{1,0,\epsilon}(\cdot), Y^{1,2,\epsilon}(\cdot), Y^{2,0,\epsilon}(\cdot)) \\
&= F(W^\epsilon(\cdot), B^\epsilon(\cdot), \rho^\epsilon(\cdot), X_0^\epsilon, X^{i,\epsilon}(\nu_n^{i,\epsilon}), \nu_n^{i,\epsilon}, \tilde{\nu}_n^{i,\epsilon}, i = 1, 2, n < \infty).
\end{aligned} \tag{2.9}$$

Also, $F(\cdot)$ is "non-anticipative" and the corresponding $X^\epsilon(\cdot)$ is non-anticipative and left side of (2.9) can increase only at those times where $X^{i,\epsilon}(\cdot)$ are zero.

Proof: Let $J_n^{1,\epsilon} = [\nu_n^{1,\epsilon}, \tilde{\nu}_n^{1,\epsilon}]$ denote the successive intervals of interpolated time such that $P_1^{1,\epsilon}(k) \wedge P_2^{1,\epsilon}(k) = P^{0,1,\epsilon}(k) = 1$ for $\epsilon k \in J_n^{1,\epsilon}$ and let $J_n^{2,\epsilon} = [\nu_n^{2,\epsilon}, \tilde{\nu}_n^{2,\epsilon}]$ denote the successive intervals such that $P_1^{2,\epsilon}(k) \wedge P_2^{2,\epsilon}(k) = P^{0,2,\epsilon}(k) = C^{1,2,\epsilon}(k) = 1$ for $\epsilon k \in J_n^{2,\epsilon}$. Then $Y^{i,\epsilon}(\cdot)$ can increase only on $J_n^{i,\epsilon}$. For any function $f(\cdot)$ define

$$f_{i,n}(\cdot) = f(\nu_n^{i,\epsilon} + \cdot) \cap \tilde{\nu}_n^{i,\epsilon} - f(\nu_n^{i,\epsilon}).$$

For the pieces between the control intervals by [14, Lemma 1], there is a unique continuous function $\tilde{F}(\cdot) = (\tilde{F}^1(\cdot), \tilde{F}^2(\cdot))$ (the centering in the arguments which are functions is taken to be the continuity in the topology of uniform convergence on bounded time intervals) such that

$$Y_1^{1,0,\epsilon}(n) + Y_1^{1,2,\epsilon}(n) = \tilde{F}^1(X^{1,\epsilon}(\nu_n^{1,\epsilon}), W_{1,n}^{1,\epsilon}(\cdot), B_{1,n}^{1,\epsilon}(\cdot), \rho_{1,n}^{1,\epsilon}(\cdot)). \quad (2.10)$$

$$Y_2^{2,0,\epsilon}(n) = \tilde{F}^2(X^{2,\epsilon}(\nu_n^{2,\epsilon}), W_{2,n}^{2,\epsilon}(\cdot), B_{2,n}^{2,\epsilon}(\cdot), Y_{2,n}^{1,0,\epsilon}(\cdot) + Y_{2,n}^{1,2,\epsilon}(\cdot), \rho_{2,n}^{2,\epsilon}(\cdot)).$$

Also, $\tilde{F}(\cdot)$ is non-anticipative. That is there is a unique continuous function $F(\cdot)$ such that

$$\begin{aligned} & (Y^{1,0,\epsilon}(\cdot) + Y^{1,2,\epsilon}(\cdot) + Y^{2,0,\epsilon}(\cdot)) \\ & = F(W^\epsilon(\cdot), B^\epsilon(\cdot), \rho^\epsilon(\cdot), X_0^\epsilon, X^{i,\epsilon}(\nu_n^{i,\epsilon}), \nu_n^{i,\epsilon}, \tilde{\nu}_n^{i,\epsilon}, i = 1, 2, n < \infty) \end{aligned}$$

where $F(\cdot)$ has all the properties of $\tilde{F}(\cdot)$ and hence the conclusion of the lemma. Q.E.D.

By Lemma 2.2, we need not concern with the weak convergence of the arguments of $F(\cdot)$, see Kushner [6]. Formally, let the arguments of $F(\cdot)$ converge to $W(\cdot)$, $B(\cdot)$, ν_n^i , $\tilde{\nu}_n^i$ and let $Y^i(\cdot)$ be the limit of $Y^{i,\epsilon}(\cdot)$. Then on each bounded interval, the complement of $\{(\nu_n^i, \tilde{\nu}_n^i), n < \infty\}$ will just be a finite set of points, and the controls will be impulses acting at these points. The magnitude of the impulses will be depending on one or two servers at a station is shut off.

Using assumed convergence, we will have

$$X^1(t) = X^1(0) + W^1(t) + B^1(t) + (1 - p_{11})Y^1(t) - U^{0,1}(t) + U^1(t) \quad (2.11.a)$$

$$\begin{aligned} X^2(t) &= X^2(0) + W^2(t) + B^2(t) + (1 - p_{22})Y^2(t) \\ &\quad - p_{12}Y^1(t) - U^{0,2}(t) - U_c^{1,2}(t) \end{aligned} \quad (2.11.b)$$

We will now specify each term of (2.11). Using (2.7) in (2.11), the limit $W^i(\cdot)$ can be decomposed as

$$W^1(\cdot) = \tilde{A}^1(\cdot) + W_d^1(\cdot), \quad W_d^1(\cdot) = -\tilde{D}^{1,0}(\cdot) - \tilde{D}^{1,2}(\cdot)$$

$$W^2(\cdot) = \tilde{A}^2(\cdot) + W_d^2(\cdot), \quad W_d^2(\cdot) = -\tilde{D}^{2,0}(\cdot) + \tilde{D}^{1,2}(\cdot).$$

All the terms here are continuous martingales with

$$\tilde{A}^1(\cdot), \tilde{A}^2(\cdot), \tilde{D}^{2,0}(\cdot), \text{ and } (\tilde{D}^{1,0}(\cdot), \tilde{D}^{1,2}(\cdot))$$

being mutually orthogonal. The quadratic variation of $\tilde{A}^i(\cdot)$ is $\int_0^t g_{a_i}^3 \sigma_{a_i}^2(X(s)) ds$ and that $W_d((\cdot) = (W_d^1(\cdot), W_d^2(\cdot))$ is $\Sigma(t) = \{\Sigma_{ij}(t)\}$ where

$$\Sigma_{11}(t) = g_{d,1}[p_{11}(1 - p_{11})t + g_{d,1}^2(1 - p_{11})^2 \int_0^t (\sigma_{d,1}^2(X(s)) ds]$$

$$\begin{aligned}
\Sigma_{12}(t) &= -g_{d,1}^3 p_{12}(1 - p_{11}) \int_0^t (\sigma_{d,1}^2(X(s)) ds - p_{12} p_{11} g_{d,1} t \\
\Sigma_{22}(t) &= g_{d,2} [p_{20}(1 - p_{20})t + p_{20}^2 g_{d,2}^2 \int_0^t (\sigma_{d,2}(X(s)) ds] \\
&\quad + g_{d,1} [p_{12}(1 - p_{12})t + p_{12}^2 g_{d,1}^2 \int_0^t \sigma_{d,1}^2(X(s)) ds]
\end{aligned} \tag{2.12}$$

where $g_{d,i}$ are given in (A2) and $\sigma_{d,i}^2 = \sigma_{d_1^i}^2 + \sigma_{d_2^i}^2$.

From (2.12) and the orthogonality properties, there are mutually independent Wiener processes $W_a^i(\cdot)$, $W_d^i(\cdot)$, $W_d^{2,0}(\cdot)$, $\{W_d^{1,1}(\cdot), W_d^{1,2}(\cdot)\}$, where each scalar valued process is standard and with respect to which $X(\cdot)$ is non-anticipative and

$$E[W_d^{1,1}(\cdot)W_d^{1,2}(t)] = -\left(\frac{p_{12}p_{11}}{(1-p_{11})(1-p_{22})}\right)^{1/2} t$$

and

$$\begin{aligned}
\tilde{A}^i(t) &= g_{a,i}^{\frac{3}{2}} \int_0^t \sigma_{a,i}(X(s)) dW_a^i(s) \\
W_d^1(t) &+ [g_{d,1} p_{11}(1 - p_{11})]^{1/2} W_d^{1,1}(t) + (1 - p_{11}) g_{d,1}^{3/2} \int_0^t \sigma_{a,1}(W(s)) dW_d^1(s) \\
W_d^2(t) &= [g_{d,2} p_{20}(1 - p_{20})]^{1/2} W_d^{2,0}(t) + [g_{d,1} p_{12}(1 - p_{12})]^{1/2} W_d^{1,2}(t) \\
&\quad + p_{2,0} g_{d,2}^{3/2} \int_0^t \sigma_{d,2}(X(s)) dW_d^2(s) - p_{12} g_{d,1}^{3/2} \int_0^t \sigma_{d,1}(X(s)) dW_d^1(s).
\end{aligned} \tag{2.13}$$

The terms involving $W_d^{i,j}(\cdot)$ are due to the variations in routing, where as the terms involving $W_d^2(\cdot)$ are due to variations in service times. The drift terms are

$$B^1(t) = \int_0^t [a^1(X(s)) - (1 - p_{22})d_1(X(s))] ds$$

$$B^2(t) = \int_0^t [a^2(X(s)) - (1 - p_{22})(d_2(X(s))p_{12}(d_1(X(s))))] ds.$$

Then the limit problem is defined by (2.11).

Given $W(\cdot)$, $B(\cdot)$, $U(\cdot)$, there are unique processes $X(\cdot)$ and $Y(\cdot)$ such that $Y^i(\cdot)$ increase only when $X^i(t) = 0$ and when $X^i(t) \geq 0$, (2.11) holds, as in Reiman [14]. The main difference here is that $B(\cdot)$ and $W(\cdot)$ might depend on $X(\cdot)$, so it is not a priori that (2.13) has a unique solution. If $\sigma_{a,i}$, $\sigma_{d,i,j}$, B^i do not depend on X^i , then we have

uniqueness of the solution to (2.11) for each admissible control policy. For an admissible policy the limitcost function is

$$\begin{aligned} V(\pi, x, P) &= E_x^\pi \int_0^\infty e^{-\beta t} k(X(t)) dt \\ &\quad + E_x^\pi \sum_{i=1}^2 \sum_n (k_1^i e^{-\beta \nu_1^i(n)} + k_2^i e^{-\beta \nu_2^i(n)}) + E_x^\pi \sum_n c_{1,2} e^{-\beta t^{1,2}(n)} \\ &\quad + E_x^\pi \int_0^\infty e^{-\beta t} \left[\sum_{i=1}^2 q_{0,1} dU^{0,i}(t) + q_{1,2} d(U_c^{1,2}(t) - U^{1,2}(t)) \right] \end{aligned}$$

3. Weak convergence

In this section, we state and prove the result on weak convergence of the queue length processes and all of its components.

Theorem 3.1: Assume (A1)-(A7), and let

$$\sup_\epsilon V^\epsilon(\pi^\epsilon, X_0^\epsilon) < \infty \text{ for } \pi^\epsilon = \{R_n^\epsilon, \tau_n^\epsilon, \delta U_n^\epsilon, n < \infty\} \text{ admissible.}$$

Then

$$\{\tilde{A}^{1,\epsilon}(\cdot), \tilde{A}^{2,\epsilon}(\cdot), (\tilde{D}^{1,0,\epsilon}(\cdot), \tilde{D}^{1,2,\epsilon}(\cdot)), \tilde{D}^{2,0,\epsilon}(\cdot)\}$$

is tight in $D^5[0, \infty)$ (with Skorohod topology), and the limit of any weakly convergent subsequence of the four sets are orthogonal continuous martingales. On each $[0, t]$, the mean number of control actions is finite and the set of intervals on which some control is active converges to a finite set of points. The pieces of $X^\epsilon(\cdot)$ on the intervals where no controls are active are tight, and the weak limits of these ‘pieces’ are continuous. Let ϵ index a weakly convergent subsequence of

$$R^\epsilon = \{X_0^\epsilon, \tilde{A}^{i,\epsilon}(\cdot), \tilde{D}^{i,j,\epsilon}(\cdot), B^\epsilon(\cdot), R_n^\epsilon, \tau_n^\epsilon, \delta U_n^\epsilon, i, j, n\}.$$

Define the process $R(\cdot)$ from the limit processes by

$$R(t) = (X_0, \tilde{A}^i(t), \tilde{D}^{i,j}(t), B(t), i, j, (R_n, \tau_n, \delta U_n), I_{\{\tau_n \leq t\}}, n < \infty).$$

Then $\tilde{A}^i(\cdot)$ and $\tilde{D}^{i,j}(\cdot)$ are martingales on the filtration generated by $R(t)$, with the quadratic variations given by (2.12). The limit policy $\pi = \{R_n, \tau_n, \delta U_n\}$ is admissible for $X(\cdot)$. Except at points where there is control actions, (2.11) holds. We define $X(t)$ by (2.11) even at the points of control actions.

Proof: We will show the tightness of $\{W^\epsilon(\cdot)\}$ and that its limits are continuous. Thus, with the representation in Lemma 2.2 implies that for any weakly convergent subsequence, the $Y^{i,\epsilon}(\cdot)$ converges in the Skorohod topology to a continuous process. We also have $\epsilon S_{\alpha, t/\epsilon}^{i,\epsilon}$ and $\tilde{S}_{\alpha}^{i,\epsilon}(t)$ converging weakly to the processes $S_{\alpha}^i(\cdot)$ and $\tilde{S}_{\alpha}^i(\cdot)$ with values $\frac{t}{g_{\alpha, i}}$

and $g_{\alpha,i}t$ respectively. $\epsilon \sum_1^{t/\epsilon} [\alpha_n^{i,\epsilon} - \bar{\alpha}_n^{i,\epsilon}]$ has orthogonal increments and its variance tends to zero as $\epsilon \rightarrow 0$, the increments of each $\tilde{A}_0^{i,\epsilon}(\cdot)$ and $\tilde{D}_0^{i,j,\epsilon}(\cdot)$ are also orthogonal. Due to the uniform integrability in (A3), those processes are tight and all weak limits are continuous martingales. The asserted mutual orthogonality follows by a similar proof as in [9]. Hence, we have all the assertion of theorem except the non-anticipativity and the quadratic variation values.

Owing to the mutual orthogonality of the four processes $\tilde{A}^{i,\epsilon}(\cdot)$ etc., and to (A3), we can calculate for the limit process, the quadratic variation and prove martingale property w.r.t. the σ -algebra generated by $R(\cdot)$ separately for each component. We do it only for $(\tilde{D}^{1,0,\epsilon}(\cdot), \tilde{D}^{1,2,\epsilon}(\cdot))$. Let ϵ index a weakly convergent subsequence of R^ϵ and define $R(\cdot)$ as in the theorem statement. Consider two real valued functions $h(\cdot)$, continuous and bounded and $f(\cdot)$ smooth with compact support. Let $t, t+s$ and $t_k \leq t$ be the points such that the probability $P\{\tau_n = t, t+s \text{ or } t_k\} = 0$, for each n, k . Define

$$\delta\psi^{i,j,\epsilon}(n) = I^{i,j,\epsilon}(n) - p_{ij} \left(\frac{\Delta_1^{i,\epsilon}(n) + \Delta_2^{i,\epsilon}(n)}{\bar{\Delta}_1^{i,\epsilon}(n) + \bar{\Delta}_2^{i,\epsilon}(n)} \right)$$

Using a truncated Taylor series expansion and uniform integrability,

$$\begin{aligned} & Eh(X^\epsilon(t_k), \tilde{A}^{i,\epsilon}(t_k), \tilde{D}^{i,j,\epsilon}(t_k), B^{i,\epsilon}(t_k), (R_n^\epsilon, \tau_n^\epsilon, \delta\bar{U}_n^\epsilon)I_{\{\tau_n^\epsilon \leq t\}}, k, n) \\ & \times [f(\tilde{D}^{1,0,\epsilon}(t+s), \tilde{D}^{1,2,\epsilon}(t+s)) - f(\tilde{D}^{1,0,\epsilon}(t), \tilde{D}^{1,2,\epsilon}(t))] \\ & - \sqrt{\epsilon} \sum_{\alpha=0,2} \sum_{\epsilon n = \bar{S}_d^{1,\epsilon}(t)}^{\bar{S}_d^{1,\epsilon}(t+s)} f_{x_\alpha}(\sqrt{\epsilon} \sum_1^{n-1} \delta\psi^{1,0}(k), \sqrt{\epsilon} \sum_1^{n-1} \delta\psi^{1,2}(k)) \delta\psi^{1,\alpha}(k) \\ & - \frac{1}{2} \epsilon \sum_{\alpha=0,2} \sum_{\epsilon n = \bar{S}_d^{1,\epsilon}(t)}^{\bar{S}_d^{1,\epsilon}(t+s)} f_{x_\alpha x_\beta}(\sqrt{\epsilon} \sum_1^{n-1} \delta\psi^{1,2}(k)) \delta\psi^{1,\alpha}(k) \delta\psi^{1,\alpha}(n) \delta\psi^{1,\beta}(n) \xrightarrow{\epsilon} 0. \end{aligned} \quad (3.1)$$

Now using the centering of $\delta\psi^{1,\alpha}(k)$ and (A6), we can replace $\delta\psi^{1,\alpha}$ in (3.1) by zero and $\delta\psi^{1,\alpha}(n)\delta\psi^{1,\beta}(n)$ by $E_{d,n-1}^{1,\epsilon} \delta\psi^{1,\alpha}(n)\delta\psi^{1,\beta}(n)$. This latter quantity is

$$\begin{aligned} & E_{d,k-1}^{1,\epsilon} [I^{1,\alpha}(k) - p_{12} \frac{\Delta_1^1(k) + \Delta_2^1(k)}{\bar{\Delta}_1^1(k) + \bar{\Delta}_2^1(k)}] \\ & = p_{1,\alpha} \delta_{\alpha\beta} - 2p_{1\alpha} p_{1\beta} + p_{1\alpha} p_{1\beta} var \frac{(\Delta_1^1(k) + \Delta_2^1(k))}{(\bar{\Delta}_1^1(k) + \bar{\Delta}_2^1(k))^2} \\ & = p_{1,\alpha} \delta_{\alpha\beta} - 2p_{1\alpha} p_{1\beta} + p_{1\alpha} p_{1\beta} g_{d,1}^2 \sigma_{d,1}^2 (X^\epsilon(S_d^{1,\epsilon}(n))) + \text{negligible terms} \end{aligned} \quad (3.2)$$

The limit (as $\epsilon \rightarrow 0$) of the double sum in (3.1) is

$$\frac{1}{2} \sum_{\alpha,\beta=0,2} \int_{\bar{S}_d^{1,\epsilon}(t)}^{\bar{S}_d^{1,\epsilon}(t+s)} f_{x_\alpha x_\beta}(\tilde{D}^{1,0}(\tau), \tilde{D}^{1,2}(\tau)) \hat{\Sigma}_{\alpha\beta}(\tau) d\tau$$

where

$$\begin{aligned}
\hat{\Sigma}_{0,0} &= p_{10} - 2p_{10}^2 + p_{10}g_{d,1}^2\sigma_{d,1}^2(X(\frac{t}{g_{d,1}})) \\
\hat{\Sigma}_{0,2}(t) &= -2p_{10}p_{12} + p_{10}p_{12}g_{d,1}^2\sigma_{d,1}^2(X(\frac{t}{g_{d,1}})) \\
\hat{\Sigma}_{22}(t) &= p_{22} - 2p_{12}^2 + p_{12}^2g_{d,1}^2\sigma_{d,1}^2(X(\frac{t}{g_{d,1}})).
\end{aligned} \tag{3.3}$$

Noticing that $\bar{S}_d^i(t) = g_{d,i}t$, and taking limit in (3.1) yields

$$\begin{aligned}
& Eh(X(t_k), \tilde{A}^i(t_k), \tilde{D}^{i,j}(t_k), B^i(t_k), (R_n, \tau_n, \delta U_n)I_{\{\tau_n \leq t_k\}}, n, k) \\
& [f(\tilde{D}^{1,0}(t+s), \tilde{D}^{1,2}(t+s)) - f(\tilde{D}^{1,0}(t), \tilde{D}^{1,2}(t)) \\
& - \frac{1}{2} \sum_{\alpha\beta=0,2} \int_{tg_{d,i}}^{(s+t)g_{d,i}} f_{x_{\alpha}x_{\beta}}(\tilde{D}^{1,0}(\tau), \tilde{D}^{1,2}(\tau)) \hat{\Sigma}_{\alpha\beta}(\tau) d\tau] = 0.
\end{aligned} \tag{3.4}$$

The arbitrariness of $h(\cdot)$, $f(\cdot)$ and $t, t+s, \{t_j\}$ imply that $(\tilde{D}^{1,0}(\cdot), \tilde{D}^{1,2}(\cdot))$ is a martingale w.r.t. the asserted filtration.

The quadratic variation can be obtained from (3.4) via a change of variables and is $\int_0^t \bar{\Sigma}(\tau) d\tau$, where $\bar{\Sigma}(\cdot) = \{\bar{\Sigma}_{\alpha\beta}(\cdot), \alpha, \beta = 0, 2\}$ and

$$\begin{aligned}
\Sigma_{0,0}(t) &= g_{d,1}[p_{10} - 2p_{10}^2 + p_{10}^2g_{d,1}^2\sigma_{d,1}^2(X(t))] \\
\Sigma_{0,2}(t) &= g_{d,1}[-2p_{10}p_{12} + p_{10}p_{12}g_{d,1}^2\sigma_{d,1}^2(X(t))] \\
\Sigma_{2,2}(t) &= g_{d,1}[p_{22} - 2p_{12}^2 + p_{12}^2g_{d,1}^2\sigma_{d,1}^2(X(t))].
\end{aligned}$$

With a similar arguments for $\tilde{D}^{2,0,\epsilon}(\cdot)$, and for the $\tilde{A}^{i,\epsilon}(\cdot)$, we get quadratic variations for $W_{\alpha}^i(\cdot)$, $\tilde{A}^i(\cdot)$, $\tilde{D}^{i,j}(\cdot)$ as given in (2.12).

The limit policy $\{\tau_n, R_n, \delta U_n\}$ is admissible in the sense that it corresponds to admissible sequences of off/on controls. By the above argument, the limit policy $\{\tau_n, R_n, \delta U_n\}$ is ‘non-anticipative’ w.r.t. the martingales or their generating Wiener Processes. Q.E.D.

4. Comparison and limit controls

We define a ‘comparison’ control policy called the Δ_0 -boundary policy. Let $\Delta_0 \in (0, \min(B_1, B_2)/4)$. If $X^{2,\epsilon} = B_2$ then shut off all the inputs to P_2 until $X^{2,\epsilon}$ reaches $B_2 - \Delta_0$. Then turn them back on. If at the end of that time $B_1 - \Delta_0 < X^{1,\epsilon} < B_1$, shut $P_{0,1}$ off until $X^{1,\epsilon} = B_1 - \Delta_0$. If $X^{1,\epsilon} = B_1$ then shut $P_{0,1}$ off until $X^{1,\epsilon}$ reaches $B_1 - \Delta_0$. Then turn $P_{0,1}$ back on. In general, if ever $X^{\epsilon}(\cdot)$ or $X(\cdot)$ hits the outer boundary, we control it to a distance at least Δ_0 (in each coordinate) from the outer boundary.

Theorem 4.1: Assume (A2) to (A6). Then for the Δ_0 -boundary control and for each $k < \infty$,

$$\sup_{\epsilon \text{ small}, \alpha, x, n} E_x |N^{\alpha,\epsilon}(N+1) - N^{\alpha,\epsilon}(n)|^k < \infty, \text{ all } \alpha \tag{4.1}$$

and similarly for the ‘jump numbers’ of the limit process $X(\cdot)$.

For a proof, we refer to [13] in similar situation.

The Theorem 4.1 tells us that there is at least one control policy for which $\{N^{\alpha,\epsilon}(n+1) - N^{\alpha,\epsilon}(n), \alpha, n < \infty\}$ is uniformly integrable; which is necessary for the convergence of cost functionals.

Theorem 4.2 Assume (A1)-(A7), and let ϵ index a weakly convergent subsequence with limit $R(\cdot)$. Then for any P

$$\liminf_{\epsilon} V^{\epsilon}(\pi^{\epsilon}, x, P) \geq V(\pi, x, P). \quad (4.2)$$

Let $N^{\alpha,\epsilon}(t)$ be the number of actions of the control P_{α} on the interval $[0, t]$. If $\{N^{\alpha,\epsilon}(n+1) - N^{\alpha,\epsilon}(n), \alpha, n < \infty\}$ is uniformly integrable, then

$$V^{\epsilon}(\pi^{\epsilon}, x, P) \rightarrow V(\pi, x, P). \quad (4.3)$$

Proof: (4.2) follows from Fatou’s Lemma and the weak convergence. The proof of (4.3) is essentially same as in [9, Theorem 5.2].

(A8) For each $\epsilon > 0$, there is a δ -optimal admissible policy π_{δ} for $X(\cdot)$ and admissible policies π_{δ}^{ϵ} for $X^{\epsilon}(\cdot)$ such that $X^{\epsilon}(\cdot)$ (under π_{δ}) $\Rightarrow$ $X(\cdot)$ (under π_{δ}), and the associated costs converge.

We would like to remark here that the assumption is necessitated by lack of results on optimal or δ optimal policies even for the limit control problem. Also, using the uniform integrability of comparison controls, we can show that any δ -optimal control for $X^{\epsilon}(\cdot)$ the convergence of costs follow.

Next theorem says that a ‘nice’ control which is almost optimal for $X(\cdot)$ will also be almost optimal for $X^{\epsilon}(\cdot)$. This essentially justifies the use of the limit approximations for the purposes of getting good or nearly optimal controls.

Theorem 4.3: Assume (A2)-(A8). Let π_{δ}^{ϵ} be the policy in (A8) adapted to $X^{\epsilon}(\cdot)$. Then

$$V^{\epsilon}(\pi_{\delta}^{\epsilon}, x, P) \rightarrow V(\pi_{\delta}, x, P). \quad (4.4)$$

uniformly in x . For admissible π^{ϵ} and small ϵ ,

$$\sup_{\{\pi^{\epsilon}\}} \sup_x [V^{\epsilon}(\pi_{\delta}^{\epsilon}, x, P) - V^{\epsilon}(\pi^{\epsilon}, x, P)] \leq 2\delta. \quad (4.5)$$

Note: (4.5) tells us that whatever be the optimal policy for the real queueing process, when we adapt the optimal policy for the limit process, the difference of the value functions is very small. This result justifies the use of optimal policy for the limit processes to the real system.

Proof. When π_{δ}^{ϵ} -control policy is used, we have the uniform integrability of $\{N^{\delta,\epsilon}(n+1) - N^{\delta,\epsilon}(n), \delta, n < \infty\}$. Where $N^{\delta,\epsilon}(t)$ is the number of actions of the control π_{δ}^{ϵ} . Because

there is a $M_\delta < \infty$, such that if we switch to Δ_0 -boundary control when the number of actions exceeds M_δ , then for this policy we have uniform integrability by Theorem 4.1, and the value function still remains finite. Thus (4.4) follows by Theorem 4.2. Let $\bar{\pi}^\epsilon$ be the control policy for which $V^\epsilon(\pi^\epsilon, x, P)$ attains its supremum in the class of admissible controls for $X^\epsilon(\cdot)$. Then by the similar arguments as above, we get the uniform integrability and hence the weak convergence. By the definition of δ -optimality,

$$\begin{aligned} \delta + \liminf_{\epsilon} \sup_{\pi^\epsilon} \sup_x V^\epsilon(\pi^\epsilon, x, P) &\geq \liminf_{\epsilon} \sup_x V^\epsilon(\bar{\pi}^\epsilon, x, P) \\ &\geq \sup_{\pi} \sup_x V(\pi, x, P) \geq V(x, P) - \delta. \end{aligned}$$

Hence (4.5). Q.E.D.

5. Conclusion

In this work, we have considered control of tandem queueing network with heterogeneous servers. Control of a general feedforward network with heterogeneous servers can be analyzed by the method of this work. For simple situation, one can use dynamic programming arguments to numerically obtain the optimal control for the limit process [8], [13]. However, due to the dimensionality problem of dynamic programming, one wants to develop some more efficient algorithms for the computational purposes. It is interesting to obtain similar results for several customer classes with heterogeneous servers. Here, the control policy has to be suitably modified.

Acknowledgement. The author wants to thank Prof. George Yin for his valuable help in preparation of this paper. Part of this work was completed when the author was visiting the IMA, at the University of Minnesota.

References

- [1] C.A. Courcoubetis and M.I. Reiman, Optimal control of queueing systems with simultaneous service requirements, IEEE Tran. on Automatic Control, AC-32(1987).
- [2] T. Crabill, D. Gross and M.J. Magazine, A classified bibliography of research on optimal design and control of queues, Operat. Res., vol. 25, 1977.
- [3] M. El-Arabaty, Mathematical modelling for large systems with applications to service systems, Proc. 24th CDC, 1985.
- [4] J.M. Harrison, Brownian models of queueing networks with heterogeneous customer population, Proc. Stochastic Differential Equations, Springer-Verlag, 1987.
- [5] P.R. Kumar and J. Warland, Individually optimal routing in parallel systems, J. Appl. Prob., vol. 22, 1985.
- [6] H.J. Kushner, *Probability Methods for Approximation in Stochastic Control and Elliptic Equations*, Academic Press, New York, 1977.
- [7] H.J. Kushner, *Approximation and Weak Convergence Methods for Random Processes with Applications to Stochastic Systems Theory*, MIT Press, Cambridge, MA. 1984.
- [8] H.J. Kushner and A.J. Kleinman, Accelerated procedures for the solution of discrete markov control problem, IEEE Trans. on Automatic Control, AC-16, 1971.

- [9] H.J. Kushner and K.M. Ramachandran, Optimal and approximately optimal control policies for queues in heavy traffic, LCDS #87-24, Brown Univ., 1987, also submitted to SIAM J. on Control and Optimization.
- [10] H.J. Kushner and K.M. Ramachandran, Nearly optimal singular controls for wideband noise driven systems, SIAM J. on Control and Optimization, vol. 26, 1988.
- [11] A.J. Lemoine, Networks of queues-A survey of weak convergence results, Management Sciences, vol. 24, 1978.
- [12] W. Lin and P.R. Kumar, Optimal control of queueing systems with heterogeneous servers, IEEE Trans. on Automatic Control, AC-29, 1984.
- [13] K.M. Ramachandran, Nearly optimal controls for wideband noise driven systems and queueing processes, Ph.D dissertation, Division of Applied Mathematics, Brown University, July, 1987.
- [14] M.I. Reiman, Open queueing networks in heavy traffic, Math. Operations Research, vol. 9, 1984.
- [15] S. Stidham and Prabhu, Optimal control of queueing systems, *Mathematical Methods in Queueing Theory*, Lecture notes in Economics and Math. Systems, Springer-Verlag, vol. 98, 1974.