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LOEB ECONOMIES

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## ABSTRACT

We examine a class of perfectly competitive economies called Loeb economies, i.e., economies defined on a measure space of agents where the cardinality of the underlying set of agents may be either a finite integer or an "infinite non-standard number". The particular attraction of Loeb economies is that in addition to providing a class of models consistent with both the classical Arrow-Debreu-McKenzie model and the atomless measure theoretic model of Aumann, they allow one to exploit results for finite economies in obtaining analogous results for the atomless models. In particular, we will demonstrate how some of the major results for the atomless measure theoretic model can be obtained as corollaries of results proved in a fixed finite economy framework.

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## INTRODUCTION

An exchange economy consists of a set of economic agents, each of whom is characterized by an endowment of goods and a "preference ordering" over all possible combinations of goods, who engage in the exchange of commodities so as to make themselves as well off as possible. This process of exchange is said to take place under perfect competition if, and only if, no agent is able to influence the terms of trade in other agents' transactions. If the terms of trade are expressed via prices for the commodities, and if agents' actions result in "market clearing", i.e., in equality between the aggregate demand and aggregate supply for each commodity, then a competitive equilibrium is said to exist.

The classical model of exchange under perfect competition (see, e.g., McKenzie (1954) and Arrow-Debreu (1954)) was formulated in terms of a finite set of rational agents taking prices as given and engaging in the sale and purchase of commodities. Such a model is clearly at odds with itself as the finitude of agents means that each agent is able to exercise some influence on the prices at which goods are bought and sold. To resolve this contradiction, Aumann (1964) modeled a perfectly competitive exchange economy as a map on an atomless measure space of agents to their characteristics. Within such a framework, the influence of an individual agent is seen to be "negligible" as every agent is of measure zero. An alternative resolution of the problem was given by Brown and Robinson (1975), who introduced the notion of a nonstandard exchange economy. A nonstandard exchange economy is similar to the classical model of exchange under perfect competition, but the cardinality of the collection of economic agents is taken to be an infinite nonstandard number. This has the effect of rendering each agent's commodity endowment an "infinitesimal" part of the market and thus renders his/her influence on price formation negligible.

The present paper examines a class of measure theoretic exchange economies that is closely related to the nonstandard economies of Brown and Robinson (1975). Specifically, we examine the class of Loeb economies, i.e., economies defined on a measure space of agents where the cardinality of the underlying set of agents may be either a finite integer or an "infinite nonstandard integer". The earliest reference to such economies is Rashid (1979). The class of Loeb economies is broad enough to contain both the measure theoretic versions of the classical models and a class of atomless measure theoretic economies.

The particular attraction of Loeb economies is that in addition to providing a class of models consistent with both the classical and atomless measure theoretic models, they allow one to exploit results for finite economies in obtaining analogous results for the atomless models. The mathematical muscle in this task is the work of Loeb (1975) and Anderson (1976, 1982). Recognition of the usefulness of this work for economic analysis may be attributed to Rashid (1979)<sup>0/</sup>; the ideas underlying this work were crystallized in Emmons (1984) and were also employed by Yannelis (1983, 1984). To demonstrate the usefulness of this approach, we will show that versions of Aumann's (1964, 1966) theorems on core equivalence and the existence of a competitive equilibrium, and Robert's (1972) theorem on the existence of Lindahl equilibria, can be derived very simply from results concerning finite economies, viz., Anderson's (1978) core equivalence theorem, the Anderson-Khan-Rashid (1982) approximate equilibrium existence theorem, and a version of the Gale-Mas-Colell (1975) existence result, respectively.

The remainder of the paper may be outlined as follows. Section 0 provides the basic notation employed. Section 1 provides an introduction to nonstandard analysis. For an alternative account of this material, see Loeb (1979); for a more complete presentation, see Robinson (1974) or Stroyan and Luxemburg (1976).

Section 2 presents some of the basic properties of the nonstandard real number system. Section 3 is an introduction to Loeb measure spaces; alternative presentations of this material may be found in Loeb (1975) and Cutland (1983). Section 4 presents the definitions needed from economics. Section 5 contains the statements of the results demonstrated and a discussion of their significance. Section 6 consists of a statement of the basic technique underlying each of the proofs. Finally Section 7 contains the proofs.

## 0. NOTATION

$\mathbb{R}$  denotes the set of real numbers

$\mathbb{R}^{\ell}$  denotes the  $\ell$ -fold Cartesian product of  $\mathbb{R}$

$\mathbb{R}_+^{\ell}$  denotes the positive cone of  $\mathbb{R}^{\ell}$

For any  $x, y$  in  $\mathbb{R}^{\ell}$  (or  ${}^*\mathbb{R}^{\ell}$ )

$$x \geq y \Rightarrow x_i \geq y_i \quad \text{all } i$$

$$x > y \Rightarrow x \geq y \quad \text{and} \quad x \neq y$$

$$x \gg y \Rightarrow x_i > y_i \quad \text{all } i$$

$$x \cdot y = \sum_{i=1}^{\ell} x_i y_i$$

$$\|x\| = \sum_{i=1}^{\ell} |x_i|$$

$$\|x\|_{\infty} = \max_{1 \leq i \leq \ell} |x_i|$$

$u \in \mathbb{R}^{\ell}$  denotes  $u = (1, 1, \dots, 1)$

$\text{int } A$  denotes the interior of the set  $A$

$|S|$  denotes the number of elements in the set  $A$

For any  $x, y$  in  ${}^*\mathbb{R}^{\ell}$

$x \approx y \Rightarrow x$  differs from  $y$  by an infinitesimal amount

$x \succ y \Rightarrow x = y$  or  $x > y$

$x \succcurlyeq y \Rightarrow x$  is greater than  $y$  by a noninfinitesimal amount and not less in any other;

$x \succcurlyeq\!\!\succcurlyeq y \Rightarrow x$  is noninfinitesimally greater than  $y$  in all coordinates.



## 1. INTRODUCTION TO NON-STANDARD ANALYSIS

At the heart of non-standard analysis lies an extension,  ${}^*\mathbb{R}$ , of the ordered field of real numbers. As a first description, the important features of this field are:

- i)  ${}^*\mathbb{R}$  properly contains  $\mathbb{R}$ ;
- ii)  ${}^*\mathbb{R}$  is non-Archimedean, i.e., it contains "infinite" numbers;
- iii) the structure surrounding  ${}^*\mathbb{R}$  is "role preserving", thus, e.g., given  $S \subset \mathbb{R}$  there exists a set  ${}^*S \subset {}^*\mathbb{R}$  which plays a role in  ${}^*\mathbb{R}$  analogous to that played by  $S$  in  $\mathbb{R}$ ; and
- iv) statements that are true regarding  $\mathbb{R}$  and the relations defined with respect to  $\mathbb{R}$  are also true in relationship to  ${}^*\mathbb{R}$  and the relations that are defined with respect to  ${}^*\mathbb{R}$ , provided they are "suitably interpreted".

A better understanding of the above description may be had by introducing the notion of superstructure. Intuitively a superstructure represents a "universe of discourse", i.e., it is a set large enough to contain all the mathematical objects under study in some mathematical inquiry. More formally, let  $S$  be a set. Let  $V_0(S) \equiv S$ , and let  $V_n(S) \equiv V_{n-1}(S) \cup P(V_{n-1}(S))$  for all  $n \in \mathbb{N}$ , where  $P(\ )$  denotes the power set of the set in parentheses. Let  $V(S) \equiv \bigcup_{n=0}^{\infty} V_n(S)$ . Then the superstructure over  $S$  is the set  $V(S)$  together with the notions of equality,  $=$ , and membership,  $\in$ , on the elements of  $V(S)$ . We shall denote the superstructure over  $S$  simply as  $V(S)$ . The elements of  $S$  are called atoms; no  $s$  in  $S$  has members. The elements of  $V(S) - S$  are called entities.

Given a superstructure as defined above, there are a great many mathematical statements that can be formed using the atoms and entities of  $V(S)$ . A

formal language provides the means by which to identify those statements about  $V(S)$  which are true, and those which are not. (Note that we said "the means"; there will, in general, be a great many statements about  $V(S)$  whose truth is not known). Our description of a formal language will be in two parts. We shall first describe the symbols of a formal language,  $L$ , and then we shall describe the process by which sentences are formed. When we have finished with these we shall go on to describe how such a language is interpreted.

The atomic symbols of  $L$  are: (i) The (constants)  $a, b', 0, 1, \dots$ . The set of constants is arbitrary but fixed. It is a set of symbols large enough to be placed into one-to-one correspondence with the elements of whatever structure(s) is(are) under consideration. (ii) The variables  $x, y', \dots$ . The set of variables is countably infinite. (iii) The connectives  $\wedge$  ("and"),  $\vee$  ("or"),  $\rightarrow$  ("implies"),  $\leftrightarrow$  ("if and only if"), and  $\neg$  ("not"). (iv) The quantifiers  $\forall$  (universal) and  $\exists$  (existential). (v) The separation symbols  $[$  and  $]$ . The separation symbols are used to group formulas. (vi) The basic predicates  $\in$  ("is an element of") and  $=$  ("is equal to").

Atomic formulas for  $L$  are constructs obtained by combining  $\in$  or  $=$  (in the usual manner) with two atomic symbols which may be either constants or variables. Thus, e.g.,  $a = b$ ,  $a \in x$ ,  $x \in z$  are all atomic formulas. Well-formed formulas for  $L$  (more briefly, wff) are defined inductively. If  $V$  is an atomic formula, then  $[V]$  is a wff. If  $V$  is a wff then  $\neg[V]$  is a wff. If  $V$  and  $W$  are wff, then  $[V \wedge W]$ ,  $[V \vee W]$ ,  $[V \rightarrow W]$ , and  $[V \leftrightarrow W]$  are wff. If  $V$  is a wff and  $x$  is an arbitrary variable, then  $[(\forall x)V]$  and  $[(\exists x)V]$  are wff provided  $V$  does not already contain one of  $(\forall x)$  or  $(\exists x)$ .

A sentence for  $L$  is a wff,  $V$ , in which every variable  $x$  contained in  $V$  is within the scope of some quantifier, i.e., it is within a wff  $W$  contained in  $V$  which starts with the left bracket immediately following a quan-

tifier and ends with the corresponding right bracket. It can be demonstrated that every sentence for  $L$  has an equivalent statement of the form  $(Q_1 x_1) \dots (Q_n x_n) W$ , where for  $i=1, \dots, n$ ,  $Q_i$  denotes either  $\forall$  or  $\exists$ , and  $W$  is a wff without quantifiers. Accordingly we shall reserve the expression "sentence" for sentences of this form. A sentence is bounded when its quantifiers always appear in the forms:

$[(\forall x)[[x \in A] \rightarrow \dots]]$  for the universal quantifier,

$[(\exists x)[[x \in B] \wedge \dots]]$  for the existential quantifier,

where  $A$  and  $B$  are constants of  $L$ .

Now that we have described the basic elements of a formal language, we turn our attention to providing an interpretation of such a language in set theory. Let  $L$  be a formal language, and let  $V(S)$  denote the superstructure over the nonempty set  $S$ . A one-to-one mapping,  $I$ , of a subset of the set of constants of  $L$  into  $V(S)$  will be called an interpretation map of  $L$  in set theory. The prepositional phrase "in set theory" may now be explained as follows. Each constant in  $\text{dom}(I)$  is interpreted as its image under  $I$ . Atomic formulas  $[\alpha = \beta]$  and  $[\alpha \in \beta]$ , whose constants belong to  $\text{dom}(I)$ , are interpreted as  $I[\alpha = \beta]$ , "the atom or entity  $I(\alpha)$  equals the atom or entity  $I(\beta)$ ", and  $I[\alpha \in \beta]$ , "the atom or entity  $I(\alpha)$  is an element of the atom or entity  $I(\beta)$ ", respectively. Thus the basic predicates  $\in$  and  $=$  are interpreted in the usual set-theoretical way. The process of interpretation may be completed by noting that the connectives for  $L$  are interpreted in the usual fashion;  $\forall = (\forall x)[x \in \alpha \rightarrow W]$  is interpreted as  $I\forall =$  "for all  $x$  elements of  $I(\alpha)$ , the statement  $I_W(x)$ ", where  $I_W(x)$  denotes the portion of the formula already interpreted where occurrences of  $x$  other than as the variable of a quantifier or as a variable within the scope of a quantifier are replaced by the elements of  $I(\alpha)$ ;

$V = (\exists x)[x \in \alpha \wedge W]$  is interpreted as  $I_V =$  "there is an  $x$  in  $I(\alpha)$  such that  $I_W(x)$ ", where  $W(x)$  is as above; arbitrary sentences in  $L$  are interpreted by proceeding "component-wise" according to the rules above.

An interpretation,  $I$ , for the language  $L$  is said to provide a model for a set,  $K$ , of sentences in  $L$  if all of the constants occurring in the sentences of  $K$  are in  $\text{dom}(I)$ , and  $I_V$  is true for each  $V$  in  $K$ . Now let  $V(S)$  be a superstructure over the nonempty set  $S$ , and let  $L$  be a formal language with a set of constants,  $C$ , of greater cardinality than the cardinality of  $V(S)$ .  $I$  is a standard interpretation map if it maps  $C$  onto  $V(S)$ . Let  $K_0$  denote the set of sentences,  $V$ , such that  $I_V$  holds in  $V(S)$ . It is easy to show that the set of all constants occurring in the sentences of  $K_0$  is equal to  $\text{dom}(I)$ , i.e., that  $I$  provides a model for  $K_0$ . Now suppose that  $I'$  maps  $C$  into the superstructure  $V(T)$ , and provides a model  $K_0$ . If  $I' \circ I \equiv *$  is not onto, we say that  $V(*S) \equiv V(T)$  is a non-standard model of  $V(S)$ .

Given  $S$  and  $*S$ , such that  $S \subset *S$  and  $V(*S)$  is a non-standard model of  $V(S)$ , the collection of internal objects of  $V(*S)$  is defined to be  $*V(S) = \{T: T = *B \text{ for some entity } B \text{ in } V(S)\}$ . (As we shall see momentarily,  $*V(S)$  provides the key to the proviso "when suitably interpreted" in our initial description of non-standard analysis). Any object in  $V(*S)$  that is not internal is called external. It is an important fact that if the cardinality of  $S$  is infinite and  $V(*S)$  is a non-standard model for  $V(S)$ , then the set of internal objects,  $*V(S)$ , is properly contained in  $V(*S)$ .

## 2. THE NON-STANDARD REALS

Having defined what a non-standard model is, we now postulate the existence of a non-standard model,  $V(*\mathbb{R})$ , of the reals satisfying the following conditions:

- 1)  $*\mathbb{R} \supset \mathbb{R}$  and  $*\mathbb{R} \neq \mathbb{R}$ ;
- 2)  $*$ :  $V(\mathbb{R}) \rightarrow V(*\mathbb{R})$  satisfies:
  - i)  $*r = r$  for all  $r \in \mathbb{R}$ ,
  - ii) (Transfer Principle) For every  $A_1, \dots, A_n$  in  $V(\mathbb{R})$  and every bounded sentence  $V(x_1, \dots, x_n)$ ,  $V(A_1, \dots, A_n)$  holds in  $V(\mathbb{R})$  if and only if  $V(*A_1, \dots, *A_n)$  holds in  $V(*\mathbb{R})$ ;
- 3) (Denumerably Comprehensive) For every internal set  $A$  and every function  $f: N \rightarrow A$ , there is an internal function  $f^*: *N \rightarrow A$  that extends  $f$ .
- 4) (Enlargement) If  $(A_j)_{j \in J}$  is a collection of entities in  $V(\mathbb{R})$  having the finite intersection property, then  $\bigcap_j *A_j \neq \emptyset$ .<sup>1/</sup>

A couple of comments are in order at this point. First, note that by (2i) we are assuming that  $\mathbb{R}$  is imbedded in  $*\mathbb{R}$ . As  $\mathbb{R}$  is an infinite set, the remarks made at the end of the last section indicate that there exist external sets in this model. Some examples of external sets in  $V(*\mathbb{R})$  are  $*N - N$ , and  $N$ . Secondly, the Transfer Principle provides the technical rationale for our earlier description of  $*\mathbb{R}$  as being "role preserving" and having the property that true statements regarding  $\mathbb{R}$  were also true regarding  $*\mathbb{R}$  when "suitably interpreted". The reference to suitable interpretation stems from the fact that given the true statement  $V(A_1, \dots, A_n)$ , it is the statement  $V$  made with regard to the internal sets  $*A_1, \dots, *A_n$  that is true. The description "role preserving" stems from the fact that when we refer to some property regarding a subset

of  $\mathbb{R}$ , we are able to use the Transfer Principle to obtain a statement referring to the same property in regard to an internal subset of  ${}^*\mathbb{R}$ . Consider, for example, the Archimedean Property of  $\mathbb{N}$  in  $\mathbb{R}$ , which may be stated as 'For all  $r \in \mathbb{R}$ , there exists  $n \in \mathbb{N}$  such that  $n > |r|$ '; however,  ${}^*\mathbb{N}$  plays the "role" of  $\mathbb{N}$  in that by the Transfer Principle the statement 'For all  $r \in {}^*\mathbb{R}$ , there exists  $n \in {}^*\mathbb{N}$  such that  $n > |r|$ '.  ${}^*\mathbb{R} \not\supseteq \mathbb{R}$ , may be shown to be non-Archimedean, i.e., it is false that "For all  $r \in {}^*\mathbb{R}$ , there exists  $n \in \mathbb{N}$  such that  $n > |r|$ "; however,  ${}^*\mathbb{N}$  plays the "role" of  $\mathbb{N}$  in that by the Transfer Principle the statement 'For all  $r \in {}^*\mathbb{R}$ , there exists  $n \in {}^*\mathbb{N}$  such that  $n > |r|$ ' is true.<sup>2/</sup> In the future statements regarding  ${}^*\mathbb{R}$  that are made by appealing to the Transfer Principle will be said to have been obtained by transfer.

From the discussion above, it should be clear that it is extremely desirable to know just what objects in  $V({}^*\mathbb{R})$  are internal and which are external. The following result is often useful in regard to this problem.

Theorem (The Internal Definition Principle): A set  $A$  is internal if and only if it can be described as

$$\{x: x \in B \ \& \ V(x, A_1, \dots, A_n) \text{ holds in } V({}^*\mathbb{R})\},$$

where  $B, A_1, \dots, A_n$  are internal and  $V(x, x_1, \dots, x_n)$  is a bounded sentence.

We conclude this section with some terminology and results that we shall need later on. The element  $r$  in  ${}^*\mathbb{R}$  is said to be finite if there exists  $n \in \mathbb{N}$  such that  $|r| < n$ , otherwise  $r$  is said to be infinite. The element  $r$  in  ${}^*\mathbb{R}$  is said to be infinitesimal if  $|r| < 1/n$  for all  $n \in \mathbb{N}$ . The elements  $r_1$  and  $r_2$  in  ${}^*\mathbb{R}$  are said to be infinitely close if  $r_1 - r_2$  is infinitesimal. In the event that  $r_1$  and  $r_2$  are infinitely close, we shall write  $r_1 = r_2$ .

It can be shown that for every finite  $r$  in  ${}^*\mathbb{R}$ , there exists a unique element, denoted  ${}^\circ r$ , in  $\mathbb{R}$  such that  ${}^\circ r = r$ ;  ${}^\circ r$  is said to be the standard part of  $r$ . For every  $r \in \mathbb{R}$ , the monad of  $r$ , denoted  $mo(r)$ , is given by  $mo(A) \equiv \{x \in {}^*\mathbb{R} : {}^\circ x = r \text{ for some } r \in A\}$ . It may be shown that  $A \subset \mathbb{R}$  is compact iff  $mo({}^*A) = A$ , i.e., iff every point of  ${}^*A$  is finite. Any subset of  ${}^*\mathbb{R}$  having the property that all of its elements are finite will be called near-standard.

Now  $N$ , an infinite set, implies that  $N$  is properly contained in its extension  ${}^*N$ . The next result shows that  $N$  contains all of the finite elements of  ${}^*\mathbb{N}$  and that  ${}^*N - N$  contains all of the infinite elements.

Theorem 2.1: An element of  ${}^*N$  is finite iff  $n \in N$ .

Proof: That  $n \in N$  implies  $n$  is finite is obvious from the definition. Let  $n \in {}^*N$  be finite. Then there exists  $m \in N$  such that  $0 < n < m$ . Now the following statement is true about  $\mathbb{R}$ :

$$[(\forall x)[x \in N][[x \leq m] \leftrightarrow [[x = 1] \vee [x = 2] \vee \dots \vee [x = m]]]].$$

The transfer of this statement is:

$$[(\forall x)[x \in {}^*N][[x \leq m] \leftrightarrow [[x = 1] \vee [x = 2] \vee \dots \vee [x = m]]]].$$

It follows that  $n \in N$ .

Considering the initial segments of  ${}^*N$ , which are internal, analogy with the initial segments of  $N$  motivates the last definition for this section. An entity  $A$  is said to be \*-finite if there exists  $\omega \in {}^*N$  and an internal bijection from  $\{1, 2, \dots, \omega\}$  to  $A$ . The \*-finite subsets of  ${}^*\mathbb{R}$  behave as finite subsets of  $\mathbb{R}$  do. It is a fact that every \*-finite collection of internal objects is internal. Moreover, it is a fact that every \*-finite subset

of  $\mathbb{R}$  has both a smallest and a largest element.



### 3. INTRODUCTION TO LOEB MEASURES

Throughout this section we will be working within the framework of the denumerably comprehensive enlargement,  $V(^*\mathbb{R})$ , of  $V(\mathbb{R})$  that we established in the previous section.

Let  $X$  denote an internal set in  $V(^*\mathbb{R})$ . Let  $\Psi$  be an internal algebra of subsets of  $X$ , i.e., an internal collection, of internal subsets, satisfying  $A \cup B$  and  $X - A$  are elements of  $\Psi$  whenever  $A$  and  $B$  are elements of  $\Psi$ . Let  $\nu: \Psi \rightarrow ^*\mathbb{R}$  be an internal finitely additive measure satisfying  $\nu(X) < \infty$ . We will call such an ordered triple an internal measure space. Let  $(X, \Psi, \nu)$  be an internal measure space, a function  $f: (X, \Psi, \nu) \rightarrow ^*\mathbb{R}$  is said to be  $\nu$ -measurable if  $f$  is internal and for each  $a \in ^*\mathbb{R}$ ,  $\{x \in X: f(x) < a\} \in \Psi$  and  $\{x \in X: f(x) \leq a\} \in \Psi$ . Let the map  $\nu_0: \Psi \rightarrow \mathbb{R}$  be defined by  $\nu_0(A) = {}^\circ(\nu(A))$  for all  $A \in \Psi$ , and let  $\sigma(\Psi)$  denote the  $\sigma$ -algebra generated by  $\Psi$ . The following result is fundamental to our study, and has served as the foundation for a large number of applications of non-standard analysis to other fields as well.

Theorem 3.1 [Loeb, 1975]: Let  $(X, \Psi, \nu)$ ,  $\nu_0$ , and  $\sigma(\Psi)$  be defined as above. Then  $\nu_0$  has a unique, standard,  $\sigma$ -additive extension  $\nu_0$ , to  $\sigma(\Psi)$ . For each  $B \in \sigma(\Psi)$ , the value of this extension at  $B$  is given by

$$\nu_0(B) = \inf_{A \in \Psi, B \subset A} \nu_0(A) = \sup_{C \in \Psi, B \supset C} \nu_0(C).$$

Moreover, for all  $B \in \sigma(\Psi)$  there exists  $B' \in \Psi$  with  $\nu_0(B \Delta B') = 0$ , where  $\Delta$  denotes the symmetric difference operator.

Proof: Let  $(A_n)$  be a sequence of sets in  $\Psi$ , such that  $B \subset \bigcup_{n=1}^{\infty} A_n$ . Let  $(A'_n)$ ,  $n \in ^*\mathbb{N}$ , be an internal extension of the sequence  $(A_n)$ . Consider the

set  $\{m: m \in {}^*\mathbb{N} \ \& \ B \subset \bigcup_{n=1}^m A_n\}$ . It may be seen from the Internal Definition Principle that this set is internal. From this and the fact that it is non-empty (it clearly contains  ${}^*\mathbb{N} - \mathbb{N}$ ), it may be concluded that it has a smallest element. But  ${}^*\mathbb{N} - \mathbb{N}$  has no first element; hence it must be that there exists  $m \in \mathbb{N}$  such that  $B \subset \bigcup_{n=1}^m A_n$ . It follows that  $\nu_0$  is  $\sigma$ -additive on  $\Psi$ . The Hahn Extension Theorem [Dunford-Schwartz (1957), p. 136] now implies that  $\nu_0$  has a unique,  $\sigma$ -additive extension, which we shall also denote as  $\nu_0$ , to  $\sigma(\Psi)$ .

Now let  $B \in \sigma(\Psi)$ , and let  $\epsilon > 0$  in  $\mathbb{R}$  be given. Then there exists an increasing sequence of measurable sets (in  $\Psi$ )  $(A_n)$  such that  $B \subset \bigcup_{n=1} A_n$  and  $\nu_0(\bigcup_{n=1} A_n) < \nu_0(B) + \epsilon$ . Let  $(A_n)_{n \in {}^*\mathbb{N}}$  denote an internal extension of this sequence. Then there exists  $\omega \in {}^*\mathbb{N}$  such that  $\forall n \in {}^*\mathbb{N}$ ,  $2 \leq n \leq \omega$ ,  $A_{n-1} \subset A_n$ ,  $A_n \in \Psi$ , and  $\nu_0(A_n) < \nu_0(B) + \epsilon$ . As  $\epsilon > 0$  was arbitrary we have that  $\nu_0(B) = \inf_{A \in \Psi, B \supset A} \nu_0(A)$ . Consideration of  $A_\omega - B$  yields  $\nu_0(B) = \sup_{C \in \Psi, B \supset C} \nu_0(C)$ . From the argument above it is clear that for all  $n$  in  $\mathbb{N}$  there exist sets  $A_n$  and  $C_n$  in  $\Psi$  with

$$A_{n-1} \subset A_n \subset B \subset C_n \subset C_{n-1}$$

and  $\nu_0(C_n - A_n) \leq 1/n$ . Considering the internal extensions of the two sequences  $(A_n)$  and  $(C_n)$ , we see that there exists  $\omega \in {}^*\mathbb{N}$  such that  $A_{\omega-1} \subset A_\omega \subset B \subset C_\omega \subset C_{\omega-1}$  and  $\nu_0(C_\omega - A_\omega) \leq 1/\omega = 0$ , i.e.,  $\nu_0(C_\omega - A_\omega) = 0$ . It follows that there exists  $B' \in \Psi$  such that  $\nu_0(B \Delta B') = 0$ .

The importance of this result is that it takes an internal measure space and shows how to convert it into a standard measure space. Internal measure spaces in general will only be finitely additive so that prior to Loeb's result there was little that non-standard analysis had to offer to the study of measure theory (which is not to say that there were not contributions to measure theory

before Loeb's theorem). This result alone merely says that there exists a class of standard measure spaces that could be generated using non-standard analysis. What makes this result so valuable are the following three theorems of Loeb (1975) (and their subsequent extensions), which essentially say that the measure spaces  $(X, \sigma(\Psi), \nu_0)$  can be studied internally,<sup>3/</sup> i.e., by looking at the non-standard spaces that underlie them.

Theorem 3.3: Let  $(X, \Psi, \nu)$  be an internal measure space. Let  $f: X \rightarrow {}^*[-n, n]$ ,  $n \in \mathbb{N}$ , be  $\Psi$ -measurable. Then, for each  $A \in \Psi$ ,  $\int_A f d\nu = \int_A {}^\circ f d\nu_0$ .

Theorem 3.4: Let  $(X, \Psi, \nu)$  be an internal measure space. If  $g: X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  is  $\sigma(\Psi)$ -measurable, then there is an  $f: X \rightarrow {}^*\mathbb{R}$  which is  $\Psi$ -measurable such that  ${}^\circ f = g$ ,  $\nu_0$ -a.e.

Following Anderson (1976) we shall call the completion of the space  $(X, \sigma(\Psi), \nu_0)$  the Loeb space of the internal measure space  $(X, \Psi, \nu)$ , and we shall denote it by  $(X, L(\Psi), L(\nu))$ . An important class of Loeb measure spaces is the collection of Loeb spaces based on an underlying space  $(X, \Psi, \nu)$  such that  $X$  is  $*$ -finite,  $\Psi$  is the  $*$ -finite set  ${}^*\mathcal{P}(X)$ , and  $\nu$  is the counting measure on  ${}^*\mathcal{P}(X)$ , i.e.,  $\nu(A) = |A|/|X|$  for all  $A$ . We shall call these measure spaces " $*$ -finite Loeb measure spaces" and shall denote them by  $(T, L(\tau), L(\mu))$ , with  $|T|$  denoted by  $\omega$ . A function  $f: (X, \Psi, \nu) \rightarrow {}^*\mathbb{R}$  is said to be S-integrable if (1)  $f$  is  $\Psi$ -measurable; (2)  ${}^\circ(\int_X |f| d\nu) < \infty$ ; and (3)  $Z \in \Psi$ ,  $\nu(Z) = 0$ , implies  $\int_Z |f| d\nu = 0$ . The following two theorems of Anderson (1976) serve to show that the results of Loeb above may be sharpened to say that integration is "preserved in moving between" a Loeb space and its underlying internal measure space. In the case of  $*$ -finite Loeb space  $(T, L(\tau), L(\mu))$

we get the additional dividend that integration with respect to  $\mu$  is really  $\ast$ -finite summation which, of course, can be manipulated in precisely the same ways that finite summation can.

Theorem 3.5: Let  $(X, L(\Psi), L(\nu))$  be the Loeb space of the internal measure space  $(X, \Psi, \nu)$ . Let  $g: X \rightarrow \mathbb{R}$  be  $L(\nu)$ -integrable. Then there exists an  $S$ -integrable function  $f: X \rightarrow \ast\mathbb{R}$  such that  $\circ f = g, L(\nu)$ -a.e. Moreover,  $\circ f$  is  $L(\nu)$ -integrable and  $\int |f| d\nu \approx \int |\circ f| dL(\nu)$ .

Theorem 3.6: Let  $(X, L(\Psi), L(\nu))$  be the Loeb space of the internal measure space  $(X, \Psi, \nu)$ . Let  $f: X \rightarrow \ast\mathbb{R}$  be  $S$ -integrable. Then  $\circ f$  is  $L(\nu)$ -integrable and  $\circ\left(\int_S f d\nu\right) = \int_S \circ f dL(\nu)$  for all  $S$  in  $\Psi$ .

The last of our mathematical preliminaries is the following theorem of Anderson (1982). The significance of this result for our work will be discussed in Section 6.

Theorem 3.7: Let  $(X, L(\Psi), L(\nu))$  be the Loeb space of the internal measure space  $(X, \Psi, \nu)$ . Let  $Y$  be a second-countable Hausdorff space. If  $g: X \rightarrow Y$  is  $L(\nu)$ -measurable, then there exists an internal map  $f, f: X \rightarrow \ast Y$ ,  $f$   $\Psi$ -measurable, such that  $\circ f(x) = g(x), L(\nu)$ -a.e. In particular,  $\circ f$  is  $L(\nu)$ -measurable and has the same distribution as  $g$ .

## 4. DEFINITIONS

In this section we will present the basic definitions for the various models to be considered. The presentation will necessarily be brief; readers desiring more in-depth explanations of this material are directed to the references at the end of the paper.

### 4.1 Preferences

Let  $\mathcal{P}^m$  denote the set of all binary relations  $\succ$  on  $\mathbb{R}_+^m$ .  $\mathcal{P}^m$  represents the class of all potential preference orderings on the commodity space  $\mathbb{R}_+^m$ . A preference ordering is an ordinal ranking of collections of commodities. In what follows we shall have need to restrict our attention to certain subclasses of  $\mathcal{P}^m$ , each possessing some of the properties listed below.

#### Properties of Preferences:

- 1) irreflexivity:  $x \not\succeq x$  for all  $x \in \mathbb{R}_+^m$ ;
- 2) transitivity: [ $x \succ y$  and  $y \succ z$ ] implies  $x \succ z$ ;
- 3) free disposal (or weak transitivity): [ $x \gg y$  and  $y \succ z$ ] implies  $x \succ z$ ;
- 4) continuity:  $\succ$  has open graph in  $\mathbb{R}_+^m \times \mathbb{R}_+^m$ ;
- 5) monotonicity:  $x \succ y$  implies  $x \succ y$ ;
- 6) convexity: for all  $x \in \mathbb{R}_+^m$ ,  $P(x) \equiv \{y \in \mathbb{R}_+^m: y \succ x\}$  is convex.

Let  $\mathcal{P}_1^m$  denote the subclass of  $\succ$  in  $\mathcal{P}^m$  satisfying (3), (4), and (5). Let  $\mathcal{P}_2^m$  denote the subclass of  $\succ$  in  $\mathcal{P}^m$  satisfying (1), (2), (4), (5). Finally, let  $\mathcal{P}_2^m$  denote the subclass of  $\mathcal{P}^m$  satisfying (1), and (4)-(6).

### 4.2 Measure Theoretic Exchange Economies

A measure theoretic exchange economy,  $E$ , is a measurable mapping  $E: (X, \mathcal{M}, \nu) \rightarrow \mathcal{P}^l \times \mathbb{R}_+^l$ , such that  $\int_X \text{pr} \circ E d\nu$  is finite, where  $\text{pr}$  denotes the

projection map from  $P^\ell \times \mathbb{R}_+^\ell$  to  $\mathbb{R}_+^\ell$ . The interpretation of the measure space  $(X, M, \nu)$  is as follows.  $X$  denotes the set of agents;  $M$ , a  $\sigma$ -algebra of subsets of  $X$ , denotes the class of all permissible coalitions of agents; and the measure  $\nu$  provides a measure of the size of the coalitions in  $M$ , relative to  $X$ . Let  $\succ_t$  be the projection of  $E(t)$  onto  $P^\ell$ , and let  $e(t)$  be the projection of  $E(t)$  onto  $\mathbb{R}_+^\ell$ . Then  $\succ_t$  represents the preference ordering of agent  $t$  and  $e(t)$  represents  $t$ 's initial endowment of commodities. An assignment for  $E$ ,  $f$ , is an integrable function of  $X$  into  $\mathbb{R}_+^\ell$ . An allocation for  $E$  is an assignment  $f: X \rightarrow \mathbb{R}_+^\ell$  such that  $\int_X f d\nu = \int_X e d\nu$ . A price system for  $E$  is a vector  $p$  in  $\mathbb{R}_+^\ell$ . A competitive equilibrium for the economy  $E$  is an ordered pair  $(p, f)$  such that  $p$  is a price system for  $E$  and  $f$  is an allocation for  $E$  with the property that  $f(t)$  is maximal for the ordering  $\succ_t$  in  $\{x \in \mathbb{R}_+^\ell: p \cdot x \leq p \cdot e(t)\}$ ,  $\nu$ -a.e. in  $T$ . An assignment  $f$  is blocked by a coalition  $S \in M$  if there exists an assignment  $x: T \rightarrow \mathbb{R}_+^\ell$  such that  $x(t) \succ_t f(t)$ ,  $\nu$ -a.e. in  $S$  and  $\int_S x d\nu = \int_S e d\nu$ . The set of all allocations that cannot be blocked by any coalition in  $M$  is called the core of the economy  $E$  and will be denoted by  $C(E)$ .

### 4.3 Measure Theoretic Public Goods Economies

In addition to the simple exchange economy described above, we will examine an economy having  $\ell$  private goods,  $q$  public goods, production possibilities, and a measure space of agents. Let  $P^{\ell+q}$  denote the set of binary relations on  $\mathbb{R}_+^{\ell+q}$  (the dimension of the consumers' consumption sets being expanded from  $\ell$  to  $\ell+q$  due to the presence of the  $q$  public goods). A measure theoretic public goods economy,  $E$ , is a measurable mapping  $E: (X, M, \nu) \rightarrow P^{\ell+q} \times \mathbb{R}_+^\ell$ , such that  $\int_X p \cdot e d\nu$  is finite, together with a set  $A \subset \mathbb{R}^{\ell+q}$ , and a non-negative Radon-Nikodym derivative,  $\phi$ , of a measure  $\theta$  on  $M$ , where  $\theta$  is absolutely continuous with respect to  $\nu$  and  $\theta(X) = 1$ . The interpretation of

$(X, M, v)$  is that of Section 4.2, and  $e(t)$  and  $\succ_t$  shall retain their interpretations as well. Note that consumers are taken to be endowed only with private commodities, not public ones. The set  $A$  denotes the aggregate production possibilities set for the economy and  $\phi$  assigns profit shares to the agents in  $X$ . An allocation for a public goods economy  $E$  is an ordered pair  $(x, y)$  where  $x$  is an integrable function of  $X$  into  $\mathbb{R}_+^{\ell}$ ,  $y \in \mathbb{R}^q$ , and  $(x(t), y) \in \mathbb{R}_+^{\ell+q}$ ,  $v$ -a.e. in  $X$ . A feasible allocation for the public goods economy  $E$  is an assignment  $(x, y)$  such that  $(\int_X (x-e)dv, y) \in A$ . A price system,  $p$ , for the public goods economy  $E$  is an ordered pair  $(p_x, p_y)$  where  $p_x \in \Delta = \{x \in \mathbb{R}_+^{\ell} : \sum_{i=1}^{\ell} x_i = 1\}$  and  $p_y$  is an integrable function from  $X$  into  $\mathbb{R}_+^q$ . A Lindahl equilibrium for the public goods economy  $E$  is an ordered pair  $((x, y), p)$ , where  $(x, y)$  is a feasible allocation for  $E$  and  $p$  is a price system for  $E$  such that:

- i)  $(p_x, \int_X p_y dv) \cdot (\int_X (x-e)dv, y) \geq (p_x, \int_X p_y dv) \cdot z$  for all  $z$  in  $A$ ; and
- ii)  $(x(t), y)$  is maximal for  $\succ_t$  in  $B(p, t)$ ,  $v$ -a.e., where  $B(p, t) = \{z \in \mathbb{R}_+^{\ell+q} : (p_x, p_y(t)) \cdot z \leq p_x \cdot e(t) + \phi(t)[(p_x, \int_X p_y dv) \cdot (\int_X (x-e)dv, y)]\}$ .

#### 4.4 Nonstandard Exchange Economies

A nonstandard exchange economy,  $E$ , is an internal mapping  $E: T \rightarrow *P^{\ell} \times *R_+^{\ell}$ , where  $T$  is an internal set of agents with cardinality  $\omega \in *N$ . Let  $\succ_t$  be the projection of  $E(t)$  into  $*P^{\ell}$  and  $e(t)$  the projection of  $E(t)$  onto  $*R_+^{\ell}$ . The interpretation of  $\succ_t$  and  $e(t)$  is the same as it was for standard exchange economies. An assignment for the nonstandard exchange economy  $E$  is an internal function  $f$  of  $T$  into  $*R_+^{\ell}$ . An allocation for the nonstandard exchange economy  $E$  is an assignment  $f: T \rightarrow *R_+^{\ell}$  such that  $\frac{1}{\omega} \sum_T f(t) = \frac{1}{\omega} \sum_T *e(t)$ . A price system for the nonstandard exchange economy  $E$  is a vector

$p$  in  ${}^*\mathbb{R}_+^{\ell}$ . A competitive equilibrium for the nonstandard exchange economy is an ordered pair  $(p, f)$  such that  $p$  is a price system for  $E$ , and  $f$  is an allocation for  $E$  such that  $f(t)$  is maximal for  $\succ_t$  in  $\{x \in {}^*\mathbb{R}_+^{\ell} : p \cdot x \leq p \cdot e(t)\}$  for all  $t \in K$ , where  $K$  is an internal set of agents such that  $|K|/\omega = 1$ . An assignment  $f$  is blocked by an internal coalition  $S \subset T$ , if there exists an assignment  $x: T \rightarrow {}^*\mathbb{R}_+^{\ell}$  such that  $x(t) \succ_t f(t)$  for all  $t \in S$  and  $\frac{1}{\omega} \sum_S x(t) = \frac{1}{\omega} \sum_S e(t)$ . As before, the set of all allocations which cannot be blocked by any coalition in the economy  $E$  is called the core of  $E$  and is denoted by  $C(E)$ .

#### 4.5 Nonstandard Public Goods Economies

A nonstandard public goods economy,  $E$ , is an internal mapping  $E: T \rightarrow {}^*P^{\ell+q} \times {}^*\mathbb{R}^{\ell}$ , together with a set  $A \subset {}^*\mathbb{R}^{\ell+q}$ , and an internal mapping  $\phi: T \rightarrow {}^*\mathbb{R}$  such that  $\frac{1}{\omega} \sum_T \phi(t) = 1$ , where  $|T| = \omega \in {}^*\mathbb{N}$ . The interpretation of the maps  $E$  and  $\phi$ , and the set  $A$  is as above. An allocation for the nonstandard public goods economy  $E$  is an ordered pair  $(x, y)$ , where  $x$  is an internal mapping from  $T$  into  ${}^*\mathbb{R}_+^{\ell}$ ,  $y \in {}^*\mathbb{R}^{\ell}$ , and  $(x(t), y) \in {}^*\mathbb{R}_+^{\ell+q}$  for all  $t$  in  $T$ . A feasible allocation for the nonstandard public goods economy  $E$  is an allocation  $(x, y)$  for  $E$  such that  $(\frac{1}{\omega} \sum_T (x(t) - e(t)), y) \in A$ . A price system for the nonstandard public goods economy  $E$  is an ordered pair  $(p_x, p_y)$  where  $p_x \in {}^*\Delta = \{x \in {}^*\mathbb{R}_+^{\ell} : \sum_{i=1}^{\ell} x_i = 1\}$  and  $p_y$  is an internal mapping from  $T$  into  ${}^*\mathbb{R}_+^{\ell}$ . A Lindahl equilibrium for the nonstandard public goods economy  $E$  is an ordered pair  $((x, y), p)$  where  $(x, y)$  is an allocation for  $E$  and  $p$  is a price system for  $E$  such that:

- i)  $(p_x, \frac{1}{\omega} \sum_T p_y(t)) \cdot (\frac{1}{\omega} \sum_T (x(t) - e(t)), y) \geq (p_x, \frac{1}{\omega} \sum_T p_y(t)) \cdot z$  for all  $z \in A$ ; and
- ii)  $(x(t), y)$  is maximal for  $\succ_t$  in  $B(p, t)$  for all  $t \in T$ , where



$$B(p, t) = \{z \in {}^*R_+^{l+q} : (p_x, p_y(t)) \cdot z \leq p_x \cdot e(t) + \phi(t) \left[ (p_x, \frac{1}{\omega} \sum_T p(t)) \cdot \left( \frac{1}{\omega} \sum_T (x(t) - e(t)), y \right) \right] \}.$$

## 5. THE MAIN RESULTS

We can now state the main results of the paper.

Theorem 5.1: Let  $E: (T, L(\tau), L(\mu)) \rightarrow P_1^\ell \times \mathbb{R}_+^\ell$  be an atomless (Loeb) measure theoretic exchange economy satisfying:

- i)  $\int_T e dL(\mu) \gg 0$ ; and
- ii) for all  $t \in T$ ,  $\succ_t \in K \subset P_1^\ell$ , where  $K$  is compact in the topology of closed convergence.

Then if  $f \in C(E)$ , there exists a price system  $p$  for  $E$  such that  $(p, f)$  is a competitive equilibrium for  $E$ .

Theorem 5.2: Let  $E: (T, L(\tau), L(\mu)) \rightarrow P_{1,2}^\ell \times \mathbb{R}_+^\ell$  be an atomless (Loeb) measure theoretic exchange economy satisfying:

- i)  $\int_T e dL(\mu) \gg 0$ ; and
- ii) for all  $t \in T$ ,  $\succ_t \in K' \subset P_{2,2}^\ell$ , where  $K'$  is compact in the topology of closed convergence.

Then there exists a competitive equilibrium for  $E$ .

Theorem 5.3: Let  $E: (T, L(\tau), L(\mu)) \rightarrow P_3^{\ell+q} \times \mathbb{R}_+^\ell$  be a Loeb measure theoretic public goods economy satisfying:

- A0) For all  $t \in T$ ,  $\succ_t \in K'' \subset P_3^{\ell+q}$ , where  $K''$  is compact in the topology of closed convergence;
- A1)  $e(t) \gg 0$  for all  $t \in T$ ;
- A2)  $A$  is a closed convex cone with vertex  $0$ ;
- A3)  $A \supset -\mathbb{R}_+^{\ell+q}$  and  $A \cap \mathbb{R}_+^{\ell+q} = \{0\}$ ; and
- A4) there exists  $(\hat{x}, \hat{y}) \in A$  such that  $\hat{y} \gg 0$ .

Then there exists a Lindahl equilibrium for  $E$ .

Notice that Theorems (5.1)-(5.3) are versions of the core equivalence result of Aumann (1964), the existence of a competitive equilibrium result of Aumann (1966) (and Schmeidler (1969)) and the existence of Lindahl equilibrium theorem of Roberts (1972) respectively. In that sense our results are not really new (although some of our assumptions are weaker than those of the above authors, and therefore are not implied by theirs). Moreover, there is a restriction as far as the measure space of agents is concerned. Indeed, our space of agents in a Loeb space, contrary to the arbitrary atomless measure spaces which can be used in the above cited papers. However, our objective is not to provide the most general versions of the above theorems but rather to illustrate a method of obtaining results for mathematically consistent models of perfect competition in a very simple and intuitive way. We must emphasize that we do not need the Fatou Lemma (in one dimension or several dimensions) and the complicated lengthy process adopted in Aumann (1966), Schmeidler (1969), and Roberts (1972). Moreover, we do not need the Lyapunov theorem; only a simple version of the Shapley-Folkman theorem is needed. As a matter of fact, our results turn out to be corollaries of theorems proved in a fixed finite economy framework, and therefore have been obtained in a very simple and natural way. Finally, we believe that the restriction involved in the measure space of economic agents is not really important. Indeed, if the purpose of an arbitrary atomless measure space of agents is to capture the meaning of perfect competition, then Loeb spaces serve precisely this purpose.

## 6. STRATEGY OF THE PROOFS

The theorems stated above are concerned with economies defined on a \*-finite Loeb space of agents. Henceforth we shall refer to such an economy as a Loeb economy. Now the essence of Theorem 3.7 (Anderson's Lifting Theorem) is that given a Loeb economy  $E: (X, L(\tau), L(\mu)) \rightarrow P^m \times R_+^m$ , there exists a nonstandard economy  $E': X \rightarrow {}^*P^m \times {}^*R_+^m$  having essentially identical preferences and endowments. More precisely, given the Loeb economy  $E$ , there exists a nonstandard economy,  $E'$ , having the property that  $\circ(E'(t)) = E(t), L(\mu)$ -a.e. It will then be noted that  $E'$  inherits properties similar to those stated (for  $E$ ) in the hypotheses of the theorem to be proved. These, in turn, will be sufficient to ensure that the analogous result for nonstandard economies may be had by transferring a known finite result.

The proofs will then be completed by "pushing back down" to the Loeb economy that we began with. This "pushing down" will consist, in large part, of composing the standard part mapping with the nonstandard solution and utilizing the strong similarities that exist between  $E$  and  $E'$ . The  $S$ -integrability of allocations of private goods in the nonstandard solution will play an important part at this stage of the proof. In particular,  $S$ -integrability will be necessary for demonstrating needed integrability properties in the solution of the theorem being proved.

## 7. PROOFS OF THE MAIN RESULTS

Proof of Theorem 5.2: By Theorems 3.7 and 3.5, there exists an internal nonstandard exchange economy  $*E: T \rightarrow *P_2^\ell \times *R_+^\ell$  such that:

$$\circ(*E(t)) = E(t), L(\mu)\text{-a.e.}; \text{ and}$$

$$\frac{1}{\omega} \sum_S *e(t) = \int_S e dL(\mu) \text{ for all } S \in \tau.$$

We will need to make use of the following result of Anderson-Khan-Rashid (1982).

Theorem 7.1: Let  $E_F: T \rightarrow P_2^\ell \times R_+^\ell$  be a finite exchange economy ( $|T| = n \in \mathbb{N}$ ). Then there exists  $f: T \rightarrow R_+^\ell$  and  $p \in \text{int } R_+^\ell$  such that

i)  $f(t)$  is maximal for  $>_t$  in  $\{x \in R_+^\ell: p \cdot x \leq p \cdot e(t)\}$  for all  $t \in T$ ,

and

$$\text{ii) } \frac{1}{n} \sum_{i=1}^{\ell} \max \left\{ \sum_T f_i(t) - \sum_T e_i(t), 0 \right\} \leq \frac{(\ell+1)}{\sqrt{n}} \max_{t \in T} \|e(t)\|.$$

Now the nonstandard exchange economy  $*E$  satisfies:

i) for all  $t \in T$ ,  $\frac{y}{t} \in *K \subset *P_2^\ell$ , where  $*K$  is near standard in the topology of closed convergence; and

$$\text{ii) } \frac{1}{\omega} \sum_T *e(t) \gg_{\omega} 0.$$

It follows from the transfer of Theorem 7.1 that there exists an ordered pair  $(p, f)$  such that:

$$\begin{aligned} * (7.1) \quad f(t) \text{ is maximal for } * >_t \text{ in } \{x \in *R_+^\ell: p \cdot x \leq p \cdot *e(t)\} \text{ for all } \\ t \in K, \text{ where } K \text{ is an internal set of agents such that} \\ |K|/|T| = 1, \text{ where } |T| = \omega \in *N - N. \end{aligned}$$

$$\begin{aligned} * (7.2) \quad \frac{1}{\omega} \sum_{i=1}^{\ell} \max \left\{ \sum_T f_i(t) - \sum_T *e_i(t), 0 \right\} \leq (\ell+1) \max_{t \in T} \|*e(t)\|/\sqrt{\omega} = 0, \text{ i.e.,} \\ \frac{1}{\omega} \sum_T f(t) \leq \frac{1}{\omega} \sum_T *e(t). \end{aligned}$$

Note that in  $* (7.2)$ , the bound  $(\ell+1) \max_{t \in T} \|*e(t)\|/\sqrt{\omega}$  is now infinitesimal

since  $*e$  is  $S$ -integrable. Given the fact that preferences are monotone, we can strengthen \*(7.2) to  $\frac{1}{\omega} \sum_T f(t) \approx \frac{1}{\omega} \sum_T *e(t)$ . Hence, we have shown that a competitive equilibrium,  $(p, f)$ , for  $*E$  exists. It remains to be shown that the existence of a competitive equilibrium for  $*E$  implies the existence of a competitive equilibrium for  $E$ . The following lemma will be needed in this task.

Lemma 7.1:  $f: T \rightarrow *R_+^k$  is  $S$ -integrable.

Proof: Note that  $p \cdot f(t) \leq p \cdot *e(t)$  for all  $t$  in  $T$ . Therefore, for any internal set of agents  $S$  having  $|S|/\omega \approx 0$ , it follows that:

$$p \cdot \frac{1}{\omega} \sum_S f(t) = \frac{1}{\omega} \sum_S p \cdot f(t) \leq \frac{1}{\omega} \sum_S p \cdot *e(t) = p \cdot \sum_S *e(t)/\omega \approx 0,$$

by the  $S$ -integrability of  $*e$ . On the other hand, since  $p \gg_{\neq} 0$  (see, e.g., Khan (1975)) it follows that  $\frac{1}{\omega} \sum_S f(t) \approx 0$ .

The proof of Theorem 5.2 can now be completed by verifying that  $({}^{\circ}p, {}^{\circ}f)$  is a competitive equilibrium for  $E$ . The proposition below shows that an even stronger relationship exists between the competitive equilibria for  $E$  and the competitive equilibria for  $*E$ .

Proposition 7.2: If the pair  $(p, f)$  is a competitive equilibrium for  $*E$  if and only if  $({}^{\circ}p, {}^{\circ}f)$  is a competitive equilibrium for  $E$ .

Proof: ( $\Rightarrow$ ): Let  $(p, f)$  be a competitive equilibrium for  $*E$  and  $({}^{\circ}p, {}^{\circ}f)$  is not a competitive equilibrium for  $E$ . Then there exists  $y: T \rightarrow R_+^k$ ,  $L(\mu)$ -integrable such that

$$(7.1) \quad \int_T y dL(\mu) = \int_T e dL(\mu),$$

and

$$(7.2) \quad {}^\circ p, y(t) \leq {}^\circ p \cdot e(t) \text{ and } y(t) \underset{t}{\succ} {}^\circ f(t) \text{ for all } t \in S, L(\mu)(S) > 0.$$

By Theorem 3.5 there exists a function  $x: T \rightarrow {}^*\mathbb{R}_+^k$  such that  $x$  is  $S$ -integrable and  ${}^\circ x = y$ ,  $L(\mu)$ -a.e. From (7.2) it follows that  ${}^\circ p \cdot {}^\circ x \leq {}^\circ p \cdot e(t)$  and  ${}^\circ x \underset{t}{\succ} f(t)$  for all  $t \in S$ ,  $L(\mu)(S) > 0$ . Since  ${}^\circ x$  is  $L(\mu)$ -integrable and  ${}^\circ x = y$ ,  $L(\mu)$ -a.e.  $\int_T {}^\circ x dL(\mu) = \int_T y dL(\mu) = \int_T e dL(\mu)$ . By Theorem 3.5 we have  $\int_T {}^\circ x dL(\mu) = \frac{1}{\omega} \sum_T x(t)$ . Since  $\int_T e dL(\mu) = \frac{1}{\omega} \sum_T {}^*e(t)$  by (7.1) we have  $\frac{1}{\omega} \sum_T x(t) = \frac{1}{\omega} \sum_T {}^*e(t)$ . Moreover, it follows from (7.2) that  $p \cdot x \leq p \cdot {}^*e(t)$  and  $x \underset{t}{\succ} f(t)$  for all  $t \in S$ ,  $|S|/\omega \neq 0$ , a contradiction to the fact that  $(p, f)$  is a competitive equilibrium for  ${}^*E$ .

( $\Leftarrow$ ): Let  $({}^\circ p, {}^\circ f)$  be a competitive equilibrium for  $E$  and  $(p, f)$  is not a competitive equilibrium for  ${}^*E$ . Then there exists an internal function  $y: T \rightarrow {}^*\mathbb{R}_+^k$  such that

$$(7.1)' \quad p \cdot y \leq p \cdot {}^*e(t) \text{ and } y \underset{t}{\succ} f(t) \text{ for all } t \in S, |S| \text{ internal, } |S|/\omega \neq 0.$$

It follows from (7.1)' that for any infinitesimal set  $V$ ,  $p \cdot \frac{1}{\omega} \sum_V y(t) = \frac{1}{\omega} \sum_V p \cdot y(t) \leq \frac{1}{\omega} \sum_V p \cdot {}^*e(t) = 0$  since  ${}^*e$  is  $S$ -integrable. Since  $p \gg 0$ ,  $\frac{1}{\omega} \sum_V y(t) = 0$  and therefore  $y$  is  $S$ -integrable. By Theorem 3.6,  ${}^\circ y$  in  $L(\mu)$ -integrable and from (7.1)' it follows that  ${}^\circ p \cdot {}^\circ y \leq {}^\circ p \cdot e(t)$  and  ${}^\circ y \underset{t}{\succ} f(t)$  for all  $t \in S$ ,  $L(\mu)(S) > 0$ , a contradiction to the fact that  $({}^\circ p, {}^\circ f)$  is a competitive equilibrium for  $E$ . This completes the proof of the proposition.

Proof of Theorem 5.1: By Theorem 3.7, there exists an internal nonstandard

exchange economy  $*E: T \rightarrow *P_1^\ell \times *R_+^\ell$  such that  $o(*E(t)) = E(t)$ ,  $L(u)$ -a.e. We now make use of the following theorem due to Anderson (1978).

Theorem 7.2: Let  $E_F: T \rightarrow P_1^\ell \times R_+^\ell$ , be a finite exchange economy, ( $|T| = n \in N$ ). Let  $M = \sup \{\|e(t_1) + \dots + e(t_\ell)\|_\infty : t_i \in T, 1 \leq i \leq \ell\}$ . If  $f \in C(E_F)$ , then there exists  $p \in \{q \in R_+^\ell : \sum_{i=1}^{\ell} q_i = 1\}$  such that

$$(7.3) \quad \frac{1}{n} \sum_T |p \cdot (f(t) - e(t))| \leq \frac{2M}{n},$$

and

$$(7.4) \quad \frac{1}{n} \sum_T |\inf \{p \cdot (x - e(t)) : x \succ_t f(t)\}| \leq \frac{2M}{n}.$$

The nonstandard exchange economy  $*E$  will satisfy the following assumption,

- i)  $*\succ_t \in *K' \subset *P_1^\ell, *K'$  is near standard in the topology of closed convergence.
- ii)  $\frac{1}{\omega} \sum_T *e(t) \gg_{K'} 0$ .

By the transfer of Theorem 7.2 we have that if  $f: T \rightarrow *R_+^\ell$  is an allocation such that  $f \in C(*E)$ , then there exists  $p \in \{y \in *R_+^\ell : \sum_{i=1}^{\ell} y_i = 1\}$  such that

$$*(7.3) \quad \frac{1}{\omega} \sum_T |p \cdot (f(t) - *e(t))| \leq \frac{2M}{\omega} \approx 0 \text{ or}$$

$$p \cdot \frac{1}{\omega} \sum_T f(t) \approx p \cdot \frac{1}{\omega} \sum_T *e(t), \text{ and}$$

$$*(7.4) \quad \frac{1}{\omega} \sum_T |\inf \{p \cdot (x - *e(t)) : x \succ_t^* f(t)\}| \leq \frac{2M}{\omega} \approx 0, \text{ i.e.,}$$

$$p \cdot (f) \approx p \cdot *e(t) \approx \inf \{p \cdot x : x \succ_t^* f(t)\} \text{ for all } t \in K$$

where  $K$  is an internal set of agents such that  $|K|/\omega \approx 1$ .

It follows from Assumption (ii) and the continuity of preferences that \*(7.4) can be strengthened to  $f(t)$  is maximal for  $\succ_t$  in  $\{x : p \cdot x \leq p \cdot *e(t)\}$



for all  $t \in K$ , where  $K$  is an internal set of agents so that  $|K|/\omega = 1$ .

Lemma 7.2:  $f: T \rightarrow {}^*\mathbb{R}_+^l$  is  $S$ -integrable.

Proof: Same as in Lemma 7.1.

The proof of Theorem 5.1 may now be completed by noting the following proposition.

Proposition 7.2:  $f \in C(*E) \Leftrightarrow {}^\circ f \in C(E)$ .

Proof: ( $\Rightarrow$ ): Let  $f \in C(*E)$  and  ${}^\circ f \notin C(E)$ . Then there exists  $y: T \rightarrow \mathbb{R}_+^l$   $L(\mu)$ -integrable such that

$$(7.5) \quad y(t) \underset{t}{\succ} {}^\circ f(t), \text{ for all } t \in S, \quad L(\mu)(S) > 0$$

$$(7.6) \quad \int_S y dL(\mu) = \int_S e dL(\mu).$$

Since  $y$  is  $L(\mu)$ -integrable by Theorem 3.5, there exists  $x: T \rightarrow {}^*\mathbb{R}_+^l$   $S$ -integrable such that  ${}^\circ x = y$ ,  $L(\mu)$ -a.e. Moreover, by the same Theorem,  ${}^\circ x$  is  $L(\mu)$ -integrable. Hence,  $\int_S {}^\circ x dL(\mu) = \int_S y dL(\mu)$  and by (7.6)  $\int_S {}^\circ x dL(\mu) = \int_S e dL(\mu)$ . It follows from (7.5) that  $x(t) \underset{t}{\succ}^* f(t)$  for all  $t \in S$ ,  $|S|/\omega \neq 0$ . Furthermore, since  $\int_S {}^\circ x dL(\mu) = \frac{1}{\omega} \sum_S x(t)$  and  $\int_S e dL(\mu) = \frac{1}{\omega} \sum_S *e(t)$  we have then  $\frac{1}{\omega} \sum_S x(t) = \frac{1}{\omega} \sum_S *e(t)$ , a contradiction to the fact that  $f \in C(*E)$ .

( $\Leftarrow$ ): Let  ${}^\circ f \in C(E)$  and  $f \notin C(*E)$ . Then there exists an internal function  $y: T \rightarrow {}^*\mathbb{R}_+^l$  such that

$$(7.7) \quad y(t) \underset{t}{\prec}^* f(t) \text{ for all } t \in S, \quad |S|/\omega \neq 0,$$

and

$$(7.8) \quad \frac{1}{\omega} \sum_S y(t) = \frac{1}{\omega} \sum_S *e(t).$$

Since  $*e$  is  $S$ -integrable for any internal set  $V$ ,  $|V|/\omega = 0$  we have

$\frac{1}{\omega} \int_V y(t) = \frac{1}{\omega} \int_V *e(t) = 0$ . Therefore  $y$  is  $S$ -integrable. By Theorem 3.5  ${}^{\circ}y$  is  $L(\mu)$ -integrable and  ${}^{\circ}(\frac{1}{\omega} \int_T y(t)) = \int_T {}^{\circ}y dL(\mu)$ . Since  $\int_T {}^{\circ}y dL(\mu) = \frac{1}{\omega} \int_T y(t)$  and  $\int_T e dL(\mu) = \frac{1}{\omega} \int_T *e(t)$  from (7.8), it follows that  $\int_S {}^{\circ}y dL(\mu) = \int_S e dL(\mu)$ . Moreover, it follows from (7.7) that  ${}^{\circ}y(t) \succ_t {}^{\circ}f(t)$  for all  $t \in S$ ,  $L(\mu)(S^{\circ}) > 0$ , a contradiction to the fact that  ${}^{\circ}f \in C(E)$ . This completes the proof of Proposition 7.2.

It follows from Propositions (7.1) and (7.2) that the core equivalence theorem is true for  $*E$  if and only if it is true in  $E$ . This completes the proof of Theorem 5.1.

Sketch of the Proof of Theorem 5.3: By Anderson's Lifting Theorem and Theorem 3.5, there exist internal mappings  $E': T \rightarrow *P_3^{\ell+q} \times *R_+^{\ell}$  and  $\phi': T \rightarrow *R$  such that

$$(7.9) \quad E'(t) \approx E(t) \quad \text{and} \quad \phi'(t) \approx \phi(t), L(\mu)\text{-a.e.},$$

and

$$(7.10) \quad \int_S e'(t)/\omega \approx \int_S e dL(\mu) \quad \text{and} \quad \int_S \phi(t)/\omega \approx \int_S \phi(t) dL(\mu) \quad \forall S \in \tau.$$

Note that  $*A$ ,  $E'$ , and  $\phi'$  comprise a nonstandard public goods economy. Note also that Foley's (1970) technique may be used in conjunction with the Gale-Mas-Colell (1975) existence result for economies without ordered preferences to obtain the following:

Lemma 7.3: Let  $E_F: T \rightarrow P_3^{\ell+q} \times R_+^{\ell}$  be a finite public goods economy satisfying:

- 1)  $e(t) \gg 0$ ;
- 2)  $A_F$  is a closed convex cone with vertex  $0$ ;

- 3)  $A_F \supset -\mathbb{R}_+^{\ell+q}$  and  $A_F \cap \mathbb{R}_+^{\ell+q} = \{0\}$ ; and  
 4) there exists  $(x,y) \in A$  such that  $y \gg 0$ .

Then there exists a Lindahl equilibrium for  $E$ .

Now the nonstandard public goods economy  $E'$  satisfies:

- N0) for all  $t \in T$ ,  $\frac{e^t}{\omega} \in *K'' \subset *P_3^{\ell+q}$ , where  $*K''$  is near-standard in the topology of closed convergence;  
 N1)  $e^t(t) \gg_{\mu} 0$  for all  $t \in T$ ;  
 N2)  $*A$  is a closed,  $*$ -convex,  $*$ -cone with vertex  $0$ ;  
 N3)  $*A \supset -*\mathbb{R}_+^{\ell+q}$  and  $*A \cap *\mathbb{R}_+^{\ell+q} = \{0\}$ ; and  
 N4)  $(\hat{x}, \hat{y}) \in *A$  and  $\hat{y} \gg_{\mu} 0$ .

Hence, by the transfer of Lemma 7.3,  $E'$  has a Lindahl equilibrium,  $((\bar{x}, \bar{y}), \bar{p})$ .

Lemma 7.4:  $(\sum_T \bar{x}(t)/\omega, \bar{y})$  is near-standard.

Proof: It is a standard exercise to show that (A1)-(A3) imply that  $F_1 \equiv (\mathbb{R}_+^{\ell+q} \cap (A + (\int_T e dL(\mu), 0)))$  is compact. It follows that  $mo(F_1)$  is near-standard. Now  $(\sum_T \bar{x}(t)/\omega, \bar{y}) \in F_2 \equiv (*\mathbb{R}_+^{\ell+q} \cap (*A + (\sum_T e^t(t)/\omega, 0)))$ , and (7.9) implies  $F_2 \subset mo(F_1)$ .

Lemma 7.5: For all  $n \in *N - N$ ,  $|S^n|/\omega = 1$ , where  $S^n \equiv \{t \in T: x(t) \leq nu\}$ .

Proof: Supposition to the contrary contradicts Lemma 7.4.

Lemma 7.6:  $\bar{x}$  is  $S$ -integrable.

Proof: All that really requires demonstration is Condition 3 of the definition. Let  $Z \in \tau$  satisfy  $|Z|/\omega = 0$ .  $*A$  a  $*$ -cone (N2) implies that equilibrium profits for  $E'$  are zero. That, together with the monotonicity of the  $\frac{e^t}{\omega}$ , implies

$$(7.11) \quad \bar{p}_x \cdot \bar{x}(t) + \bar{p}_y'(t) \cdot \bar{y} = \bar{p}_x \cdot e'(t) \quad \text{for all } t \in T.$$

Aggregation of (7.11) over  $Z$ , together with (7.10), yields

$$(7.12) \quad \bar{p}_x \cdot \left( \sum_Z \bar{x}(t)/\omega \right) = 0,$$

$$(7.13) \quad \left( \sum_Z \bar{p}_y(t)/\omega \right) \cdot \bar{y} = 0.$$

The properties of the preferences in  $*K''$  may now be used to show that all prices are noninfinitesimally greater than zero. In particular  $p \gg 0$ , and the conclusion follows from (7.12).

Lemma 7.7:  $\sum_T \bar{p}_y(t)/\omega$  is near-standard.

Proof: Suppose to the contrary. Then (N2) and (N4) imply that the profits associated with  $((\bar{x}, \bar{y}), \bar{p})$  are not bounded, a contradiction to the equilibrium conditions for  $E'$ .

Lemma 7.8: There exists an internal set  $G \subset T$ ,  $|G|/\omega = 1$ , such that  $\bar{p}_y(t)$  is near-standard for all  $t \in G$ .

Proof: Supposition to the contrary contradicts Lemma 7.7. Let  $J = \{j \in \{1, \dots, q\} : \bar{p}_{y_j} \text{ is not } S\text{-integrable}\}$ . If  $J = \emptyset$ , then straightforward argument shows that  $((\bar{x}, \bar{y}), \bar{p})$  is a Lindahl equilibrium for  $E$ . For the remainder of the proof we consider the case  $J \neq \emptyset$ . Let the map  $\hat{p}_y: T \rightarrow *R$  be defined by

$$(7.14) \quad \hat{p}_{y_j}(t) = \begin{cases} 0 & (j, t) \in J \times (T-G) \\ \bar{p}_{y_j}(t) & (j, t) \notin J \times (T-G) \end{cases}$$

then  $\hat{p}_y$  is  $S$ -integrable. Moreover,  $\hat{p}_y(t)$  is near-standard for all  $t \in T$ .

Lemma 7.9:  $({}^\circ\bar{x}(t), {}^\circ\bar{y})$  is  $\succ_t$ -maximal on  $B(t, ({}^\circ\bar{p}_x, {}^\circ\bar{p}_y)), L(\mu)$ -a.e.

Proof: From equation (7.11) we have

$$(7.15) \quad (\bar{p}_x, \sum_T \bar{p}_y(t)/\omega) \cdot (\sum_T (\bar{x}(t) - e'(t))/\omega, \bar{y}) = 0.$$

The left-hand side of (7.15) may be rewritten as

$$\bar{p}_x \cdot (\sum_T \bar{x}(t)/\omega) + \sum_{j \notin J} ((\sum_T \bar{p}_{y_j}(t)/\omega) \cdot \bar{y}_j) + \sum_{j \in J} ((\sum_T \bar{p}_{y_j}(t)/\omega) \cdot \bar{y}_j)$$

which, from equations (7.13) and (7.14) is infinitesimally close to

$$\bar{p}_x \cdot (\sum_T \bar{x}(t)/\omega) + \sum_{j \notin J} ((\sum_T \hat{p}_{y_j}(t)/\omega) \cdot \bar{y}) + \sum_J ((\sum_T \hat{p}_{y_j}(t)/\omega) \cdot \bar{y}_j).$$

So that  $B(t, ({}^\circ\bar{p}_x, {}^\circ\hat{p}_y(t))) = B(t, ({}^\circ\bar{p}_x, {}^\circ\bar{p}_y(t)))$  for all  $t \in G$ . Finally, the supposition, to the contrary, that there exists  $\hat{t} \in G$  for whom there exists  $z \in B(\hat{t}, ({}^\circ\bar{p}_x, {}^\circ\bar{p}_y(\hat{t})))$  preferred to  $({}^\circ\bar{x}(\hat{t}), {}^\circ\bar{y})$  can be shown to contradict the  $\succ_t$ -maximality of  $(\bar{x}(\hat{t}), \bar{y})$ .

At this point  $({}^\circ(\bar{x}, \bar{y}), ({}^\circ\bar{p}_x, \hat{p}_y))$  is a natural candidate for an equilibrium for  $E$ . Unfortunately, the production optimality condition cannot be demonstrated due to the change from  $\bar{p}_y$  to  $\hat{p}_y$ . This situation may be remedied however by employing a transformation due to Roberts (1972). The essence of this transformation is that it raises  $p_y$  to a level which  $\bar{p}_y$  suggests should be (and in fact is) consistent with production optimality. Moreover, it does this without altering consumers' choices. Let the price system  $r$  (for  $E$ ) be defined by

$$\begin{cases} r_x = {}^\circ\bar{p}_x \\ r_y(t) = {}^\circ\hat{p}_y(t) + [{}^\circ(\sum_T \bar{p}_{y_j}(t)/\omega) - {}^\circ(\sum_T \hat{p}_{y_j}(t)/\omega)] \text{ for all } t \in T. \end{cases}$$

$\hat{p}_y$  S-integrable, Lemma 7.7, and the fact that  $\hat{p}_y(t) \leq \bar{p}_y(t)$ , for all  $t \in T$ , imply that  $r$  is  $L(\mu)$ -integrable. Moreover  $\int_T r_y dL(\mu) = \circ(\sum_T \bar{p}_y(t)/\omega)$ .

Straightforward argument will now show (1) that  $(\circ\bar{x}, \circ\bar{y})$  is a feasible allocation for  $E$ , and (2) that  $(\circ\bar{x}, \circ\bar{y})$  is  $\lambda_t$ -maximal on  $B(t, (r_x, r_y))$  a.e. in  $T$ . The maximality of profits may be had through a simple argument by contradiction.

## FOOTNOTES

0. Brown-Lewis (1981) employ similar ideas but no "lifting" and "pushing down" arguments were used.
1. The existence of such a model can be demonstrated by means of an ultrapower construction. For details see Stroyan-Luxemburg (1976).
2. Functions on  $\ast\mathbb{R}$  which are extensions of a given function  $f$ , on  $\mathbb{R}$  are denoted by the same symbol,  $f$ .
3. Pun intended.

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