**MODELLING OF COMPOSITIONAL FLOW
IN NATURALLY FRACTURED RESERVOIRS**

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Abstract. A double porosity model of multidimensional, multicomponent, three-phase flow in naturally fractured reservoirs is derived first on the basis of physical intuition and then by the mathematical theory of homogenization. A fully compositional model is considered when there are N chemical components, each of which may exist in any or all of the three phases: gas, oil, and water. The equations of the interaction between matrix and fracture systems are obtained from the mass, momentum, and energy balance laws and the entropy condition, and a mechanical potential tensor is introduced to describe the matrix boundary condition. Various types of flow in naturally fractured reservoirs are treated as particular cases of the present techniques.

Key words: Double porosity, fractured reservoir, compositional flow, homogenization

AMS(MOS) subject classification(1985 revision). 35K60, 35K65, 76S05, 76T05

1. Introduction. The purpose of this paper is to model the simultaneous flow of three fluids with change of phase in a naturally fractured reservoir. We consider the very general case where there are N chemical species, or components, each being able to exist in any or all of the three phases: gas, oil, and water. With this generality, the distribution of hydrocarbon components between the oil and gas phases are not only included, but also a component (such as carbon dioxide) is allowed to dissolve in the water phase. It also allows for vaporization of water into the gas phase, which plays an important role in a steam drive process. Of course, three-phase immiscible flow without mass transfer between the oil and gas phases and the water phase is included as well.

The model will be derived first on the basis of physical intuition by applying the concept of double porosity [5], [16], [26], and then by the mathematical theory of homogenization [18]. The double porosity concept has been used to describe the flow of fluids in a naturally fractured reservoir [4], [5], [8], [23], [26]. The reservoir is supposed to possess two porous structures, one associated with the matrix blocks and the other with the system of fractures. Then sets of equations governing the motion of fluids are required for the two systems, and the interactions between them must be appropriately modeled.

It has been known that, to model the flow of fluids in a naturally fractured reservoir, the interactions between the matrix and fracture systems are crucial. Various suggestions have

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been made for these interactions [10], [12], [20], [24], [25]. In these papers, the matrix-fracture interactions have been modeled by introducing various *ad hoc* parameters and the matrix boundary conditions have not been incorporated in any general fashion. More recently, the double porosity concept has been exploited to model single-phase and two-phase, incompressible, immiscible flows [3], [4], [7], [8], where the interactions are treated through boundary conditions on the matrix blocks.

In this paper the dynamical equations of the interactions between the fractures and matrix blocks will be explicitly derived by homogenization from the mass, momentum, and energy balance laws and the entropy condition. The entropy condition is needed to guarantee that the fracture flow essentially reaches local thermodynamic equilibrium with respect to the matrix flow on the block surfaces. A mechanical potential tensor is introduced to describe the matrix boundary condition. Moreover, it is shown that, on the matrix-fracture interface, continuity of chemical potentials is a physically reasonable boundary condition for the matrix concentrations.

Various types of flow in naturally fractured reservoirs are considered through the present techniques. In particular, a single component in a single phase, a single mixture of multi-components, and two-phase immiscible flows are treated as particular cases of the present general model. Therefore, the various matrix boundary conditions proposed earlier for these flows are confirmed by our techniques.

The rest of the paper is organized as follows. In the next section we shall briefly review the differential equations which describe compositional flow in an ordinary porous medium. For the subject of flow of fluids through porous media, we refer to [6], [15], [19]. In §3 we shall derive a double porosity model of general compositional flow in a naturally fractured reservoir on physical grounds and then rederive it by homogenization in §4. Finally, in §5, the interactions between the matrix and fracture systems will be modeled through homogenization.

We end this section with a remark on the notation. Generally speaking, fracture quantities are represented by upper case letters, while the corresponding matrix quantities are indicated by the corresponding lower case letters.

2. Compositional flow in an ordinary porous medium. In this section we consider multiphase flow with mass interchange between phases in a porous medium. Specifically, the fluid phases will be gas, oil, and water, and we shall use the subscripts g , o , and w to refer to them. Denote the phase density and viscosity by ρ_α and μ_α , respectively, and the pressure in the α -phase by p_α , $\alpha = g, o, w$. Then, Darcy's law for the volumetric flow rates v_α in multiphase flow in a single porosity system takes the form

$$(2.1) \quad v_\alpha = -\frac{k_\alpha}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha g), \quad \alpha = g, o, w,$$

where k_α is the effective permeability for flow for the α -phase and g is the gravitational, downward-pointing, constant vector.

To state equations for the conservation of mass, we consider, as mentioned in the introduction, the general case where there are N chemical species, or components, each of

which may exist in any or all of the three phases. Let $c_{i\alpha}$ be the mass fraction of the i th component in the α -phase, $i = 1, \dots, N$. The mass of each phase is not conserved because of the possibility of transfer of various components between the phases. Instead, the total mass of each component must be conserved:

$$(2.2) \quad q_{\text{ext},i} = \phi \frac{\partial}{\partial t} \sum_{\alpha=g}^w c_{i\alpha} \rho_{\alpha} s_{\alpha} + \nabla \cdot \sum_{\alpha=g}^w c_{i\alpha} \rho_{\alpha} v_{\alpha} + \nabla \cdot \sum_{\alpha=g}^w j_{i\alpha}, \quad i = 1, 2, \dots, N,$$

where ϕ is the porosity of the system, s_{α} indicates the α -saturation, and $j_{i\alpha}$ represents the diffusive flux of the i th component in the α -phase. Here for expositional convenience, we present wells as external source terms of the form $q_{\text{ext},i}$ and take the sum over the three phases. Note that the mass fractions must add up to one:

$$(2.3) \quad \sum_{i=1}^N c_{i\alpha} = 1, \quad \alpha = g, o, w,$$

and that the diffusive fluxes satisfy the relation

$$(2.4) \quad \sum_{i=1}^N j_{i\alpha} = 0, \quad \alpha = g, o, w.$$

The least well understood term in (2.2) is that involving the diffusive fluxes $j_{i\alpha}$. The precise constitutive relations for these quantities remain unknown at this time; however, from a practical point of view the following straightforward extension of the single-phase Fick's law to multiphase flow has been in widespread use [14], [19]:

$$(2.5) \quad j_{i\alpha} = -\rho_{\alpha} d_{i\alpha} \nabla c_{i\alpha}, \quad i = 1, 2, \dots, N, \quad \alpha = g, o, w,$$

where $d_{i\alpha}$ is the diffusion coefficient of component i in the α -phase. For completeness, we shall adopt this definition for $j_{i\alpha}$ in the next two sections.

We remark that, under the hypotheses that the gas phase is missing, $c_{2o} = c_{3w} = 1, c_{2w} = c_{3o} = 0$, and ρ_o and ρ_w are constant, the equations reduce to the usual model for incompressible two-phase flow. Impose the customary property that the fluid fills the volume:

$$(2.6) \quad \sum_{\alpha=g}^w s_{\alpha} = 1.$$

Also, define the capillary pressure functions by

$$(2.7) \quad p_g - p_o = p_{c_{go}}(s_g, s_o, s_w),$$

$$(2.8) \quad p_o - p_w = p_{c_{ow}}(s_g, s_o, s_w).$$

It is not necessary to define a third capillary pressure, as this would be a simple combination of the other two:

$$p_{c_{gw}} = p_g - p_w = p_{c_{go}} + p_{c_{ow}}.$$

It should be noted that there are more dependent variables than there are differential relations; there are $3N + 18$ dependent variables: $c_{i\alpha}, i = 1, \dots, N, v_\alpha, p_\alpha, s_\alpha, \rho_\alpha, \mu_\alpha$, and $k_\alpha, \alpha = g, o, w$. It is then necessary to have $3N + 18$ independent relations in order to determine a solution of the system. The density ρ_α is assumed to be determined by $c_{1\alpha}, \dots, c_{N\alpha}$, and $p_\alpha : \rho_\alpha = \rho_\alpha(c_{1\alpha}, \dots, c_{N\alpha}, p_\alpha), \alpha = g, o, w$. The viscosities are assumed known in exactly the same fashion; the permeabilities are assumed known in terms of the saturations. Equations (2.1)–(2.8) provide $N + 9$ independent relations, differential or algebraic; the additional $2N$ relations are provided by the equilibrium relations needed to relate the mass fractions $c_{i\alpha}$ to mole fractions. If the system is made up of pure components, it is then trivial to convert reversibly from mass fractions to mole fractions. In reservoir simulation it is often computationally convenient (or necessary) to group several similar physical components into one mathematical component. In this case a pseudo-molecular weight, which is not a constant, is developed for this grouped component. This induces more unknowns and more relations into the problem [17].

Let m_i be the mass of component i :

$$(2.9) \quad m_i = \sum_{\alpha=g}^w m_{i\alpha}, \quad m_{i\alpha} = c_{i\alpha} \rho_\alpha s_\alpha.$$

Then, the total mole density in the system is given at a fixed time and location by

$$(2.10) \quad n = \sum_{i=1}^N n_i, \quad n_i = m_i / w_i,$$

where w_i is the molecular weight of the i th-component. The mole fraction of each component is calculated by

$$(2.11) \quad \omega_i = n_i / n.$$

We denote the fraction of the system mole density in the α -phase by u_α . We now perform a flash calculation to obtain the fractions u_α , the mole fractions $\omega_{i\alpha}$ in the α -phase, and the mass fractions $c_{i\alpha}, \alpha = g, o, w$. First, we have the restrictions

$$(2.12) \quad u_\alpha = \sum_{i=1}^N c_{i\alpha} \rho_\alpha s_\alpha / (n w_i),$$

$$(2.13) \quad \sum_{i=1}^N \omega_{i\alpha} = 1, \quad \alpha = g, o, w.$$

Next, we have $N - 1$ mole balances:

$$(2.14) \quad \sum_{\alpha=g}^w u_{\alpha} \omega_{i\alpha} = \omega_i, \quad i = 1, \dots, N - 1.$$

Finally, there are $2N$ conditions of phase equilibrium ([2], [15], [17], and the bibliographies therein):

$$(2.15) \quad \omega_{ig} = K_{igo} \omega_{io}, \quad i = 1, \dots, N,$$

$$(2.16) \quad \omega_{ig} = K_{igw} \omega_{iw}, \quad i = 1, \dots, N,$$

where the equilibrium ratios K_{igo} and K_{igw} are calculated from the pressures and the (assumed constant) temperature:

$$(2.17) \quad K_{igo} = K_{igo}(T, p_g, p_o), \quad K_{igw} = K_{igw}(T, p_g, p_w).$$

We now can compute the mass fractions by the formula

$$(2.18) \quad c_{i\alpha} = \omega_{i\alpha} w_i / \sum_{j=1}^N (w_j \omega_{j\alpha}), \quad i = 1, \dots, N, \quad \alpha = g, o, w.$$

Now, we see that there are as many relations as there are dependent variables; the system above is solvable if proper expressions of the equilibrium ratios are given [15], [17].

The phase permeabilities are calculated by use of the relations

$$(2.19) \quad k_{\alpha} = k k_{r\alpha}, \quad \alpha = g, o, w,$$

where $k_{r\alpha}$ is the classic relative permeability for the α -phase and k denotes the absolute permeability of the system.

Since three-phase relative permeabilities can be functions of the saturation of all three phases, their direct experimental determination is extremely difficult. This can make it necessary to obtain estimates from two-phase data measured on the water-oil and gas-oil subsystems (see, e.g., [1], [21], [22]); we shall not address the difficulties introduced by the use of any such correlations when the capillary pressures are ignored. Typical relative permeabilities as well as capillary pressure functions are shown in Figures 1, 2, and 3. In the system shown here, both imbibition and drainage occur simultaneously as oil is displaced by both water and gas. The relative permeability to the water phase is assumed to be a function only of its own saturation; similarly, the relative permeability to the gas phase is assumed to be a function of its saturation alone. The relative permeability to oil is calculated from two-phase data; it is a function of two independent saturations. The water-oil capillary pressure is taken to be independent of gas saturation and to be the same

function of water saturation as in the water-oil imbibition system. However, the gas-oil capillary pressure is allowed in general to be a function of both water and gas saturation [21], [22]. In summary, we assume the relations

$$(2.20) \quad k_{rw} = k_{rw}(s_w), \quad k_{rg} = k_{rg}(s_g), \quad k_{ro} = k_{ro}(s_w, s_g),$$

$$(2.21) \quad p_{c_{ow}} = p_{c_{ow}}(s_w), \quad p_{c_{go}} = p_{c_{go}}(s_w, s_g).$$

Water saturation
FIG. 1 Typical water-oil imbibition
relative permeability functions.

Gas saturation
FIG. 2 Typical gas-oil drainage
relative permeability function.

Displacing phase saturation
FIG. 3 Typical capillary pressure functions (psi).

3. Modelling of flow in a fractured medium by physical arguments. We shall indicate fracture quantities by upper case letters and matrix quantities by corresponding

lower case letters, as mentioned in the introduction, with the exception of the fracture densities, which are denoted by $\tilde{\rho}_\alpha$, $\alpha = g, o, w$.

Equations (2.1)–(2.2) are the starting point for describing the flow in a naturally fractured reservoir. We denote the fracture domain by $\mathcal{D}$ and the matrix domain by $\mathcal{D}_m$. Assume that $\mathcal{D}_m$ is a union of disjoint matrix blocks $\mathcal{D}_j$; i.e., $\mathcal{D}_m = \bigcup_j \mathcal{D}_j \subset \mathcal{D}$. Furthermore, we idealize the reservoir by assuming that the fractures form three sets of parallel, equally-spaced planes (see Figure 4), so that all matrix blocks are identical and the reservoir has a periodic structure. Let Q be a cell of this structure, containing a matrix block Q_m and half of the surrounding fracture Q_f (Figure 5).

FIG. 4. The periodic reservoir $\mathcal{D}$.

FIG. 5. The reservoir cell Q .

As in [3], [4], [7], [8], we shall assume that fluid does not flow directly from one matrix block to another; rather, it first flows into the fractures, and then it can pass into another block or remain in the fractures. The equations that describe the flow in the fracture system will contain source terms $q_{m,i}(x, t)$, which represent the flow of fluid from the matrix to the fractures; this term will be macroscopically distributed over the entire reservoir, as is the fracture flow itself.

Capillary pressure and relative permeability functions are somewhat different in the fractures than in the matrix blocks. Generally, one assumes that the fractures are essentially like spaces between two parallel planes [8], so that $S_{w,\min} = S_{o,\min} = S_{g,\min} = 0$ and $S_{w,\max} = S_{o,\max} = S_{g,\max} = 1$. Typical examples are shown in Figures 6, 7, and 8.

Water saturation

FIG. 6. Typical water-oil relative permeabilities in fractures.

Gas saturation

FIG. 7. Typical gas-oil relative permeabilities in fractures.

Displacing phase saturation

FIG. 8. Typical capillary pressure functions in fractures (psi).

Flow in the fractures for $x \in \mathcal{D}$, $t > 0$, and $i = 1, \dots, N$ is described by

$$(3.1) \quad q_{\text{ext},i} + q_{m,i} = \Phi \frac{\partial}{\partial t} \sum_{\alpha=g}^w C_{i\alpha} \tilde{\rho}_\alpha S_\alpha - \nabla \cdot \sum_{\alpha=g}^w C_{i\alpha} \tilde{\rho}_\alpha \Lambda_\alpha (\nabla P_\alpha - \tilde{\rho}_\alpha g) \\ - \nabla \cdot \sum_{\alpha=g}^w \tilde{\rho}_\alpha D_{i\alpha} \nabla C_{i\alpha},$$

$$(3.2) \quad \sum_{\alpha=g}^w S_\alpha = 1,$$

$$(3.3) \quad P_g - P_o = P_{c_{go}}(S_w, S_g),$$

$$(3.4) \quad P_o - P_w = P_{c_{ow}}(S_w),$$

where the phase mobilities, Λ_α , are defined by

$$(3.5) \quad \Lambda_\alpha = K_\alpha / \mu_\alpha, \quad \alpha = g, o, w,$$

and

$$(3.6) \quad \sum_i \tilde{\rho}_\alpha D_{i\alpha} \nabla C_{i\alpha} = 0, \quad \alpha = g, o, w.$$

The equations on the matrix block $\mathcal{D}_j$ are given for $t > 0$ by

$$(3.7) \quad 0 = \phi \frac{\partial}{\partial t} \sum_{\alpha=g}^w c_{i\alpha} \rho_\alpha s_\alpha - \nabla \cdot \sum_{\alpha=g}^w c_{i\alpha} \rho_\alpha \lambda_\alpha (\nabla p_\alpha - \rho_\alpha g) \\ - \nabla \cdot \sum_{\alpha=g}^w \rho_\alpha d_{i\alpha} \nabla c_{i\alpha},$$

$$(3.8) \quad \sum_{\alpha=g}^w s_\alpha = 1,$$

$$(3.9) \quad p_g - p_o = p_{c_{go}}(s_w, s_g),$$

$$(3.10) \quad p_o - p_w = p_{c_{ow}}(s_w),$$

where

$$(3.11) \quad \lambda_\alpha = k_\alpha / \mu_\alpha, \quad \sum_i \rho_\alpha d_{i\alpha} \nabla c_{i\alpha} = 0, \quad \alpha = g, o, w,$$

and we have assumed that the external sources do not affect the matrix system.

The mass fractions in the matrix and fracture systems can be computed as follows. In the matrix system,

$$(3.12) \quad m_i = \sum_{\alpha=g}^w c_{i\alpha} \rho_\alpha s_\alpha, \quad i = 1, \dots, N,$$

$$(3.13) \quad n = \sum_{j=1}^N n_j, \quad n_j = m_j / w_j,$$

$$(3.14) \quad \omega_i = n_i / n, \quad i = 1, \dots, N,$$

$$(3.15) \quad u_\alpha = \sum_{i=1}^N c_{i\alpha} \rho_\alpha s_\alpha / (n w_i), \quad \alpha = g, o, w$$

$$(3.16) \quad \sum_{i=1}^N \omega_{i\alpha} = 1, \quad \alpha = g, o, w,$$

$$(3.17) \quad \sum_{\alpha=g}^w u_\alpha \omega_{i\alpha} = \omega_i, \quad i = 1, \dots, N-1,$$

$$(3.18) \quad \omega_{ig} = K_{ig\alpha}(T, p_g, p_\alpha) \omega_{i\alpha}, \quad i = 1, \dots, N, \quad \alpha = o, w,$$

$$(3.19) \quad c_{i\alpha} = \omega_{i\alpha} w_i / \sum_{j=1}^N (w_j \omega_{j\alpha}), \quad i = 1, \dots, N, \quad \alpha = g, o, w,$$

$$(3.20) \quad \sum_{j=1}^N c_{j\alpha} = 1, \quad \alpha = g, o, w;$$

analogous relations hold for the fracture system.

The matrix source term can be defined as follows. The mass of component i leaving the j th block is

$$\int_{\partial \mathcal{D}_j} \sum_{\alpha=g}^w [c_{i\alpha} \rho_\alpha v_\alpha + j_{i\alpha}] \cdot \nu da(x),$$

where ν is the outer normal unit vector to the surface. The divergence theorem then implies that

$$\begin{aligned} \int_{\partial \mathcal{D}_j} \sum_{\alpha=g}^w [c_{i\alpha} \rho_\alpha v_\alpha + j_{i\alpha}] \cdot \nu da(x) &= \int_{\mathcal{D}_j} \nabla \cdot \sum_{\alpha=g}^w [c_{i\alpha} \rho_\alpha v_\alpha + j_{i\alpha}] dx \\ &= - \int_{\mathcal{D}_j} \phi \frac{\partial}{\partial t} \sum_{\alpha=g}^w c_{i\alpha} \rho_\alpha s_\alpha dx; \end{aligned}$$

thus, let

$$q_{m,i,j}(t) = -\frac{1}{|\mathcal{D}_j|} \int_{\mathcal{D}_j} \phi \frac{\partial}{\partial t} \sum_{\alpha=g}^w c_{i\alpha} \rho_{\alpha} s_{\alpha} dx,$$

and

$$(3.21) \quad q_{m,i} = \sum_j \chi_j(x) q_{m,i,j}(t), \quad x \in \mathcal{D}, t > 0, \quad i = 1, \dots, N,$$

where $|\mathcal{D}_j|$ is the volume of $\mathcal{D}_j$ and $\chi_j(x)$ is the characteristic function of the domain consisting of $\mathcal{D}_j$ together with half of the surrounding fractures.

We shall now consider the boundary conditions for the matrix problem. It is not easy to see how to impose them due to the complexity of multiphase flow processes. Recall that the theory of classical thermostatics makes use of an assumption of local thermodynamic equilibrium in order to model fluid phase behavior. The mathematical conditions for the local thermodynamic equilibrium of two coexisting fluids are that the pressures and chemical potentials in the two phases are equal. Namely, the so-called mechanical and chemical equilibrium conditions are satisfied [2], [9]. This observation will lead us to make a similar assumption. Since the matrix sees the fractures only at the surfaces of the blocks, we can treat the matrix and fracture fluids macroscopically as two coexisting phases and assume the fracture flow essentially to reach thermodynamic equilibrium with respect to the matrix flow on the block surfaces. Accordingly, we shall require continuity of phase pressures and chemical potentials on the matrix–fracture interface. However, the influence of gravity on the matrix flow must be taken into account.

For an individual period Q_j such that $\mathcal{D}_j \subset Q_j$, let $x_j^0 = (x_{j1}^0, x_{j2}^0, x_{j3}^0)$ be its centroid and define the vector function $g_{\alpha} = (g_{\alpha 1}, g_{\alpha 2}, g_{\alpha 3})$ by

$$\begin{aligned} g_{\alpha 1}(x_1, x_2, x_3) &= g_1 \int_{x_{j1}^0}^{x_1} \rho_{\alpha}(z, x_2, x_3) dz / \rho_{\alpha}(x_1, x_2, x_3), \\ g_{\alpha 2}(x_1, x_2, x_3) &= g_2 \int_{x_{j2}^0}^{x_2} \rho_{\alpha}(x_1, z, x_3) dz / \rho_{\alpha}(x_1, x_2, x_3), \\ g_{\alpha 3}(x_1, x_2, x_3) &= g_3 \int_{x_{j3}^0}^{x_3} \rho_{\alpha}(x_1, x_2, z) dz / \rho_{\alpha}(x_1, x_2, x_3), \end{aligned}$$

for any $(x_1, x_2, x_3) \in \mathcal{D}_j$, where $g = (g_1, g_2, g_3)$ and we have used the expression

$$\rho_{\alpha}(x_1, x_2, x_3) = \rho_{\alpha}(c_{1\alpha}(x_1, x_2, x_3), \dots, c_{N\alpha}(x_1, x_2, x_3), p_{\alpha}(x_1, x_2, x_3)).$$

Then, for each j , introduce the diagonal matrix

$$\text{diag}(g_{\alpha}) = \begin{pmatrix} g_{\alpha 1} & 0 & 0 \\ 0 & g_{\alpha 2} & 0 \\ 0 & 0 & g_{\alpha 3} \end{pmatrix},$$

and the matrix “mechanical potential” tensor

$$\psi_\alpha = p_\alpha I - \rho_\alpha \text{diag}(g_\alpha), \quad \alpha = g, o, w,$$

where I is the identity tensor. Analogously, on each period Q_j , we define the fracture vector $G_\alpha = (G_{\alpha 1}, G_{\alpha 2}, G_{\alpha 3})$ and the mechanical potential tensor by

$$\begin{aligned} G_{\alpha 1}(x_1, x_2, x_3) &= g_1 \int_{x_{j1}^0}^{x_1} \tilde{\rho}_\alpha(z, x_2, x_3) dz / \tilde{\rho}_\alpha(x_1, x_2, x_3), \\ G_{\alpha 2}(x_1, x_2, x_3) &= g_2 \int_{x_{j2}^0}^{x_2} \tilde{\rho}_\alpha(x_1, z, x_3) dz / \tilde{\rho}_\alpha(x_1, x_2, x_3), \\ G_{\alpha 3}(x_1, x_2, x_3) &= g_3 \int_{x_{j3}^0}^{x_3} \tilde{\rho}_\alpha(x_1, x_2, z) dz / \tilde{\rho}_\alpha(x_1, x_2, x_3), \\ \Psi_\alpha &= P_\alpha I - \tilde{\rho}_\alpha \text{diag}(G_\alpha), \quad \alpha = g, o, w, \end{aligned}$$

for $x = (x_1, x_2, x_3) \in Q_j$.

Now, we take, for each j , the boundary condition for the matrix flow

$$(3.22) \quad \psi_\alpha(x, t) \cdot \nu = \frac{1}{|\mathcal{D}_j|} \int_{\mathcal{D}_j} \Psi_\alpha(y, t) dy \cdot \nu, \quad x \in \partial \mathcal{D}_j, t > 0, \quad \alpha = g, o, w,$$

where “ ∂ ” indicates the boundary of the set.

As a remark, we can easily see that, for incompressible two-phase flow, equation (3.22) reduces to the usual boundary condition for matrix flow [8].

We now turn to the boundary condition of the matrix concentrations. In general, the chemical potentials are very complex functions of pressures and concentrations [2], [9]. The boundary condition for the matrix concentrations is a nonlinear set of algebraic equations if we require continuity of the chemical potentials on the matrix–fracture interface. From a practical and computational point of view, however, we may simply impose the condition

$$(3.23) \quad c_{i\alpha}(x, t) = \frac{1}{|\mathcal{D}_j|} \int_{\mathcal{D}_j} C_{i\alpha}(y, t) dy, \quad x \in \partial \mathcal{D}_j, t > 0, i = 1, \dots, N, \quad \alpha = g, o, w;$$

i.e., continuity of the concentrations. But, it should be emphasized that the physically reasonable boundary condition is much more complicated, as will be shown in §5.

The model will be completed by specifying the external boundary conditions and initial conditions for the system. For the case of no flow across the external boundary,

$$(3.24) \quad \sum_{\alpha=g}^w [C_{i\alpha} \tilde{\rho}_\alpha \Lambda_\alpha (\nabla P_\alpha - \tilde{\rho}_\alpha g) + \tilde{\rho}_\alpha D_{i\alpha} \nabla C_{i\alpha}] \cdot \nu = 0, \quad x \in \partial \mathcal{D}, \quad t > 0,$$

for $i = 1, 2, \dots, N$. Initial pressures and concentrations must be specified:

$$(3.25) \quad P_\alpha(x, 0) = P_{\text{init}, \alpha}(x), \quad C_{i\alpha}(x, 0) = C_{\text{init}, i\alpha}, \quad x \in \mathcal{D},$$

$$(3.26) \quad p_\alpha(x, 0) = p_{\text{init}, \alpha}(x), \quad c_{i\alpha}(x, 0) = c_{\text{init}, i\alpha}, \quad x \in \mathcal{D}_m,$$

with $i = 1, \dots, N$, $\alpha = g, o, w$. Initial equilibrium requires that (3.22) and (3.23) hold at $t = 0$.

4. Modelling of flow in a fractured reservoir by homogenization. We shall rederive the model obtained in the previous section by homogenization. To this end let us scale the periodic structure $\mathcal{D}$ by a parameter ε such that $0 < \varepsilon \leq 1$. When $\varepsilon = 1$, we have the original microscopic model. For $\varepsilon < 1$, we have a fractured reservoir filling the original reservoir with translates of cells $Q^\varepsilon = \varepsilon Q$ having side lengths ε times the original lengths. Let $\mathcal{D}_m^\varepsilon$ be the matrix part of the reservoir and $\mathcal{D}_f^\varepsilon$ the fracture part. We shall ignore the outer boundary of the reservoir; a rigorous homogenization could not, but the point here is to understand the flow in the interior of the reservoir, since (3.24) will again be imposed on the outer boundary.

In the homogenization process to be performed, our starting point is a microscopic model consisting of the proper governing equations in a porous medium posed over all of $\mathcal{D}$. The parameters in the equations on $\mathcal{D}_m^\varepsilon$ will be scaled by appropriate powers of ε to conserve flow in a sense to be specified below; consequently, the porous medium equations do not degenerate as ε tends to zero and the form of the matrix equations of the microscopic model and of the homogenized macroscopic model will be shown to coincide. This technique has been used in other papers (see, e.g., [3], [4], [8], [11]).

Let Φ^* be the porosity of an individual fracture; that is, the fracture porosity defined on a finer scale than the fracture width. Consequently, $\Phi^* \approx 1$. Let K^* be the absolute permeability of the fractures on the scale of the pores; since the fractures are treated as the space between parallel plates, K^* should be scalar. Finally, we shall assume that only molecular diffusion occurs in the fractures. Namely, the diffusion coefficients are assumed to be of the form $D_{i\alpha}^*(x)$.

The microscopic model is presented below. In the fractures, for $i = 1, \dots, N$, $\alpha = g, o, w$, and $x \in \mathcal{D}_f^\varepsilon$,

$$(4.1) \quad V_\alpha^\varepsilon = -\Lambda_\alpha^*(\nabla P_\alpha^\varepsilon - \tilde{\rho}_\alpha^\varepsilon g), \quad t > 0,$$

$$(4.2) \quad \tilde{q}_{i\alpha} = \Phi^* \frac{\partial}{\partial t} (C_{i\alpha}^\varepsilon \tilde{\rho}_\alpha^\varepsilon S_\alpha^\varepsilon) + \nabla \cdot (C_{i\alpha}^\varepsilon \tilde{\rho}_\alpha^\varepsilon V_\alpha^\varepsilon) - \nabla \cdot (\tilde{\rho}_\alpha^\varepsilon D_{i\alpha}^* \nabla C_{i\alpha}^\varepsilon), \quad t > 0,$$

$$(4.3) \quad P_\alpha^\varepsilon = P_{\text{init}, \alpha}, \quad C_{i\alpha}^\varepsilon = C_{\text{init}, i\alpha}, \quad t = 0,$$

and, for $x \in \partial \mathcal{D}_m^\varepsilon$, $t > 0$,

$$(4.4) \quad (C_{i\alpha}^\varepsilon \tilde{\rho}_\alpha^\varepsilon V_\alpha^\varepsilon - \tilde{\rho}_\alpha^\varepsilon D_{i\alpha}^* \nabla C_{i\alpha}^\varepsilon) \cdot \nu = \varepsilon (c_{i\alpha}^\varepsilon \rho_\alpha^\varepsilon v_\alpha^\varepsilon - \varepsilon \rho_\alpha^\varepsilon d_{i\alpha}^\varepsilon \nabla c_{i\alpha}^\varepsilon) \cdot \nu.$$

Here, $q_{i\alpha\beta}$ denotes the rate of mass transfer of component i from the β -phase to the α -phase and $\tilde{q}_{i\alpha} = q_{\text{ext}, i\alpha} + (q_{i\alpha\beta} - q_{i\beta\alpha}) + (q_{i\alpha\gamma} - q_{i\gamma\alpha})$. Also, set $\sum_{\alpha=g}^w q_{\text{ext}, i\alpha} = q_{\text{ext}, i}$. In the matrix, for $x \in \mathcal{D}_m^\varepsilon$ and $t > 0$,

$$(4.5) \quad v_\alpha^\varepsilon = -\lambda_\alpha^\varepsilon (\varepsilon \nabla p_\alpha^\varepsilon - \rho_\alpha^\varepsilon g),$$

$$(4.6) \quad q_{\text{ext}, i} = \phi^\varepsilon \frac{\partial}{\partial t} \sum_{\beta=g}^w c_{i\beta}^\varepsilon \rho_\beta^\varepsilon S_\beta^\varepsilon + \varepsilon \nabla \cdot \sum_{\beta=g}^w c_{i\beta}^\varepsilon \rho_\beta^\varepsilon v_\beta^\varepsilon - \varepsilon^2 \nabla \cdot \sum_{\beta=g}^w \rho_\beta^\varepsilon d_{i\beta}^\varepsilon \nabla c_{i\beta}^\varepsilon,$$

and

$$(4.7) \quad c_{i\alpha}^\varepsilon = c_{\text{init},i\alpha}, \quad t = 0,$$

$$(4.8) \quad p_\alpha^\varepsilon = p_{\text{init},\alpha}, \quad t = 0.$$

To these equations we add a collection of functional relations. For $x \in \mathcal{D}_f^\varepsilon, t > 0$, the phase saturations and the capillary pressures satisfy

$$(4.9) \quad \sum_{\alpha=g}^w S_\alpha^\varepsilon = 1,$$

$$(4.10) \quad P_g^\varepsilon - P_o^\varepsilon = P_{c_{go}}(S_w^\varepsilon, S_g^\varepsilon),$$

$$(4.11) \quad P_o^\varepsilon - P_w^\varepsilon = P_{c_{ow}}(S_w^\varepsilon),$$

while for $x \in \mathcal{D}_m^\varepsilon, t > 0$,

$$(4.12) \quad \sum_{\alpha=g}^w s_\alpha^\varepsilon = 1,$$

$$(4.13) \quad p_g^\varepsilon - p_o^\varepsilon = p_{c_{go}}(s_w^\varepsilon, s_g^\varepsilon),$$

$$(4.14) \quad p_o^\varepsilon - p_w^\varepsilon = p_{c_{ow}}(s_w^\varepsilon),$$

Finally, we have the following phase behavior relations. For $x \in \mathcal{D}_f^\varepsilon, t > 0$,

$$(4.15) \quad M_i^\varepsilon = \sum_{\alpha=g}^w C_{i\alpha}^\varepsilon \tilde{\rho}_\alpha^\varepsilon S_\alpha^\varepsilon, \quad i = 1, \dots, N,$$

$$(4.16) \quad N^\varepsilon = \sum_{j=1}^N N_j^\varepsilon, \quad N_j^\varepsilon = M_j^\varepsilon / W_j,$$

$$(4.17) \quad \Omega_i^\varepsilon = N_i^\varepsilon / N^\varepsilon, \quad i = 1, \dots, N,$$

$$(4.18) \quad U_\alpha^\varepsilon = \sum_{i=1}^N C_{i\alpha}^\varepsilon \tilde{\rho}_\alpha^\varepsilon S_\alpha^\varepsilon / (N^\varepsilon W_i), \quad \alpha = g, o, w$$

$$(4.19) \quad \sum_{i=1}^N \Omega_{i\alpha}^\varepsilon = 1, \quad \alpha = g, o, w,$$

$$(4.20) \quad \sum_{\alpha=g}^w U_\alpha^\varepsilon \Omega_{i\alpha}^\varepsilon = \Omega_i^\varepsilon, \quad i = 1, \dots, N-1,$$

$$(4.21) \quad \Omega_{ig}^\varepsilon = K_{ig\alpha}(T, P_g^\varepsilon, P_\alpha^\varepsilon) \Omega_{i\alpha}^\varepsilon, \quad i = 1, \dots, N, \quad \alpha = o, w,$$

$$(4.22) \quad C_{i\alpha}^\varepsilon = \Omega_{i\alpha}^\varepsilon W_i / \sum_{j=1}^N (W_j \Omega_{j\alpha}^\varepsilon), \quad i = 1, \dots, N, \quad \alpha = g, o, w,$$

$$(4.23) \quad \sum_{j=1}^N C_{j\alpha}^\varepsilon = 1, \quad \alpha = g, o, w.$$

Similarly, for $x \in \mathcal{D}_m^\varepsilon, t > 0$,

$$(4.24) \quad m_i^\varepsilon = \sum_{\alpha=g}^w c_{i\alpha}^\varepsilon \rho_\alpha^\varepsilon s_\alpha^\varepsilon, \quad i = 1, \dots, N,$$

$$(4.25) \quad n^\varepsilon = \sum_{j=1}^N n_j^\varepsilon, \quad n_j^\varepsilon = m_j^\varepsilon / w_j,$$

$$(4.26) \quad \omega_i^\varepsilon = n_i^\varepsilon / n^\varepsilon, \quad i = 1, \dots, N,$$

$$(4.27) \quad u_\alpha^\varepsilon = \sum_{i=1}^N c_{i\alpha}^\varepsilon \rho_\alpha^\varepsilon s_\alpha^\varepsilon / (n^\varepsilon w_i), \quad \alpha = g, o, w$$

$$(4.28) \quad \sum_{i=1}^N \omega_{i\alpha}^\varepsilon = 1, \quad \alpha = g, o, w,$$

$$(4.29) \quad \sum_{\alpha=g}^w u_\alpha^\varepsilon \omega_{i\alpha}^\varepsilon = \omega_i^\varepsilon, \quad i = 1, \dots, N-1,$$

$$(4.30) \quad \omega_{ig}^\varepsilon = K_{ig\alpha}(T, p_g^\varepsilon, p_\alpha^\varepsilon) \omega_{i\alpha}^\varepsilon, \quad i = 1, \dots, N, \quad \alpha = o, w,$$

$$(4.31) \quad c_{i\alpha}^\varepsilon = \omega_{i\alpha}^\varepsilon w_i / \sum_{j=1}^N (w_j \omega_{j\alpha}^\varepsilon), \quad i = 1, \dots, N, \quad \alpha = g, o, w,$$

$$(4.32) \quad \sum_{j=1}^N c_{j\alpha}^\varepsilon = 1, \quad \alpha = g, o, w.$$

Notice that we have the following equations of state for $\alpha = g, o, w$:

$$(4.33) \quad \tilde{\rho}_\alpha^\varepsilon = \tilde{\rho}_\alpha(C_{1\alpha}^\varepsilon, \dots, C_{N\alpha}^\varepsilon, P_\alpha^\varepsilon), \quad x \in \mathcal{D}_f^\varepsilon,$$

$$(4.34) \quad \rho_\alpha^\varepsilon = \rho_\alpha(c_{1\alpha}^\varepsilon, \dots, c_{N\alpha}^\varepsilon, p_\alpha^\varepsilon), \quad x \in \mathcal{D}_m^\varepsilon;$$

the density functions are the same in the fractures and matrix blocks, but their arguments are different.

It is convenient to introduce

$$(4.35a) \quad \mathcal{G}_\alpha(C_{1\alpha}^\varepsilon, \dots, C_{N\alpha}^\varepsilon, P_\alpha^\varepsilon) = \tilde{\rho}_\alpha(C_{1\alpha}^\varepsilon, \dots, C_{N\alpha}^\varepsilon, P_\alpha^\varepsilon)g,$$

$$(4.35b) \quad \hat{g}_\alpha(c_{1\alpha}^\varepsilon, \dots, c_{N\alpha}^\varepsilon, p_\alpha^\varepsilon) = \rho_\alpha(c_{1\alpha}^\varepsilon, \dots, c_{N\alpha}^\varepsilon, p_\alpha^\varepsilon)g,$$

for $\alpha = g, o, w$. Finally, as in (2.4), we require that

$$(4.36) \quad \sum_{j=1}^N \tilde{\rho}_\alpha^\varepsilon D_{i\alpha}^* \nabla C_{i\alpha}^\varepsilon = 0, \quad x \in \mathcal{D}_f^\varepsilon, t > 0, \quad \alpha = g, o, w,$$

$$(4.37) \quad \sum_{j=1}^N \rho_\alpha^\varepsilon d_{i\alpha}^\varepsilon \nabla c_{i\alpha}^\varepsilon = 0, \quad x \in \mathcal{D}_m^\varepsilon, t > 0, \quad \alpha = g, o, w.$$

The ε in (4.4)–(4.6) properly scales the matrix flux term as mentioned above; v_α^ε is defined by (4.5), so that the matrix permeability has been scaled by ε^2 , while gravity in the matrix has been scaled by ε^{-1} . This essentially preserves the total mass interchange between the matrix blocks and the fractures within a fixed volume of the reservoir as ε varies. See [4] and [8].

We now perform homogenization heuristically. Assume that any point is described by two variables: $x \in \mathcal{D}$ giving the general location of the point and $y \in Q$ the location of the point within an ε -cell εQ . Clearly, x and y are related by the scale

$$y \sim \varepsilon^{-1} x,$$

up to a translation. Consequently,

$$(4.38) \quad \nabla \sim \varepsilon^{-1} \nabla_y + \nabla_x,$$

where ∇_y and ∇_x are the gradient operators with respect to y and x , respectively. The solutions are then assumed to have the asymptotic form [18]

$$(4.39) \quad \Psi^\varepsilon(x, t) \sim \sum_{k=0}^{\infty} \varepsilon^k \Psi^k(x, y, t),$$

$$(4.40) \quad \psi^\varepsilon(x, t) \sim \sum_{k=0}^{\infty} \varepsilon^k \psi^k(x, y, t),$$

for generic functions Ψ^ε and ψ^ε associated with the fracture and matrix systems, respectively. Each of the functions Ψ^k is assumed to be periodic in y in Q_f . Next, functions of Ψ^ε or ψ^ε can be expanded by Taylor's Theorem as

$$(4.41) \quad \varphi(\xi^\varepsilon) = \varphi(\xi^0) + \varphi'(\xi^\varepsilon - \xi^0) + \cdots \equiv \varphi(\xi^0) + \varepsilon \varphi^1 + \varepsilon^2 \varphi^2 + \cdots, \quad \xi = \Psi \text{ or } \psi,$$

for some $\varphi^1, \varphi^2, \dots$. Finally, if $\zeta^\varepsilon(x) = \zeta(\varepsilon^{-1}x)$ is periodic of period εQ , then,

$$(4.42) \quad \zeta^\varepsilon(x) \sim \zeta(y).$$

We shall substitute the formal asymptotic expansions given by (4.38)–(4.42) into (4.1)–(4.37) and equate coefficients of like powers of ε . The process will be rather long and somewhat tedious, but it will lead to our macroscopic model, which will be the set of equations for the leading terms in the expansions.

(Note that the matrix blocks have been assumed to possess identical properties, so that ϕ and k are periodic. More generally, these may vary slowly over the reservoir: $\phi^\varepsilon(x) = \phi(x, \varepsilon^{-1}x)$, $k^\varepsilon(x) = k(x, \varepsilon^{-1}x)$.)

From (4.1), for $(x, y) \in \mathcal{D} \times Q_f$ and $t > 0$, the ε^{-1} and ε^0 terms give

$$(4.1/-1) \quad 0 = -\Lambda_\alpha^*(S_g^0, S_o^0, S_w^0) \nabla_y P_\alpha^0, \quad \alpha = g, o, w,$$

$$(4.1/0) \quad V_\alpha^0 = -\Lambda_\alpha^*(S_g^0, S_o^0, S_w^0)(\nabla_y P_\alpha^1 + \nabla_x P_\alpha^0 - \mathcal{G}_\alpha^0) + \Lambda_\alpha^1 \nabla_y P_\alpha^0, \quad \alpha = g, o, w,$$

where $\mathcal{G}_\alpha^0 = \mathcal{G}_\alpha(C_{1\alpha}^0, \dots, C_{N\alpha}^0, P_\alpha^0)$. From (4.2), for $(x, y) \in \mathcal{D} \times Q_f$ and $t > 0$, the ε^{-2} , ε^{-1} , and ε^0 terms lead to

$$(4.2/-2) \quad 0 = -\nabla_y \cdot (\tilde{\rho}_\alpha^0 D_{i\alpha}^* \nabla_y C_{i\alpha}^0),$$

$$(4.2/-1) \quad 0 = \nabla_y \cdot (C_{i\alpha}^0 \tilde{\rho}_\alpha^0 V_\alpha^0) - \nabla_x \cdot (\tilde{\rho}_\alpha^0 D_{i\alpha}^* \nabla_y C_{i\alpha}^0) \\ - \nabla_y \cdot (D_{i\alpha}^* [\tilde{\rho}_\alpha^0 \nabla_x C_{i\alpha}^0 + \tilde{\rho}_\alpha^0 \nabla_y C_{i\alpha}^1 + \tilde{\rho}_\alpha^1 \nabla_y C_{i\alpha}^0]),$$

$$(4.2/0) \quad \tilde{q}_{i\alpha} = \Phi^* \frac{\partial}{\partial t} (C_{i\alpha}^0 \tilde{\rho}_\alpha^0 S_\alpha^0) + \nabla_y \cdot (C_{i\alpha}^0 \tilde{\rho}_\alpha^0 V_\alpha^1 + C_{i\alpha}^0 \tilde{\rho}_\alpha^1 V_\alpha^0 + C_{i\alpha}^1 \tilde{\rho}_\alpha^0 V_\alpha^0) \\ + \nabla_x \cdot (C_{i\alpha}^0 \tilde{\rho}_\alpha^0 V_\alpha^0) - \nabla_y \cdot (D_{i\alpha}^* [\tilde{\rho}_\alpha^0 \nabla_y C_{i\alpha}^2 + \tilde{\rho}_\alpha^1 \nabla_y C_{i\alpha}^1 + \tilde{\rho}_\alpha^2 \nabla_y C_{i\alpha}^0 \\ + \tilde{\rho}_\alpha^0 \nabla_x C_{i\alpha}^1 + \tilde{\rho}_\alpha^1 \nabla_x C_{i\alpha}^0]) - \nabla_x \cdot (D_{i\alpha}^* [\tilde{\rho}_\alpha^0 \nabla_x C_{i\alpha}^0 \\ + \tilde{\rho}_\alpha^0 \nabla_y C_{i\alpha}^1 + \tilde{\rho}_\alpha^1 \nabla_y C_{i\alpha}^0]),$$

for $i = 1, 2, \dots, N$, $\alpha = g, o, w$. For $(x, y) \in \mathcal{D} \times \partial Q_m$ and $t > 0$, it follows from the ε^{-1} , ε^0 , and ε^1 terms of (4.4) that

$$(4.4/-1) \quad \tilde{\rho}_\alpha^0 D_{i\alpha}^* \nabla_y C_{i\alpha}^0 \cdot \nu = 0,$$

$$(4.4/0) \quad (C_{i\alpha}^0 \tilde{\rho}_\alpha^0 V_\alpha^0 - D_{i\alpha}^* [\tilde{\rho}_\alpha^0 \nabla_y C_{i\alpha}^1 + \tilde{\rho}_\alpha^1 \nabla_y C_{i\alpha}^0] + \tilde{\rho}_\alpha^0 \nabla_x C_{i\alpha}^0) \cdot \nu = 0,$$

$$(4.4/1) \quad (C_{i\alpha}^0 \tilde{\rho}_\alpha^0 V_\alpha^1 + C_{i\alpha}^0 \tilde{\rho}_\alpha^1 V_\alpha^0 + C_{i\alpha}^1 \tilde{\rho}_\alpha^0 V_\alpha^0 - D_{i\alpha}^* [\tilde{\rho}_\alpha^0 \nabla_x C_{i\alpha}^1 + \tilde{\rho}_\alpha^1 \nabla_x C_{i\alpha}^0 \\ + \tilde{\rho}_\alpha^0 \nabla_y C_{i\alpha}^2 + \tilde{\rho}_\alpha^1 \nabla_y C_{i\alpha}^1 + \tilde{\rho}_\alpha^2 \nabla_y C_{i\alpha}^0]) \cdot \nu = (c_{i\alpha}^0 \rho_\alpha^0 v_\alpha^0 - \rho_\alpha^0 d_{i\alpha}^0 \nabla_y c_{i\alpha}^0) \cdot \nu,$$

with $i = 1, \dots, N$ and $\alpha = g, o, w$. The ε^0 term of (4.3) gives

$$(4.3/0) \quad P_\alpha^0 = P_{\text{init}, \alpha}, \quad C_{i\alpha}^0 = C_{\text{init}, i\alpha}, \quad (x, y) \in \mathcal{D} \times Q_f, t = 0.$$

From (4.5) and (4.6), for $(x, y) \in \mathcal{D} \times Q_m$ and $t > 0$, the ε^0 terms yield

$$(4.5/0) \quad v_\alpha^0 = -\lambda_\alpha(y, s_g^0, s_o^0, s_w^0)(\nabla_y p_\alpha^0 - \hat{g}_\alpha^0), \quad \alpha = g, o, w,$$

$$(4.6/0) \quad q_{\text{ext}, i} = \phi(y) \frac{\partial}{\partial t} \sum_{\alpha=g}^w c_{i\alpha}^0 \rho_\alpha^0 s_\alpha^0 + \nabla_y \cdot \sum_{\alpha=g}^w c_{i\alpha}^0 \rho_\alpha^0 v_\alpha^0 - \nabla_y \cdot \sum_{\alpha=g}^w \rho_\alpha^0 d_{i\alpha}^0 \nabla_y c_{i\alpha}^0,$$

where $\widehat{g}_\alpha^0 = \widehat{g}_\alpha(c_{1\alpha}^0, \dots, c_{N\alpha}^0, p_\alpha^0)$. From (4.7) and (4.8), the ε^0 terms give

$$(4.7/0) \quad c_{i\alpha}^0 = c_{\text{init}, i\alpha}, \quad (x, y) \in \mathcal{D} \times Q_m, \quad t = 0, \quad \alpha = g, o, w,$$

$$(4.8/0) \quad p_\alpha^0 = p_{\text{init}, \alpha}, \quad (x, y) \in \mathcal{D} \times Q_m, \quad t = 0, \quad \alpha = g, o, w.$$

From (4.9)–(4.11), for $(x, y) \in \mathcal{D} \times Q_f$ and $t > 0$, the ε^0 terms lead to

$$(4.9/0) \quad \sum_{\alpha=g}^w S_\alpha^0 = 1,$$

$$(4.10/0) \quad P_g^0 - P_o^0 = P_{c_{go}}(S_w^0, S_g^0),$$

$$(4.11/0) \quad P_o^0 - P_w^0 = P_{c_{ow}}(S_w^0);$$

correspondingly, from (4.12)–(4.14), for $(x, y) \in \mathcal{D} \times Q_m$ and $t > 0$,

$$(4.12/0) \quad \sum_{\alpha=g}^w s_\alpha^0 = 1,$$

$$(4.13/0) \quad p_g^0 - p_o^0 = p_{c_{go}}(s_w^0, s_g^0),$$

$$(4.14/0) \quad p_o^0 - p_w^0 = p_{c_{ow}}(s_w^0).$$

Next, for $i = 1, \dots, N$, and for $(x, y) \in \mathcal{D} \times Q_f$ and $t > 0$, the phase relations in the fractures, (4.15)–(4.32), lead to

$$(4.15/0) \quad M_i^0 = \sum_{\alpha=g}^w C_{i\alpha}^0 \widehat{\rho}_\alpha^0 S_\alpha^0,$$

$$(4.16/0) \quad N^0 = \sum_{j=1}^N N_j^0, \quad N_j^0 = M_j^0 / W_j,$$

$$(4.17/0) \quad \Omega_i^0 = N_i^0 / N^0,$$

$$(4.18/0) \quad U_\alpha^0 = \sum_{i=1}^N C_{i\alpha}^0 \widehat{\rho}_\alpha^0 S_\alpha^0 / (N^0 W_i), \quad \alpha = g, o, w,$$

$$(4.19/0) \quad \sum_{i=1}^N \Omega_{i\alpha}^0 = 1, \quad \alpha = g, o, w,$$

$$(4.20/0) \quad \sum_{\alpha=g}^w U_\alpha^0 \Omega_{i\alpha}^0 = \Omega_i^0,$$

$$(4.21/0) \quad \Omega_{ig}^0 = K_{ig\alpha}(T, P_g^0, P_\alpha^0) \Omega_{i\alpha}^0, \quad \alpha = o, w,$$

$$(4.22/0) \quad C_{i\alpha}^0 = \Omega_{i\alpha}^0 W_i / \sum_{j=1}^N (W_j \Omega_{j\alpha}^0), \quad \alpha = g, o, w,$$

$$(4.23/0) \quad \sum_{j=1}^N C_{j\alpha}^0 = 1, \quad \alpha = g, o, w,$$

and, for $(x, y) \in \mathcal{D} \times Q_m$ and $t > 0$, those in the matrix blocks imply that

$$(4.24/0) \quad m_i^0 = \sum_{\alpha=g}^w c_{i\alpha}^0 \rho_\alpha s_\alpha^0,$$

$$(4.25/0) \quad n^0 = \sum_{j=1}^N n_j^0, \quad n_j^0 = m_j^0 / w_j,$$

$$(4.26/0) \quad \omega_i^0 = n_i^0 / n^0,$$

$$(4.27/0) \quad u_\alpha^0 = \sum_{i=1}^N c_{i\alpha}^0 \rho_\alpha s_\alpha^0 / (n^0 w_i), \quad \alpha = g, o, w,$$

$$(4.28/0) \quad \sum_{i=1}^N \omega_{i\alpha}^0 = 1, \quad \alpha = g, o, w,$$

$$(4.29/0) \quad \sum_{\alpha=g}^w u_\alpha^0 \omega_{i\alpha}^0 = \omega_i^0,$$

$$(4.30/0) \quad \omega_{ig}^0 = K_{ig\alpha}(T, p_g^0, p_\alpha^0) \omega_{i\alpha}^0, \quad \alpha = o, w,$$

$$(4.31/0) \quad c_{i\alpha}^0 = \omega_{i\alpha}^0 w_i / \sum_{j=1}^N (w_j \omega_{j\alpha}^0), \quad \alpha = g, o, w,$$

$$(4.32/0) \quad \sum_{j=1}^N c_{j\alpha}^0 = 1, \quad \alpha = g, o, w.$$

The diffusion terms in the fractures can be treated by use of (4.36); for $(x, y) \in \mathcal{D} \times Q_f$ and $t > 0$, the ε^0 term implies that

$$(4.36/0) \quad \sum_{i=1}^N D_{i\alpha}^* (\tilde{\rho}_\alpha^0 \nabla_y C_{i\alpha}^1 + \tilde{\rho}_\alpha^1 \nabla_y C_{i\alpha}^0 + \tilde{\rho}_\alpha^0 \nabla_x C_{i\alpha}^0) = 0.$$

Similarly, by (4.37), for $(x, y) \in \mathcal{D} \times Q_m$ and $t > 0$, the ε^{-1} term gives the diffusion relations in the blocks:

$$(4.37/-1) \quad \sum_{i=1}^N \rho_\alpha^0 d_{i\alpha}^0 \nabla_y c_{i\alpha}^0 = 0.$$

We shall now analyze these asymptotic relations. First, we would like to show that

$$(4.43) \quad P_\alpha^0 = P_\alpha^0(x, t) \text{ only, } \alpha = g, o, w.$$

This is not easy to see; system (4.1/-1) is degenerate, since Λ_α^* can be zero. However, if we consider the system shown in Figures 1 and 2; i.e.,

$$\begin{aligned} \Lambda_g^* &= K^*(x)K_{rg}(S_g^0)/\mu_g, \\ \Lambda_o^* &= K^*(x)K_{ro}(S_w^0, S_g^0)/\mu_o, \\ \Lambda_w^* &= K^*(x)K_{rw}(S_w^0)/\mu_w, \end{aligned}$$

we can easily conclude that one of the following must hold:

- i) $S_g^0 = S_{g,\min}, \quad P_o^0 = P_o^0(x, t) \text{ only,} \quad \text{and } S_w^0 = S_{w,\min},$
- ii) $S_g^0 = S_{g,\min}, \quad P_o^0 = P_o^0(x, t) \text{ only,} \quad \text{and } P_w^0 = P_w^0(x, t) \text{ only,}$
- iii) $S_w^0 = S_{w,\min}, \quad P_o^0 = P_o^0(x, t) \text{ only,} \quad \text{and } P_g^0 = P_g^0(x, t) \text{ only,}$
- iv) (4.43) holds,
- v) $S_o^0 = S_{o,\min}, \quad S_g^0 = S_{g,\min}, \quad \text{and } P_w^0 = P_w^0(x, t) \text{ only,}$
- vi) $S_o^0 = S_{o,\min}, \quad S_w^0 = S_{w,\min}, \quad \text{and } P_g^0 = P_g^0(x, t) \text{ only,}$
- vii) $S_o^0 = S_{o,\min}, \quad P_g^0 = P_g^0(x, t) \text{ only,} \quad \text{and } P_w^0 = P_w^0(x, t) \text{ only.}$

But, (4.9/0)–(4.11/0) imply that if one of i), ii), iii), v), or vi) is valid, (4.43) holds and

$$(4.44) \quad S_\alpha^0 = S_\alpha^0(x, t) \text{ only, } \alpha = g, o, w.$$

Now we consider case vii) and assume that $P_{c_{go}}$ is a function of gas fraction of the hydrocarbon volume as shown in Figure 6. Add (4.10/0) and (4.11/0) and then differentiate the resulting equation to obtain

$$0 = P'_{c_{go}}(S_g^0/(S_o^0 + S_g^0))S_o^0 \nabla_y S_g^0 / (S_o^0 + S_g^0)^2 + P'_{c_{ow}}(S_w^0) \nabla_y S_w^0,$$

so that it follows from (4.9/0) that

$$0 = [P'_{c_{go}}(S_g^0/(S_o^0 + S_g^0))S_o^0/(S_o^0 + S_g^0)^2 - P'_{c_{ow}}(S_w^0)] \nabla_y S_g^0.$$

This shows that S_g^0 is independent of y , since the first term in the bracket above is nonnegative and the second term is positive. Hence, (4.43) and (4.44) hold, as required. Thus, all terms containing $\nabla_y P_\alpha^0$ and $\nabla_y S_\alpha^0$ drop out.

Also, by (4.2/-2), (4.4/-1), and the assumption on $D_{i\alpha}^*$, we have

$$(4.45) \quad C_{i\alpha}^0 = C_{i\alpha}^0(x, t) \text{ only, } i = 1, \dots, N, \quad \alpha = g, o, w;$$

consequently,

$$(4.46) \quad \tilde{\rho}_\alpha^0 = \tilde{\rho}_\alpha^0(C_\alpha^0, P_\alpha^0) \quad \alpha = g, o, w,$$

is independent of y . Hence, all terms containing $\nabla_y \tilde{\rho}_\alpha^0$ and $\nabla_y C_{i\alpha}^0$ drop out as well.

Next, we relate P_α^1 to P_α^0 . To this end, note that P_α^1 satisfies (4.1/0), (4.2/-1), and (4.4/0), which by summation on i , (4.23/0), and (4.36/0), can be rewritten as

$$\begin{aligned} -\nabla_y \cdot (\Lambda_\alpha^* \nabla_y P_\alpha^1) &= 0, & (x, y) \in \mathcal{D} \times Q_f, \quad t > 0, \\ \Lambda_\alpha^* (\nabla_y P_\alpha^1 + \nabla_x P_\alpha^0 - \mathcal{G}_\alpha^0) \cdot \nu &= 0, & (x, y) \in \mathcal{D} \times \partial Q_m, \quad t > 0, \end{aligned}$$

for $\alpha = g, o, w$. This is a linear system for the P_α^1 , $\alpha = g, o, w$. The P_α^1 are determined by this system up to additive functions of x and t , at least if $\Lambda_\alpha^* \neq 0$, $\alpha = g, o, w$. Following Sanchez-Palencia [18], we now define auxiliary functions $a_j(y)$, $j = 1, 2, 3$, as Q_f -periodic solutions to the problems

$$(4.47) \quad \nabla_y^2 a_j(y) = 0, \quad y \in Q_f, \quad j = 1, 2, 3,$$

$$(4.48) \quad \nabla_y a_j \cdot \nu = -e_j \cdot \nu, \quad y \in \partial Q_m, \quad j = 1, 2, 3,$$

where e_j is the unit vector in the j th direction. Since P_α^0 , S_α^0 , and $\tilde{\rho}_\alpha^0$ are independent of y , P_α^1 can be expressed in terms of P_α^0 by

$$(4.49) \quad P_\alpha^1(x, y, t) = \sum_{j=1}^3 a_j(y) [\partial_j P_\alpha^0(x, t) - \mathcal{G}_{\alpha j}^0] + \theta_\alpha(x, t), \quad \alpha = g, o, w,$$

where $\partial_j = \partial/\partial x_j$, $\mathcal{G}_{\alpha j}^0$ is the j th component of $\mathcal{G}_\alpha^0$ and θ_α are a function of x and t ; since only y -derivatives of P_α^1 will be required below, θ_α need not be evaluated.

Similarly, from (4.2/-1) and (4.4/0), we find that

$$(4.50) \quad C_{i\alpha}^1 = \sum_{j=1}^3 a_j(y) \partial_j C_{i\alpha}^0 + \gamma_{i\alpha}(x, t), \quad i = 1, \dots, N, \quad \alpha = g, o, w,$$

by (4.45) and (4.46), where $\gamma_{i\alpha}(x, t)$ are again functions of x and t whose values will not affect the argument below.

We shall derive a macroscopic Darcy's law for the fracture flow. Applying (4.49), (4.1/0) becomes

$$(4.51) \quad V_\alpha^0 = -\Lambda_\alpha^* [I + \mathcal{A}] [\nabla_x P_\alpha^0 - \mathcal{G}_\alpha^0], \quad \alpha = g, o, w,$$

where $\mathcal{A}$ is the tensor whose (k, l) -component is $\partial a_l / \partial y_k$. We define a local average

$$(4.52) \quad \bar{\varphi} = \frac{1}{|Q|} \int_{Q_f} \varphi(y) dy, \quad \text{for a function or tensor } \varphi,$$

and locally average equation (4.51) to obtain

$$(4.53) \quad \bar{V}_\alpha^0 = -\Lambda_\alpha^* \left[\frac{|Q_f|}{|Q|} I + \bar{\mathcal{A}} \right] [\nabla_x P_\alpha^0 - \mathcal{G}_\alpha^0], \quad \alpha = g, o, w.$$

We now define the fracture permeability tensor and the fracture mobilities by

$$(4.54) \quad \begin{aligned} K &= K^* [|Q_f|I/|Q| + \bar{\mathcal{A}}], \\ \Lambda_\alpha &= K K_{r\alpha} / \mu_\alpha, \quad \alpha = g, o, w; \end{aligned}$$

consequently, (4.53) takes the form

$$(4.55) \quad \bar{V}_\alpha^0 = -\Lambda_\alpha (\nabla P_\alpha^0 - \tilde{\rho}_\alpha^0 g), \quad x \in \mathcal{D}, \quad t > 0, \quad \alpha = g, o, w,$$

since $\mathcal{G}_\alpha^0 = \tilde{\rho}_\alpha^0 g$. This is Darcy's law for the macroscopic volumetric flow rate $\bar{V}_\alpha^0$ in the fractures.

We shall next verify that the total mass of each component is conserved by locally averaging equation (4.2/0) and summing on α to obtain

$$(4.56) \quad \begin{aligned} \frac{|Q_f|}{|Q|} \Phi^* \frac{\partial}{\partial t} \sum_{\alpha=g}^w C_{i\alpha}^0 \tilde{\rho}_\alpha^0 S_\alpha^0 &+ \frac{1}{|Q|} \int_{Q_f} \nabla_y \cdot \sum_{\alpha=g}^w (C_{i\alpha}^0 \tilde{\rho}_\alpha^0 V_\alpha^1 + C_{i\alpha}^0 \tilde{\rho}_\alpha^1 V_\alpha^0 + C_{i\alpha}^1 \tilde{\rho}_\alpha^0 V_\alpha^0 \\ &- D_{i\alpha}^* [\tilde{\rho}_\alpha^0 \nabla_y C_{i\alpha}^2 + \tilde{\rho}_\alpha^1 \nabla_y C_{i\alpha}^1 + \tilde{\rho}_\alpha^0 \nabla_x C_{i\alpha}^1 + \tilde{\rho}_\alpha^1 \nabla_x C_{i\alpha}^0]) dy \\ &- \frac{1}{|Q|} \int_{Q_f} \nabla_x \cdot \sum_{\alpha=g}^w D_{i\alpha}^* (\tilde{\rho}_\alpha^0 \nabla_y C_{i\alpha}^1 + \tilde{\rho}_\alpha^1 \nabla_x C_{i\alpha}^0) dy \\ &+ \nabla_x \cdot \sum_{\alpha=g}^w C_{i\alpha}^0 \tilde{\rho}_\alpha^0 \bar{V}_\alpha^0 = \frac{|Q_f|}{|Q|} q_{\text{ext},i}, \quad i = 1, \dots, N. \end{aligned}$$

Apply the divergence theorem to the first integral above, use (4.4/1) and (4.6/0), and make a second application of the divergence theorem; then we find that

$$\begin{aligned} &\frac{1}{|Q|} \int_{Q_f} \nabla_y \cdot \sum_{\alpha=g}^w (C_{i\alpha}^0 \tilde{\rho}_\alpha^0 V_\alpha^1 + C_{i\alpha}^0 \tilde{\rho}_\alpha^1 V_\alpha^0 + C_{i\alpha}^1 \tilde{\rho}_\alpha^0 V_\alpha^0 \\ &- D_{i\alpha}^* [\tilde{\rho}_\alpha^0 \nabla_y C_{i\alpha}^2 + \tilde{\rho}_\alpha^1 \nabla_y C_{i\alpha}^1 + \tilde{\rho}_\alpha^0 \nabla_x C_{i\alpha}^1 + \tilde{\rho}_\alpha^1 \nabla_x C_{i\alpha}^0]) dy \\ &= \frac{1}{|Q|} \int_{\partial Q_f} \sum_{\alpha=g}^w (C_{i\alpha}^0 \tilde{\rho}_\alpha^0 V_\alpha^1 + C_{i\alpha}^0 \tilde{\rho}_\alpha^1 V_\alpha^0 + C_{i\alpha}^1 \tilde{\rho}_\alpha^0 V_\alpha^0 \\ &- D_{i\alpha}^* [\tilde{\rho}_\alpha^0 \nabla_y C_{i\alpha}^2 + \tilde{\rho}_\alpha^1 \nabla_y C_{i\alpha}^1 + \tilde{\rho}_\alpha^0 \nabla_x C_{i\alpha}^1 + \tilde{\rho}_\alpha^1 \nabla_x C_{i\alpha}^0]) \cdot \nu \, ds(y) \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{|Q|} \int_{\partial Q_m} \sum_{\alpha=g}^w (c_{i\alpha}^0 \rho_\alpha^0 v_\alpha^0 - \rho_\alpha^0 d_{i\alpha}^0 \nabla_y c_{i\alpha}^0) \cdot \nu \, ds(y) \\
&= -\frac{1}{|Q|} \int_{Q_m} \nabla \cdot \sum_{\alpha=g}^w (c_{i\alpha}^0 \rho_\alpha^0 v_\alpha^0 - \rho_\alpha^0 d_{i\alpha}^0 \nabla_y c_{i\alpha}^0) dy \\
&= -\frac{1}{|Q|} \int_{Q_m} (q_{\text{ext},i} - \phi \frac{\partial}{\partial t} \sum_{\alpha=g}^w c_{i\alpha}^0 \rho_\alpha^0 s_\alpha^0) dy.
\end{aligned}$$

Here, we have used periodicity to see that no contribution arises from the integral over ∂Q , the outer boundary of Q_f , and the fact that the outer normal to ∂Q_f is opposite to that of ∂Q_m . By (4.50), the first part of the second integral of (4.56) can be reduced to

$$\begin{aligned}
&\int_{Q_f} \nabla_x \cdot \sum_{\alpha=g}^w \tilde{\rho}_\alpha^0 D_{i\alpha}^* \nabla_y C_{i\alpha}^1 dy \\
&= \int_{Q_f} \nabla_x \cdot \sum_{\alpha=g}^w \tilde{\rho}_\alpha^0 D_{i\alpha}^* \nabla_y \left(\sum_{j=1}^3 a_j(y) \partial_j C_{i\alpha}^0 \right) dy \\
&= |Q| \nabla_x \cdot \sum_{\alpha=g}^w \sum_{j=1}^3 \tilde{\rho}_\alpha^0 D_{i\alpha}^* \overline{\nabla_y a_j(y)} \partial_j C_{i\alpha}^0.
\end{aligned}$$

With these, (4.56) then becomes

$$\begin{aligned}
(4.57) \quad &\Phi \frac{\partial}{\partial t} \sum_{\alpha=g}^w C_{i\alpha}^0 \tilde{\rho}_\alpha^0 S_\alpha^0 + \nabla \cdot \sum_{\alpha=g}^w C_{i\alpha}^0 \tilde{\rho}_\alpha^0 \overline{V}_\alpha^0 - \nabla \cdot \sum_{\alpha=g}^w \tilde{\rho}_\alpha^0 D_{i\alpha}^* \nabla C_{i\alpha}^0 \\
&= q_{\text{ext},i} + q_{m,i}, \quad x \in \mathcal{D}, \quad t > 0, \quad i = 1, \dots, N,
\end{aligned}$$

where $\Phi = |Q_f| \Phi^* / |Q|$ is clearly the macroscopic fracture porosity,

$$(4.58) \quad D_{i\alpha} = D_{i\alpha}^* \left[\frac{|Q_f|}{|Q|} I + \overline{\mathcal{A}} \right],$$

and

$$(4.59) \quad q_{m,i} = -\frac{1}{|Q|} \int_{Q_m} \phi \frac{\partial}{\partial t} \sum_{\alpha=g}^w c_{i\alpha}^0 \rho_\alpha^0 s_\alpha^0 dy, \quad i = 1, \dots, N,$$

is the matrix source term. Finally, it follows from (4.45), (4.50), (4.58), and (4.36/0) that

$$(4.60) \quad \sum_{i=1}^N \tilde{\rho}_\alpha^0 D_{i\alpha}^* \nabla C_{i\alpha}^0 = 0, \quad x \in \mathcal{D}, \quad \alpha = g, o, w.$$

Substituting (4.55) into (4.57) and (4.5/0) into (4.6/0), respectively, we see that

$$(4.61) \quad \begin{aligned} \Phi \frac{\partial}{\partial t} \sum_{\alpha=g}^w C_{i\alpha}^0 \tilde{\rho}_\alpha^0 S_\alpha^0 - \nabla \cdot \sum_{\alpha=g}^w C_{i\alpha}^0 \tilde{\rho}_\alpha^0 \Lambda_\alpha (\nabla P_\alpha^0 - \tilde{\rho}_\alpha^0 g) - \nabla \cdot \sum_{\alpha=g}^w \tilde{\rho}_\alpha^0 D_{i\alpha} \nabla C_{i\alpha}^0 \\ = q_{\text{ext},i} + q_{m,i}, \quad x \in \mathcal{D}, \quad t > 0, \quad i = 1, \dots, N, \end{aligned}$$

$$(4.62) \quad \begin{aligned} \phi(y) \frac{\partial}{\partial t} \sum_{\alpha=g}^w c_{i\alpha}^0 \rho_\alpha^0 s_\alpha^0 - \nabla_y \cdot \sum_{\alpha=g}^w c_{i\alpha}^0 \rho_\alpha^0 \lambda_\alpha (\nabla_y p_\alpha^0 - \rho_\alpha^0 g) - \nabla_y \cdot \sum_{\alpha=g}^w \rho_\alpha^0 d_{i\alpha}^0 \nabla_y c_{i\alpha}^0 \\ = q_{\text{ext},i}, \quad (x, y) \in \mathcal{D} \times Q_m, \quad t > 0, \quad i = 1, \dots, N. \end{aligned}$$

The homogenized, macroscopic model consists of equations (4.58)–(4.62), (4.3/0), (4.7/0)–(4.32/0), and (4.37/-1), together with an outer boundary condition corresponding to (3.23) and matrix boundary conditions on the matrix–fracture interface; the interface conditions will be developed in the next section.

As observed in [8] for both single–phase and two–phase immiscible flow, the two versions of the model obtained through the two derivations in §3 and §4 differ slightly. The only noticeable difference is that the external source term appears in the matrix equations of the homogenized model but not in the physically defined one. This term was left out of the latter version deliberately; it is not very significant and it would be appropriate to delete it from the homogenized version. The other difference is that the physically defined version of the model has a finite number of matrix blocks, whereas the homogenized version has an infinite number of blocks, one for each $x \in \mathcal{D}$. As a consequence, the matrix source terms seem to differ, but this difference has no consequence in practice, as we must restrict to a number of blocks that is much smaller than the actual number in order to approximate the solution numerically. What should be emphasized is that the form of the equations is the same. In particular, the matrix source term is defined to be the total amount of fluid leaving the matrix blocks per unit volume.

5. Modelling of the interactions between the matrix and fracture systems. In this section we shall derive the equations of the interactions between the matrix and fracture systems by homogenization; i.e., we derive the interface boundary conditions. We shall first characterize the mass, momentum, and internal energy balance laws and the entropy condition. Then, we shall develop their flux equations on the matrix–fracture interface, from which we can derive the matrix boundary conditions by means of the techniques exploited in the last section. Again, for completeness, the diffusion fluxes $j_{i\alpha}$ will be included, but they will not be restricted to the special form (2.5).

We first derive the boundary conditions for the pressures. From single–mixture theory [13], [19], the momentum balance law governing fluid phase velocities in the fracture system can be written as

$$(5.1) \quad \frac{\partial}{\partial t} (\tilde{\rho}_\alpha^\varepsilon V_\alpha^\varepsilon) + \nabla \cdot (\mathcal{V}_\alpha^\varepsilon \tilde{\rho}_\alpha^\varepsilon V_\alpha^\varepsilon) - \nabla \cdot T_\alpha^\varepsilon - \tilde{\rho}_\alpha^\varepsilon g = M_\alpha,$$

for $x \in \mathcal{D}_f^\varepsilon, t > 0$, $\alpha = g, o, w$, where $\mathcal{V}_\alpha^\varepsilon$ is the α -phase velocity relating to the Darcy velocity V_α^ε by

$$V_\alpha^\varepsilon = \Phi^* S_\alpha^\varepsilon \mathcal{V}_\alpha^\varepsilon,$$

T_α^ε represents the α -phase stress tensor, and M_α indicates the net exchange of momentum into phase α from other phases, subject to the restriction that

$$\sum_{\alpha=g}^w M_\alpha = 0.$$

Similarly, for $x \in \mathcal{D}_m^\varepsilon, t > 0$, and $\alpha = g, o, w$,

$$(5.2) \quad \frac{\partial}{\partial t}(\rho_\alpha^\varepsilon v_\alpha^\varepsilon) + \varepsilon \nabla \cdot (\nu_\alpha^\varepsilon \rho_\alpha^\varepsilon v_\alpha^\varepsilon) - \nabla \cdot t_\alpha^\varepsilon - \rho_\alpha^\varepsilon g = m_\alpha,$$

where

$$\nu_\alpha^\varepsilon = \phi^\varepsilon s_\alpha^\varepsilon v_\alpha^\varepsilon, \quad \sum_{\alpha=g}^w m_\alpha = 0.$$

We shall assume that the momentum transfer via shear stresses is negligible; thus, the stress tensors are restricted to be of the form

$$(5.3) \quad T_\alpha^\varepsilon = -P_\alpha^\varepsilon I, \quad t_\alpha^\varepsilon = -p_\alpha^\varepsilon I.$$

For expositional convenience, let the centroid of the cell Q be the origin and the period at point $x \in \mathcal{D}$ be indicated by $Q^\varepsilon(x)$. Then, having introduced the vectors g_α^ε and G_α^ε as in §3, on each period $Q^\varepsilon(x)$, continuity of momentum fluxes on the interface $\partial \mathcal{D}_m^\varepsilon$ is expressed by

$$(5.4) \quad (\mathcal{V}_\alpha^\varepsilon \tilde{\rho}_\alpha^\varepsilon V_\alpha^\varepsilon + P_\alpha^\varepsilon I - \tilde{\rho}_\alpha^\varepsilon \text{diag}(G_\alpha^\varepsilon)) \cdot \nu \\ = (\varepsilon \nu_\alpha^\varepsilon \rho_\alpha^\varepsilon v_\alpha^\varepsilon + p_\alpha^\varepsilon I - \rho_\alpha^\varepsilon \text{diag}(g_\alpha^\varepsilon)) \cdot \nu, \quad x \in \partial \mathcal{D}_m^\varepsilon, t > 0, \quad \alpha = g, o, w.$$

The mass flux continuity equation (4.4) will be also needed:

$$(5.5) \quad (\tilde{\rho}_\alpha^\varepsilon C_{i\alpha}^\varepsilon V_\alpha^\varepsilon + J_{i\alpha}^\varepsilon) \cdot \nu = \varepsilon(\rho_\alpha^\varepsilon c_{i\alpha}^\varepsilon v_\alpha^\varepsilon + j_{i\alpha}^\varepsilon) \cdot \nu,$$

for $x \in \partial \mathcal{D}_m^\varepsilon, t > 0, \alpha = g, o, w, i = 1, \dots, N$. Here, as before, we require that

$$(5.6) \quad \sum_{i=1}^N J_{i\alpha}^\varepsilon = 0,$$

$$(5.7) \quad \sum_{i=1}^N j_{i\alpha}^\varepsilon = 0, \quad \alpha = g, o, w.$$

Since the point here is to derive the equation of the interactions between the two systems, we homogenize only on (5.4) and (5.5). It follows from (5.4), for $(x, y) \in \mathcal{D} \times \partial Q_m$ and $t > 0$, by considering the ε^0 term that

$$(5.4/0) \quad (V_\alpha^0 \tilde{\rho}_\alpha^0 V_\alpha^0 + P_\alpha^0 I - \tilde{\rho}_\alpha^0 \text{diag}(G_\alpha^0)) \cdot \nu = (p_\alpha^0 I - \rho_\alpha^0 \text{diag}(g_\alpha^0)) \cdot \nu, \quad \alpha = g, o, w.$$

Similarly, in (5.5) the ε^0 and ε^1 terms lead to

$$(5.5/0) \quad (\tilde{\rho}_\alpha^0 C_{i\alpha}^0 V_\alpha^0 + J_{i\alpha}^0) \cdot \nu = 0,$$

$$(5.5/1) \quad (\tilde{\rho}_\alpha^0 C_{i\alpha}^0 V_\alpha^1 + \tilde{\rho}_\alpha^0 C_{i\alpha}^1 V_\alpha^0 + \tilde{\rho}_\alpha^1 C_{i\alpha}^0 V_\alpha^0 + J_{i\alpha}^1) \cdot \nu (\rho_\alpha^0 c_{i\alpha}^0 v_\alpha^0 + j_{i\alpha}^0) \cdot \nu,$$

for $i = 1, 2, \dots, N$, $\alpha = g, o, w$, where

$$\text{diag}(G_\alpha^0) = \begin{pmatrix} g_1 y_1 & 0 & 0 \\ 0 & g_2 y_2 & 0 \\ 0 & 0 & g_3 y_3 \end{pmatrix},$$

and

$$\text{diag}(g_\alpha^0) = \begin{pmatrix} g_{\alpha 1}^0 & 0 & 0 \\ 0 & g_{\alpha 2}^0 & 0 \\ 0 & 0 & g_{\alpha 3}^0 \end{pmatrix},$$

with

$$\begin{aligned} g_{\alpha 1}^0 &= g_1 \int_0^{y_1} \rho_\alpha^0(x, z, y_2, y_3) dz / \rho_\alpha^0(x, y), \\ g_{\alpha 2}^0 &= g_2 \int_0^{y_2} \rho_\alpha^0(x, y_1, z, y_3) dz / \rho_\alpha^0(x, y), \\ g_{\alpha 3}^0 &= g_3 \int_0^{y_3} \rho_\alpha^0(x, y_1, y_2, z) dz / \rho_\alpha^0(x, y), \end{aligned}$$

where $y = (y_1, y_2, y_3)$. Also, from (5.6), the ε^0 term yields

$$(5.6/0) \quad \sum_{i=1}^N J_{i\alpha}^0 = 0, \quad (x, y) \in \mathcal{D} \times Q_f, \quad t > 0, \quad \alpha = g, o, w,$$

and, from (5.7),

$$(5.7/0) \quad \sum_{i=1}^N j_{i\alpha}^0 = 0, \quad (x, y) \in \mathcal{D} \times Q_m, \quad t > 0, \quad \alpha = g, o, w.$$

Sum on i in (5.5/0) and use (4.23/0) and (5.6/0) to obtain

$$\tilde{\rho}_\alpha^0 V_\alpha^0 \cdot \nu = 0, \quad (x, y) \in \mathcal{D} \times \partial Q_m, \quad t > 0, \quad \alpha = g, o, w.$$

Substitute this into (5.4/0) to find that

$$(5.8) \quad (P_\alpha^0 I - \tilde{\rho}_\alpha^0 \text{diag}(G_\alpha^0)) \cdot \nu = (p_\alpha^0 I - \rho_\alpha^0 \text{diag}(g_\alpha^0)) \cdot \nu,$$

for $(x, y) \in \mathcal{D} \times \partial Q_m, t > 0, \alpha = g, o, w$, which is the boundary condition of the pressures on the matrix–fracture interface.

We now turn to deriving the boundary condition for the matrix concentrations. To do this, we shall first develop the internal energy balance equations and entropy conditions for the fracture and matrix systems. Let $E_{i\alpha}^\varepsilon$ and $\Pi_{i\alpha}^\varepsilon$ denote the internal energy and entropy of the component i in the α -phase, respectively, in the fracture system; the corresponding matrix quantities are indicated by $e_{i\alpha}^\varepsilon$ and $\pi_{i\alpha}^\varepsilon$. Then, with an extension of the balance law of internal energy for a single mixture [13] to multiphase flow, we have, for $x \in \mathcal{D}_f^\varepsilon$ and $t > 0$,

$$(5.9) \quad \begin{aligned} & \Phi^* \frac{\partial}{\partial t} (S_\alpha^\varepsilon \tilde{\rho}_\alpha^\varepsilon C_{i\alpha}^\varepsilon E_{i\alpha}^\varepsilon) + \nabla \cdot (\tilde{\rho}_\alpha^\varepsilon C_{i\alpha}^\varepsilon E_{i\alpha}^\varepsilon V_\alpha^\varepsilon + E_{i\alpha}^\varepsilon J_{i\alpha}^\varepsilon) \\ & - \nabla \cdot (\hat{T}_{i\alpha}^\varepsilon [\tilde{\rho}_\alpha^\varepsilon C_{i\alpha}^\varepsilon V_\alpha^\varepsilon + J_{i\alpha}^\varepsilon]) = \tilde{E}_{i\alpha}, \quad i = 1, \dots, N, \end{aligned}$$

where $\tilde{E}_{i\alpha} = E_{\text{ext},i\alpha} + (\tilde{E}_{i\alpha\beta} - \tilde{E}_{i\beta\alpha}) + (\tilde{E}_{i\alpha\gamma} - \tilde{E}_{i\gamma\alpha})$ with $E_{\text{ext},i\alpha}$ being an external energy source and $\tilde{E}_{i\alpha\beta}$ the rate of energy transfer in component i from the β -phase to the α -phase, and

$$(5.10) \quad \hat{T}_{i\alpha}^\varepsilon = T_{i\alpha}^\varepsilon / \tilde{\rho}_{i\alpha}^\varepsilon, \quad \tilde{\rho}_{i\alpha}^\varepsilon = \Phi^* \tilde{\rho}_\alpha^\varepsilon C_{i\alpha}^\varepsilon S_\alpha^\varepsilon,$$

with $T_{i\alpha}^\varepsilon$ being the stress tensor of the i th component in the α -phase. Analogously, for $x \in \mathcal{D}_m^\varepsilon$ and $t > 0$,

$$(5.11) \quad \begin{aligned} & \phi^\varepsilon \frac{\partial}{\partial t} (s_\alpha^\varepsilon \rho_\alpha^\varepsilon c_{i\alpha}^\varepsilon e_{i\alpha}^\varepsilon) + \varepsilon \nabla \cdot (\rho_\alpha^\varepsilon c_{i\alpha}^\varepsilon e_{i\alpha}^\varepsilon v_\alpha^\varepsilon + e_{i\alpha}^\varepsilon j_{i\alpha}^\varepsilon) \\ & - \varepsilon \nabla \cdot (\hat{t}_{i\alpha}^\varepsilon [\rho_\alpha^\varepsilon c_{i\alpha}^\varepsilon v_\alpha^\varepsilon + j_{i\alpha}^\varepsilon]) = \tilde{e}_{i\alpha}, \quad i = 1, \dots, N, \end{aligned}$$

where $\hat{t}_{i\alpha}^\varepsilon$ has a similar meaning to $\hat{T}_{i\alpha}^\varepsilon$. We shall again use the assumption (5.3); i.e.,

$$(5.12) \quad T_{i\alpha}^\varepsilon = -P_{i\alpha}^\varepsilon I, \quad t_{i\alpha}^\varepsilon = -p_{i\alpha}^\varepsilon I.$$

Then, by (5.9)–(5.12), continuity of internal energy fluxes on the interface $\partial \mathcal{D}_m^\varepsilon$ takes the form

$$(5.13) \quad (E_{i\alpha}^\varepsilon + \hat{P}_{i\alpha}^\varepsilon)(\tilde{\rho}_\alpha^\varepsilon C_{i\alpha}^\varepsilon V_\alpha^\varepsilon + J_{i\alpha}^\varepsilon) \cdot \nu = \varepsilon(e_{i\alpha}^\varepsilon + \hat{p}_{i\alpha}^\varepsilon)(\rho_\alpha^\varepsilon c_{i\alpha}^\varepsilon v_\alpha^\varepsilon + j_{i\alpha}^\varepsilon) \cdot \nu,$$

for $i = 1, 2, \dots, N, \alpha = g, o, w$, where

$$\hat{P}_{i\alpha}^\varepsilon = P_{i\alpha}^\varepsilon / \tilde{\rho}_{i\alpha}^\varepsilon, \quad \hat{p}_{i\alpha}^\varepsilon = p_{i\alpha}^\varepsilon / \tilde{\rho}_{i\alpha}^\varepsilon.$$

Next, we introduce the entropy inequalities in the fracture and matrix domains:

$$(5.14) \quad \Phi^* \frac{\partial}{\partial t} (S_\alpha^\varepsilon \tilde{\rho}_\alpha^\varepsilon C_{i\alpha}^\varepsilon \Pi_{i\alpha}^\varepsilon) + \nabla \cdot (\Pi_{i\alpha}^\varepsilon [\tilde{\rho}_\alpha^\varepsilon C_{i\alpha}^\varepsilon V_\alpha^\varepsilon + J_{i\alpha}^\varepsilon]) \geq 0, \quad x \in \mathcal{D}_f^\varepsilon, \quad t > 0,$$

$$(5.15) \quad \phi^\varepsilon \frac{\partial}{\partial t} (s_\alpha^\varepsilon \rho_\alpha^\varepsilon c_{i\alpha}^\varepsilon \pi_{i\alpha}^\varepsilon) + \nabla \cdot (\pi_{i\alpha}^\varepsilon [\rho_\alpha^\varepsilon c_{i\alpha}^\varepsilon v_\alpha^\varepsilon + j_{i\alpha}^\varepsilon]) \geq 0, \quad x \in \mathcal{D}_m^\varepsilon, \quad t > 0.$$

In general, the entropy should increase across a shock [19] like the interface $\partial \mathcal{D}_m^\varepsilon$. But, since it has been assumed that the fracture flow reaches equilibrium with respect to the matrix flow on the block surfaces, the entropy inequality becomes an equality on $\partial \mathcal{D}_m^\varepsilon$; i.e.,

$$(5.16) \quad \Pi_{i\alpha}^\varepsilon (\tilde{\rho}_\alpha^\varepsilon C_{i\alpha}^\varepsilon V_\alpha^\varepsilon + J_{i\alpha}^\varepsilon) \cdot \nu = \varepsilon \pi_{i\alpha}^\varepsilon (\rho_\alpha^\varepsilon c_{i\alpha}^\varepsilon v_\alpha^\varepsilon + j_{i\alpha}^\varepsilon) \cdot \nu,$$

for $x \in \mathcal{D}_m^\varepsilon, t > 0, i = 1, \dots, N, \alpha = g, o, w$. Homogenize (5.13) and (5.16) to obtain

$$(5.13/0) \quad (E_{i\alpha}^0 + \hat{P}_{i\alpha}^0) (\tilde{\rho}_\alpha^0 C_{i\alpha}^0 V_\alpha^0 + J_{i\alpha}^0) \cdot \nu = 0,$$

$$(5.13/1) \quad ([E_{i\alpha}^0 + \hat{P}_{i\alpha}^0] [\tilde{\rho}_\alpha^0 C_{i\alpha}^0 V_\alpha^1 + \tilde{\rho}_\alpha^0 C_{i\alpha}^1 V_\alpha^0 + \tilde{\rho}_\alpha^1 C_{i\alpha}^0 V_\alpha^0 + J_{i\alpha}^1] + [E_{i\alpha}^1 + \hat{P}_{i\alpha}^1] [\tilde{\rho}_\alpha^0 C_{i\alpha}^0 V_\alpha^0 + J_{i\alpha}^0]) \cdot \nu = (e_{i\alpha}^0 + \hat{p}_{i\alpha}^0) (\rho_\alpha^0 c_{i\alpha}^0 v_\alpha^0 + j_{i\alpha}^0) \cdot \nu,$$

$$(5.16/0) \quad \Pi_{i\alpha}^0 (\tilde{\rho}_\alpha^0 C_{i\alpha}^0 V_\alpha^0 + J_{i\alpha}^0) \cdot \nu = 0,$$

$$(5.16/1) \quad (\Pi_{i\alpha}^0 [\tilde{\rho}_\alpha^0 C_{i\alpha}^0 V_\alpha^1 + \tilde{\rho}_\alpha^0 C_{i\alpha}^1 V_\alpha^0 + \tilde{\rho}_\alpha^1 C_{i\alpha}^0 V_\alpha^0 + J_{i\alpha}^0] + \Pi_{i\alpha}^1 [\tilde{\rho}_\alpha^0 C_{i\alpha}^0 V_\alpha^0 + J_{i\alpha}^0]) \cdot \nu = \pi_{i\alpha}^0 (\rho_\alpha^0 c_{i\alpha}^0 v_\alpha^0 + j_{i\alpha}^0) \cdot \nu,$$

with $(x, y) \in \mathcal{D} \times \partial Q_m, t > 0, i = 1, \dots, N, \alpha = g, o, w$. Then, apply (5.5/0)–(5.5/1), (5.13/0)–(5.13/1), and (5.16/0)–(5.16/1) to get

$$(5.17) \quad (E_{i\alpha}^0 + \hat{P}_{i\alpha}^0) (\rho_\alpha^0 c_{i\alpha}^0 v_\alpha^0 + j_{i\alpha}^0) \cdot \nu = (e_{i\alpha}^0 + \hat{p}_{i\alpha}^0) (\rho_\alpha^0 c_{i\alpha}^0 v_\alpha^0 + j_{i\alpha}^0) \cdot \nu,$$

$$(5.18) \quad \Pi_{i\alpha}^0 (\rho_\alpha^0 c_{i\alpha}^0 v_\alpha^0 + j_{i\alpha}^0) \cdot \nu = \pi_{i\alpha}^0 (\rho_\alpha^0 c_{i\alpha}^0 v_\alpha^0 + j_{i\alpha}^0) \cdot \nu.$$

Finally, from the definition of chemical potentials $\varsigma_{i\alpha}^\varepsilon$ and $\Sigma_{i\alpha}^\varepsilon$ [19],

$$\Sigma_{i\alpha}^\varepsilon = E_{i\alpha}^\varepsilon - T \Pi_{i\alpha}^\varepsilon + \hat{P}_{i\alpha}^\varepsilon,$$

$$\varsigma_{i\alpha}^\varepsilon = e_{i\alpha}^\varepsilon - T \pi_{i\alpha}^\varepsilon + \hat{p}_{i\alpha}^\varepsilon,$$

we have

$$(5.19) \quad \Sigma_{i\alpha}^0 = E_{i\alpha}^0 - T \Pi_{i\alpha}^0 + \hat{P}_{i\alpha}^0,$$

$$(5.20) \quad \varsigma_{i\alpha}^0 = e_{i\alpha}^0 - T \pi_{i\alpha}^0 + \hat{p}_{i\alpha}^0.$$

Now, combine (5.17)–(5.20) to yield

$$(5.21) \quad (\Sigma_{i\alpha}^0 - \varsigma_{i\alpha}^0) (\rho_\alpha^0 c_{i\alpha}^0 v_\alpha^0 + j_{i\alpha}^0) \cdot \nu = 0, \quad (x, y) \in \mathcal{D} \times \partial Q_m, \quad t > 0,$$

for $i = 1, \dots, N$, $\alpha = g, o, w$. It is immediate from this equation that either

$$(5.22) \quad (\rho_\alpha^0 c_{i\alpha}^0 v_\alpha^0 + j_{i\alpha}^0) \cdot \nu = 0, \quad (x, y) \in \mathcal{D} \times \partial Q_m, \quad t > 0,$$

or

$$(5.23) \quad \Sigma_{i\alpha}(C_{1\alpha}^0, \dots, C_{N\alpha}^0, P_\alpha^0) = \varsigma_{i\alpha}(c_{i\alpha}^0, \dots, c_{N\alpha}^0, p_\alpha^0), \quad (x, y) \in \mathcal{D} \times \partial Q_m, \quad t > 0.$$

The relation (5.22) means that there is no flow for the i th component in the α -phase across the matrix–fracture interface, while (5.23) requires continuity of chemical potentials on this interface when that component actually flows across it. Hence, we conclude from the argument above that continuity of the chemical potentials on the block surfaces as the matrix boundary condition is physically reasonable.

There are slight differences between the two versions of the boundary condition derived in §3 and this section. When the block has a fixed size, the boundary value on each block is explicitly a local average; in the homogenized case, the boundary condition is simply the value at the point where the block is. Practically, these conditions are the same [8].

The chemical potentials, as mentioned in §3, are in general very complex functions of pressures and concentrations. The condition (5.23) is thus a nonlinear system of algebraic equations of these variables. We shall now consider some special cases in which (5.23) can be dramatically simplified; the point of these examples is to show the consistency of the very complicated model of this paper with the models for much simpler situations.

Example 1. A single component in a single phase. The simplest example is the flow of a single ideal gas in a naturally fractured reservoir. In this case, the chemical potential ς is given by Dalton's law [13]

$$(5.24) \quad \varsigma(\rho, T) = \mathcal{C}_1(w, T) + \mathcal{C}_2(w, T) \ln \rho,$$

for constants $\mathcal{C}_i(w, T)$, $i = 1, 2$, dependent upon the molecular weight w and temperature T of the gas. Then, it is easy to see that condition (5.23) reduces to continuity of density on the matrix–fracture interface. This is the boundary condition that has been imposed in [3] and [8] for the flow of a single phase.

Example 2. A mixture of ideal gases. In this example, we consider a particular case of a simple mixture, the mixture of ideal gases. Again, the chemical potentials are defined by the Dalton law [13]

$$(5.25) \quad \varsigma_i(c_i, T) = \mathcal{C}_{1i}(w_i, T) + \mathcal{C}_{2i}(w_i, T) \ln(\rho c_i),$$

where c_i is the mass fraction of component i and ρ denotes the mass density of the mixture. The concentration of component i in the mixture is given by

$$\hat{c}_i = \rho c_i.$$

Then, using (5.25), condition (5.23) implies the continuity of concentrations

$$\hat{c}_i^0 = \hat{C}_i^0, \quad (x, y) \in \mathcal{D} \times \partial Q_m, \quad t > 0, \quad i = 1, 2, \dots, N.$$

Under the assumption of incompressibility of the fluids and $N = 2$, the equation has been used to model the flow of two incompressible, completely miscible fluids [4].

Example 3. Incompressible, immiscible flow. Consider now two-phase, incompressible, immiscible flow in a naturally fractured reservoir. In this case, equation (5.23) is always valid since the chemical potentials are constants. Since the fluids are incompressible, relation (5.8) reduces to continuity of the potentials in each phase:

$$P_\alpha^0 = p_\alpha^0, \quad (x, y) \in \mathcal{D} \times \partial Q_m, \quad t > 0, \quad \alpha = o, w,$$

which has been studied for two-phase, incompressible, immiscible flow [4], [8].

Remark. While the chemical potentials are generally complicated functions of the pressures and concentrations, the form given in (5.25) for a mixture of ideal gases can be applied to more general cases with very good success [13]. Hence, continuity of the concentrations may be imposed on the block surfaces as a mathematical approximation to condition (5.23). It is this condition that we introduced for the matrix flow in §3.

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