COMPUTING CENTRE CONDITIONS FOR CERTAIN CUBIC SYSTEMS

By

N.G. Lloyd

and

J.M. Pearson

IMA Preprint Series # 772

February 1991

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370
URL: http://www.ima.umn.edu

Computing centre conditions for certain cubic systems

N G Lloyd
J M Pearson

Department of Mathematics
The University College of Wales
Aberystwyth
Dyfed
SY23 3BZ

1. Introduction

We consider systems of differential equations

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y) \tag{1.1}$$

in which p and q are polynomials, and seek conditions under which the origin is a centre (that is, a critical point in a neighbourhood of which all orbits are closed). The derivation of conditions for a centre is a difficult and long-standing problem in the theory of Nonlinear Differential Equations; necessary and sufficient conditions are known for very few classes of systems. There are well known conditions for quadratic systems (see [4]) and the problem has been resolved for systems in which p and q are cubic polynomials without quadratic terms ([7]), but it is only recently that conditions have been obtained for other classes of cubic systems [3].

Our interest in 'the problem of the centre' arose as part of our investigation of Hilbert's sixteenth problem. Hilbert's problem is to determine the maximum possible number of limit cycles of polynomial systems (1.1) in terms of the degrees of p and q. To have a realistic chance of making progress particular classes of systems are investigated and various kinds of bifurcation considered. We are interested in the number of limit cycles which bifurcate from the origin under perturbation of the coefficients in p and q; these are called *small-amplitude limit cycles*. Much of our recent work has been concerned with such limit cycles in cubic systems (see [10,11] for instance, and the survey articles [8,9]).

In 1944 Kukles [6] proposed necessary and sufficient conditions for the origin to be a centre for systems of the form

$$\left.\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^4 + a_5x^2y + a_6xy^2 + a_7y^3 \end{aligned}\right\} \tag{1.2}$$

(see [NS, page 124]). Kukles stated that the origin is a centre if and only if one of the following holds:

(K1) $M_1 = M_2 = M_3 = M_4 = 0$;

(K2) $a_7 = M_1 = M_2 = M_3 = 0$;

(K3) $a_7 = a_5 = a_2 = 0$;

(K4) $a_7 = a_5 = a_3 = a_1 = 0$;

where

$$M_1 = a_4a_2^2 + a_5m,$$

$$M_2 = (3a_7m + m^2 + a_6a_2^2)a_5 - 3a_7m^2 - a_6a_2^2m,$$

$$M_3 = m + a_1a_2 + a_5,$$

$$M_4 = 9a_6a_2^2 + 2a_2^4 + 9m^2 + 27a_7m$$

and

$$m = 3a_7 + a_2a_3.$$

Our interest in (1.2) arose when Jin and Wang [5] reported an example of a system for which the origin appeared to be a centre but which was not covered by any of the conditions (K1)-(K4). Their example has $a_2 = 0, a_3 = -2a_1, a_4 = -\frac{1}{3}a_1^2$, $a_5 = -3a_7, a_6 = 0$ and $18a_7^2 = a_1^4$. It was proved in [2] that the origin is then indeed a centre. In [2] we also considered the class of systems (1.2) with $a_7 = 0$. We showed that at most five limit cycles bifurcate from the origin and that the origin is a centre if and only if one of the following conditions holds:

$$\left.\begin{array}{l} \text{(i) } a_2 = a_5 = 0, \\ \text{(ii) } a_1 = a_3 = a_5 = 0, \\ \text{(iii) } a_4 = a_5 = a_6 = 0, a_1 + a_3 = 0, \\ \text{(iv) } a_4 = (a_1 + a_3)a_3, a_5 = -(a_1 + a_3)a_2, a_6 = -(a_1 + a_3)a_3^2(a_1 + 2a_3)^{-1}. \end{array}\right\} \quad (1.3)$$

It follows that the Kukles conditions (K2), (K3), (K4) are in fact complete for this subclass.

To study the full system (1.2) we scale the variables by a_2. Let $X = a_2 x$ and $Y = a_2 y$; the system becomes

$$\left.\begin{array}{l} \dot{X} = Y, \\ \dot{X} = -X + A_1X^2 + XY + A_3Y^2 + A_4X^3 + A_5X^2Y + A_6XY^2 + A_7Y^3, \end{array}\right\} \quad (1.4)$$

where $A_i = a_i/a_2 \quad (i = 1, 3)$ and $A_i = a_i/a_2^2 \quad (i = 4, 5, 6, 7)$. Thus the case $a_2 = 0$ has to be considered separately. We did this in [11] where we proved that at most six limit cycles bifurcate from the origin and that if $a_7 \neq 0$ the origin is a centre if and only if

$$a_3 = -2a_1, a_4 = -\tfrac{1}{3}a_1^2, a_5 = -3a_7, a_6 = 0, 18a_7^2 = a_1^4. \quad (1.5)$$

We note that the example of Jin and Wang is covered by (1.5).

The necessity of the conditions which we derive and their sufficiency are proved independently. For the necessity we exploit the fact that there exists a function V defined in a neighbourhood of the origin such that $\dot{V}$, its rate of change along orbits, is of the form $\dot{V} = \eta_2 r^2 + \eta_4 r^4 + \ldots$, where $r^2 = x^2 + y^2$. The coefficients η_{2k} are the *focal values* and are polynomials in the coefficients arising in p and q. The origin is a centre if and only if all the focal values vanish. By Hilbert's basis theorem the set of focal values has a finite basis in the ring of polynomials in the coefficients arising in p and q.

Since the origin has to be a critical point of focus type, the system (1.1) can be written in canonical coordinates as

$$\dot{x} = \lambda x + y + \tilde{p}(x,y), \quad \dot{y} = -x + \lambda y + \tilde{q}(x,y).$$

It is easily shown that $\eta_2 = \lambda$ and we recall that the origin is a *fine focus* if $\lambda = 0$. Since our aim is to derive conditions under which all the focal values vanish, after calculating a focal value, η_{2k} say, we 'reduce' it by means of substitutions from the relations $\eta_4 = \ldots = \eta_{2k-2} = 0$.

The sufficiency of the conditions we give for a centre is proved by a technique recently developed by Christopher [3]. Invariant algebraic curves are sought and appropriate Dulac functions constructed. In this way explicit first integrals are obtained and this was the procedure used in [2] to confirm the conjecture of Jin and Wang.

Explicitly it was shown that under conditions (1.5), with the equations scaled such that $a_1 = 1$, the orbits are the level curves of the function

$$(y^2(x+1)+\sqrt{2}yx(x-2)+6(3x-5))(x(\sqrt{2}y+x)+3(1-x))^{-2}\exp(x(1-\tfrac{1}{2}x))+\phi(x)$$

where $\phi(x) = \int_0^x \exp(u(1-\frac{1}{2}u))du$. Under conditions (iv) of (1.3) the orbits are the level curves of

$$(y-\sigma_+(\kappa x+1))^{\sigma_+}(y+\sigma_-(\kappa x+1))^{\sigma_-}\exp\{(\rho^2(1+\rho)^{-1}\kappa^2x^2+2\kappa\rho x)\sigma_2\}$$

where $\sigma_\pm = -\sigma_1 \pm \sigma_2$, $\sigma_1 = a_2(1+\rho)/2\rho^3\kappa^2$, $\sigma_2^2 = \sigma_1^2 + (1+\rho)/\rho^3\kappa^2$, $\kappa = a_1 + a_3$ and $\rho = a_3(a_1+a_3)^{-1}$ (see [1]).

The work which we describe in this paper is heavily dependent on the use of an appropriate Computer Algebra system, and indeed would not be possible without such facilities. Large-scale computing is involved and much of the interest of the work derives from this. We use REDUCE and most of the computing was done on the Amdahl 5890/30 at the Manchester Computing Centre. However, we soon encountered the 64 mbyte limit imposed under the VM/XA CMS operating system and difficulties also arose because the maximum cpu time allowed per job was only 2500 seconds. The final calculations were performed on the Cray 2 at the Minnesota Supercomputer Center at Minneapolis. The computing problems arise because of the need to manipulate large polynomials. The intermediate expression swell together with the size of the integers occurring caused severe difficulties; for instance, it was not possible to set up some of the polynomials within 64 mbytes.

Even with large-scale computing facilities it was only possible to obtain necessary and sufficient conditions for persistent centres. We say that a centre is *persistent* relative to a class $\mathcal{S}$ of systems if it is not destroyed by all perturbations within $\mathcal{S}$; we also describe the corresponding conditions as persistent. Thus non-persistent centres are given by isolated points in parameter space, and are consequently of less interest to us than persistent centres. We remark that all the known conditions for a centre for quadratic and cubic systems are in fact persistent. We identify all conditions for a centre for (1.4) except possibly for a finite set of points in the space of coefficients $A_1, A_3, A_4, A_5, A_6, A_7$ and find that there is exactly one condition for a centre not covered by (K1)-(K4) and (1.5). Throughout the paper 'persistent' means persistent relative to the class of systems of the form (1.4). We conjecture that there are no non-persistent centres.

2. The Kukles System

As explained in Section 1 we consider systems of the form

$$\left.\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x + A_1x^2 + xy + A_3y^2 + A_4x^3 + A_5x^2y + A_6xy^2 + A_7y^3.\end{aligned}\right\} \qquad (2.1)$$

The focal values for (2.1) were calculated using FINDETA, our computer algorithm for determining focal values of systems of differential equations of the form (1.1); we used an improved version of that described in [12]. The reduced focal values are then obtained by making a sequence of rational substitutions. Here we are concerned only with the zero sets of the reduced focal values - in contrast to the situation when bifurcating limit cycles are investigated, when it is necessary to keep track of signs. We therefore need retain only the numerators of the reduced focal values, and these we denote by $L(k)$; constant multiplicative factors are also ignored. At each rational substitution, the zeros of the denominator are excluded and considered separately.

For convenience, we let $\kappa = A_1 + A_3$, $\mu = A_4 + A_6$ and $\nu = A_3 + 3A_7$. Computation of η_4 gives

$$L(1) = \kappa + A_5 + 3A_7.$$

Now we certainly require $L(1) = 0$ for a centre; so we take

$$A_5 = -\kappa - 3A_7. \qquad (2.2)$$

Substituting this into η_6 gives

$$L(2) = -15\kappa^2\nu + 15\kappa\mu + 6\kappa A_3\nu - 12\kappa A_6 + 6\mu A_3 - 9\mu\nu + 2A_3 + 6\nu A_6 - 2\nu.$$

From $L(2) = 0$ we take

$$A_6 = (15\kappa^2\nu - 15\kappa\mu - 6\kappa A_3\nu - 6\mu A_3 + 9\mu\nu - 2A_3 + 2\nu)/6(\nu - 2\kappa) \qquad (2.3)$$

provided that $\nu - 2\kappa \neq 0$. The case $\nu - 2\kappa = 0$ will be considered separately.

The substitution (2.3) for A_6 gives

$$L(3) = c_2\mu^2 + c_1\mu + c_0$$

where

$$c_2 = 9(20\kappa^3 - 29\kappa^2\nu + 50\kappa^2 A_3 + 18\kappa\nu^2 - 82\kappa\nu A_3 + 64\kappa A_3^2 - 3\nu^3 + 18\nu^2 A_3$$
$$- 8\nu A_3^2 - 8A_3^3)$$

$$c_1 = 3(-210\kappa^5 + 195\kappa^4\nu - 420\kappa^4 A_3 + 57\kappa^3\nu^2 - 402\kappa^3\nu A_3 + 408\kappa^3 A_3^2 + 24\kappa^2\nu^3$$
$$+ 180\kappa^2\nu^2 A_3 - 132\kappa^2\nu A_3^2 - 40\kappa^2\nu - 48\kappa^2 A_3^3 + 40\kappa^2 A_3 - 132\kappa\nu^3 A_3$$
$$+ 108\kappa\nu^2 A_3^2 + 64\kappa\nu^2 + 24\kappa\nu A_3^3 - 160\kappa\nu A_3 + 96\kappa A_3^2 + 24\nu^3 A_3^2 - 12\nu^3$$
$$- 24\nu^2 A_3^3 + 16\nu^2 A_3 + 12\nu A_3^2 - 16A_3^3),$$

$$c_0 = 630\kappa^6\nu - 1395\kappa^5\nu^2 + 1260\kappa^5\nu A_3 + 1665\kappa^4\nu^3 - 504\kappa^4\nu^2 A_3 - 1224\kappa^4\nu A_3^2$$
$$+ 660\kappa^4\nu - 660\kappa^4 A_3 - 1188\kappa^3\nu^3 A_3 + 1044\kappa^3\nu^2 A_3^2 + 150\kappa^3\nu^2 + 144\kappa^3\nu A_3^3$$
$$- 582\kappa^3\nu A_3 + 432\kappa^3 A_3^2 + 144\kappa^2\nu^3 A_3^2 - 324\kappa^2\nu^3 - 144\kappa^2\nu^2 A_3^3$$
$$+ 708\kappa^2\nu^2 A_3 - 432\kappa^2\nu A_3^2 + 48\kappa^2 A_3^3 + 216\kappa\nu^4 - 504\kappa\nu^3 A_3 + 360\kappa\nu^2 A_3^2$$
$$+ 32\kappa\nu^2 - 72\kappa\nu A_3^3 - 64\kappa\nu A_3 + 32\kappa A_3^2 - 12\nu^5 + 24\nu^4 A_3 - 12\nu^3 A_3^2 + 4\nu^3$$
$$- 16\nu^2 A_3 + 20\nu A_3^2 - 8A_3^3.$$

Suppose for now that $c_2 \neq 0$; we take

$$\mu^2 = -(c_1\mu + c_0)/c_2 \tag{2.4}$$

from $L(3) = 0$ and obtain $L(4) = C\mu + D$, where C and D are polynomials in κ, ν and A_3 with 184 and 255 terms respectively. Now $L(4) = 0$ if we take $\mu = -D/C$ with $C \neq 0$; the case $C = 0$ will be considered later. For consistency we require μ from $L(4) = 0$ to satisfy $L(3) = 0$ also. Substituting $\mu = -D/C$ into (2.4) and factorising the expression so obtained, we have

$$(\nu - 2\kappa)^2 c_2^2(\nu - A_3)(2\kappa + 9\nu^3 + 4\nu - 2A_3)f(\kappa, \nu, A_3) = 0 \tag{2.5}$$

where f is a polynomial of degree 19. With $\mu = -D/C$ and μ^2 given by (2.4), further computation gives

$$L(5) = (\nu - 2\kappa)^4 c_2^2(\nu - A_3)(2\kappa + 9\nu^3 + 4\nu - 2A_3)g(\kappa, \nu, A_3) \tag{2.6}$$

and

$$L(6) = (\nu - 2\kappa)^5 c_2^2(\nu - A_3)(2\kappa + 9\nu^3 + 4\nu - 2A_3)h(\kappa, \nu, A_3) \tag{2.7}$$

where g and h are polynomials of degrees 20 and 25 respectively. We have excluded the possibilities $\nu - 2\kappa = 0$ and $c_2 = 0$, so the origin may be a centre if $\nu - A_3 = 0$ or $2\kappa + 9\nu^3 + 4\nu - 2A_3 = 0$ or $f = g = h = 0$.

Now $\nu - A_3 = 0$ is equivalent to $a_7 = 0$, which was investigated fully in [2], so we exclude the possibility that $\nu - A_3 = 0$ in the following.

Consider the possibility that $2\kappa + 9\nu^3 + 4\nu - 2A_3 = 0$ with A_5 and A_6 given by (2.2) and (2.3) respectively, and $\mu = -D/C$. Restoring the original coefficients a_i of equation (1.2) and after some manipulation we obtain the conditions given in the following result.

THEOREM 2.1 *Suppose that* $a_2 \neq 0$ *and that*

$$a_1 = -m(9m^2 + 4a_2^4)/2a_2^5,$$

$$a_4 = -m^2(9m^2 + 2a_2^4)/2a_2^6,$$

$$a_5 = m(9m^2 + 2a_2^4)/2a_2^4,$$

$$a_6 = -(18m^2 - 9ma_2a_3 + 2a_2^4)/9a_2^2,$$

where $m = a_2a_3 + 3a_7$. *Then the origin is a centre. The orbits are the level curves of the function*

$$\int_0^x ((9m^3 + 2ma_2^4)u + 2a_2^5)(mu + a_2)^{-2}a_2^{-3}e^{s(u)}du$$

$$+9a_2^3y^2(3mx - a_2^2y + 3a_2)^{-2}(mx + a_2)^{-1}e^{s(x)}$$

where

$$s(x) = \frac{1}{18}(9m^2 - 18ma_2a_3 - 2a_2^4)a_2^{-2}x^2 + (3m - 2a_2a_3)a_2^{-1}x.$$

The sufficiency of these conditions follows from the existence of an invariant line and was proved in [1]. They are in fact equivalent to (K1), the first of the Kukles conditions.

Referring back to (2.5), (2.6) and (2.7) the other alternative for the origin to be a centre is that $f = g = h = 0$. Now f, g and h are inhomogeneous polynomials in κ, ν and A_3; we aim to eliminate one of these variables from f, g and h. This, however, is a non-trivial task as f has degree 19 with 458 terms and g is of degree 20 with 539 terms. The straightforward application of the RESULTANT function of REDUCE even to *bivariate* polynomials of degree more than about 10 can exceed the time limit of 2500 seconds on the Amdahl 5890. Clearly we needed an efficient strategy to eliminate a chosen variable and we used a procedure which breaks down the calculation into manageable portions. First we simplify f, g and h by returning to the variables A_1, A_3 and A_7. Recalling that $A_7 \neq 0$, we set $A_1 = A_{17}A_7$ and $A_3 = A_{37}A_7$. Then both f and g are quintics in A_7^2 with coefficients that are polynomials in A_{17} and A_{37}, while h is of degree six in A_7^2.

Let $z = A_7^{-2}$ and define $F = z^5 f,\ G = z^5 g,\ H = z^6 h$. We write

$$F = F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4 + F_5 z^5,$$

$$G = G_0 + G_1 z + G_2 z^2 + G_3 z^3 + G_4 z^4 + G_5 z^5$$

and

$$H = H_0 + H_1 z + H_2 z^2 + H_3 z^3 + H_4 z^4 + H_5 z^5 + H_6 z^6.$$

To eliminate z from the relations $F = 0$ and $G = 0$ we construct a sequence of polynomials of successively lower degree in z and at each stage remove all common factors of the coefficients. The first step is to calculate the coefficients p_k of $p = F_5 G - G_5 F$, which is quartic in z. The bivariate coefficients p_k are computed and factorised using REDUCE; they have two common factors, namely c_2 and $\psi = 4A_{37} + 5A_{17} + 12$. Define $P = c_2^{-1}\psi^{-1}p = \sum_{k=0}^{4} P_k z^k$; the degrees of P_k range from 8 to 20. Then $F = G = 0$ if and only if $F = P = 0$, provided that $F_5 c_2 \psi \neq 0$. We also factorise F_5 itself, and find that $F_5 = -192\psi\chi$, where

$$\chi = 125A_{17}^3 + 450A_{17}^2 A_{37} + 1800A_{17}^2 + 525A_{17}A_{37}^2 + 4140A_{17}A_{37}$$

$$5400A_{17} + 196A_{37}^3 + 2268A_{37}^2 + 6480A_{37} - 5400.$$

The next step is to substitute $z^4 = -P_4^{-1}\sum_{k=0}^{3} P_k z^k$ into $F = 0$ (provided that $P_4 \neq 0$). This gives a cubic in z: $q = \sum_{k=0}^{3} q_k z^k$, say. Now it can be proved by some straightforward algebra that χ is a common factor of the q_k. So we let $Q = \chi^{-1}q = \sum_{k=0}^{3} Q_k z^k$; the degrees of the Q_k range from 24 to 32. Thus $F = G = 0$ if and only if $P = Q = 0$ as long as $c_3\psi\chi P_4 \neq 0$. From $Q = 0$ we substitute for z^3 into the relation $P = 0$, provided, of course, that $Q_3 \neq 0$. We obtain a quadratic in z: $t = \sum_{k=0}^{2} t_k z^k$. It can be shown, again by straightforward algebra, that P_4^2 is a common factor of the t_k, and computation shows that $(A_{17} + A_{37})^3$ is also a common factor. We have already supposed that $P_4 \neq 0$, and the case $A_{17} + A_{37} = 0$ will be considered in Section 3. Let $T = P_4^{-2}(A_{17} + A_{37})^{-3}t = \sum_{k=0}^{2} T_k z^k$; the degrees of the T_k are 47,44 and 41. From $T=0$ we take $z^2 = T_2^{-1}(T_1 z + T_0)$ provided that $T_2 \neq 0$. Substituting this into $Q=0$ gives a linear relation $w = w_0 + w_1 z = 0$.

Because of the large expression swell involved, it was not possible to set up w_0 and w_1 as such in the 64 mbytes available. Instead we exploited the fact that both w_0 and w_1 are divisible by $Q_3^2 P_4^2$ (this can again be confirmed by straightforward algebra). For example, one can write w_0 in the form $Q_3^2(\xi_2 P_4^2 + \xi_1 P_4 + \xi_0)$. Since w_0 is divisible by P_4^2, $\xi_0 = 0$ or ξ_0 is divisible by P_4; in fact ξ_0 is divisible by P_4^2 and similarly ξ_1 is divisible by P_4. In this way the expressions $\tilde{w}_k = P_4^{-2} Q_3^{-2} w_k$ $(k = 0, 1)$ were computed. It was then found that $(A_{17} + A_{37})^9$ is a common factor of $\tilde{w}_0$ and $\tilde{w}_1$; let $W = (A_{17} + A_{37})^{-9}(\tilde{w}_0 + \tilde{w}_1 z) = W_0 + W_1 z$. Both W_0 (of degree 65) and W_1 (of degree 62) are irreducible over $\mathbf{Q}$.

Now we consider the relation $H = 0$. The above substitutions from P, Q and T are used to reduce H and this leads (not without considerable effort) to another relation $B = B_0 + B_1 z = 0$, where B_0 is of degree 54 and B_1 is of degree 51. Common factors are again removed at each step of the procedure, but none arise which have not occurred previously. Thus $F = G = H = 0$ if and only if $T = W = B = 0$ provided that $\psi, \chi, P_4, Q_3, T_2, W_1$ and $A_{17} + A_{37}$ are non-zero (and, of course, $\nu - 2\kappa, c_2, C \neq 0$).

The next stage is to make the substitution $z = -W_0/W_1$, from $W = 0$, in B and T to form $\beta = B_0 W_1 - B_1 W_0$ and $\tau = T_2 W_0^2 - T_1 W_0 W_1 + T_0 W_1^2$ respectively.

Unfortunately the resources available at the Manchester Computing Centre were not adequate to set up these expressions, let alone to factorise them. These tasks were performed on the Cray 2 at the Minnesota Supercomputer Center. The factorisation of β, a bivariate polynomial of degree 116, took 7 hours and 10 minutes cpu time.

After factorisation β and τ are of the form

$$\beta = T_2 ESL, \qquad \tau = T_2^2 E^2 SU$$

where E, L, U are polynomials of degree 27,44,30 respectively and

$$S = 4A_{37}^4 + 17A_{37}^3 A_{17} + 24A_{37}^3 + 27A_{37}^2 A_{17}^2 + 90A_{37}^2 A_{17} 36A_{37}^2 + 19A_{37} A_{17}^3$$
$$+ 108A_{37} A_{17}^2 + 117A_{37} A_{17} 5A_{17}^4 + 42A_{17}^3 + 90A_{17}^2.$$

Having expressed the polynomials C and D, which arose in the expression for $L(4)$ derived early in this section, in terms of A_{17}, A_{37} and A_7 we found that E is a factor of the resultant of C and D with respect to A_7^2. Hence the possibility that $E = 0$ is contained in the case $C = D = 0$ which will be discussed in Section 3. Thus the origin can be a centre only if $S = 0$ or $L = U = 0$ or one of $C, c_2, \psi, \chi, P_4, Q_3, T_2, W_1$, $A_{17} + A_{37}$ and $\nu - 2\kappa$ vanishes.

THEOREM 2.2 *Suppose that* $S = 0$, $\mu = -D/C$ *and* A_5, A_6 *are given by (2.2),(2.3) respectively. Then the origin is a centre.*

PROOF: Under the conditions of the theorem we have seen that $L(k) = 0$ for $k \le 6$. The sufficiency of these conditions is proved in [3] and follows from the existence of an invariant conic. The orbits are the level curves of

$$9y^2 e^{\theta(x)}(3A_4x^2 + A_5xy + 3A_1x + y - 3)^{-2}(\omega(x))^{-1} - \int_0^x (\omega(u))^{-2} e^{\theta(u)} du$$

where

$$\theta(x) = v_1x + v_2x^2,\ v_1 = 3A_1 + 2A_3,\ v_2 = (18A_4 + 6A_6 + 9A_1^2 + 2)/6$$

and $\omega(x) = A_4x^2 + A_1x - 1$. □

The conditions given in Theorem 2.2 can be expressed in terms of the original coefficients a_k as follows:

$$a_3 = 2(18a_2^2\gamma - 4a_2^4 - 27a_2^2a_1^2 - 81a_1^4)/81a_1^3,$$

$$a_4 = -(36a_2^2\gamma + 8a_2^4 + 90a_2^2a_1^2 + 243a_1^4)/9\delta,$$

$$a_5 = a_2a_1(27\gamma - 2a_2^2 - 9a_1^2)/\delta,$$

$$a_6 = -2a_2^2(144a_2^2\gamma + 243a_1^2 - 32a_2^4 - 270a_2^2a_1^2 - 567a_1^4)/81a_1^2\delta,$$

$$a_7 = -\frac{1}{3}a_2(a_3\delta + 27\gamma + 14a_2^2 + 72a_1^2)/\delta,$$

where $\gamma^2 = (2a_2^2 + 9a_1^2)^3/162a_2^2$ and $\delta = 16a_2^2 + 81a_1^2$.

These conditions for a centre are again not covered by the Kukles conditions (K1)-(K4). We proceed to confirm that there are no other conditions for a persistent centre.

Consider first the possibility that $L = U = 0$. Both L and U are irreducible over $\mathbf{Q}$, and to confirm that they do not have a common factor over $\mathbf{R}$ we use the following straightforward result.

LEMMA 2.3 *Suppose that k is an infinite field and K is an extension of k. Suppose that $f, g \in k[x, y]$ and that $\ell \in K[x, y]$ is a common factor of f and g. Then either $\ell \in K[x] \cup K[y]$ or f and g have a common factor in $k[x, y] \backslash (k[x] \cup k[y])$.*

Lemma 2.3 is proved by applying the Euclidean algorithm to f and g considered as elements of $K(x)[y]$ and noting that all polynomials arising are elements of $k(x)[y]$.

To check that L and U do not have a common factor in just one of the variables A_{17}, A_{37} we evaluate L and U for specific values of one variable at a time. We chose to consider the three cases $A_{17} = 1, A_{17} = 0$ and $A_{17} = -A_{37}$; let the corresponding expressions for U be U_1, U_2 and U_3. We found that $U_2 = A_{37}^{12} U_4$ where $\partial U_4 = 18$ and

$$U_1 = A_{37}^{12}(3A_{37} - 1)^2(A_{37} - 30)(A_{37} - 3)U_5 U_6,$$

where $\partial U_5 = 3$ and $\partial U_6 = 4$; U_3, U_4, U_5 and U_6 are irreducible over $\mathbf{Q}$. We computed the resultants of U_4 and U_5, and of U_4 and U_6: both are non-zero. It follows that there are no (real) factors common to U_1, U_2 and U_3, and so U has no single variable factors in A_{37}. Similarly, there are no factors involving A_{17} alone.

Hence L and U certainly cannot have a common factor, and so the case $L = U = 0$ does not give rise to conditions for a persistent centre.

It remains to consider the cases which have been excluded in the course of the argument. We illustrate our approach by considering the case $F_5 = 0$; recall that F_5 is a constant multiple of $\chi\psi$. First suppose that $\chi = 0$. This case leads to a persistent centre only if χ and $R(F,G)$, the resultant of F and G with respect to z, have a common factor. Of course, $R(F,G)$ cannot be computed explicitly. We know that χ is irreducible over $\mathbf{Q}$; as for U above we show that χ does not have single-variable factors and so, by Lemma 2.3, the only possibility is that χ is itself a factor of $R(F,G)$. We show that this is not so by finding explicit values of A_{17} and A_{37}, namely $A_{17} = -6$ and $A_{37} = 0$, for which $\chi = 0$ but $R(F,G) \neq 0$. Finding such pairs of values is, of course, an experimental process. Turning to $\psi = 0$ there are obviously no single variable factors, and we find that $\psi = 0$ but $R(F,G) \neq 0$ when $A_{17} = 0$ and $A_{37} = -3$.

The excluded cases $P_4 = 0$, $Q_3 = 0$ and $T_2 = 0$ are treated in a similar way. It was confirmed that none of them is a factor of $R(F,G)$ and that they have no single variable factors ; therefore persistent centre conditions do not arise. The case $P_4 = 0$ was slightly harder than the others, for simple integer values of A_{17} and A_{37} such that $P_4 = 0$ were not found; however the relation $A_{17} = -4A_{37}/5$ simplified P_4 sufficiently to enable us to show that P_4 is not a factor of $R(F,G)$ in this case.

It follows without restriction that P_4 is not a factor of $R(F,G)$.

For the case $W_1 = 0$ we must have $W_0 = 0$ also (because $W = 0$). Both W_1 and W_0 are irreducible over $\mathbf{Q}$ and we use Lemma 2.3 to show that this case cannot give rise to persistent centre conditions.

3. Excluded Cases

In this section we cover the four remaining cases, namely

$$\text{(i) } \nu - 2\kappa = 0, \quad \text{(ii) } c_2 = 0, \quad \text{(iii) } C = 0 \text{ and (iv) } A_1 + A_3 = 0.$$

In (i) and (iv) we also prove that non-persistent centres do not occur. We again suppose that $A_7 \neq 0$.

(i) Suppose that $\nu - 2\kappa = 0$. In terms of the A_i, we have

$$A_7 = (2A_1 + A_3)/3 \tag{3.1}$$

and $L(1) = 3A_1 + 2A_3 + A_5$. We take

$$A_3 = -\tfrac{1}{2}(3A_1 + A_5) \tag{3.2}$$

whence $L(1) = 0$ and

$$L(2) = -30A_6A_1 - 6A_6A_5 - 3A_1^3 + 3A_1^2A_5 + 15A_1A_5^2 - 30A_1A_4$$
$$- 4A_1 + 9A_5^3 - 6A_5A_4 + 4A_5.$$

Now let

$$A_6 = (-3A_1^3 + 3A_1^2A_5 + 15A_1A_5^2 - 30A_1A_4 - 4A_1 + 9A_5^3$$
$$- 6A_5A_4 + 4A_5)/6(5A_1 + A_5) \tag{3.3}$$

where we suppose that $5A_1 + A_5 \neq 0$. Then

$$L(3) = d_2 A_4^2 + d_1 A_4 + d_0 \tag{3.4}$$

where $d_2 = -288(5A_1 + A_5)^2(A_1 + 2A_5)$ and d_1, d_0 are polynomials in A_1 and A_5 of degree 5,7 respectively.

Suppose that $A_1 + 2A_5 \neq 0$ and let $A_4^2 = -(d_1 A_4 + d_0)/d_2$. Then $L(4) = e_0 + e_1 A_4$ where $e_0 = (5A_1 + A_5)^4 \tilde{e}_0$ and $e_1 = (5A_1 + A_5)^5 \tilde{e}_1$; $\tilde{e}_0$ and $\tilde{e}_1$ are polynomials in A_1 and A_5 of degree 12 and 9 respectively. If $e_1 \neq 0$ let

$$A_4 = -e_0/e_1. \tag{3.5}$$

The cases $e_1 = 0$ and $d_2 = 0$ will be considered later. We now have

$$L(5) = (5A_1 + A_5)^7 (A_1 + 2A_5)^2 (A_1 - A_5) K Z_1$$

and

$$L(6) = (5A_1 + A_5)^9 (A_1 + 2A_5)^2 (A_1 - A_5) K Z_2$$

where

$$K = 9A_1^3 + 27A_1^2 A_5 + 27 A_1 A_5^2 + 2A_1 + 9A_5^3 + 4A_5,$$

Z_1 is a polynomial of degree 19 in A_1 and A_5, and Z_2 is of degree 23. For consistency we substitute $A_4 = -e_0/e_1$ into (3.3) and obtain

$$(5A_1 + A_5)(A_1 + 2A_5)^2 (A_1 - A_5) K Z_3 = 0$$

where Z_3 is of degree 19. We have already supposed that $5A_1 + A_5$ and $A_1 + 2A_5$ are non-zero, and if $A_1 - A_5 = 0$ then $A_7 = 0$. If $K = 0$ and A_7, A_3, A_6, A_4 are given by (3.1),(3.2),(3.3) and (3.5) respectively, we have a particular case of the conditions given in Theorem 2.1 (which are also covered by the Kukles condition $(K1)$).

The remaining possibility is that $Z_1 = Z_2 = Z_3 = 0$ with $5A_1 + A_5$, $A_1 + 2A_5$ and e_1 non-zero. If $e_1 = 0$ then we must have $e_0 = 0$ also. We suppose that $A_5 \neq 0$ and set $A_1 = A_{15}A_5$. (When $A_5 = 0$ we have $Z_1 = A_1^9\hat{Z}_1$ and $Z_3 = A_1^9\hat{Z}_3$, where $\hat{Z}_1$ and $\hat{Z}_3$ are polynomials in A_1^2 whose resultant is non-zero, and since $5A_1 + A_5 \neq 0$ we cannot have $A_1 = 0$.) We compute the following resultants with respect to A_5^2:

$$R(e_0, e_1) = (5A_{15} + 1)^9(A_{15} - 1)^6(A_{15} + 1)^2(A_{15} + 2)\Phi_1\Phi_2,$$

$$R(Z_1, Z_3) = (5A_{15} + 1)^{25}(A_{15} - 1)^{25}(A_{15} + 1)^{16}(A_{15} + 2)^5(A_{15} - 4)\Phi_2^2\Phi_3\Phi_5,$$

$$R(Z_1, Z_2) = (5A_{15} + 1)^{30}(A_{15} - 1)^{25}(A_{15} + 1)^{16}(A_{15} + 2)^5(A_{15} - 4)\Phi_2\Phi_4\Phi_5,$$

where $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ are irreducible polynomials of degree 7, 17, 30, 71 respectively, and $\Phi_5 = A_{15}^4 + 2A_{15}^3 + 3A_{15}^2 + 2A_{15} + 4$ which is positive definite. We require $R(Z_1, Z_3) = R(Z_1, Z_2) = 0$ while $R(e_0, e_1) \neq 0$. Thus we need to consider the two possibilities $A_{15} = 4$ and $\Phi_3 = \Phi_4 = 0$. When $A_{15} = 4$ we find that $Z_1 = A_5^2\tilde{Z}_1$ and $Z_3 = A_5^2\tilde{Z}_3$. Now $A_5 \neq 0$ and $R(\tilde{Z}_1, \tilde{Z}_3) \neq 0$; hence $A_{15} = 4$ does not give rise to a centre. The polynomials Φ_3 and Φ_4 are univariate and irreducible over $\mathbf{Q}$. They cannot have a common zero, for suppose that α is such a zero and let p_α be its minimum polynomial (that is, the monic polynomial with rational coefficients of minimum degree of which α is a zero). By minimality, p_α divides all polynomials of which α is a zero. Because Φ_3 is irreducible, $p_\alpha = \Phi_3$; then Φ_3 divides Φ_4, contradicting the fact that Φ_4 is irreducible.

We now turn to the possibility that $5A_1+A_5=0$. Taking A_7 and A_3 as defined by (3.1) and (3.2) with $A_1=-A_5/5$ we have $L(2)=A_5(32A_5^2+25)$. Thus the only possibility is that $A_5=0$ in which case $A_7=0$.

If $A_1+2A_5=0$, again with A_7 and A_3 satisfying (3.1) and (3.2), then $L(2)=A_5(18A_6+5A_5^2+18A_4+4)$. Since $A_5=0$ implies that $A_7=0$ we take $A_6=-(5A_5^2+18A_4+4)/18$ and obtain

$$L(3)=A_5(-475A_5^4+567A_5^2A_4+304A_5^2+288A_4).$$

Now take $A_4=-19A_5^2(-25A_5^2+16)/9(63A_5^2+32)$; then $L(4)=A_5^3\Theta_1$ and $L(5)=A_5\Theta_2$ where Θ_1 and Θ_2 are polynomials in A_5 whose resultant is non-zero. Again no centre conditions arise.

The remaining situation to be considered is that $e_0=e_1=0$. In this case $L(4)=0$ and $L(5)=(5A_1+A_5)^6(f_0+f_1A_4)$, $L(6)=(5A_1+A_5)^8(g_0+g_1A_4)$ where f_0, f_1, g_0, g_1 are polynomials in A_1, A_5. We now take $A_4=-f_0/f_1$ with $f_1\neq 0$ and consider $L(6)=0$ together with $e_0=e_1=0$. Since $5A_1+A_5\neq 0$ we have $J=g_0f_1-g_1f_0=0$. Now

$$J=(A_1+2A_5)^3(5A_1+A_5)(A_1-A_5)J_1J_2$$

where J_1 and J_2 are polynomials of degree 26 and 3 respectively. If $J_2=0$ we obtain a particular case of the centre conditions given in Theorem 2.1. We compute the resultant with respect to A_5^2, $R(e_1, J_1) = (5A_{15}+1)^{21}(A_{15}-1)^{10}(A_{15}+1)^2\Psi_1$, where Ψ_1 is of degree 61. We supposed that $5A_{15}+1 \neq 0$ and $A_{15}+2 \neq 0$. Both when $A_{15} = 1$ and $A_{15} = -1$ we find that $e_0 = e_1 = 0$ only if $A_5 = 0$, a contradiction. Since Ψ_1, Φ_1 and Φ_2 are irreducible over $\mathbf{Q}$, they cannot have common roots (as explained above). Hence $e_0 = e_1 = J_1 = 0$ cannot occur.

We must still look at the possibility $f_1 = 0$. We compute

$$R(e_1, f_1) = (5A_{15}+1)^{23}(A_{15}-1)^6(A_{15}+1)^2(A_{15}+2)^3\Psi_2$$

where Ψ_2 is of degree 24. The cases not already considered are $\Phi_1 = \Psi_2 = 0$ and $\Phi_2 = \Psi_2 = 0$. Neither can happen because the three polynomials are irreducible and distinct.

This completes case (i) - there are no new centre conditions of any kind.

(ii) Suppose that $c_2 = 0$, and let A_5 and A_6 be as given in (2.2) and (2.3). From $L(3) = 0$ we take $\mu = -c_0/c_1$, provided that $c_1 \neq 0$, and then

$$L(4) = A_7 z^6 \sum_{k=0}^{6} r_k z^k, \quad L(5) = A_7 z^8 \sum_{k=0}^{8} s_k z^k$$

where $z = A_7^2$ and r_k, s_k are polynomials in A_{17} and A_{37}. Since $A_7 \neq 0$, the resultant $\mathcal{R}$ of $L(4)$ and $L(5)$ with respect to z is a function of A_{17} and A_{37} only, as is c_2. If $\mathcal{R}$ is irreducible over $\mathbf{Q}$ then we can have conditions for a persistent centre only if c_2 and $\mathcal{R}$ have a common factor involving A_{17} or A_{37} alone. We verify that c_2 has no such factors. If $\mathcal{R}$ factorises over $\mathbf{Q}$ then a persistent centre can arise only if c_2 is itself a factor of $\mathcal{R}$. To show that c_2 is not a factor of $\mathcal{R}$ we evaluate $\mathcal{R}$ for specific values of A_{17} and A_{37} such that $c_2 = 0$ but $\mathcal{R} \neq 0$. If on the other hand $c_1 = 0$ then we require $c_0 = c_1 = c_2 = 0$. The resultant of c_0 and c_1 with respect to z is again a function of A_{17} and A_{37}, and c_2 is not a factor of it. Hence there are no persistent centre conditions.

(iii) For the case $C = D = 0$ we express C and D in terms of A_{17}, A_{37} and $z = A_7^2$. Let $C = C_0 + C_1 z + C_2 z^2$ and $D = D_0 + D_1 z + D_2 z^2 + D_3 z^3$. The resultant of C and D with respect to z has four factors, namely $\nu - 2\kappa$ and c_2 which are non-zero, and Γ_1 and Γ_2, say, which are bivariate polynomials of degree 14 and 27 respectively. So $L(4) = C\mu + D = 0$ if $\Gamma_1 = 0$ or $\Gamma_2 = 0$ provided that D_3 and C_2 are not both zero. Using A_5 and A_6 as given by (2.2) and (2.3) let $\mu^2 = -(c_0 + c_1\mu)/c_2$, so that $L(5) = \theta_0 + \mu\theta_1$, $L(6) = \phi_0 + \mu\phi_1$ and $L(7) = \psi_0 + \mu\psi_1$, where θ_i, ϕ_i and ψ_i are polynomials in κ, ν and A_3. When $\theta_1 \neq 0$ let $\mu = -\theta_0/\theta_1$ then $L(6) = \phi_0\theta_1 - \phi_1\theta_0$ and $L(7) = \psi_0\theta_1 - \psi_1\theta_0$. The evaluation of the resultant $\mathcal{T}$ of $L(6)$ and $L(7)$ to eliminate one of the three variables is non-trivial, so we aim to show that Γ_1 and Γ_2 are not factors of $\mathcal{T}$. Persistent centre conditions can arise only if Γ_1 or Γ_2 is a factor of $\mathcal{T}$ or either has a common factor with $\mathcal{T}$ in a single variable. We confirmed that Γ_1 and Γ_2 cannot have factors in either of the variables alone. We also showed that neither Γ_1 nor Γ_2 can be a factor of $\mathcal{T}$. When $\theta_1 = 0$

we showed that Γ_1 and Γ_2 are not factors of the resultant of θ_0 and θ_1 with respect to z. Hence in this case again there are no persistent centre conditions.

(iv) We now consider the case $A_1 + A_3 = 0$. Here $L(1) = 3A_7 + A_5$ and we take $A_1 = -A_3$, $A_7 = -\frac{1}{3}A_5$ to give

$$L(2) = 3A_3A_6 - 3A_3A_4 + 3A_6A_5 + 9A_5A_4 + 2A_5.$$

Now suppose that $3A_6 + 9A_4 + 2 \neq 0$ and let $A_5 = 3A_3(A_4 - A_6)/(3A_6 + 9A_4 + 2)$. Then $L(3) = b_2A_3^2 + b_0$ where $b_2 = 12(A_6 - A_4)b_3$, $b_0 = (3A_6 + 9A_4 + 2)^2 b_4$ and both b_3 and b_4 are cubic polynomials in A_4 and A_6. Let $A_3^2 = -b_0/b_2$, where $b_2 \neq 0$, then

$$L(4) = A_3(A_6 - A_4)^2(A_6 + 3A_4)^2(3A_6 + 9A_4 + 2)^2\tilde{K}N_1,$$

$$L(5) = A_3(A_6 - A_4)^2(A_6 + 3A_4)^2(3A_6 + 9A_4 + 2)^4\tilde{K}N_2,$$

$$L(6) = A_3(A_6 - A_4)^2(A_6 + 3A_4)^2(3A_6 + 9A_4 + 2)^4\tilde{K}N_3,$$

where $\tilde{K} = 9A_6^2 + 18A_6A_4 + 2A_6 + 9A_4^2 + 4A_4$ and the N_i are polynomials in A_4 and A_6. For $L(4) = L(5) = L(6) = 0$ there are several subcases to be considered.

If $A_3 = 0$ then $A_7 = 0$ which we are supposing is not the case. If $A_6 - A_4 = 0$ then $b_2 = 0$, a contradiction. If $A_6 + 3A_4 = 0$ then $A_3^2 = -\frac{1}{4}$ which we obviously do not consider. If $\tilde{K} = 0$ and A_7, A_5, A_3^2 are as defined above then the centre conditions given in Theorem 2.1 are satisfied. We have verified that N_1, N_2 and N_3 cannot be simultaneously zero without violating our hypotheses.

In the case $3A_6 + 9A_4 + 2 = 0$, we take $A_4 = -(3A_6 + 2)/9$ then $L(2) = A_3(6A_6 + 1)$. If $A_3 = 0$ then $L(3) = A_5(9A_6^2 + 12A_6 + 9A_5^2 - 8)$ and if $A_5 = 0$ then $A_7 = 0$. With $A_5 \neq 0$ the quantities $L(4)$ and $L(5)$ are zero if and only if $A_6 = \frac{2}{3}$, but then $A_5^2 = -\frac{4}{9}$. If $A_3 \neq 0$ and $L(2) = 0$ then $A_6 = -\frac{1}{6}$, so that $L(3), L(4)$ and $L(5)$ are functions of A_3 and A_5. We consider the resultant of $L(3)$ and $L(4)$ with respect to A_5, $\mathcal{R}_1$ say, and of $L(3)$ and $L(5)$, $\mathcal{T}_1$ say. We have $\mathcal{R}_1 = A_3(6A_3 - 5)(6A_3 + 5)\Gamma_3$ and $\mathcal{T}_1 = A_3(6A_3 - 5)(6A_3 + 5)\Gamma_4$. The resultant of Γ_3 and Γ_4 is non-zero so we consider only $A_3 = \pm\frac{5}{6}$ and it is easily shown that both lead to centre conditions which are particular cases of Theorem 2.1.

We return to the case $b_2 = 0$. Recall that $L(3) = b_0 + b_2A_3^2$ so if $b_2 = 0$ then we require $b_0 = 0$ also. We note that $3A_6 + 9A_4 + 2 \neq 0$ and if $A_6 = A_4$ then A_5 and hence A_7 are zero. We showed that all other possibilities are also inconsistent with our hypotheses.

4. Conclusions

We noted at the beginning of the paper that if $a_7 = 0$ the origin is a centre for system (1.2) if and only if the conditions (1.3) hold, and that if $a_2 = 0$ then the origin is a centre if and only if (1.5) holds. The conclusions of this paper can be summarised as follows.

THEOREM 4.1 *Suppose that* $a_7 \neq 0$. *The origin is a persistent centre for (1.4) if and only if the conditions given in Theorem 2.1 or Theorem 2.2 are satisfied. Non-persistent centres can occur only if* $L = U = 0$ *or one of* $C, c_2, \psi, \chi, P_4, Q_3, T_2, W_1$ *is zero.*

Non-persistent centres occur only for a finite number of points in the space of coefficients. We have shown that none occur if $A_1 + A_3 = 0$ or $2A_1 + A_3 - 3A_7 = 0$, two cases which were excluded in the course of the argument in Section 2. We conjecture that there are no non-persistent centres at all.

CONJECTURE *Let* $a_7 \neq 0$. *The origin is a centre for (1.4) if and only if the conditions given in Theorem 2.1 or Theorem 2.2 are satisfied.*

The possibility of deriving necessary and sufficient conditions for a centre by computing a Gröbner basis for the set of focal values is one which we have considered. We are grateful to Herbert Melenk and Winfried Neun at the Konrad Zuse Center, Berlin for helping to explore this approach. Using their efficient REDUCE Gröbner basis code on a specially configured DEC 3100 workstation they obtained Gröbner bases in the two cases $a_7 = 0$ and $a_2 = 0$. The case $a_7 = 0$ proved to be straightforward, but the case $a_2 = 0$ took over 7 days cpu time. So far, the general case (the subject of this paper) has proved intractable. These computations enabled the results of [2] and [11] to be confirmed.

Acknowledgement: We are very grateful to the Institute for Mathematics and Applications, the University of Minnesota for its hospitality and especially for enabling us to use the Cray 2 computer at the Minnesota Supercomputer Center.

References

[1] C J CHRISTOPHER, 'Invariant algebraic curves in polynomial differential systems', PhD thesis, The University College of Wales, Aberystwyth.

[2] C J CHRISTOPHER AND N G LLOYD, 'On the paper of Jin and Wang concerning the conditions for a centre in certain cubic systems', *Bull. London Math. Soc.* **22** (1990) 5-12.

[3] C J CHRISTOPHER AND N G LLOYD, 'Invariant algebraic curves and conditions for a centre', preprint, The University College of Wales, Aberystwyth, 1989.

[4] J GUCKENHEIMER, R RAND AND D SCHLOMIUK, 'Degenerate homoclinic cycles in perturbations of quadratic Hamiltonian systems', *Nonlinearity* **2** (1989) 405-418.

[5] JIN XIAOFAN AND WANG DONGMIN, 'On Kukles' conditions for the existence of a centre', *Bull. London Math. Soc.* **22** (1990) 1-4.

[6] I S KUKLES, 'Sur quelques cas de distinction entre un foyer et un centre', *Dokl. Akad. Nauk. SSSR* **42** (1944) 208-211.

[7] V A LUNKEVICH AND K S SIBIRSKII, 'On the conditions for a centre', *Differencial'nye Uravneniya* **1** (1965) 53-66.

[8] N G LLOYD, 'Limit cycles of polynomial systems' New directions in dynamical systems, *London Math. Soc. Lecture Notes Series No.127* (ed. T Bedford and J Swift, Cambridge University Press, 1988) pp.192-234.

[9] N G LLOYD, 'The number of limit cycles of polynomial systems in the plane', *Bull. Inst. Math. Appl.* **24** (1988) 161-165.

[10] N G LLOYD, T R BLOWS AND M C KALENGE, 'Some cubic systems with several limit cycles', *Nonlinearity* **1** (1988) 653-669.

[11] N G LLOYD AND J M PEARSON, 'Conditions for a centre and the bifurcation of limit cycles in a class of cubic systems', to appear in 'Bifurcations and periodic orbits of planar vector fields', ed. J P Francoise and R Roussarie (Springer-Verlag).

[12] N G LLOYD AND J M PEARSON, 'REDUCE and the bifurcation of limit cycles', *J. Symbolic Comput.* **9** (1990) 215-224.

[13] V V NEMYTSKII AND V V STEPANOV, Qualitative theory of differential equations, (Princeton University Press, 1960).