Influence of Bit Design on the Stability of the Rotary Drilling System

by

Kaixiao Tian

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Doctoral Committee:

Professor Emmanuel Detournay, Advisor

Professor Joseph Labuz, Chair

Associate Professor Stefano Gonella

Professor Yoichiro Mori

Kaixiao Tian

tianx274@umn.edu

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Dedication

For Yushuo, our parents and grandparents.

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Abstract

Rotary drilling systems used to drill deep boreholes for hydrocarbon exploration and production experience self-excited vibrations, which can lead to drilling dysfunctions such as bit bounce, stick-slip and whirling. These dysfunctions cause premature failure of drill bits and other components within the system, ultimately leading to increased cost of the drilling operation. It is of interest, therefore, to understand the mechanism of self-excited vibration, with the objective of mitigating the drilling dysfunctions.

This thesis focuses on establishing the link between the stability of the system in regard to coupled axial-torsional vibrations and features of PDC bits, which consist of multiple fixed blades or cutters mounted on a bit body. It has been established that a state-dependent delay system can depict the mechanism of the stick-slip drillstring vibrations. The complex arrangement of cutters of a realistic PDC bit, however, are responsible not only for multiple delays but also for a delay distribution that depends on the bit motion history. Since the delay distribution controls the stability of the bit, it is imperative to develop a model capable of capturing the exact design of PDC bits. This additional consideration, although greatly complicating the analysis, is expected to give more realistic insights into the mechanism of the stick-slip vibrations of a rotary drilling system.

Table of Contents

Dedication	ii
Acknowledgements	iii
Abstract	iv
List of Figures	ix
List of Tables	xv
Chapter 1: Introduction	1
1.1 Context	1
1.1.1 Rotary drilling system	1
1.1.2 PDC bits	2
1.2 Motivations	4
1.3 Organizations	7
Chapter 2: Literature Review	8
2.1 Rock cutting mechanics	8
2.2 Modeling stick-slip	9
2.2.1 The classical approach	9
2.2.2 The novel approach	10
2.3 Stick-slip mitigation, field applications	12
2.3.1 Overview	12
2.3.2 Influence of bit design on the stick-slips	13
2.3.3 Stability chart in the space of WOB and RPM	13
Chapter 3: Model Description	14
3.1 Drillstring model	14
3.2 Bit/rock interaction law for Class I Bits	15
3.2.1 Coordinates definition	16

3.2.2	Cutter engagement surface	17
3.2.3	Weight-on-bit and torque-on-bit	19
3.3	Bit/rock interaction law for Class II Bits	21
3.3.1	Coordinate systems	21
3.3.2	Projection of cutter faces	22
3.3.3	Engagement surface and cutting edge for each cutter	25
3.3.4	Polygon clipping technique	30
3.3.5	Boundaries of an engagement surface	30
3.3.5.1	Computing weight-on-bit and torque-on-bit	33
3.4	Scaling	34
Chapter 4:	Time Simulation	38
4.1	Treatment of delayed terms	38
4.1.1	Perspective of rock profile	38
4.1.2	Perspective of cutter	42
4.2	Delay term treatment	44
4.2.1	Time-based algorithm	44
4.2.2	Angle-based algorithm	46
4.2.3	Rock-profile-based algorithm	48
4.3	Treatment of drilling dysfunctions	48
4.3.1	Base algorithm for the RGD bits	48
4.3.1.1	Drilling modes	49
4.3.1.2	Normal drilling to axial stick/loss of wearflat-rock contact . .	50
4.3.1.3	Axial stick to axial slip	51
4.3.1.4	Vanish of Ω during normal drilling	52
4.3.1.5	Vanishing of Ω during axial stick	54
4.3.1.6	Axial-torsional stick	55
4.3.2	Modifications for realistic bits	57
4.4	“Complete” algorithm for this work	59
Chapter 5:	Stability	61
5.1	Partitioning of the nominal cutter engagement surface for Class I Bits	62
5.2	Linearized dynamics	64
5.3	Critical angular velocity	68
5.3.1	Approach	68
5.3.2	Crossing poles and crossing direction	70

5.3.3	Summary	71
5.4	Stability analysis for realistic bits	72
5.4.1	Preamble	72
5.4.2	Stability analysis of “real” bits with round cutters	73
Chapter 6:	Results	79
6.1	Symmetric and asymmetric full-blade bits	79
6.1.1	Critical angular velocity	80
6.1.2	Time-dependent response	83
6.2	Fishtail bits	85
6.2.1	Critical frequencies	87
6.2.2	Stability map for fishtail bits	88
6.2.3	Frequency of the decoupled axial oscillations	89
6.3	Bits with partial blades	90
6.3.1	Critical frequencies	91
6.3.2	Stability map	92
6.3.3	Verification with time simulations	95
6.4	Results for Bit A (realistic bit)	95
6.4.1	Geometric properties	96
6.4.2	Stability map for the decoupled axial dynamics	98
6.4.3	Time simulation verification	99
6.4.3.1	Verification of the stability map	99
6.4.3.2	Transient axial response	101
6.4.4	Parameter β and the torsional response	104
6.4.5	Parameter κ_W and the torsional response	107
6.5	Evaluation of stability properties of existing designs	108
6.5.1	Comparison of nominal responses	108
6.5.2	Comparison of linear stability	111
6.5.3	Bit responses	111
6.6	Geometric extension of the model	115
6.6.1	Scribe cutters	115
6.6.2	Worn cutters	115
Chapter 7:	Conclusion and Future Work	120
7.1	Contributions	120
7.2	Engineering applications	121
7.3	Future work	122

List of Figures

Figure 1.1	Schematic of the rotary system, from [1]	2
Figure 1.2	GE's Stratapax Cutters	5
Figure 1.3	Illustration of a modern PDC bit	6
Figure 2.1	Stability in RPM (represented by dimensionless ω_o) and ROP (represented by dimensionless δ_o) after 2. In this work, maximum frictional WOB is assumed constant; thus δ_o can be viewed as a proxy for WOB	11
Figure 2.2	Stability chart in RPM-WOB space	12
Figure 3.1	Simplified model for the drillstring in the model of [3]	15
Figure 3.2	An example of an asymmetric bit (each rectangle represents a cutter) and the coordinate systems related to this model . . .	18
Figure 3.3	Illustration of the bit/rock interaction considering the extended bit geometry (general motion)	19
Figure 3.4	(a) 3-D representation of the cutter faces (b) Rotate the bit without axial advance, and project cutter faces onto the same half-plane $\theta = 0, r \geq 0$ to get the bit profile	23
Figure 3.5	General procedures for cutter face projections: the coordinates of the k -th polygon nodes on a cutter i , $\mathbf{q}_{i,k}$ is transferred into the coordinates $(r_{i,k}^{proj}, z_{i,k}^{proj})$ on the cutter projection plane. .	26
Figure 3.6	Blue circles: non-zero entries of table $\mathbf{P}$ depict the potential interactions between cutter i and j ; Red circles: non-zero entries of table $\mathbf{Q}$ depict the actual interactions between cutters (presented in figure corresponds to Bit A with a nominal bit motion at $d_o = 3$ mm)	27
Figure 3.7	A specific illustration: get the cutter engagement shape for a cutter of interest	29

Figure 3.8	(a)-(c) Examples of $\mathbf{C}_i$, Γ_i and Γ_{ji} for Bit A at $d_o = 3$ mm. (d) Sketches (overlaid on bit profile, i.e. aligned cutter projections) of projected cutter face (black ellipses), engagement surface $\mathcal{S}_i$ (cyan crescent-like shapes) and cutting edge $\mathcal{L}_i$ (red curves with “x” as starting point and “o” as end point) for each cutter i .	32
Figure 3.9	Geometric parameters defined for a specific $\mathcal{S}_i$ (or $\mathbf{C}_i$)	34
Figure 4.1	Different classes of bit geometry	39
Figure 4.2	Predefined bit motion: evolution of axial (a) and angular position (b)	39
Figure 4.3	Motion trajectories of cutters (center points) in 3-D view: the crossings indicate the starting locations; the red, blue, and green curves correspond to cutter 1, 2 and 3, respectively . .	40
Figure 4.4	Motion trajectories of cutters in 2-D view: the angular position is converted to the range of $[0, 2\pi)$; the red, blue, and green curves correspond to cutter 1, 2 and 3, respectively	40
Figure 4.5	The process of updating and finding local depth of cut: each subplot depicts $\Phi_r - Z$ relations depicting cutter motion trajectories and rock profile	41
Figure 4.6	Evolution of local depth of cut δ_i with respect of angular position Φ (Dots: results from the perspective of rock; Circles: results from the perspective of cutters)	42
Figure 4.7	Determination of d_{ji}	45
Figure 4.8	Illustration of various drilling modes and their transitions . .	49
Figure 4.9	Simple interpolation for state variables at the vanishing of V	50
Figure 4.10	Simple interpolation for state variables at the vanishing of V	52
Figure 5.1	Illustration of the bit/rock interaction considering the extended bit geometry (nominal motion).	64
Figure 5.2	A PDC bit can be viewed as a superposition of many groups of rectangular cutters	73
Figure 5.3	Definition of the number sets $\{\alpha_{ji}^c\}$ and $\{\alpha_{ji}^f\}$	77
Figure 5.4	Local entries of $\{\alpha_{ji}^f\}$ and corresponding vectors at corner points for Bit A at $d_o = 3$ mm.	78
Figure 6.1	Bit A with 3 symmetric full blades and Bit B with as 3 asymmetric full blades.	79

Figure 6.2	Graphical representation for determining $\mathbf{h}_c$ for Bit A. The arrow at each intersection in the top subplot indicates that the crossing direction of a pole pair is always right-going. The bottom subplot presents the $\mathbf{h} - n_u$ relationship.	81
Figure 6.3	Graphical representation for determining $\mathbf{h}_c$ for Bit B. The arrow at each intersection in the top subplot indicates the crossing direction of the critical pole. Note that the vertical dash means that the pole trajectory is tangential to the imaginary axis, and for this case particularly, the tangent point is a local minimum/maximum, thus resulting in $\Delta n_u = 0$ at those critical poles. The bottom subplot presents the $\mathbf{h} - n_u$ relation. . .	82
Figure 6.4	Time simulation results for Bit A: (i) evolution of axial velocity (blue curve) and angular velocity (red curve) for $\omega_o = 8$ (ii) evolutions of axial velocity (blue curve) and angular velocity (red curve) for $\omega_o = 14$	83
Figure 6.5	Time simulation results for Bit B: (i) and (i') show the early and developed evolution of velocities for $\omega_o = 1.5$ (U-regime). A steady-state limit cycle can be observed with axial stick-slips but no torsional stick-slip. For $\omega_0 = 2.15$ (S-regime), (ii) shows a quasi-periodic response. (iii) and (iii') show the early and developed evolutions of velocities for $\omega_o = 6$ (U-regime). A steady-state limit cycle corresponding to both axial and torsional stick-slip is observed. (iv) Corresponds to $\omega_0 = 11$, similar to (ii), a quasi-periodic response is observed (S-regime).	84
Figure 6.6	Limit cycles for the time responses: (i) $\omega_o = 1.5$, limit cycle without torsional stick-slip ; (ii) $\omega_o = 6$, limit cycle with stick-slip.	85
Figure 6.7	Frequency spectra of $v(\tau)$ within axial limit cycles and the root-locus plots for several ω_o in the U-regime (the red circles represent the poles corresponding to a given ω_o). The resonance frequency $f = 1/2\pi$ can clearly be observed. In all cases, there is a single unstable pole s_a and the high frequency components of $v(\tau)$ are related to $\Im(s_a)$	86
Figure 6.8	General fishtail bits under consideration: each bit can be characterized by two set of numbers $\{k_m\} = \{\chi/\nu, (1 - \chi)/\nu\}$ and $\{\alpha_m\} = \{\nu, \nu\}$. Note that the range of χ reads $\chi \in [0, \frac{1}{2\nu}]$	87

Figure 6.9	Stability map for fishtail bits with 4 blades ($\nu = 2$): a. the intersecting $\omega(m, \nu, \chi)$ curves and the critical pole-crossing direction (blue: right-going as h increases or ω decreases; red: left-going as h decreases or ω increases). b. The final stability map and some highlighted features. c. Determinations of the infinitely many smaller S-regions.	89
Figure 6.10	Time simulations are conducted for fishtail bits with $\chi = \frac{1}{4}$. For different bits, the same configuration ($\psi = 50$, $\beta = 0.3$, $\zeta = 0.5$, $\mathcal{W}_{fo} = 5$, $\mathcal{W}_{co} = 2$) is considered together with a nominal rotation speed $0.9\sqrt{\frac{\psi}{\nu}}$. Once a steady-state limit cycle is reached, a frequency spectrum is generated from the axial motion for each bit. For various ν the frequency spectra are show in subplot (i). It is verified in (ii) that the dominant higher frequency of the axial oscillations is proportional to $\sqrt{\nu}$	90
Figure 6.11	Bits with ν full blades and ν partial blades: each bit can also be viewed as a combination of two symmetric bits. The blades inside inner circle of radius of $a\xi$ corresponds to a symmetric bit with ν blades; The blades outside the circle of radius of $a\xi$ to a symmetric core barrel with 2ν blades.	91
Figure 6.13	Time simulation results for case U_I , $\omega_o = 0.6241$: (i) (ii) show the early and developed evolution of velocities, (iii) shows a steady-state limit cycle with torsional stick-slip	93
Figure 6.12	$\omega_c - \xi$ plot for bits with 3 symmetric full blades and 3 symmetric partial blades ($\nu = 3$)	93
Figure 6.14	Time simulation results for case U_{II} , $\omega_o = 0.7503$: (i) (ii) show the early and developed evolution of velocities, (iii) shows a steady-state limit cycle with no torsional stick-slip (but with axial stick-slip)	94
Figure 6.15	Time simulation results: (i) case S_I , $\omega_o = 0.6841$ and (ii) S_{II} , $\omega_o = 1.3150$. For each case, a quasi-periodic response is observed.	94
Figure 6.16	Top view of Bit A (looking toward $-Z$ direction)	96
Figure 6.17	Geometric properties of Bit A	98

Figure 6.18	Stability map for Bit A (shaded area represents U -regime in contrast to the S -regime): (a) wearflat width $w = 0.46$ mm (b) with no variation of W_f (c) wearflat width $w = 0.92$ mm (d) stacked stability boundaries	100
Figure 6.19	Number sets $\{\alpha_r\}$ and $\{k_r\}$ for the characteristic function P_a of Bit A at $d_o = 2.5$ mm	100
Figure 6.20	Verification of the stability map of Bit A. Top: stability boundaries of Bit A around $d_o = 2.5$ mm and the cases being show-cased: (a) $\omega = 2.4$, (b1)-(b2) $\omega = 1.6$, (c) $\omega = 1.15$, (d) $\omega = 0.8$	102
Figure 6.21	Verification of the stability map of Bit A (continued): (e1)-(e2) $\omega = 0.55$, (f1)-(f2) $\omega = 0.495$ and (g1)-(g2) $\omega = 0.4$	103
Figure 6.22	Evolution of pole trajectories for Bit A at $d_o = 2.5$ mm. Snapshots of the trajectories are taken with decreasing ω (or increasing h , nominal bit revolution period in axial time scale t_v): (a) $\omega = 1.9180$ (b) $\omega = 1.2370$ (c) $\omega = 1.0150$ (d) $\omega = 0.6700$ (e) $\omega = 0.5880$ (f) $\omega = 0.5210$ (g) $\omega = 0.4610$ (h) $\omega = 0.4830$ (i) $\omega = 0.1630$ (j) $\omega = 0.0100$. The red curves represent the pole trajectories appearing within the current ω interval, while the blue curves represent the counterparts in all previous ω intervals. In (c)(d)(g), the generation of a new branch is highlighted in a green circle.	105
Figure 6.23	For Bit A at $d_o = 2.5$ mm, with respect to ω , (a) the evolution of the real part of rightmost pole (b) the evolution of the number of unstable pole pairs n_u	106
Figure 6.24	Influence of pole location on the transient response of the coupled axial dynamics	106
Figure 6.25	Comparison of time responses of Bit A ($d_o = 2.5$ mm, $\omega = 0.319$): (a) $\beta < 1$ and (b) $\beta > 1$	107
Figure 6.26	Parametric study of a reference RGD model ($n = 6$) with various magnitudes of external perturbation, parameters chosen follow Figure 5 of Richard et al. [3]. The bit responses are achieved by applying an external perturbing torque to a prescribed nominal motion, the duration for applying $\mathcal{T}_p$ is $\Delta\tau_p = 1$	108

Figure 6.27	Comparison of time responses of Bit A ($\omega = 0.319$): (a) $d_o = 2.5$ mm and (b) $d_o = 3$ mm	109
Figure 6.28	Bottom views of Bit 4 and Bit 20.	109
Figure 6.29	Bottom views of Bit 4 and Bit 20.	110
Figure 6.30	Comparison of geometric properties of Bit 4 (blue) and Bit 20 (red)	111
Figure 6.31	Comparison of geometric properties of Bit 4 with backup cutters (blue) and Bit 20 with backup cutters (red). The dashed lines correspond to the results without considering backup cutters, identical to the plots in Fig. 6.30.	112
Figure 6.32	Stability maps for Bit 4 (with or without backup cutters), Bit 20 (with or without backup cutters)	113
Figure 6.33	Time simulation results for bit 4	114
Figure 6.34	Cutter face of a scribe cutter (upward is the positive direction for bit axial advance)	116
Figure 6.35	Bottom views and bit profile of Bit B and Bit C.	116
Figure 6.36	Comparison of geometric properties of Bit B, Bit C and Bit 20	117
Figure 6.37	Stability maps for Bit 20, Bit B and Bit C	117
Figure 6.38	Modeling of cutter wear: examples are shown for both round and scribe cutters	118
Figure 6.39	Comparison of geometric properties of Bit 4 (intact, two different wearing degrees)	119
Figure 6.40	Stability maps for Bit 4 with various degrees of wearing . . .	119
Figure 7.1	Different types of “limit cycles” observed in the current stick-slip model: (A) axial stick-slips without torsional stick-slips, (B) axial stick-slips with torsional stick-slips, (C) a quasi-periodic motion and (D) axial stick-slips with damped torsional oscillation. For the responses (C) and (D), with the addition of damping, one can observe stationary torsional oscillations. . .	124

List of Tables

Table 6.1	List of some parameters considered in the model	95
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Chapter 1

Introduction

1.1 Context

1.1.1 Rotary drilling system

In petroleum engineering, boreholes are drilled for exploration and production of hydrocarbons. All the tools for creating boreholes are located on a drilling rig. Although having various forms and appearances, drilling rigs share the following components: a power system to provide energy, a hoisting system to transport drillstrings, casings and other equipments into (or out of) the borehole, a circulating system to bring rock debris to the surface while drilling, a rotary drilling system to rotate the bit, and finally, well monitoring and control systems.

We are interested in the rotary drilling system, whose schematic is shown in Fig 1.1. The drillstring mainly consists of drill pipes, each about 9 meters long, fastened together; the lower part of the drillstring is composed of heavy drill collars and the drill bit is attached at the end. Drill collars are also referred to as bottom-hole assembly (BHA). Typically, the length of the drillstring is of order $O(1000\text{m})$, and for BHA $O(100\text{m})$. The swivel hoists the drillstring, which is gripped by the rotary table such that the power from the rotary drive can be transmitted to ensure bit rotation.

Figure 1.1 – Schematic of the rotary system, from [1]

The instability in the drilling process causes drillstring vibrations, which are commonly observed [4–6]. Those self-excited vibrations mainly result from the drilling process at the bit/rock interface. The lateral, axial and torsional modes of vibration may degenerate into bit whirling, bit bouncing and stick-slip, respectively. These dysfunctions cause premature failure of drill bits and other components within the system, ultimately leading to increased cost of the drilling operation. It is of interest, therefore, to understand the mechanism of self-excited vibrations, with the objective of mitigating the drilling dysfunctions. It is well established that the stick-slip vibrations in the drillstring can be bit-induced and/or drillstring-induced: the former one is related to the bit/rock interaction while the later one is related to the drillpipe/borehole interaction (more commonly observed in inclined or tortuous wells). Specifically, this work focuses on the contribution of the design of drag bits (most commonly, PDC bits) to the bit induced drillstring stick-slip vibrations.

1.1.2 PDC bits

There are two major categories of bits applied in the oil industry, different in their rock removal mechanism: roller cone bits operate by either chipping/crushing (for hard formation) or gouging/scraping (for soft formation), and drag bits (PDC bits, natural diamond bits) by shearing. PDC bits (1.3), composed of many fixed cutters

mounted on a metal body, are most commonly used nowadays in the industry due to their better drilling efficiency.

Roller cone bits, introduced by the Hughes Tool Co in 1909, were widely used for oil well drilling. In 1971, General Electric (GE) invented the carbide-supported polycrystalline diamond cutter (PDC). Shortly afterwards, in late 1972, GE Carbology first introduced PDC bit technology at Hughes Tool Co. The early cutter, COMPAX Diamond Drill Black, is composed of a carbide disk with a thin layer of PDC that was without chamfer. For easier attachment into the steel bit body, a COMPAX/Slug system was introduced where the cutter was brazed to a carbide stud. The braze alloy (Silver Alloy 45) was chosen to keep the cutter temperature below the thermal degradation temperature of the PDC layer.

In July 1973, GE had the first run of a PDC bit at an oil well operated by Exxon in South Texas. After various test runs of prototypical PDC bits from 1973 - 1975, the design of PDC was refined and incorporated into Stratapax, introduced by GE in December 1976 (Fig. 1.2). The Stratapax product line included cutters of two sizes (8 mm and 13 mm in diameter) with various profiles, a longer carbide substrate and provided more impact resistant cutter.

From the late 1970s to the early 1980s, PDC bits were experimented and developed by many companies including Diamant Boart, Drilling & Service, Stratabit and Eastman Christensen. In 1984, Valeron successfully developed the non-planer interface cutter which resolved the issue of residual stress problems that are responsible for the table delamination of cutters. They also introduced the 19 mm cutter, which is an industry standard now. DeBeers Industrial Diamond Division pioneered the development of the layered diamond table cutter which is composed of a layer of wear-resistant fine-grained diamond on the cutter face and a deeper layer of tougher coarse diamond. Such a feature has become the norm since then. These developments in PDC bit technology were backed by the rapid growth in drilling until mid 1980s. The footage drilled by PDC bits saw a steady increase, up to 5% in the market by 1990. However, the drilling market was in recession in 1990 and experienced a near collapse in 1999. Many of the drilling companies went bankrupt and bit companies accordingly consolidated and continued to develop and introduce improvements in PDC bits in a very depressed market. By 2000, PDC bits was used for 24% of the drilling footage.

Since 2000, PDC bits saw a rapid expansion in application. Formations with hard

interbedded streaks which were “undrillable” can be drilled economically by PDC bits. Bit companies introduced application-specific bits, with the process of laboratory testing to confirm design objectives before shipping to the customer. Researchers in the modeling of drilling dynamics also helped the management of drillstring oscillations which extended the lifespan of PDC bits. In 2015, PDC bits comprise 80% of the revenue of the bit industry and cover more than 90% of the footage.

1.2 Motivations

One approach to analyze the occurrence of stick-slip torsional vibrations in rotary drilling system equipped with PDC bits is based on assuming a velocity-weakening bit/rock interaction law and considering only the torsional dynamics [5, 7–11]. Although consistent with field observations [8], the velocity-weakening model does not properly describe the interface laws at time scales relevant for a dynamic analysis of the system response. Moreover, this model conflicts with results of kinematically-controlled laboratory tests conducted with a single cutter [12] or with a full bit [13], as experimental results do not exhibit any significant dependence of the force on the velocity. An alternative approach [2, 3, 14–20] considers both axial and torsional dynamics, which are coupled via a rate-independent bit/rock interface law that accounts for cutting by PDC inserts as well as frictional contact on the cutter wearflats [21]. The original model in that class [3, 22] is simply referred to as the RGD model in this paper.

The RGD model reduces the drillstring to a two degrees-of-freedom coupled axial-torsional system and simplifies the bit geometry to several identical blades that are evenly distributed around the bit axis of rotation. Given the rotary motion of the drill bit, the cutting process is subject to a “regenerative effect”, as in metal machining processes [23–26]. Due to the assumed simplified bit geometry, the instantaneous depth of cut is simply related to the bit axial positions at the current time and at a previous time; the difference between these two time instances, or time delay, is the time it takes for the bit to cover the angular offset between two blades. The time delay is a priori unknown, however, since the angular motion of the bit is not uniform. In summary, the regenerative effect introduces a single, discrete, and state-dependent delay into the equations governing the bit dynamics in the RGD model [3].

Two conclusions with practical implications have been reached by analyzing the stability of the RGD model and conducting numerical simulations [3, 15, 16, 27].

STRATAPAX^{*} DRILL BLANKS

PRODUCT LINE

Product number	Product shape	Nominal dimensions
2325		A Dimension = $3.7 \pm 0.3\text{mm}$
2330		Diameter = $8.2 \pm 0.3\text{mm}$ Overall thickness = $3.3 \pm 0.1\text{mm}$ Diamond Thickness = 0.5mm (min)
2525		A Dimension = $6.2 \pm 0.6\text{mm}$
2530		Diameter = $13.3 \pm 0.2\text{mm}$ Overall thickness = $3.3 \pm 0.1\text{mm}$ Diamond thickness = 0.5mm (min)
2532		B Dimension = $11.1 \pm 0.2\text{mm}$
2534		B Dimension = $11.1 \pm 0.2\text{mm}$
2536		C Dimension = $9.4 \pm 0.2\text{mm}$
2538		C Dimension = $9.4 \pm 0.2\text{mm}$

PRODUCT 2530-NC (STRATAPAX DRILL BLANK DESIGNED FOR FABRICATOR BONDING TO CARBIDE SUBSTRATES).

PRODUCT 2541 (STRATAPAX DRILL BLANK 2530-NC ATTACHED TO CARBIDE SUBSTRATE).

PRODUCT 2540 (STRATAPAX DRILL BLANK 2530-NC ATTACHED TO CARBIDE SUBSTRATE).

PRODUCT 2542 (STRATAPAX BLANK STUD CUTTER ASSEMBLY USING 2530-NC DRILL BLANK).

Thickness of the polycrystalline diamond layer of all STRATAPAX blanks is 0.5mm minimum. All blanks except 2530-NC, 2540, 2541 and 2542 have a $45^\circ \times 0.7 \pm 0.2\text{mm}$ chamfer on the round carbide edge. The 2540 and 2541 blanks have a $15^\circ \times 1.0\text{mm}$ chamfer on the back of the substrate. The 2542 blank has a $15^\circ \times 1.27\text{mm}$ chamfer on the base of the supporting stud.

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Figure 1.2 – GE's Stratapax Cutters

Figure 1.3 – Illustration of a modern PDC bit

First, there exists a critical rotation speed separating two regimes of instability [27]: above this rotation speed, the torsional dynamics is marginally unstable and thus experiences a slow growth of the self-excited vibrations; below this rotation speed, the axial dynamics is unstable and perturbations quickly degenerate into stick-slip limit cycles or bit bouncing. Second, the observed velocity-weakening torque-on-bit is a consequence of the self-excited vibrations rather than their root cause [3, 15, 16].

In the original RGD model [3, 15, 22], the drillstring is reduced to a simple discrete system only characterized by a torsional stiffness, moment of inertia, and mass. This simple representation of the drillstring has since been enriched by other 2 DOF's models [16, 17, 19, 27–29], multi-DOF systems [2, 18–20, 30–32] and continuum representation [33, 34]. However, the simplified bit geometry considered in the original model has remained untouched. This idealized cutter layout entails that the critical angular velocity for two DOF's models is proportional to $1/\sqrt{n}$, where n is the number of symmetric blades.

In reality, PDC bits have a complex arrangement of cutters (Fig. 1.3), which are responsible not only for multiple delays but also for a delay distribution that depends on the bit motion history. Since the delay distribution controls the stability of the bit, it is imperative to develop a model capable of capturing the exact design of PDC bits. This additional consideration, although greatly complicating the analysis, is expected to give more realistic insights into the mechanism of the stick-slip vibrations of a rotary drilling system. As of now, few works have tried to incorporate the effect

of bit designs into the class of the RGD model [35–37].

This thesis compiles the author’s PhD work on the project “Advanced Simulation of PDC Bits”. There are two major technical achievements reported in this thesis. First, a methodology to characterize bit geometry and to account for the complex regenerative effect associated with bit designs. Second, the methodology is incorporated into a 2 DOF dynamical model of the drilling structure. With the necessary analyzing tools ready, the research aims to seek the factors of a PDC bit cutter layout that may influence the stability, thus shedding some light on the practical engineering designs.

A more complex cutter layout introduces multiple delays to the system and is expected to affect the stability of the bit in regard to the self-excited vibrations. Recent works [35–37] have started to investigate its effect on the stability of rotary drilling systems.

1.3 Organizations

Chapter 2 first gives a brief review of the literature on the bit/rock interaction models, and then shifts to the review of works on the stick-slip vibrations of the rotary drilling systems. Chapter 3 gives the model description which includes two parts (i) a simple lumped drillstring model [3] and (ii) a bit/rock interaction model that leads to the computation of the instantaneous weight-on-bit and torque-on-bit; governing equations in dimensionless form are given at the end of the chapter. Chapter 4 focuses on the time simulation for the model, particularly the equations of motion are integrated for normal drilling and various drilling dysfunctions. Chapter 5 presents the formulation of linear stability analyses considering the real PDC cutter layout. Chapter 6 shows multiple applications of the methodology developed. The “variable” being the bit geometry, results are shown for symmetric/asymmetric full-blade bits, fishtail bits, bits with partial blades, PDC bits with round cutters (with/without backup cutters), PDC bits with scribe cutters and PDC bits with worn cutters. Finally Chapter 7 concludes with the main insights of this research and propose some of the future work.

Chapter 2

Literature Review

2.1 Rock cutting mechanics

A key effort of this research is to model the bit/rock interaction in a more realistic manner. As drag bits are composed of fixed cutters, one should extend a proper cutter/rock interaction model to the full bit scale.

Fairhurst and Lacabanne [38] first suggest that the drilling action is composed of two co-existing processes: rock cutting and frictional contact. Detournay and Defourny [21] present a phenomenological model (the “DD model”) for the reaction forces exerted on a single PDC cutter. The cutting force is proportional to the cross-sectional area of the rock removed and the inclination of this cutting force is contact; the frictional force on the wearflat is subject to the classic law of Coulomb friction, with a constant friction coefficient assumed. One major conclusion of this model is that, for a blunt cutter, there is a linear constraint on the relationship between the specific energy E and the drilling strength S (defined respectively as the ratios between the shear/normal component of the cutter force and cross-sectional area of the cut). This linear relationship corresponds to a friction line on the $E - S$ diagram. The DD model was initially applied to some experiments with a single cutter [39], where three different types of sandstone are tested with 3 sets of cutters: one sharp cutter and two blunt cutters with different wearflat areas. While validating the independent cutting and frictional processes mentioned before, this work provides a workflow to estimate the independent parameters that are capable of depicting single cutter action.

This single-cutter/rock interaction law is supported by a series of experimental efforts [40–48]. The general rock cutting mechanics with a single PDC insert is summarized as follows. First, two cutting mechanisms are separated by a critical magnitude of depth of cut d : a ductile regime exists for a smaller d and a brittle regime for a larger d [46, 49–52]. Furthermore, two distinct ductile regimes can be recognized [53] which

can be summarized with a bilinear relationship between the force exerted on the cutter and d . As d increases from 0, forces on both the cutting face and the wearflat increase linearly with respect to d until the a critical d is reached beyond which the wearflat force saturates and remains constant while the cutting force maintain the same linear relationship with d . Zhou and Detournay [54] analyze the wearflat force and concludes that with a positive attack angle, the frictional contact force on a blunt cutter always saturates. A practical application of this rock cutting mechanics is the “scratch test” which can be used to evaluate rock properties such as the uniaxial compressive strength [55]. Coudyzer and Richard [56] investigate the influence of the rake angles on the cutter force using test results for Lens limestone a rock with high homogeneity and medium strength. Specifically, cutter force is influenced by the back rake angle and its side component (in the transversal direction perpendicular to cutter velocity) is mostly invariant with respect to the side rake angle. Dagrain [57] investigates the correlation between the groove geometry and the cutter force and validates the result with experiments.

Chen et al. [58] correlated the total cutter force with the area and shape of the cutter/rock engagement. For the brittle regime, they incorporated the rock chip model by Yan [59]. Finally, they extended the cutter force formulation to some full PDC bits and verified the results with test runs.

2.2 Modeling stick-slip

2.2.1 The classical approach

During drilling operations, the drillstring may experience stick-slip vibrations which is detrimental to the drilling equipment [5, 60]. During the stick phase, the bit is stationary while the torque is building up in the drillstring until the bit slips. As the bit is released, it may accelerates to an angular velocity around two times of the prescribed RPM applied at the rotary table on the surface [61]. Research on the mechanism of stick-slips started early by Belokobylskii and Prokopov [62] who modeled the rotary drilling system as a torsional pendulum with a Coulomb friction. Later, the non-smooth effect correlated to the dry friction was introduced to the modeling of drilling dynamics by Dawson et al. [63]. Subsequently, one approach to analyze the occurrence of stick-slip torsional vibrations in rotary drilling system equipped with PDC bits is based on assuming a velocity-weakening bit/rock interaction law and considering only the torsional dynamics [5, 7–11]. Although stick-slip motion can be replicated with this approach, the premise of the models, a rate-dependent bit/rock

interaction is not supported by experiments [12, 13].

2.2.2 The novel approach

In the study of machining vibrations, it is well-established that the cutting process is subject to a “regenerative effect” [23–26], which simply refers to the regeneration of the workpiece’s surface depends not only on the current tool position but also on its past position. In the 1990s, the regenerative effect was first introduced to the studies of drilling dynamics by Dareing et al. [64]. However, this model in fact considered the axial dynamics only, without including the variation of the rotation speed.

Later in the 2000s, Richard et al. [3, 22] considered both axial and torsional dynamics, which are coupled via a rate-independent bit/rock interface law that accounts for cutting by PDC inserts as well as frictional contact on the cutter wearflats [21]. Following the RGD model, a class of models have been developed for the stick-slip vibrations of the rotary drilling system. The common features include a rate-independent bit/rock interaction law [21], which couples the axial and torsional modes of vibrations, and leads to state-dependent time-delay effects; meanwhile other nonlinearities are also considered such as the non-smooth effect due to dry friction, loss of contact (cutting and/or frictional). Germaey et al. [30] shows that higher modes can be excited by the self-excited vibrations using a finite element model for the drillstring. Besselink et al. [16] considers a 2 DOF drillstring model with axial stiffness and damping but without torsional damping. The stability of the system is evaluated in the parameter space of some scaled axial damping and viscous friction. Liu et al. [31] studies an 8 DOF model, where dry friction, loss of contact and a single state-dependent time delay is considered for the axial-torsional-lateral coupled drilling dynamics. Various drilling dysfunctions are found such as stick-slip motions, forward and backward whirling. Although the cutting bit force is subject to the regenerative process, the frictional bit force is rate-dependent in the model. Nandakumar and Wiercigroch [17] achieved analytically a stability diagram for the fully coupled axial-torsional dynamics using the D-subdivision method. Later, by applying the semi-discretization method [26, 65], the stability map is extended to the parametric space of rotation speed (RPM), rate of penetration (ROP) and rock strength parameters [2], see Fig. 2.1. Note that the near-vertical left boundary in this figure agrees with the single critical rotation speed predicted from the original RGD model. In this work, the y -axis δ_o is the bit penetration per revolution d_o scaled by a characteristic length $2\pi d_*$, where $d_* \sim O(1 \text{ mm})$ is the characteristic length for

Figure 2.1 – Stability in RPM (represented by dimensionless ω_o) and ROP (represented by dimensionless δ_o) after 2. In this work, maximum frictional WOB is assumed constant; thus δ_o can be viewed as a proxy for WOB

the RGD model. Since d_o is usually a few millimeters, the map reported is realistic only for a rather small δ_o value. In a separate work, Liu et al. [18] analyses a coupled axial-torsional dynamics with a finite-element drillstring model, which shows that axial-torsional wave propagation along the drillstring. Also it is shown in this work that increasing the discretization resolution results in the shrinking of the stable region in the parametric space of RPM and ROP. Aarsnes and Aamo [33], Aarsnes and de Wouw [34] considers a distributed drillstring model where a stability chart is achieved based on the Nyquist stability criterion. Bakhtiari-Nejad and Hosseinzadeh [66] considered a boundary condition of constant hanging force at the surface and constructed a stability map. Yan and Wiercigroch [35] studied a system with angularly nonuniform blade distribution which leads to multiple time delays. Tian et al. [36] analyzes idealized drag bits characterized either by multiple angular offsets between the blades or by a combination of full and partial blades. Tian and Detournay [37] further relaxes the restriction of bit geometry and considers realistic PDC cutter layout; a stability chart in terms of ROP and RPM can be calculated for a particular bit design.

Gupta and Wahi [19] formulated the rate-independent bit/rock interaction (the frictional drilling force is not included) from the perspective of rock while considering the simplified bit geometry as in the RGD model (which results in a single state-dependent

Figure 2.2 – Stability chart in RPM-WOB space

discrete time delay). The key point of this new model is the introduction of rock profile a rock function. Consequently the time delay system is equivalently written as a partial differential equation, which is solved using the Galerkin procedures [67]. Zhang and Detournay [68] extends this alternative formulation to bits with full and partial blades which introduces 2 time delays to the system. The benefit of this formulation lies in the direct application of PDE solvers of MATLAB which speed up significantly the solving process .

2.3 Stick-slip mitigation, field applications

2.3.1 Overview

In the era of popular applications of roller cone bits, drillstring vibration is a subject of concern in the Industry [69]. After the common application of PDC Bits, it was discovered that this “new” bit technology could lead to lateral drillstring vibrations that may degenerate into backward bit whirl which is detrimental to the drilling equipment [70, 71]. For a while, researchers are focusing on mitigating backward whirl through the so-called anti-whirl bit designs. In recent years, it is discovered that stick-slip vibration is the main cause of bit failure in the rotary drilling applications [72].

The easiest method for stick-slip mitigation is to adjust WOB (weight-on-bit) and bit RPM. Specifically, lowering WOB and increasing RPM can eliminate the stick-slip vibrations. However, tradeoffs have to be made as reducing WOB simply decreases ROP (rate-of-penetration) and increasing RPM may lead to backward whirl. Rudat

and Dashevskiy [73] presents an algorithm that can be installed in the firmware of downhole measurement tool to estimate stick-slip instability zone in real time; the instability zone in the RPM-WOB space is derived using a model that tracks the change of energy. Another common technique to combat stick-slips is through a control system that modulates the top-drive of the rotary drilling system [74]. Other techniques include the application of downhole devices such as downhole motors (decoupling the bit from the drillstring) and shock-sub (add damping to the system).

2.3.2 Influence of bit design on the stick-slips

There are several theories of mitigating stick-slips via bit design. Fear and Abbassian [75] suggests that anti-whirl bit resists stick-slips. Outlined by Warren et al. [71], anti-whirl bits arrange cutters such that a high imbalance force is induced to push a low friction pad to contact the borehole which helps producing a forward motion.

It is commonly believed that the bit aggressiveness is a key factor that influence the torsional drillstring vibrations Jaggi et al. [76]. The bit aggressiveness is referred to the magnitude of cutter/rock engagement, which can be controlled by the back rake angles, chamfer size and cutter layout. In the light of this theory, a depth of cut controller technology is developed and integrated in modern PDC bit designs. In the work of Richard et al. [77], it is mentioned that the wearflat distribution can potentially mitigate stick-slips, which will be explored in more detail in this work. Finally, [8] hypothesized that worn bits exhibit more stick-slips.

2.3.3 Stability chart in the space of WOB and RPM

In the Industry, there are several works focusing on the stability chart in the space of RPM and WOB together with its verification. One representing work is reported by Wu et al. [78], see Fig. 2.2. They claim that the boundaries in this figure are achieved with an unspecified model that considers the coupling between torsional stick-slips and bit whirls. For our interest, the boundary for stick slips is a curve concaving up, scissored by a constant WOB that triggers buckling. This general trend of this curve is in agreement with the field/laboratory observations [79, 80].

Chapter 3

Model Description

The RGD model class can be viewed as a combination of two components: a discrete model of the drillstring and a bit/rock interface law representing the bottom-hole boundary conditions. After a brief review of the discrete drillstring model, we formulate the bit/rock interaction for two classes of bit geometry: (1) an arbitrary cutter layout under the restrictions that the cutters are rectangular and oriented radially, with their lower edge in the same plane orthogonal to the bit axis (Class I) and (2) bit geometry with realistic cutter shape and cutter layout (Class II). The governing equations are formulated and presented for each class in dimensionless form after scaling.

3.1 Drillstring model

In the RGD model, the drillstring is simplified as a two degrees-of-freedom system, which is expressed in terms of the axial and angular motion of the bit. The model is also constructed on the following assumptions: (i) the bit does not experience any lateral motion, the wellbore is vertical and there is no friction between the drillstring and the borehole wall, and (ii) a constant hook load H_o and angular velocity Ω_o are imposed at the ground surface. Furthermore, the drillpipes are treated as a single torsional spring with a torsional stiffness C , and the bottom-hole assembly (BHA) is lumped at a point with mass M and moment of inertia I (see Fig. 3.1).

The vertical and angular displacements of the bit, functions of time t , are denoted by $U(t)$ and $\Phi(t)$, respectively. The equations of motion read

Figure 3.1 – Simplified model for the drillstring in the model of [3]

$$\begin{aligned}
 M \frac{d^2 U}{dt^2} &\in W_o - W({}_0^t \Phi, {}_0^t U), \\
 I \frac{d^2 \Phi}{dt^2} + C (\Phi - \Omega_o) &\in T_o - T({}_0^t \Phi, {}_0^t U)
 \end{aligned} \tag{3.1}$$

where W_o and T_o are the reaction axial force and torque acting on the bit corresponding to the equilibrium solution, when the bit is experiencing a uniform motion corresponding to angular velocity Ω_o and rate of penetration V_o . The nominal weight-on-bit W_o (as it is referred to) is the difference between the submerged weight of the drillstring W_s and the hook load H_o . The bit/rock interface laws provide relationships between the equilibrium quantities of the form $W_o(V_o/\Omega_o)$ and $T_o(V_o/\Omega_o)$. The actual weight-on-bit W and torque-on-bit T are functions of the respective history of Φ (denoted by ${}_0^t \Phi$) and U (denoted by ${}_0^t U$). Finally, the equations of motions are differential inclusions due to the discontinuous nature of the right-hand side. The quantities introduced and their sign conventions are shown in Fig. 3.1.

3.2 Bit/rock interaction law for Class I Bits

The rate-independent cutter/rock interface law describes the interaction between a cutter and a rock as a combination of cutting and frictional contact processes. A simple cutter/rock interaction model supported by experimental evidence assumes that the cutting force is proportional to the cross-sectional area of the groove traced by the cutter [21], and that the frictional contact force transmitted by the cutter

wearflat has saturated if the angle of attack – the small angle between the wearflat and the cutter velocity – is positive [54]. Extending the cutter/rock interaction law to a bit requires information on the bit geometry [21, 53]. To this end, we consider simple bits of radius a consisting of N rectangular cutters, which are restricted to be oriented radially with their lower edge located in the same plane orthogonal to the bit axis. Figure 3.2 shows such a bit with three cutters.

3.2.1 Coordinates definition

To describe the extended bit/rock interaction, two coordinate systems are introduced: an Eulerian one fixed in space and a Lagrangian one attached to the bit. The Eulerian reference system can be introduced either as a Cartesian coordinate system (x, y, z) with the z -axis being the bit rotation axis and pointing in the direction of axial bit advance, or as a cylindrical coordinates system (r, θ, z) , with $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. Similarly, the Lagrangian reference system can also be defined either as a Cartesian coordinate system (X, Y, Z) or as a cylindrical one (R, Θ, Z) , with $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan(Y/X)$. Lagrangian coordinate Θ is restricted to the interval $(0, 2\pi]$, but there is no such restriction on θ , which is used to track the angular displacement of the cutters. Axes z and Z are coaxial on account that the bit is aligned along the axis of a straight borehole. The origin of the coordinates systems is chosen so that the plane $Z = 0$ contains the lower edges of the cutters. The two coordinates systems are illustrated in Fig. 3.2; it is required that the two systems coincide at time $t = 0$.

The Lagrangian coordinate system is used to describe the bit geometry. The cutter layout of the bits considered in this study is fully described by the location of the lower edge $\mathcal{L}_i$ of cutter i , $i = 1, 2, \dots, N$ (Fig. 3.2). Each $\mathcal{L}_i$ is defined by its two end points with cylindrical coordinates $(R_i - b_i/2, \Theta_i, 0)$ and $(R_i + b_i/2, \Theta_i, 0)$, where $(R_i, \Theta_i, 0)$ are the coordinates of the midpoint of $\mathcal{L}_i$ (also referred to as the reference point of the cutter i) and b_i is the length of $\mathcal{L}_i$, referred to as cutter width in the following. Hence a set of $3N$ numbers $\{R_i, \Theta_i, b_i\}$ describes the bit geometry.

The Eulerian coordinate system is used to describe the motion of the cutters on the bit. The bit axial and angular displacements, $U(t)$ and $\Phi(t)$, can be tracked by following the motion of a fixed point on the bit, chosen to be located on the X -axis ($\Theta = 0, Z = 0$) on the bit. During drilling, each cutter spirals around the bit axis; the cutter motion trajectories are identical, but only shifted in Φ (not in U) as there are no

axial offsets between cutters. Since the cutters are assumed to be rectangular and positioned radially, the location of cutter i is determined by the coordinates of its reference point in the eulerian cylindrical coordinate system, i.e.,

$$r = R_i \quad \theta = \Phi(t) + \Theta_i \quad z = U(t) \quad (3.2)$$

Finally, the bit profile is obtained by rotating the bit without advancing it and projecting the lower edge of each cutter onto the quarter-plane corresponding to $x > 0$, $y = 0$ and $z > 0$, when its reference point, depicted by (3.2), crosses this half-plane. Clearly, all lower cutter edges are stacked at the same z , while spanning various ranges of r ; these projections are equivalent to the bit profile. The projections of the cutters should be shifted in the axial direction to determine the actual cutter/rock interaction for a given bit motion.

3.2.2 Cutter engagement surface

Define an index set $\mathcal{I} = \{1, 2, 3, \dots, N\}$ to account for each cutter on the bit. The cross-section of the groove traced by cutter i is referred to as the cutter engagement surface, $\bar{\mathcal{S}}_i$. According to the cutter/rock interaction model [21], the cutting components of the weight-on-bit and the torque-on-bit are linearly related to cutter engagement area $\bar{A}_i = \|\bar{\mathcal{S}}_i\|$. Figure 3.3 illustrates the general approach to determine a particular $\bar{\mathcal{S}}_i$. Let index j refer to any cutter that potentially interacts with cutter i ; index j belongs to index set $\mathcal{J}_i$, a subset of $\mathcal{I}$, which is defined as

$$\mathcal{J}_i = \{j \in \mathcal{I} \mid \mathcal{L}_i \cap \mathcal{L}_j \neq \emptyset\} \quad (3.3)$$

where $\emptyset$ denotes an empty set. The angular offset $\Delta\Theta_{ji}$, or angular delay, is defined as the angle by which cutter $j \in \mathcal{J}_i$ is ahead of cutter i

$$\Delta\Theta_{ji} = \begin{cases} \Theta_j - \Theta_i & \Theta_j > \Theta_i \\ 2\pi + \Theta_j - \Theta_i, & \Theta_j < \Theta_i \end{cases} \quad (3.4)$$

Since a cutter can potentially interact with itself, this case is formally defined by $\Delta\Theta_{ii} = 2\pi$. The definition of the index set $\mathcal{J}_i$ necessarily implies that $\Delta\Theta_{ji} \neq 0$, since no parts of cutter i and j can occupy the same region of space at any moment. In summary, $\Delta\Theta_{ji} \in (0, 2\pi]$. The set of $\{\Delta\Theta_{ji}\}$ reflects the cutter layout. The cutter engagement surface $\bar{\mathcal{S}}_i$ depends on past axial positions of cutters j , $j \in \mathcal{J}_i$, when they

Figure 3.2 – An example of an asymmetric bit (each rectangle represents a cutter) and the coordinate systems related to this model

occupied the current angular position of cutter i . Let $\Delta Z_{ji}(t)$ denote the axial bit advance when the bit rotates a fixed angle $\Delta\Theta_{ji}$, from $\Phi(t) - \Delta\Theta_{ji}$ to $\Phi(t)$; the time delay corresponding to $\Delta\Theta_{ji}$ reads t_{ji} :

$$\Delta Z_{ji}(t) = U(t) - U(t - t_{ji}), \quad \Delta\Theta_{ji} = \Phi(t) - \Phi(t - t_{ji}) \quad (3.5)$$

Determination of $\bar{\mathcal{S}}_i$ is most simply done by following a sequence where the interaction of cutter i with any other cutter j (including cutter i itself) is sequentially accounted for. Let $\mathcal{S}_i$ denote the cutter engagement surface resulting from the self-interaction of cutter i ; $\mathcal{S}_i$ is thus a rectangle with height ΔZ_{ii} and width b_i . Then, $\bar{\mathcal{S}}_i$ is determined by sequentially removing part of $\mathcal{S}_i$ by cutters $j \in \mathcal{J}_i$. In Fig. 3.3(ii), the L -shaped surface (shaded) is the result of one instance of trimming of $\mathcal{S}_i$ by cutter j . In other words, if there are more cutters interacting with cutter i , the shaded area should be sequentially trimmed until the actual $\bar{\mathcal{S}}_i$ is achieved.

Figure 3.3 – Illustration of the bit/rock interaction considering the extended bit geometry (general motion)

3.2.3 Weight-on-bit and torque-on-bit

Once the engagement surface $\bar{\mathcal{S}}_i$ of cutter i has been obtained, its area $\bar{A}_i$ and radial coordinate $\bar{r}_i$ of its centroid can readily be computed. Accounting for the contributions of all the cutters, the cutting components of the weight-on-bit W and torque-on-bit T , denoted by subscript c , are given by

$$W_c = \epsilon \zeta \sum_i \bar{A}_i, \quad T_c = \epsilon \sum_i \bar{A}_i \bar{r}_i, \quad (3.6)$$

while the frictional contact components of W and T , denoted by subscript f read [53]

$$W_f \in \sigma a \ell (1 - g), \quad T_f \in \frac{1}{2} \mu \sigma \gamma a^2 \ell (1 - g) \quad (3.7)$$

The total weight-on-bit and torque-on-bit are then simply the sum of the cutting and contact components, i.e.,

$$W = W_c + W_f, \quad T = T_c + T_f \quad (3.8)$$

The above interface laws introduce two groups of parameters, one to characterize the cutter/rock interaction, the other the bit bluntness. The cutter/rock interaction parameters (assumed identical for all cutters) entering expressions (3.6) and (3.7) are [21]: intrinsic specific energy ϵ (the amount of energy spent to cut a unit volume of rock), ratio ζ (the ratio between vertical and horizontal cutting force, a function of the cutter rake angle), friction coefficient μ (ratio between horizontal and vertical contact force), cutter wearflat width w , and saturation contact stress σ (the saturation normal contact force for cutter i being then $b_i w \sigma$). The bit bluntness parameters, ℓ and γ ,

characterize the size and distribution of the wearflat surfaces

$$\ell = \frac{w}{a} \sum_i b_i, \quad \gamma = \frac{2 \sum_i b_i \bar{r}_i}{a \sum_i b_i} \quad (3.9)$$

The unilateral nature of the contact force is captured by the set-valued function g in (3.7), which depends only the sign of bit axial velocity V

$$g = \begin{cases} 0 & V > 0 \\ (0, 1) & V = 0 \\ 1 & V < 0 \end{cases} \quad (3.10)$$

Under normal drilling conditions, $V > 0$ and the contact forces acting on the cutter wearflats saturate. However, if $V = 0$ during a finite time interval, the contact component of the weight-on-bit must be resolved according to equilibrium condition, as g can take any value within $(0, 1)$. When $V < 0$ and $\Omega > 0$, the bit can still cut the rock, but with $W_f = T_f = 0$ as all the cutter wearflats are disengaged from the rock surface. If no cutters are cutting rock, i.e. $\bar{A}_i = 0$ for all $i \in \mathcal{I}$, bit bounce occurs [3].

The bit/rock interface laws (3.6) and (3.7) can be simplified, when the bit motion is uniform. Indeed, it can be shown that [21]

$$W_{co} = \epsilon \zeta a d_o, \quad T_{co} = \frac{1}{2} \epsilon a^2 d_o \quad (3.11)$$

where d_o is the constant depth of cut per revolution given by

$$d_o = \frac{2\pi V_o}{\Omega_o} \quad (3.12)$$

Since $V_o > 0$, the contact components W_{fo} and T_{fo} are simply

$$W_{fo} = \sigma a \ell, \quad T_{fo} = \frac{1}{2} \mu \sigma \gamma a^2 \ell \quad (3.13)$$

In the above, the subscript groups co and fo denote, respectively, the steady-state cutting and frictional contact components of the weight-on-bit and torque-on-bit. Thus the interface laws for a bit experiencing uniform motion read

$$W_o = \epsilon \zeta a d_o + \sigma a \ell, \quad T_o = \frac{1}{2} \epsilon a^2 d_o + \frac{1}{2} \mu \sigma \ell \gamma a^2 \quad (3.14)$$

It is thus evident from the above equations (3.12) and (3.14) that V_o and T_o can be determined from the bit/rock interface laws given the boundary conditions Ω_o and W_o (a proxy for H_o).

3.3 Bit/rock interaction law for Class II Bits

The purpose of the bit/rock interaction model is to evaluate the instantaneous force applied on a PDC bit, which results in the weight-on-bit $W({}_0^t\Phi, {}_0^tU)$ and torque-on-bit $T({}_0^t\Phi, {}_0^tU)$ in (3.1). As illustrated with a family of bits with simpler geometry – rectangular full/partial blades placed at the same level, this requires extractions of various geometric information from the engagement surface of each cutter [36]. The following describes in detail how this approach is extended to real bits. It will also be shown that for a typical drag bit, the approach results in complex geometric interactions corresponding to up to one hundred delays. This naturally leads to some special numerical treatments to enable pragmatic implementations of the model, where accuracy and CPU expense are well addressed. In the following, we illustrate the model with an example PDC bit with 32 cutters in Fig. 3.4(a), where a 3D representation of cutter faces is shown. This bit is referred to as Bit A in the following discussion.

3.3.1 Coordinate systems

Similarly, two sets of coordinates are introduced: one is fixed in space (“Eulerian”) and the other is attached to the bit (“Lagrangian”). The Eulerian reference system to depict bit motion can be expressed as either Cartesian (x, y, z) or cylindrical (r, θ, z) . The common z -axis corresponds to the bit axis of rotation and points in the direction of axial bit advance. On the other hand, the Lagrangian coordinates system describes the bit geometry, as either Cartesian (X, Y, Z) or cylindrical (R, Θ, Z) , with $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan(Y, X)$. Here the principal branch of $\arctan$ is taken which returns Θ in $(-\pi, \pi]$. Practically, the domain of Θ is altered to $(0, 2\pi]$. The origin of the Lagrangian reference systems is located at an arbitrary fixed point on the bit axis. Since the borehole is assumed to be straight, axes z and Z are coaxial. These four coordinate systems are illustrated in Fig. 3.4(a). In conjunction, they can translate the bit motion to each respective cutter motion under the assumption of the bit being a rigid body. For clarity, it is required that the origin of (X, Y, Z) coincides with that of (x, y, z) at time $t = 0$.

Denote the center of i -th cutter face as C_i ($i = 1, 2, \dots, N$, N being the total number

of cutters). Given the axial and angular displacements $(U(t), \Phi(t))$ of the bit, the motion of the i -th cutter read

$$\Phi_i(t) = \Phi(t) + \Theta_i^c \quad U_i(t) = U(t) + Z_i^c \quad (3.15)$$

where $(R_i^c, \Theta_i^c, Z_i^c)$ are the time-invariant coordinates of the cutter center in the Lagrangian reference system, equivalent to (X_i^c, Y_i^c, Z_i^c) . Following the Industry convention, the cutters are numbered in an ascending order with increasing radial coordinates R_i^c . Define the cutter radial direction as

$$\mathbf{r}_i^c = \frac{1}{\sqrt{(X_i^c)^2 + (Y_i^c)^2}} \begin{pmatrix} X_i^c \\ Y_i^c \\ 0 \end{pmatrix} \quad (3.16)$$

The tangential direction of a cutter, related to pure bit rotation without axial velocity, is given by

$$\mathbf{t}_i^c = -\mathbf{r}_i^c \times \mathbf{e}_z \quad (3.17)$$

where $\mathbf{e}_z = (0, 0, 1)^T$ is the positive direction of bit axial advance. Finally, the velocity of helical motion of cutter center C_i can be expressed as

$$\mathbf{V}_i(t) = \Omega(t) \mathbf{t}_i^c R_i^c + V(t) \mathbf{e}_z \quad (3.18)$$

in which $R_i^c = \sqrt{(X_i^c)^2 + (Y_i^c)^2}$. The direction of $\mathbf{V}_i(t)$ is simply $\mathbf{e}_i(t) = \mathbf{V}_i(t)/\|\mathbf{V}_i(t)\|$.

3.3.2 Projection of cutter faces

For a PDC bit equipped with N circular inserts, its layout can be characterized by the cutter center location C_i (vector OC_i is written as $\mathbf{c}_i$), the cutter face orientation $\mathbf{n}_i^c$ which forms an acute angle with the tangential direction $\mathbf{t}_i^c$ (3.17) and cutter radius a_i . Accordingly, the cutter layout can be represented as a collection of circular faces, see Fig 3.4(a). Use a finite array of evenly spaced points to the rim of cutter i and denote each of them as point $Q_{i,k}$ where $k = 1, 2, \dots, \infty$. Find the orthonormal basis of the null space of cutter normal $\mathbf{n}_i^c$ and denote them as $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_i$; the null space is illustrated by plane $EFGH$ in Fig. 3.5(a). The coordinates of point $Q_{i,k}$ are given by

$$\mathbf{q}_{i,k} = \mathbf{c}_i + a_i \boldsymbol{\mu}_i \cos(kd\phi) + a_i \boldsymbol{\nu}_i \sin(kd\phi) \quad (3.19)$$

Figure 3.4 – (a) 3-D representation of the cutter faces (b) Rotate the bit without axial advance, and project cutter faces onto the same half-plane $\theta = 0$, $r \geq 0$ to get the bit profile

where

$$d\phi = \frac{2\pi}{K} \quad (3.20)$$

Note that K in (3.20) is a positive integer denoting the total number of points $\{Q_{i,k}\}$. The collection $\{q_{i,k}\}$ alternatively determines the bit geometry¹.

In Fig. 3.5(b), some directional vectors can be recognized. Three vectors $\mathbf{e}_i(t)$ introduced after (3.18), $\mathbf{t}_i^c$ introduced in (3.17) and cross product $\mathbf{r}_i^c \times \mathbf{e}_i(t)$ are coplanar and orthogonal to the radial direction $\mathbf{r}_i^c$ in (3.16). Readily, vectors $\mathbf{r}_i^c$ in (3.16) and $\mathbf{r}_i^c \times \mathbf{e}_i(t)$ form a pair of orthonormal bases for the plane orthogonal to $\mathbf{e}_i(t)$. Impose that this plane also contains point C_i , and refer to it as the “projection plane”, illustrated as the plane $E'F'G'H'$ in Fig. 3.5(b). Given the way the projection plane is defined, there exists one for each cutter. To determine the engagement surface of a cutter, one needs to first determine for each point $Q_{i,k}$ in Fig. 3.5(a), its projection $P_{i,k}$ on the corresponding projection plane.

Evidently the component of $\mathbf{q}_{i,k}$ that lies within the projection plane, denoted by $\mathbf{q}_{i,k}$,

¹The discrete indexing of polar angle ($\phi_k = kd\phi$) in $Q_{i,k}$ and $q_{i,k}$ facilitates the polygonal approximation introduced later. Alternatively, using proper continuum notation, for $Q_k(\phi)$

$$\mathbf{q}_k(\phi) = \mathbf{c}_i + a_i \boldsymbol{\mu}_i \cos \phi + a_i \boldsymbol{\nu}_i \sin \phi$$

where $\phi \in (0, 2\pi]$ is a polar coordinate corresponding to the plane determined by bases $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_i$.

reads

$$\mathbf{p}_{i,k}(t) = \mathbf{q}_{i,k} - \mathbf{e}_i(t) \mathbf{q}_{i,k} \cdot \mathbf{e}_i(t) \quad (3.21)$$

Given O' as the projected coordinate origin O onto the projection plane, vector $\mathbf{p}_{i,k}(t)$ connects point O' to $P_{i,k}$. In Fig. 3.5(c), the vector pointing from the cutter center C_i to the projected point $P_{i,k}$ is given by $\mathbf{p}_{i,k} - \boldsymbol{\xi}_i$, where

$$\boldsymbol{\xi}_i(t) = \mathbf{e}_i(t) \mathbf{c}_i \cdot \mathbf{e}_i(t) \quad (3.22)$$

is the component of $\mathbf{c}_i$ that is within the projection plane, as represented by $O'C_i$. Using C_i as the origin, the coordinates $(r_{i,k}^{proj}, z_{i,k}^{proj})$ for the projected $Q_{i,k}$ read

$$\begin{cases} r_{i,k}^{proj}(t) = (\mathbf{p}_{i,k}(t) - \boldsymbol{\xi}_i(t)) \cdot \mathbf{r}_i^c \\ z_{i,k}^{proj}(t) = (\mathbf{p}_{i,k}(t) - \boldsymbol{\xi}_i(t)) \cdot (\mathbf{r}_i^c \times \mathbf{e}_i(t)) \end{cases} \quad (3.23)$$

Readily, the cutter projection (3.23) is dynamic since $\mathbf{p}_{i,k}$, $\boldsymbol{\xi}_i$ and $\mathbf{e}_i$ are time-dependent; physically this indicates that the maximum amount of rock removed by a cutter varies with bit motion. Fortunately this “regenerative effect” of cutter projection is not necessary practically. Note that vectors $\mathbf{e}_i(t)$ and $\mathbf{t}_i^c$ lie in the same plane perpendicular to $\mathbf{r}_i^c$; from (3.18), $\mathbf{e}_i(t)$ and $\mathbf{t}_i^c$ are approximately collinear if

$$\Omega(t) \sqrt{(X_i^c)^2 + (Y_i^c)^2} \gg V(t) \quad (3.24)$$

Considering realistic configurations, (3.24) always holds even for cutter 1 for which $\sqrt{(X_i^c)^2 + (Y_i^c)^2}$ is minimum. Consequently, the bases of the projection plane $\mathbf{r}_i^c$ and $\mathbf{e}_z$, are approximately constant, so are the projections in (3.23).

Now that $\mathbf{e}_i(t)$ can be replaced by $\mathbf{t}_i^c$ in (3.23), rotate the bit without axial advance, there is an intersection between the circular trajectory of each cutter center C_i and the half-plane $\theta = 0, r \geq 0$. Rearrange the cutter projections according to the intersection locations, the resulting picture is the bit profile with each point given by

$$\begin{cases} r_{i,k} = (\bar{\mathbf{p}}_{i,k} - \bar{\boldsymbol{\xi}}_i) \cdot \mathbf{r}_i^c + \sqrt{(X_i^c)^2 + (Y_i^c)^2} \\ z_{i,k} = (\bar{\mathbf{p}}_{i,k} - \bar{\boldsymbol{\xi}}_i) \cdot \mathbf{e}_z + Z_i^c \end{cases} \quad (3.25)$$

where the bar $(\bar{\cdot})$ denotes the motion invariant version of the corresponding quantity

under the assumption of $\mathbf{e}_i(t) \equiv \mathbf{t}_i^c$ given that (3.24) holds. The resulting bit profile is in Fig 3.4(b), where the cutter numbering is also shown; each projection is in fact an ellipse with the length of its major axis equal to the cutter radius a_i . If there is no side/back rake angles, the bit profile will just be an array of circles. In the bit profile, the cutter center is still denoted as C_i for simplicity.

In practice, the circular cutter faces can be approximated by polygons. Accordingly, the k -th polygonal nodes of the i -th cutter projection can be collected in a $K \times 2$ table (K is a large enough integer instead of infinity):

$$\mathbf{B}_i = [\mathbf{r}_i, \mathbf{z}_i]_{K \times 2} \quad (3.26)$$

where $\mathbf{r}_i$ is a column vector achieved by concatenating $r_{i,k}$ collectively for all k , and $\mathbf{z}_i$ is defined in a similar manner. The regeneration of the rock surface is now translated to the geometrical interactions of polygons $\{\mathbf{B}_i\}$. For bit/rock interactions, the bit design is encapsulated in the arrays of $\{\mathbf{B}_i\}$ and $\{\Theta_i^c\}$, where $i = 1, 2, \dots, N$, N being the total number of cutters and Θ_i^c depicts the cutters' angular positions.

3.3.3 Engagement surface and cutting edge for each cutter

As discussed in [36], calculation of the engagement surface $\mathcal{S}_i$ of cutter i , i.e. the cross-section of the rock groove traced the cutter $\mathcal{C}_i$, depends on the past positions of the cutters that are interacting with $\mathcal{C}_i$. Define an index set I_i , which contains the cutter numbers of all the cutters $\mathcal{C}_j$ that are potentially interacting with $\mathcal{C}_i$. Here “interacting” means that during drilling cutter $\mathcal{C}_i$ may potentially cut a rock surface created by cutters $\mathcal{C}_j$, $j \in I_i$.

The index set I_i is determined by first constructing an $N \times N$ square cutter interaction matrix $\mathbf{P}$, with $P_{ij} = 1$ indicating a potential interaction between $\mathcal{C}_i$ and $\mathcal{C}_j$ and $P_{ij} = 0$ meaning no interaction. Evidently, $P_{ij} = 1$ if $\mathcal{C}_j$ is radially overlapping with $\mathcal{C}_i$ in the bit profile; hence

$$P_{ij} = \begin{cases} 0, & \bar{R}_j < \underline{R}_i \text{ or } \underline{R}_j > \bar{R}_i \\ 1, & \text{otherwise} \end{cases}, \quad j \in \mathcal{J}_i \quad (3.27)$$

where $[\underline{R}_i, \bar{R}_i]$ ($\bar{R}_i > \underline{R}_i$) gives the radial range of a cutter $\mathcal{C}_i$ in the bit profile. The sparsity pattern of $\mathbf{P}$ is shown in Fig 3.6. Evidently matrix $\mathbf{P}$ is symmetric, i.e.,

Figure 3.6 – Blue circles: non-zero entries of table $\mathbf{P}$ depict the potential interactions between cutter i and j ; Red circles: non-zero entries of table $\mathbf{Q}$ depict the actual interactions between cutters (presented in figure corresponds to Bit A with a nominal bit motion at $d_o = 3$ mm)

$P_{ij} = P_{ji}$. Thus the index set I_i is the column numbers for non-zero entries in the i -th row of $\mathbf{P}$.

For example, the set I_i for cutter 1 of the PDC bit illustrated in Fig. 3.4 is given by $I_1 = \{1, 2, 3\}$.

The angular offsets Φ_{ji} (or angular delays), defined by the angle by which cutter $\mathcal{C}_j$, $j \in I_i$ is ahead of $\mathcal{C}_i$

$$\Phi_{ji} = \begin{cases} \Theta_j^c - \Theta_i^c, & \Theta_j^c > \Theta_i^c \\ 2\pi + \Theta_j^c - \Theta_i^c, & \Theta_j^c < \Theta_i^c \end{cases} \quad (3.28)$$

where Θ_i^c , Θ_j^c are found from the cutter layout. It is required that $\Phi_{ji} \neq 0$ since two interacting cutters cannot occupy the same Θ ; when a cutter is interacting with itself, $\Phi_{ii} = 2\pi$.

To account for the engagement surface of each cutter, the projections in the bit profile need to be shifted to mimic the rock surface seen by the cutter of interest. The shift

amount d_{ji} is a delay term related to the bit motion history:

$$d_{ji}(t; t_{ji}) = U(t) - U(t - t_{ji}), \quad \Phi_{ji}(t; t_{ji}) = \Phi(t) - \Phi(t - t_{ji}) \quad (3.29)$$

Use $\mathbf{D}_{ji}$ to denote the matrix whose entries are d_{ji} . Apparently $\mathbf{D}_{ji}$ shares the same sparsity pattern as the interaction table $\mathbf{P}$. A later chapter will present a systematic way to determine $\mathbf{D}_{ji}$. Note that $\mathbf{D}_{ji}$ is not symmetric although its diagonals have identical entries corresponding to $\Phi_{ii} = 2\pi$. For a typical PDC bit, the number of unique entries of $\mathbf{D}_{ji}$ is $O(10) \sim O(100)$, which corresponds to the number of searches for state variables in the bit motion history. To accommodate, a customized inverse angle-based algorithm is used to integrate the equations of motion such that the state variables are solved at each uniform angle step [81]. Accordingly the interpolation operation related to the history search is replaced by a simple index reading.

The reconfigured bit profile $\mathbf{B}'_j$ ($j \in I_i$) is achieved by shifting polygons $\mathbf{B}_j$:

$$\mathbf{B}'_j = \mathbf{B}_j - \Delta_{ji}, \quad j \in I_i \quad (3.30)$$

where matrix Δ_{ji} matches the dimension of $\mathbf{B}_j$

$$\Delta_{ji} = d_{ji} \cdot \begin{bmatrix} \mathbf{0}_{K \times 1} & \mathbf{1}_{0 \times 1} \end{bmatrix}_{K \times 2} \quad (3.31)$$

To illustrate, we set to determine the engagement surface of cutter $\mathcal{C}_1$ in Fig 3.4. According to I_1 , the first three cutters of the bit should be considered, see Fig 3.7(a). Fig 3.7(b) shows $\mathbf{B}_j$ ($j \in I_1$) in dashed line and $\mathbf{B}'_j$ in bold lines ($j \in I_1$).

Define the surface of the rock removed by cutters $\mathcal{C}_j$ ($j \in I_i$) as the void surface $\mathcal{V}_i$, part of its boundary represents the rock surface seen by cutter $\mathcal{C}_i$. Polygon $\mathbf{V}_i$ can approximate the boundary of $\mathcal{V}_i$, which is determined by all $\mathbf{B}'_j$, $j \in I_i$ using a Boolean set operation of union. In Fig 3.7(c), the shaded surface represents $\mathcal{V}_1$ (its boundary $\partial\mathcal{V}_1$ is approximated by $\mathbf{V}_1$), which has a shape of a union of three ellipses. Finally, take the Boolean set difference between $\mathbf{B}_i$ and $\mathbf{V}_i$ to get the polygon $\mathbf{C}_i$, which approximates the cutter/rock engagement surface $\mathcal{S}_i$. Fig 3.7(d) shows the shape of $\mathcal{S}_1$ (its boundary $\partial\mathcal{S}_1$ is approximated by polygon $\mathbf{C}_1$). Polygons $\mathbf{C}_i$ are essential to the computation of weight-on-bit and torque-on-bit.

Figure 3.7 – A specific illustration: get the cutter engagement shape for a cutter of interest

3.3.4 Polygon clipping technique

The Boolean set operations mentioned earlier are implemented by the “polygon clipping” technique, developed in the field of computer graphics, where complex geometric interactions are accounted for in real-time computations. First the interacting shapes are approximated by polygons, a sequential (counterclockwise is used in this work) set of coordinates for the bounding line segments. One can “clip” one set (subject polygon) with another one (clip polygon) to find not only the intersections but also a new set of polygon as the result of a Boolean set operation, such as union, difference, intersection and so forth [82]. The mentioned Boolean set operation results in the two intersections between the two given polygons in addition to a subset of the original nodes (which subset is yielded depends on the type of operation), all sequenced counterclockwise. Among all polygon clipping algorithms, the Vatti Algorithm [83] is advantageous for being fast and versatile, as no restrictions are imposed on the subject and clip polygons. In this work, we adopt the library developed in [84], which is a robust implementation of the Vatti clipping algorithm in C++.

3.3.5 Boundaries of an engagement surface

Once the polygon $\mathbf{C}_i$ representing $\partial\mathcal{S}_i$ is determined, its geometric information such as polygonal area and centroid location can be achieved with some classic formulae, which will eventually enter the computation of W_c and T_c . For the frictional components W_f and T_f , it is necessary to solve for the cutting edge $\mathcal{L}_i$, which is defined as a segment of $\partial\mathcal{S}_i$ where the constant contact stress σ exists as the bit is engaging with rock [21]. In the bit profile plane $r - z$, $\mathcal{L}_i$ is formally defined as a collection of points P such that

$$\begin{cases} P \in \partial\mathcal{S}_i \\ -\text{sign}(V) \mathbf{e}_z \cdot \mathbf{n}_P > 0 \end{cases} \quad (3.32)$$

where unit vector $\mathbf{n}_P$ is the inward normal of $\partial\mathcal{S}_i$ at P , representing the direction of the contact force.

Within the polygon $\mathbf{C}_i$, vectors pointing from the cutter center C_i to vertices $\mathbf{p}$ can be used to evaluate its polar angle $\phi_{\mathbf{p}}$. The polygon vertices are sequenced according to ascending order of $\phi_{\mathbf{p}}$ (counterclockwise); accordingly $\mathbf{n}_P$ can be achieved by a $\pi/2$ counterclockwise rotation of the tangential direction approximated by two neighboring vertices. For $V \geq 0$, a simple polyline Γ_i (“simple” means not self-intersecting, not

closed) approximating $\mathcal{L}_i$ can be determined with

$$\Gamma_i = \{\mathbf{p} | \mathbf{p} \in \mathbf{C}_i, \mathbf{p} \in \mathbf{B}_i, \mathbf{e}_z \cdot \mathbf{n}_p < 0\} \quad (3.33)$$

Note that vertices on a polyline maintains the counterclockwise sequence. Some examples of $\mathbf{C}_i$ and Γ_i are given in Fig. 3.8(a)-(c) for Bit A with a uniform motion characterized by bit nominal penetration per revolution of 3 mm (namely nominal depth of cut $d_o = 3$ mm). Fig. 3.8(d) shows the stacked sketches of projected cutter face, engagement surface $\mathcal{S}_i$ and cutting edge $\mathcal{L}_i$ for each cutter i . Varying elliptical projections in the sketch reflects different rake angles of the cutters on Bit A. Interestingly, the engagement surfaces have a geometric evolution with respect to the radial distance r .

The rest of $\partial\mathcal{S}_i$ is useful for the linear stability analysis discussed later; these boundaries are defined as $\mathcal{L}_{ji}$ to reflect the interaction between cutters i and $j \in I_i$. Similarly, $\mathcal{L}_{ji}$ are approximated with simple polyline Γ_{ji} ,

$$\Gamma_{ji} = \{\mathbf{p} | \mathbf{p} \in \mathbf{C}_i, \mathbf{p} \in \mathbf{B}_j, j \in I_i\} \quad (3.34)$$

The polar angle ϕ_p for vertices on Γ_{ji} are defined with the shifted center of cutter j and vertex $\mathbf{p}$. Construct another interaction table $\mathbf{Q}$ with the same size as $\mathbf{P}$, let $Q_{ji} = 1$ if Γ_{ji} ($\mathcal{L}_{ji}$) is not an empty set and $Q_{ji} = 0$ if otherwise. The sparsity pattern of $\mathbf{Q}$ indicates the actual interaction between cutters, as shown by the red dots in Fig. 3.6 for Bit A at $d_o = 3$ mm. For this specific configuration, the engagement surfaces $\{\mathcal{S}_i\}$ result from interactions between 2-4 cutters. For different uniform motion, proxied by varying d_o values, the sparsity pattern for $\mathbf{Q}$ changes in contrast to $\mathbf{P}$ which is fixed irrespective of the actual bit/rock interaction.

Note that the above discussion is restricted to the case $V \geq 0$. Indeed for $V < 0$ there may exist a different frictional contact on a cutter that is negatively contributing to the weight-on-bit. Such frictional contacts occur at cutter 1 and cutters close to the outer rim of the bit, as illustrated by the black edge in Fig 3.8(b)-(d). It is assumed that these small contacts are not pivotal to the bit dynamics and thus neglected in the modeling to ensure there is no frictional bit force for $V < 0$.

Figure 3.8 – (a)-(c) Examples of C_i , Γ_i and Γ_{ji} for Bit A at $d_o = 3$ mm. (d) Sketches (overlaid on bit profile, i.e. aligned cutter projections) of projected cutter face (black ellipses), engagement surface $\mathcal{S}_i$ (cyan crescent-like shapes) and cutting edge $\mathcal{L}_i$ (red curves with “x” as starting point and “o” as end point) for each cutter i .

3.3.5.1 Computing weight-on-bit and torque-on-bit

With polygon $\mathbf{C}_i$ approximating $\partial\mathcal{S}_i$ and polyline approximating $\mathcal{L}_i$ determined, the weight-on-bit and torque-on-bit can be easily computed. In Fig 3.9, the relevant geometric parameters are A_i (the area of $\mathcal{S}_i$), ρ_i (the r -coordinate for the centroid of $\mathcal{S}_i$), L_i (the arc length of $\mathcal{L}_i$), b_i (the length of the projection of $\mathcal{L}_i$ on r -axis) and $\rho_i^{\mathcal{L}}$ (the r -coordinate for the centroid of $\mathcal{L}_i$). From single-cutter experiments [21], the following parameters are assumed identical for all cutters: ϵ (the intrinsic specific energy - the amount of energy spent to cut a unit volume of rock, with the unit of stress), ζ (the ratio between vertical and horizontal cutting forces), μ (the ratio between horizontal and vertical contact forces) and w (the wearflat width per cutter).

Thus the contributions of cutter i to the cutting and frictional components of the weight and torque-on-bit are given by

$$W_{ci} = \epsilon\zeta A_i, \quad T_{ci} = \epsilon A_i \rho_i, \quad W_{fi} \in \sigma w b_i (1 - g), \quad T_{fi} \in \mu \sigma w L_i \rho_i^{\mathcal{L}} (1 - g) \quad (3.35)$$

in which g is a set-value function depending on the sign of axial velocity:

$$g \begin{cases} = 0 & V > 0 \\ \in (0, 1) & V = 0 \\ = 1 & V < 0 \end{cases} \quad (3.36)$$

When axial velocity vanishes $V = 0$, both weight-on-bit and torque-on-bit related to contact force should be solved according to equilibrium condition, as g can take any values within $(0, 1)$. When the bit moves upward $V < 0$, the model neglects the small frictional contacts which theoretically make negative contribution to W_{fi} and a positive one to T_{fi} .

Finally, weight-on-bit and torque-on-bit read

$$W \in \sum_i [\epsilon\zeta A_i + \sigma w b_i (1 - g)], \quad T \in \sum_i [\epsilon A_i \rho_i + \mu \sigma w L_i \rho_i^{\mathcal{L}} (1 - g)] \quad (3.37)$$

When plugging back into (3.1), the geometric terms in (3.35) encapsulate the regenerative effect as the governing equation is a delay differential inclusion with multiple state-dependent time delays.

Figure 3.9 – Geometric parameters defined for a specific $\mathcal{S}_i$ (or $\mathbf{C}_i$)

3.4 Scaling

The complete governing equations of the extended model are obtained by substituting (3.7)-(3.8) into (3.1). Here, we formulate these equations in terms of perturbations with respect to the trivial motion and in a scaled form. First, we note the existence of three characteristic times in this model,

$$t_\omega = \sqrt{\frac{I}{C}}, \quad t_v = \sqrt{\frac{M}{K_r}}, \quad \bar{t}_{\text{rev}} = \frac{2\pi}{\Omega_o} \quad (3.38)$$

Time scales t_ω and t_v are related to resonance period of a torsional and an axial oscillator, respectively, and $\bar{t}_{\text{rev}}$ is the imposed period of revolution at the rig. The torsional oscillator (t_ω) is the pendulum representation of the drillstring, while the axial oscillator (t_v) consists of mass M of the BHA and pseudo-stiffness $K_r = \zeta \epsilon a$ of the bit/rock interface.

Following [3], we choose the reference scales for time, length, force, and torque as

$$t_* = t_\omega, \quad d_* = \frac{2C}{\epsilon a^2}, \quad W_* = \zeta a \epsilon d_*, \quad T_* = C \quad (3.39)$$

The introduction of these scales lead naturally to the definition of a dimensionless time, and dimensionless velocity and force variables

$$\tau = \frac{t}{t_*}, \quad v = \frac{Vt_*}{d_*}, \quad \omega = \Omega t_*, \quad \mathcal{W} = \frac{W}{W_*}, \quad \mathcal{T} = \frac{T}{T_*} \quad (3.40)$$

Evidently the equilibrium point, expressed in terms of scaled nominal velocities, is given by

$$v_o = \frac{V_o t_*}{d_*}, \quad \omega_o = \Omega_o t_* \quad (3.41)$$

The dynamical response of the system is now expressed in terms of the axial and angular perturbed displacement from the equilibrium point, denoted as $u(\tau)$ and $\varphi(\tau)$ respectively,

$$u = \frac{U}{d_*} - v_o \tau, \quad \varphi = \Phi - \omega_o \tau + \mathcal{T}_o \quad (3.42)$$

where $\mathcal{T}_o = T_o/T_*$. We also introduce the scaled total axial and angular velocity v and ω

$$v = v_o + \dot{u}, \quad \omega = \omega_o + \dot{\varphi} \quad (3.43)$$

where the dotted variable represents the derivative of that variable with respect to time τ .

Formulated in terms of the perturbed displacements $u(\tau)$ and $\varphi(\tau)$, the equations of motion (3.1) become

$$\begin{cases} \ddot{u} \in \psi [\mathcal{W}_o - (\mathcal{W}_c + \mathcal{W}_f)] \\ \ddot{\varphi} \in \mathcal{T}_o - (\mathcal{T}_c + \mathcal{T}_f) - \varphi \end{cases} \quad (3.44)$$

where $\mathcal{W}_o$ and $\mathcal{T}_o$ are the scaled weight- and torque-on-bit at equilibrium. The shifts ΔZ_{ji} in (3.5) or equivalently d_{ji} in (3.29) now become $\delta_{ji} = d_{ji}/d_*$ which reads

$$\delta_{ji} = \frac{v_o}{\omega_o} \Phi_{ji} + u(\tau) - u(\tau - \tau_{ji}) - \frac{v_o}{\omega_o} [\varphi(\tau) - \varphi(\tau - \tau_{ji})] \quad (3.45)$$

where the first term $\frac{v_o}{\omega_o} \Phi_{ji}$ is the nominal value of δ_{ji} while the rest depicts the perturbation. For the simpler bit geometry in [3] — n continuous identical blades evenly distributed around the bit axis of rotation, this quantity is nothing but the local depth of cut.

Scaled WOB and TOB for Class I bits

The actual cutting and contact components of $\mathcal{W}$ and $\mathcal{T}$ are given by the bit-rock interface laws

$$\mathcal{W}_c = \sum_i \bar{\mathcal{A}}_i, \quad \mathcal{W}_f \in \lambda(1-g) \quad \mathcal{T}_c = 2 \sum_i \bar{\mathcal{A}}_i \bar{\rho}_i \quad \mathcal{T}_f \in \beta\lambda(1-g) \quad (3.46)$$

with

$$\bar{\mathcal{A}}_i = \frac{\bar{A}_i}{ad_*}, \quad \bar{\rho}_i = \frac{\bar{r}_i}{a} \quad (3.47)$$

The quantities $\bar{\mathcal{A}}_i$ and $\bar{\mathcal{A}}_i \bar{\rho}_i$ are functions of time, and capture the regenerative nature of the cutting process.

Scaled WOB and TOB for Class II bits

The scaled geometric variables from bit/rock interaction model read

$$\tilde{A}_i = \frac{A_i}{ad_*}, \quad \tilde{\rho}_i = \frac{\rho_i}{a}, \quad \tilde{b}_i = \frac{b_i}{a}, \quad \tilde{\rho}_i^{\mathcal{L}} = \frac{\rho_i^{\mathcal{L}}}{a}, \quad \tilde{L}_i = \frac{L_i}{a} \quad (3.48)$$

Therefore, components of $\mathcal{W}$ and $\mathcal{T}$ read

$$\mathcal{W}_c = \sum_i \tilde{A}_i, \quad \mathcal{W}_f \in \lambda(1-g) \sum_i \tilde{b}_i, \quad \mathcal{T}_c = 2 \sum_i \tilde{A}_i \tilde{\rho}_i, \quad \mathcal{T}_f \in 2\check{\beta}\lambda(1-g) \sum_i \tilde{L}_i \tilde{\rho}_i^{\mathcal{L}} \quad (3.49)$$

The nominal values $\mathcal{W}_o = \mathcal{W}_{o,c} + \mathcal{W}_{o,f}$ and $\mathcal{T}_o = \mathcal{T}_{o,c} + \mathcal{T}_{o,f}$ result from geometric information of $\mathcal{S}_i$ corresponding to nominal bit motion, i.e. $\mathcal{S}_{o,i}$.

Model parameters

Three numbers appear in the above governing equations (3.44), (3.46) and (3.49), namely: number $\check{\beta}$, bluntness number λ , and system number ψ , respectively defined as

$$\check{\beta} = \mu\zeta, \quad \lambda = \frac{\sigma w}{\zeta \epsilon d_*}, \quad \psi = \left(\frac{t_w}{t_v} \right)^2 = \frac{Ia\zeta\epsilon}{MC} \quad (3.50)$$

Number ψ can be expressed in terms of the ratio between the two characteristic times t_w and t_v ; it is of order $O(10^1 - 10^3)$ for deep well drilling. In connection to [3, 21],

the bit number β

$$\beta = \breve{\beta}\gamma \quad (3.51)$$

dictates the apparent response of the bit, either velocity weakening ($0 < \beta < 1$) or velocity hardening ($\beta > 1$). The number γ depicts the wearflat distribution

$$\gamma = \frac{2 \sum_i \tilde{L}_i \tilde{\rho}_i^{\mathcal{L}}}{a \sum_i \tilde{b}_i} \quad (3.52)$$

For later use, the quantity λ can be rescaled using bit radius a instead of d_*

$$\breve{\lambda} = \frac{\sigma w}{\zeta \epsilon a} = \frac{d_*}{a} \lambda \quad (3.53)$$

Note that $\lambda \sim O(1)$ while $\breve{\lambda} \sim O(0.01)$.

The equilibrium point of the system, expressed as v_o , ω_o , $\mathcal{W}_o$ and $\mathcal{T}_o$ satisfies the bit/rock interface laws

$$\mathcal{W}_o = 2\pi \frac{v_o}{\omega_o} + \lambda, \quad \mathcal{T}_o = 2\pi \frac{v_o}{\omega_o} + \beta \lambda \quad (3.54)$$

where the first term in the right-hand side in the above equations corresponds to the cutting ($\mathcal{W}_{co}$, $\mathcal{T}_{co}$) and the second term represents the contact components ($\mathcal{W}_{fo}$, $\mathcal{T}_{fo}$) of the weight-on-bit and torque. Note that (3.54) holds for any equilibrium point irrespective of the bit design so long as the cutters span the bit radius a (the bit is not a “core barrel”).

Finally it is convenient, when analyzing the stability of the axial dynamics to use t_v as alternative time scale. All the quantities that scale with t_v are denoted by $\square$

$$\tau = \frac{t}{t_v}, \quad \mathbf{v} = \frac{V t_v}{d_*}, \quad \boldsymbol{\omega} = \Omega t_v \quad (3.55)$$

Chapter 4

Time Simulation

The time simulation based on the model described in the previous chapter requires two modules:

- Given the bit motion history, compute the instantaneous reaction weight-on-bit and torque-on-bit.
- Given the current state variables, use an explicit numerical scheme to update the state variables at the next time step.

This chapter discusses in detail how these two modules are implemented.

4.1 Treatment of delayed terms

It was illustrated in the previous chapter that the bit/interaction depends on the bit geometry. The time simulation algorithm for the model is composed of two parts: (1) a universal process of solving for the bit trajectory function [68] (2) the determination of the bit/rock engagement for a specific bit design. As the latter part has been articulated in the previous chapter, this section focuses on the first part.

4.1.1 Perspective of rock profile

Figure 4.1 presents the bit classes (I and II) mentioned in the previous chapter. A similar “clipping” process was defined to determine the bit/rock engagement for both classes. Here we illustrate the justification of such a treatment with separate bit classes (III and III’). The Class III of the bits adds axial offsets to class II and class III’ focuses a specific radial range defined by $(\underline{r}, \bar{r})$, see Fig. 4.1. Now we set to determine the local depths of cut for Class III bits.

Figure 4.1 – Different classes of bit geometry

Figure 4.2 – Predefined bit motion: evolution of axial (a) and angular position (b)

As a drill bit can be treated as a rigid body, the relative positions between cutters are fixed and the motion of any cutter can be replaced by that of its center point. Now consider a bit motion defined by the evolution of its angular and axial positions, as shown in Fig. 4.2. This bit motion can be readily translated to cutter motions, as shown in Fig. 4.3. Note that all trajectories are identical in shape, but shifted angularly and axially by constant values.

One approach to determine the depth of cut is to view the rock profile from an “unrolled” plane [19]. Introduce a new angular coordinate $\Phi_r \in [0, 2\pi)$ to depict rock profile. With this, the trajectories in Fig. 4.3 can be converted into a new form as in Fig. 4.4. In this view, angular delays and relative heights can be readily recognized.

In the $\Phi_r - Z$ plane, the rock surface profile is a continuous curve spanning $[0, 2\pi)$. We can use the curves in Fig. 4.4 to determine how the rock profile is updated for the increment of the bit angular position (Fig. 4.5). The local depth of cut of a specific

Figure 4.3 – Motion trajectories of cutters (center points) in 3-D view: the crossings indicate the starting locations; the red, blue, and green curves correspond to cutter 1, 2 and 3, respectively

Figure 4.4 – Motion trajectories of cutters in 2-D view: the angular position is converted to the range of $[0, 2\pi)$; the red, blue, and green curves correspond to cutter 1, 2 and 3, respectively

Figure 4.5 – The process of updating and finding local depth of cut: each subplot depicts $\Phi_r - Z$ relations depicting cutter motion trajectories and rock profile

cutter can be readily recognized as the height of the jump in the rock profile. The resulting local depths of cut for this bit are presented in Fig. 4.6.

Numerically, the local depth of cut for cutter i reads

$$d_i(t) = U(t) - U(t - t_{\tilde{i}}) + \Delta Z_i - \Delta Z_{\tilde{i}} > 0 \quad (4.1)$$

where index $\tilde{i}$ is determined via a comparison process to achieve

$$\max_{\tilde{i}} [U(t - t_{\tilde{i}}) + \Delta Z_{\tilde{i}}], \quad \tilde{i} \in I_i \quad (4.2)$$

Physically, $\tilde{i}$ is the index for the cutter that is responsible for the rock profile seen by cutter i .

In essence, the described method records the rock profile, which works for the Class III' bits. However, when considering more complex bit geometry, this method is counterproductive. Instead, as mentioned, it is more advantageous to account for the bit/rock engagement from the perspective of each cutter.

Figure 4.6 – Evolution of local depth of cut δ_i with respect of angular position Φ (Dots: results from the perspective of rock; Circles: results from the perspective of cutters)

4.1.2 Perspective of cutter

From the perspective of a cutter, to account for the bit/rock engagement, a crucial quantity to determine is the local shifts in (3.45). The projections of all cutters j are shifted and combined as a union, then a boolean set difference is performed between the projection of the cutter of interest i and the union of shifted cutters.

The following provides a systematic approach to determine the matrix $\mathbf{D}_{ji}$ whose entry d_{ji} depicts the axial shift amount that is essential to determine the reconfigured bit profile and accordingly the modeling of bit/rock interaction. To illustrate, an example bit with 3 cutters is considered. The angular layout of the cutters is displayed in Fig. 4.7(a). The bit profile is presented in Fig. 4.7(b); a datum $Z = Z_*$ is defined for each cutter in this view.

The bit motion can be expressed in the $\theta - z$ plane in terms of $U = U(\Phi)$ where U and Φ are the axial and angular displacements (as functions of time) respectively. Since the bit is considered as a rigid body, the fixed cutters mounted on the bit are sharing the same motion evaluated at a fixed point on the respective datum. In the “unrolled” plane, see Figure 4.7(c), the cutter motions are depicted by identical curve shifted in θ value.

Consider the bit motion history within a single revolution up to the current time

instance, which in $\theta - z$ plane can be represented by a bounded curve

$$z = f(\theta), \quad \theta \in (\Phi(t) - 2\pi, \Phi(t)] \quad (4.3)$$

which is named as bit trajectory function by Zhang and Detournay [68]. This curve can also be seen as the motion history of the first cutter, and the motion histories for other cutters within the same revolution can be represented as $z = f(\theta)$ shifted towards $-\theta$ direction by the angular delay. Formally, the bit motion history for cutter j can be written as

$$z = f(\theta + \Phi_{1j}), \quad \theta + \Phi_{1j} \in (\Phi(t) - 2\pi, \Phi(t)], \quad j \in I_1 \quad (4.4)$$

For the example in Fig. 4.7, the motion history curves for cutter 2 and 3 read

$$z = f(\theta + \Phi_{12}), \quad \theta \in (\Phi(t) - 2\pi - \Phi_{12}, \Phi(t) - \Phi_{12}] \quad (4.5)$$

and

$$z = f(\theta + \Phi_{13}), \quad \theta \in (\Phi(t) - 2\pi - \Phi_{13}, \Phi(t) - \Phi_{13}] \quad (4.6)$$

respectively.

The current position of cutter 1, $f(\Phi(t))$ can be equivalently written as $f(\theta + 2\pi)|_{\theta=\Phi(t)-2\pi}$; meanwhile the intersections between shifted curves $z = f(\theta + \Phi_{1j})$, $j \in I_1$ and vertical line $\theta = \Phi(t) - 2\pi$ physically represents the axial locations z_{j1} of cutters j (for the reference $Z = Z_*$ on each cutter) when they are at the current angular position of cutter 1; these axial locations read $z_{j1} = f(\Phi(t) - 2\pi + \Phi_{1j})$, $j \in I_1$. Each axial shift d_{j1} is the difference between $U(t)$ and z_j .

$$d_{j1}(t) = U(t) - f(\Phi(t) - \Phi_{j1}), \quad j \in I_1 \quad (4.7)$$

where the identity

$$\Phi_{ij} + \Phi_{ji} = 2\pi, \quad j \in I_i \quad (4.8)$$

is used. For (4.7), relaxing index 1 to free index i leads to the full solution to d_{ji}

$$d_{ji}(t) = U(t) - f(\Phi(t) - \Phi_{ji}), \quad j \in I_i \quad (4.9)$$

Note that the diagonal entries of matrix $\mathbf{D}_{ji}$ are identical, as it correspond to the axial advance of the bit within the current revolution. For the example bit under

consideration, the matrix $\mathbf{D}_{ji}$ belongs to $\mathbb{R}^{3 \times 3}$, and each of its entries is illustrated as coordinates z_{ji} in Fig. 4.7. For more general cases, the matrix $\mathbf{D}_{ji}$ belongs to $\mathbb{R}^{M \times M}$, where M is the total number of cutters. Equivalently, M can be seen as the number of *uniquely determined* angular offsets Φ_m – the dual indices ji in Φ_{ji} contract into m , and entries in $\{\Phi_m\}$ can be identical according to the bit design. As the diagonal entries of $\mathbf{D}_{ji}$ correspond to the identical angular offset of 2π , the maximum number of *unique* time delays is simply

$$M^2 - M + 1 \quad (4.10)$$

In numerical time simulations, at each time step, the evaluation of $f(\Phi(t) - \Phi_{ji})$ requires up to $(M^2 - M + 1)$ times of searching in the motion history. The searching of state-variables is needed since the curve $f(\theta)$ is known as an array of points instead of a closed-form function. Also from the above process, in terms of bit motion history, it is shown a segment of that history within one revolution is all that is required to solve for the instantaneous bit/rock interactions. In other words, the bit motion trajectory, achieved from the state variables in one bit revolution, encapsulates the “delayed terms” for the original system.

4.2 Delay term treatment

With the bit/rock interaction clarified, solving for the time response now requires an integrating scheme to solve for the state variables:

$$\mathbf{x}(\tau) = \begin{Bmatrix} u \\ \dot{u} \\ \varphi \\ \dot{\varphi} \end{Bmatrix}(\tau) \quad (4.11)$$

There are three approaches for this purpose.

4.2.1 Time-based algorithm

The equations of motion read

$$\begin{cases} \ddot{u}(\tau) \in \psi[\mathcal{W}_o - \mathcal{W}(\tau; \tau_1, \tau_2, \dots, \tau_M)] \\ \ddot{\varphi}(\tau) \in \mathcal{T}_o - \mathcal{T}(\tau; \tau_1, \tau_2, \dots, \tau_M) - \varphi(\tau) \end{cases} \quad (4.12)$$

Figure 4.7 – Determination of d_{ji}

where the array $\{\tau_m\}$ ($m = 1, 2, \dots, M$) represents the time delays. The quantity $\max \tau_m$ represents the “time window” where the bit motion needs to be searched to solve (4.12); this “time window” correspond to a bit rotation within one cycle (including one cycle even). Meanwhile the frictional component of the drilling force exerted on the bit depends on the sign of the axial velocity, thus giving rise to the inclusion sign in (4.12). The time-based algorithm simply relies on an explicit scheme

$$\begin{cases} \dot{u}(\tau + h) = \dot{u}(\tau) + \ddot{u}(\tau) h \\ u(\tau + h) = u(\tau) + \dot{u}(\tau) h + \frac{1}{2} \ddot{u}(\tau) h^2 \\ \dot{\varphi}(\tau + h) = \dot{\varphi}(\tau) + \ddot{\varphi}(\tau) h \\ \varphi(\tau + h) = \varphi(\tau) + \dot{\varphi}(\tau) h + \frac{1}{2} \ddot{\varphi}(\tau) h^2 \end{cases} \quad (4.13)$$

where h depicts a uniform time step. Note that (4.13) works only for the normal drilling; modifications are required when drilling dysfunctions are detected. However, according to (4.10), solving for the bit motion trajectory requires $M^2 - M + 1$ times of search and interpolation for a single cutter i at a single time step, where M is the total number of cutters.. When considering realistic bits, in conjunction with the geometric computation of polygon clipping, this scheme is computationally expensive. Thus it is imperative to speed up the integrating process to achieve the bit trajectory function.

4.2.2 Angle-based algorithm

In the light of the original document [81] on the angle-based algorithm, this section provides a different approach to integrate (4.13). Recall that

$$\varphi(\tau) + \omega_o \tau = \Phi(\tau) \quad (4.14)$$

Readily

$$\varphi(\tau + h) + \omega_o \tau + \omega_o h = \Phi(\tau) + \Delta\Phi \quad (4.15)$$

where $\Delta\Phi$ is a constant angular step. Expanding (4.15) with respect to h and ignoring $O(h^3)$ terms lead to

$$-\frac{1}{2} \ddot{\varphi}(\tau) h^2 - (\dot{\varphi}(\tau) + \omega_o) h + \Delta\Phi = 0 \quad (4.16)$$

which is the quadratic equation to solve for h corresponding to $\Delta\Phi$. Set

$$a = -\frac{1}{2}\ddot{\varphi}(\tau), \quad b = -(\dot{\varphi}(\tau) + \omega_o), \quad c = \Delta\Phi \quad (4.17)$$

Obviously, the quadratic equation becomes

$$ah^2 + bh + c = 0 \quad (4.18)$$

For (4.18), define

$$\Delta = \sqrt{b^2 - 4ac} \quad (4.19)$$

If $\Delta \in \mathbb{C}$, physically it means the dynamics does not allow $\Delta\Phi$ to be covered, accordingly no h can be solved under normal drilling condition; in fact, this condition translates to the occurrence of axial-torsional stick. In this case,

$$h = \frac{\dot{\varphi}_{i-1} + \omega_o}{-\ddot{\varphi}_{i-1}} \quad (4.20)$$

where h will not correspond to the completion of $\Delta\Phi$ (it is worth pointing out that in the following algorithm, the detection of torsional-axial stick corresponds to $\dot{\varphi}_i = -\omega_o$ exactly instead of $\dot{\varphi}_i \leq -\omega_o$, contrary to the time-based code). If $\Delta \in \mathbb{R}$, physically there is an admissible h such that $\Delta\Phi$ can be covered. If (4.18) is indeed a quadratics, i.e. $a \neq 0$, $h = (-b \pm \Delta)/2a$. It can be proven that only smaller root is physically possible

$$h = \frac{-b - \Delta}{2a} \quad (4.21)$$

Meanwhile if (4.18) is not a quadratics $a \approx 0$, then

$$h = \frac{-c}{b} \quad (4.22)$$

After h is solved, update the state variables with (4.13). The roots for the quadratics can be readily achieved when the bit is drilling normally; when the bit experiences drilling dysfunctions, special treatment is required. The benefits of this scheme is that state variables are solved at each angle step, and searching for delayed coordinate on the bit trajectory function is a trivial index subtraction where no interpolation is needed. This scheme is mostly utilized for the work reported in this thesis.

4.2.3 Rock-profile-based algorithm

An alternative approach of achieving the bit motion trajectory is presented by Gupta and Wahi [19], Zhang and Detournay [68]. If one uses a continuous function $L(\theta, \tau)$ to depict the evolution of the curve $f(\theta)$, the material derivative of $L(\theta, \tau)$ being 0 results in a partial differential equation

$$\frac{\partial L}{\partial \tau} + (\omega_o + \dot{\varphi}) \frac{\partial L}{\partial \theta} - v_o = 0 \quad (4.23)$$

As $L(\theta, \tau)$ can be represented using a Fourier series, the original SDDDE can be replaced by a system of ODEs that can be solved using conventional computational software packages. The sole restriction of this method now is that a Fourier-like approximation of $L(\theta, \tau)$ requires the function to be smooth (C^1 continuous) to work well. However, when the system experiences axial stick, $L(\theta, \tau)$ is C^0 continuous; this practically requires more terms in the Fourier expansion in the approximation process.

4.3 Treatment of drilling dysfunctions

Independent of the integrating methods, the time simulation requires treatment of the drilling dysfunctions.

4.3.1 Base algorithm for the RGD bits

For clarification, the treatment for drilling dysfunctions is first explained in detail for the RGD bits (n continuous full blades evenly distributed around the bit axis of rotation). The modifications for realistic bits is presented later.

Define the variations of cutting weight-on-bit $\mathcal{W}_c$ and frictional weight-on-bit $\mathcal{W}_f$ respectively as

$$p = \delta - \delta_o \quad (4.24)$$

$$g = 1 - \frac{\mathcal{W}_f}{\lambda} \quad (4.25)$$

The quantity $\delta = d/d_*$ is the scaled depth of cut. Note that for normal drilling $g = 0$; for loss of wear-flat/rock contact $g = 1$; and nil axial velocity (obviously, this includes axial stick) is equivalent to $g \in (0, 1)$. With $p(\tau, \bar{\tau})$ and $g(\tau)$, the equations of motion can be written in more detailed form (the quantity $\bar{\tau}$ refers to the single time delay in

Figure 4.8 – Illustration of various drilling modes and their transitions

the regenerative cutting process):

$$\begin{cases} \ddot{u} = -\psi p + \psi \lambda g \\ \ddot{\varphi} = -h + \beta \lambda g - \varphi \end{cases} \quad (4.26)$$

Note that for the RGD bit, $(\mathcal{W}_f - \lambda)_{\max} = 0$. Thus the time-based scheme for integrating the equations of motion reads

$$\begin{cases} u(\tau + \Delta\tau) = u(\tau) + \dot{u}(\tau) \Delta\tau + \frac{1}{2} [-\psi h(\tau, \bar{\tau}) + \psi \lambda g(\tau)] \cdot (\Delta\tau)^2 \\ \dot{u}(\tau + \Delta\tau) = \dot{u}(\tau) + [-\psi h(\tau, \bar{\tau}) + \psi \lambda g(\tau)] \cdot (\Delta\tau) \\ \varphi(\tau + \Delta\tau) = \varphi(\tau) + \frac{1}{2} [-h(\tau, \bar{\tau}) + \beta \lambda g(\tau) - \varphi(\tau)] \cdot (\Delta\tau)^2 \\ \dot{\varphi}(\tau + \Delta\tau) = \dot{\varphi}(\tau) + [-h(\tau, \bar{\tau}) + \beta \lambda g(\tau) - \varphi(\tau)] \cdot (\Delta\tau) \end{cases} \quad (4.27)$$

This scheme is readily applicable for the normal drilling case; for other drilling modes, the right-hand-sides should be modified.

4.3.1.1 Drilling modes

For the RGD-type models, one important aspect is to model various drilling modes, as well as the “switching” between modes. The drilling modes are listed as follows:

- $d = 0$, no cutting (bit sliding on rock surface, disengagement between wearflat

Figure 4.9 – Simple interpolation for state variables at the vanishing of V

and rock surface, or backward rotation). Without clarifications, the following discussions assume $d > 0$.

- $\Omega > 0, V > 0, d > 0$, normal drilling
- $\Omega < 0$, backward rotation
- $V < 0$, loss of wearflat/rock contact
- $\Omega > 0, V = 0, \frac{dV}{dt} = 0$, axial stick
- $\Omega = 0, \frac{d\Omega}{dt} = 0, V = 0, \frac{dV}{dt} = 0$, axial-torsional stick

The following discusses the transition between drilling modes (illustrated in Fig. (4.8)).

4.3.1.2 Normal drilling to axial stick/loss of wearflat-rock contact

During normal drilling, when $V = 0$ is detected, the first step is to approximate the time τ_{ak} at which $V = 0$ takes place. The simplest way is to perform a linear interpolation (Fig. 4.9)

$$\tau_{ak} = \tau + \frac{\dot{u}(\tau) + v_o}{\psi p(\tau, \bar{\tau})} \quad (4.28)$$

Note that at $\tau = \tau_{ak}$, the function $p(\tau)$ is continuous, while the axial acceleration and the contact force experience jump. Accordingly, from the expression of axial acceleration,

$$\ddot{u}(\tau_{ak}^-) - \psi \lambda g(\tau_{ak}^-) = \ddot{u}(\tau_{ak}^+) - \psi \lambda g(\tau_{ak}^+) = -\psi p(\tau_{ak}) \quad (4.29)$$

Since at $\tau = \tau_{ak}^-$, the bit is drilling normally, $g(\tau_{ak}^-) = 0$, the above equation becomes

$$\ddot{u}(\tau_{ak}^-) = \ddot{u}(\tau_{ak}^+) - \psi \lambda g(\tau_{ak}^+) = -\psi p(\tau_{ak}) \quad (4.30)$$

Also it is obvious that $\ddot{u}(\tau_{ak}^-) < 0$; thus it can be deduced that $p(\tau_{ak}) > 0$ or equivalently

$$\delta(\tau_{ak}) > \delta_o \quad (4.31)$$

which indicates that when $V = 0$ is detected during drilling, the bit depth of cut is greater than its nominal value.

In Fig. 4.8, $(\Omega(t_{ak}), V(t_{ak}))$ is represented by a maraschino point on Ω -axis. From this point, there are two possible transition branches:

- The bit enters axial stick mode

$$\begin{cases} \ddot{u}(\tau_{ak}^+) = 0 \\ 0 < g(\tau_{ak}^+) < 1 \end{cases} \Leftrightarrow 0 < p(\tau_{ak}) < \lambda \text{ or } -\psi \lambda < \ddot{u}(\tau_{ak}^-) < 0 \quad (4.32)$$

- The bit experiences loss of wearflat-rock contact (negative axial acceleration)

$$\begin{cases} \ddot{u}(\tau_{ak}^+) < 0 \\ g(\tau_{ak}^+) = 1 \end{cases} \Leftrightarrow p(\tau_{ak}) > \lambda \text{ or } \ddot{u}(\tau_{ak}^-) > -\psi \lambda \quad (4.33)$$

Evidently, as ψ is mostly “large” in comparison to $\ddot{u}(\tau_{ak}^-)$ for deep well drilling and $\lambda \sim O(1)$, the simulation rarely experiences negative axial velocity.

4.3.1.3 Axial stick to axial slip

If the bit enters axial stick mode ($\Omega > 0, V = 0$), the bit is still interacting with rock since it is still rotating at a fixed axial position. During axial stick phase, the following holds

$$g = \frac{p}{\lambda} \in (0, 1) \text{ or } p \in (0, \lambda) \quad (4.34)$$

Numerically p is being updated at each time step, and the above condition is evaluated; when the condition is not met, axial slip is detected. Physically, during the axial stick phase, each cutter is interacting with a slope-like rock surface, without axial advance, resulting in a shallowing bit/rock engagement. Note that the rock surface

Figure 4.10 – Simple interpolation for state variables at the vanishing of V

in-between cutters is continuous, so is the function p , and during axial stick, the frictional component balances with the cutting component, as well as their respective first-order derivatives:

$$\frac{dp}{d\tau}(\tau) = \lambda \frac{dg}{d\tau}(\tau) \quad \tau \in (\tau_{ak}, \tau_{ap}^-) \quad (4.35)$$

In the neighborhood of $\tau = \tau_{ap}$, the function p is continuous, thus

$$h(\tau_{ap}^-) = h(\tau_{ap}^+) = h(\tau_{ap}) \quad (4.36)$$

Note that

$$\mathcal{W}_f(\tau) = \lambda - h(\tau) \quad \tau \in (\tau_{ak}, \tau_{ap}) \quad (4.37)$$

which should also be continuous at $\tau = \tau_{ap}$.

Numerically, the exit of axial stick can be treated in a very simplified manner. During axial stick phase, pinpoint the time instance τ_{ap} when $p(\tau_{ap}) = \lambda$ or $p(\tau_{ap}) = 0$; for the previous case, assign a small positive axial velocity to $\mathbf{x}(\tau_{ap})$, while for the latter case, assign a small negative axial velocity to $\mathbf{x}(\tau_{ap})$.

4.3.1.4 Vanish of Ω during normal drilling

During normal drilling, when $\Omega = 0$ is detected, the event time $\tau_{\omega k}$ can be approximated by a simple linear interpolation (see Fig. (4.10))

$$\Delta\tau = \tau + \frac{\dot{\varphi}(\tau) + \omega_o}{\varphi(\tau) + \psi p(\tau)} \quad (4.38)$$

Recall that

$$p + \varphi = -\ddot{\varphi} + \beta\lambda - \beta\mathcal{W}_f \quad (4.39)$$

At $\tau = \tau_{\omega k}$, there are two possible dynamical modes for the bit:

- Backward rotation: Note that the cutting process is missing, and the frictional contact force follows the same modeling as normal drilling. The term φ should be continuous, while jumps should exist for p , $\ddot{\varphi}$ and $\mathcal{W}_f$. Particularly, $p(\tau_{\omega k}^-) \rightarrow p(\tau_{\omega k}^+) = -\delta_o$. Therefore

$$\varphi(\tau_{\omega k}) = -p(\tau_{\omega k}^-) - \ddot{\varphi}(\tau_{\omega k}^-) = \delta_o - \ddot{\varphi}(\tau_{\omega k}^+) + \beta\lambda - \beta\mathcal{W}_f(\tau_{\omega k}^+) \quad (4.40)$$

where $\mathcal{W}_f(\tau_{\omega k}^+)$ is the frictional weight-on-bit in the reverse direction. Also due to the axial stick state and absence of $\mathcal{W}_c$, it is noted that $\mathcal{W}_f(\tau_{\omega k}^+) = -\mathcal{W}_o$ (backward rotation ensures axial stick). Furthermore,

$$\begin{aligned} \ddot{\varphi}(\tau_{\omega k}^+) &< 0 \\ \Leftrightarrow \ddot{\varphi}(\tau_{\omega k}^-) &< -\delta(\tau_{\omega k}^-) - \beta\lambda - \beta\mathcal{W}_o \\ \Leftrightarrow \varphi(\tau_{\omega k}) &> [\delta_o + \beta\lambda - \beta\mathcal{W}_f^-(\tau_{\omega k}^+)] = \delta_o + \beta\lambda + \beta\mathcal{W}_o \end{aligned} \quad (4.41)$$

- Enter axial-torsional stick: At $\tau = \tau_{\omega k}$

$$\varphi(\tau_{\omega k}) - \delta_o - \beta\lambda - \omega_o\tau_{\omega k} = -\ddot{\varphi}(\tau_{\omega k}^-) - \delta(\tau_{\omega k}^-) - \beta\lambda = -\mathcal{T}(\tau_{\omega k}^+) \quad (4.42)$$

Furthermore, we have

$$\varphi(\tau_{\omega k}) = \delta_o + \beta\lambda - \mathcal{T}(\tau_{\omega k}^+) \Leftrightarrow \ddot{\varphi}(\tau_{\omega k}^-) = \mathcal{T}(\tau_{\omega k}^+) - \delta(\tau_{\omega k}^-) - \beta\lambda \quad (4.43)$$

Propose that

$$-\beta\mathcal{W}_o < \mathcal{T}(\tau_{\omega k}^+) < \delta(\tau_{\omega k}^-) + \beta\lambda \quad (4.44)$$

Therefore the condition reads

$$-\delta(\tau_{\omega k}^-) - \beta\lambda - \beta\mathcal{W}_o < \ddot{\varphi}(\tau_{\omega k}^-) < 0 \text{ or } -h(\tau_{\omega k}^-) < \varphi(\tau_{\omega k}) < \delta_o + \beta\lambda + \beta\mathcal{W}_o \quad (4.45)$$

4.3.1.5 Vanishing of Ω during axial stick

It is possible that the bit initiates axial-torsional stick within the axial stick phase. The event time $\tau_{\omega k}$ can be approximated in the same manner as before. Note that when $\tau = \tau_{\omega k}^-$, the bit is in axial stick, thus

$$\mathcal{W}_f(\tau_{\omega k}^-) = \lambda + \delta_o - \delta(\tau_{\omega k}^-) \text{ or } p(\tau_{\omega k}^-) = \lambda g(\tau_{\omega k}^-) \in (0, \lambda) \quad (4.46)$$

- Backward rotation: Assume that negative acceleration is taking place at $\tau_{\omega k}^+$, that is

$$\ddot{\varphi}(\tau_{\omega k}^+) < 0 \quad (4.47)$$

Recall that

$$\varphi = \delta_o - \delta - \ddot{\varphi} + \beta\lambda - \beta\mathcal{W}_f \quad (4.48)$$

Therefore

$$\begin{aligned} \varphi(\tau_{\omega k}) &= \delta_o + \beta\lambda - \ddot{\varphi}(\tau_{\omega k}^-) - \delta(\tau_{\omega k}^-) - \beta\mathcal{W}_f(\tau_{\omega k}^-) \\ &= \delta_o + \beta\lambda - \ddot{\varphi}(\tau_{\omega k}^+) - \delta(\tau_{\omega k}^+) - \beta\mathcal{W}_f(\tau_{\omega k}^+) \end{aligned} \quad (4.49)$$

Note that $\beta\lambda - \beta\mathcal{W}_f(\tau_{\omega k}^-) = \beta\delta(\tau_{\omega k}^-) - \beta\delta_o$, $\mathcal{W}_f(\tau_{\omega k}^+) = -\mathcal{W}_o$ and $\delta(\tau_{\omega k}^+) = 0$, thus

$$-\ddot{\varphi}(\tau_{\omega k}^-) + (\beta - 1)p(\tau_{\omega k}^-) = \delta_o + \beta(\lambda + \mathcal{W}_o) - \ddot{\varphi}(\tau_{\omega k}^+) \quad (4.50)$$

$$\ddot{\varphi}(\tau_{\omega k}^+) = \ddot{\varphi}(\tau_{\omega k}^-) + \delta_o + \beta(\lambda + \mathcal{W}_o) + (1 - \beta)h(\tau_{\omega k}^-) \quad (4.51)$$

Hence the criteria for backward rotation initiated from axial stick reads

$$\ddot{\varphi}(\tau_{\omega k}^-) + (1 - \beta)p(\tau_{\omega k}^-) < -\delta_o - \beta(\lambda + \mathcal{W}_o) \quad (4.52)$$

Equivalently, the condition for backward rotation can be expressed in terms of $\varphi(\tau_{\omega k})$

$$\varphi(\tau_{\omega k}) > \mathcal{T}_o + \beta\mathcal{W}_o \quad (4.53)$$

- Axial-torsional stick: Assume that torsional stick is taking place at $\tau_{\omega k}^+$, that is

$$\ddot{\varphi}(\tau_{\omega k}^+) = 0 \quad (4.54)$$

Similar to previous discussion on backward rotation,

$$\begin{aligned}\varphi(\tau_{\omega k}) &= \delta_o + \beta\lambda - \ddot{\varphi}(\tau_{\omega k}^-) - \delta(\tau_{\omega k}^-) - \beta\mathcal{W}_f(\tau_{\omega k}^-) \\ &= \delta_o + \beta\lambda - \mathcal{T}(\tau_{\omega k}^+)\end{aligned}\quad (4.55)$$

Note that $\beta\lambda - \beta\mathcal{W}_f(\tau_{\omega k}^-) = \beta\delta(\tau_{\omega k}^-) - \beta\delta_o$, thus

$$\begin{aligned}\varphi(\tau_{\omega k}) &= (\beta - 1)p(\tau_{\omega k}^-) - \ddot{\varphi}(\tau_{\omega k}^-) \\ &= \delta_o + \beta\lambda - \mathcal{T}(\tau_{\omega k}^+)\end{aligned}\quad (4.56)$$

Propose that

$$-\beta\mathcal{W}_o < \mathcal{T}(\tau_{\omega k}^+) < \delta(\tau_{\omega k}^-) + \beta\lambda \quad (4.57)$$

Therefore the conditions read

$$-\beta\mathcal{W}_o - \mathcal{T}_o < \ddot{\varphi}(\tau_{\omega k}^-) + (1 - \beta)\delta(\tau_{\omega k}^-) < h(\tau_{\omega k}^-) \quad (4.58)$$

or

$$-p(\tau_{\omega k}^-) < \varphi(\tau_{\omega k}) < \mathcal{T}_o + \beta\mathcal{W}_o \quad (4.59)$$

4.3.1.6 Axial-torsional stick

During the axial-torsional stick, $\tau \in [\tau_{\omega k}^+, \tau_{\omega p}^-]$, it is impossible to differentiate the forces exerted on the cutting surface and the wearflat surface. Simply, the weight-on-bit remains constant, while the torque-on-bit increases linearly with τ

$$\mathcal{T} = \mathcal{T}_o - \Phi(\tau_{\omega k}) + \omega_o\tau \quad \tau \in [\tau_{\omega k}^+, \tau_{\omega p}^-] \quad (4.60)$$

Upon slip, the bit has a positive $\ddot{\varphi}$; For the axial acceleration, note that

$$\ddot{u}(\tau_{\omega p}^+) = -\psi\delta(\tau_{\omega p}^+) - \psi\mathcal{W}_f(\tau_{\omega p}^+) + \psi\delta_o + \psi\lambda \quad (4.61)$$

Also

$$\delta(\tau_{\omega p}^+) = \delta(\tau_{\omega k}^-) \quad (4.62)$$

To determine the sign of axial velocity upon the exit axial-torsional slip, one needs to consider how the bit enters this stick phase.

If the axial-torsional stick is initiated from normal drilling:

Since the bit drills normally at $\tau = \tau_{\omega k}^-$, and recall

$$\psi \delta (\tau_{\omega k}^-) = \psi \delta_o - \ddot{u} (\tau_{\omega k}^-) \quad (4.63)$$

Therefore, using $\delta (\tau_{\omega p}^+) = \delta (\tau_{\omega k}^-)$

$$\ddot{u} (\tau_{\omega p}^+) = \ddot{u} (\tau_{\omega k}^-) + [\psi \lambda - \psi \mathcal{W}_f (\tau_{\omega p}^+)] \quad (4.64)$$

- Exit with positive $\dot{u} + v_o$: since $\ddot{u} (\tau_{\omega p}^+) > 0$, $\mathcal{W}_f (\tau_{\omega p}^+) = \lambda$,

$$\ddot{u} (\tau_{\omega k}^-) > 0 \quad (4.65)$$

- Exit with negative $\dot{u} + v_o$: since $\ddot{u} (\tau_{\omega p}^+) < 0$, $\mathcal{W}_f (\tau_{\omega p}^+) = 0$,

$$\ddot{u} (\tau_{\omega k}^-) < -\psi \lambda \quad (4.66)$$

- Exit with zero $\dot{u} + v_o$ (remain axial stick): since $\ddot{u} (\tau_{\omega p}^+) = 0$, $\mathcal{W}_f (\tau_{\omega p}^+) \in (0, \lambda)$,

$$-\psi \lambda < \ddot{u} (\tau_{\omega k}^-) < 0 \quad (4.67)$$

The duration of the axial-torsional stick (initiated from normal drilling) is determined by

$$\Delta \tau = \frac{1}{\omega_o} [\varphi (\tau_{\omega k}) + p (\tau_{\omega k}) - \beta \lambda g (\tau_{\omega p}^+)] \quad (4.68)$$

If the axial-torsional stick is initiated from axial stick:

It can be proven that if the bit enters axial-torsional stick from axial stick, upon slip, it has to stay in axial stick phase. Since the bit sticks axially at $\tau = \tau_{\omega k}^-$, (4.29) becomes

$$\psi \delta (\tau_{\omega k}^-) = \psi \delta_o + \psi \lambda g (\tau_{\omega k}^-) \quad (4.69)$$

Therefore, using $\delta (\tau_{\omega p}^+) = \delta (\tau_{\omega k}^-)$

$$\ddot{u} (\tau_{\omega p}^+) = \psi \lambda [g (\tau_{\omega p}^+) - g (\tau_{\omega k}^-)] \quad (4.70)$$

where $g (\tau_{\omega k}^-) \in (0, 1)$.

- It is impossible to exit with positive $\dot{u} + v_o$: if $\ddot{u}(\tau_{\omega p}^+) > 0$, $g(\tau_{\omega p}^+) = 0$

$$\ddot{u}(\tau_{\omega p}^+) = -\psi\lambda g(\tau_{\omega k}^-) < 0 \quad (4.71)$$

which is contradictory to the assumption.

- It is impossible to exit with negative $\dot{u} + v_o$: if $\ddot{u}(\tau_{\omega p}^+) < 0$, $g(\tau_{\omega p}^+) = 1$

$$\ddot{u}(\tau_{\omega p}^+) = \psi\lambda [1 - g(\tau_{\omega k}^-)] > 0 \quad (4.72)$$

which is contradictory to the assumption.

- It is only possible to maintain axial stick: since $\ddot{u}(\tau_{\omega p}^+) = 0$, $g(\tau_{\omega p}^+) \in (0, 1)$, we have

$$g(\tau_{\omega k}^-) = g(\tau_{\omega p}^+) \quad (4.73)$$

In other words, the bit resumes axial stick phase (with rotation) after the “interruption” of axial-torsional stick during $\tau \in [\tau_{\omega k}^+, \tau_{\omega p}^-]$.

The duration of the axial-torsional stick (initiated from axial stick) is determined by

$$\Delta\tau = \frac{1}{\omega_o} [(1 - \beta) \cdot h(\tau_{\omega k}) + \varphi(\tau_{\omega k})] \quad (4.74)$$

4.3.2 Modifications for realistic bits

The logic for the treatment of drilling dysfunctions for realistic bits remain near identical except that

- While the judgement for the entry of axial stick is still based on the predicted axial velocity at a step (uniform angular step instead of time step), the judgement for the entry of axial-torsional stick is based on the root of the quadratics (4.16).
- At each step, the bit/rock interaction gives the upper bounds for the frictional components $\mathcal{W}_f$ and $\mathcal{T}_f$.

If there is a real root (4.21), the bit can continue in either normal drilling (axial bit velocity $V > 0$), or drilling with loss of contact between the wear flats and the rock ($V < 0$, but cutters still engaged with the rock), or drilling with axial stick ($V = 0$).

If $V = 0$ is detected within an angular step, either normal drilling or loss of contact at the wear flats can take place prior to axial stick, with the condition for the occurrence of axial stick reading

$$0 < \mathcal{W}_o - \mathcal{W}_c(\tau_{ak}) < \mathcal{W}_{f,\max}(\tau_{ak}) \quad (4.75)$$

while the condition for loss of contact at the wear flats is

$$\mathcal{W}_o - \mathcal{W}_c(\tau_{ak}) < 0 \quad (4.76)$$

If axial stick takes place, the event time needs to be pinpointed and $\mathcal{W}_c + \mathcal{W}_f$ is balanced with $\mathcal{W}_o$. Nonetheless, the transitions between normal drilling ($V > 0$), loss of contact at the wear flats ($V < 0$) and axial stick ($V = 0$) are covered in the computation of WOB and TOB from the geometrical variables resulting from bit/rock interaction. Note that numerically, the case of loss of contact at the wear flats is treated with the same branch; in other words, no artificial constraint is put to stop the bit to experience negative axial velocities. A subtle point is that axial-stick treatment is not initiated if $V_{i-1} < 0, V_i \geq 0$.

Meanwhile, torsional stick happens if there is no real root to the quadratics. For axial-torsional stick-slips, there are two scenarios for axial dynamics. The axial stick can take place either before torsional stick or simultaneously. The dynamics needs to be solved for prior-stick, covering $\alpha\Delta\Phi$ and post-stick, covering $(1 - \alpha)\Delta\Phi$. When $\Omega = 0$ is detected while normal drilling or off-bottom, the condition for backward rotation is

$$\ddot{\varphi}(\tau_{\omega k}^-) + \mathcal{T}(\tau_{\omega k}^-) + \beta\mathcal{W}_o < 0 \quad (4.77)$$

If this condition is not met, the initiation of torsional stick is ensured. The duration of axial-torsional stick reads

$$\Delta\tau_{\omega k} = \frac{1}{\omega_o} [\mathcal{T}_f(\tau_{\omega p}^+) + \mathcal{T}_c(\tau_{\omega k}^-) - \mathcal{T}_o + \varphi(\tau_{\omega k}^-)] \quad (4.78)$$

Note that (4.78) is compatible with various initiations (from normal drilling, axial stick, loss of contact on the wear flats) and various terminations (normal drilling, axial stick, loss of contact on the wear flats) of axial-torsional stick-slip events.

4.4 “Complete” algorithm for this work

Finally, the pseudocode for the angle-based time simulation is presented in Algorithm 4.1. Note that the parameter α is designed to implement the adaptive stepping of $\Delta\Phi$ when stick process takes place, the details of this treatment is presented in [81]. The output of this algorithm is simply an array of state variables $\mathbf{x}$ solved at each uniform angular step, rendering the bit motion history search much less CPU expensive when solving the state-dependent delay system (4.12) [85].

Algorithm 4.1 Pseudocode for the angle-based algorithm

```
1: while  $\Phi_{end}$  is not reached do
2:   Solve the quadratics for  $\Delta\tau$  it takes to cover  $\Delta\Phi$ 
3:   Update state variables  $\mathbf{x}_i$ 
4:   if  $\dot{\varphi}_i + \omega_o \leq 0$  (in fact the effective condition is  $\dot{\varphi}_i + \omega_o = 0$ ) then
5:      $\Delta\tau_1 = \frac{\varphi_{i-1} + \omega_o}{-\ddot{\varphi}_{i-1}}$ 
6:      $\dot{u}_{\omega k} = \dot{u}_{i-1} + \ddot{u}_{i-1} \Delta\tau_1$ 
7:     if  $\dot{u}_{\omega k} + v_o \leq 0$  (axial stick takes place before torsional stick) then
8:        $\Delta\tau_a = -\frac{\dot{u}_{i-1} + v_o}{\ddot{u}_{i-1}}$ 
9:        $\dot{u}_{ak} = -v_o$ ,  $u_{ak} = u_{i-1} + \frac{\dot{u}_{ak} + \dot{u}_{i-1}}{2} \Delta\tau_a$ 
10:       $\dot{\varphi}_{ak} = \dot{\varphi}_{i-1} + \ddot{\varphi}_{i-1} \Delta\tau_a$ ,  $\varphi_{ak} = \varphi_{i-1} + \frac{\dot{\varphi}_{i-1} + \dot{\varphi}_{ak}}{2} \Delta\tau_a$ 
11:       $\alpha_1 \Delta\Phi = \frac{\dot{\varphi}_{i-1} + \dot{\varphi}_{ak} + 2\omega_o}{2} \Delta\tau_a$ 
12:      Determine WOB, TOB at  $\tau_{ak}$  using  $\mathbf{x}_{ak}$ 
13:       $\alpha_2 \Delta\Phi = \frac{(\dot{\varphi}_{ak} + \omega_o)^2}{-2\ddot{\varphi}_{ak}}$ 
14:       $\alpha = \alpha_1 + \alpha_2$ 
15:      if  $\alpha > 1$  (it takes more than  $\Delta\Phi$  to reach torsional stick) then
16:        Solve for  $\Delta\tau_{at}$  that covers  $(1 - \alpha)\Delta\Phi$ 
17:        Update  $\mathbf{x}_{at}$ , determine WOB( $\tau_{at}$ ) and TOB( $\tau_{at}$ ), define as step  $i$ 
18:        Bit bounce check, apply external perturbation if necessary
19:         $i = i + 1$ 
20:      else if  $\alpha \leq 1$  (it takes less than  $\Delta\Phi$  to reach torsional stick) then
21:         $\Delta\tau_{at} = \frac{\dot{\varphi}_{ak} + \omega_o}{-\ddot{\varphi}_{ak}}$ ,  $\Delta\tau_1 = \Delta\tau_a + \Delta\tau_{at}$ 
22:         $\varphi_{\omega k} = \varphi_{ak} + \frac{1}{2} (\dot{\varphi}_{ak} - \omega_o) \Delta\tau_{at}$ ,  $\dot{u}_{\omega k} = \dot{u}_{ak} + \ddot{u}_{ak} \Delta\tau_{at}$ 
23:         $u_{\omega k} = u_{ak} + \frac{1}{2} (\dot{u}_{ak} + \dot{u}_{\omega k}) \Delta\tau_{at}$ 
24:         $\alpha_2 \Delta\Phi = \frac{1}{2} \frac{(\dot{\varphi}_{ak} + \omega_o)^2}{-\ddot{\varphi}_{ak}}$ ,  $\alpha = \alpha_1 + \alpha_2$ 
25:      end if
26:    else (axial stick not detected before torsional stick, force axial-torsional stick)
27:       $u_{\omega k} = u_{i-1} + \frac{1}{2} (\dot{u}_{\omega k} + \dot{u}_{i-1}) \Delta\tau_1$ ,  $\varphi_{\omega k} = \varphi_{i-1} + \frac{1}{2} (-\omega_o + \dot{\varphi}_{i-1}) \Delta\tau_1$ 
28:       $\alpha \Delta\Phi = -\frac{1}{2} \ddot{\varphi}_{i-1} (\Delta\tau_1)^2$ 
29:    end if
30:     $\tau_{\omega k} = \tau_{i-1} + \Delta\tau_1$ 
31:    Determine WOB, TOB at  $\tau_{\omega k}$  using  $\mathbf{x}_{\omega k}$ 
32:    Determine the duration of axial-torsional stick  $\Delta\tau_{\omega k}$ 
33:    Slip time :  $\tau_{\omega p} = \tau_{\omega k} + \Delta\tau_{\omega p}$ 
34:    Complete  $\Delta\Phi$  ... some changes are made to this section
35:  else if  $\dot{u}_{i-1} + v_o \geq 0$  and  $\dot{u}_i + v_o < 0$  then
36:     $\Delta\tau_a = \frac{\dot{u}_{i-1} + v_o}{-\ddot{\psi}(\mathcal{W}_{i-1} - \mathcal{W}_o)}$ , determine  $u_{vk}$ ,  $\dot{\varphi}_{vk}$  and  $\varphi_{vk}$ 
37:    Determine  $\alpha \Delta\Phi$  covered from previous angular position to that of axial stick
38:    Determine WOB and TOB at axial stick
39:    Solve for  $\Delta\tau_{vk}$  that covers  $(1 - \alpha)\Delta\Phi$ 
40:     $\tau_i = \tau_{i-1} + \Delta\tau_a + \Delta\tau_{vk}$  and update state variables, WOB, TOB at step  $i$ 
41:    Bit bounce check, perturbation application, move to next step
42:  end if
43:  fill up state variables, WOB, TOB...
44: end while
```

Chapter 5

Stability

The RGD model is always unstable according to linear stability analyses that account for the dynamics of the delay [17, 27]. Two regimes of instability exist, however, separated by a critical angular velocity Ω_c : a “slow” S-regime observed for any nominal angular velocity Ω_o larger than Ω_c , and a “fast” U-regime for $\Omega_o < \Omega_c$ [27]. In the U-regime, the instability is driven by the axial dynamics; any perturbation rapidly evolves, at the time scale of the torsional dynamics, into a limit cycle characterized by axial stick-slip events. The coupled torsional dynamics simultaneously reaches a limit cycle, which include stick-slip events. In the S-regime, it is the torsional dynamics that first becomes unstable, but only marginally. As a result, both dynamics exhibit a slowly evolving quasi-periodic pattern, which eventually reaches a limit cycle with stick-slip. The time needed to attain the limit cycle is sensitive to the magnitude of the perturbation. However, the addition of a very small structural damping (ignored in the stability analysis) would be enough to stabilize the S-regime. In practical terms, this means that the RGD system can be considered stable if $\Omega_o > \Omega_c$. In the U-regime ($\Omega_o < \Omega_c$), the growing perturbation leads to bit bouncing rather than to the establishment of a stable limit cycle if the frictional dissipation is small compared to the energy expended in cutting [3].

The proof for the existence of two instability regimes relies on a linearized form of the original state-dependent delay differential equations (SD-DDEs) that govern the response of the RGD model [27]. This linearization preserves the stability property of the original SD-DDE, as proved for systems with a single time delay [26, 65, 86]. For deep drilling applications ($\psi \gg \frac{v_o}{\omega_o}$), however, the critical angular velocity is virtually identical to the one computed by assuming a constant delay [27]. Ignoring the torsional dynamics leads to a simpler characteristic equation, of a type first analyzed by Bhatt and Hsu [87, 88]. This equation can be solved to yield an explicit expression for the

critical angular velocity

$$\Omega_c = \frac{1}{t_v} \sqrt{\frac{8}{n}} \quad (5.1)$$

with n denoting the number of symmetric blades. As evidenced by (5.1), the stability of the system depends on the bit geometry. There is a qualitative difference between the two analyses, however, as a stability analysis based on assuming a constant delay predicts that the system is stable for $\Omega > \Omega_c$ [3, 15, 27].

Here, we extend the linear stability analysis for the general cutter layout described earlier, which introduces multiple time delays in the SD-DDE.

The general linear stability analyses will be first illustrated with class I bits. To apply the same linearization approach, we first partition the nominal cutter engagement surface according to each (nominal) time delay, thus enabling the case of multiple delays to be treated similarly to the single delay case. Second, the linearization process is presented for the dynamics depicted by (3.44) to achieve the equivalent linearized system; then using the assumption of $\psi \gg \frac{v_o}{\omega_o}$, the newly-achieved system is further simplified by freezing each delay to its nominal value. Finally, an analytical approach based on the D-subdivision method [89] is presented to determine the critical nominal rotation speed, similar to (5.1).

To conclude, we extend the methodology to class II bits (realistic cutter layout).

5.1 Partitioning of the nominal cutter engagement surface for Class I Bits

For a bit undergoing a uniform motion described by axial velocity V_o and angular velocity Ω_o , the union $\bar{\mathcal{S}}_o$ of the nominal engagement surface $\bar{\mathcal{S}}_{oi}$ for all the cutters is a rectangle of width a and height $2\pi V_o/\Omega_o$. Hence

$$\sum_i \bar{A}_{oi} = \frac{2\pi a V_o}{\Omega_o} \quad (5.2)$$

where $\bar{A}_{oi} = \|\bar{\mathcal{S}}_{oi}\|$. As described next, rectangle $\bar{\mathcal{S}}_o$ can alternatively be partitioned in terms of the angular delays instead of the cutter engagement surfaces $\bar{\mathcal{S}}_{oi}$. This partitioning is required to calculate the contribution of each nominal delay to the

weight- and torque-on-bit.

As an example of the methodology used to partition $\bar{\mathcal{S}}_o$, consider a bit consisting of 3 identical rectangular cutters of width b placed at the same axial position, see Fig. 5.1(i). The cutters overlap radially and collectively span bit radius a . Each cutter engagement surface $\bar{\mathcal{S}}_{oi}$ is actually a combination of some rectangles, each linked to a particular delay. As illustrated in Fig. 5.1(ii), the rectangular cutters segment the bit radius a into several contiguous regions denoted by Roman numbers. With this segmentation, the number of cutters interacting with each other is fixed within each region, and so is the angular delay distribution.

Consider first the engagement surface $\mathcal{S}_{o1}$ of cutter 1. As illustrated in the top row of Fig. 5.1(ii), $\mathcal{S}_{o1}$ is partitioned into contributions from regions I, II, and III. The contribution from region I results from the interaction of cutter 1 with itself, while the contributions from region II and III are affected by cutter 2. Considering $\mathcal{S}_{o2}$ next, it can be seen from the middle row of Fig. 5.1(ii), that it is determined from the interaction of cutter 2 with cutter 1 in region II, and with cutter 3 in regions III and IV. Finally, $\mathcal{S}_{o3}$ (bottom row of Fig. 5.1(iii)) is the result of the interaction of cutter 3 with itself (region V) and with cutter 1 (region III) and cutter 2 (region IV). Thus, region III sees the interaction between all three cutters, resulting in fixed angular delays $\{\Delta\Theta_{21}, \Delta\Theta_{32}, \Delta\Theta_{13}\}$. The summation of all angular delays within a region is evidently 2π . Denote the width of each region as Γ_P , ($P = \text{I, II, III, ...}$). The summation of all the Γ_P is a .

Figure 5.1(iii) indicates that each rectangular component of $\bar{\mathcal{S}}_{oi}$ belongs to a region and thus corresponds to a specific angular delay. For instance, $\bar{\mathcal{S}}_{o3}$ is composed of three components corresponding to $\Delta\Theta_{13}$, $\Delta\Theta_{23}$ and 2π respectively. Accordingly, $\bar{\mathcal{S}}_{o3}$ can be decomposed into three rectangles, with respective areas $\Gamma_{\text{III}}\Delta\Theta_{13}V_o/\Omega_o$, $\Gamma_{\text{IV}}\Delta\Theta_{23}V_o/\Omega_o$, $\Gamma_{\text{V}}2\pi V_o/\Omega_o$.

The procedure consists therefore of performing this segmentation on all the $\bar{\mathcal{S}}_{oi}$, then identifying the like terms of $\Delta\Theta_{ji}$ in the rectangular components of the $\bar{\mathcal{S}}_{oi}$. Assuming there are M distinct values of $\Delta\Theta_{ji}$, i.e., $\Delta\Theta_m$, $m = 1, 2, \dots, M$, $\mathcal{S}_o$ can thus be seen to be partitioned into M surfaces $\check{\mathcal{S}}_{om}$, each characterized by the same angular delay $\Delta\Theta_m$. Area $\check{A}_{om}$ of $\check{\mathcal{S}}_{om}$ is given by

$$\check{A}_{om} = \Gamma_m \times \frac{\Delta\Theta_m V_o}{\Omega_o} \quad (5.3)$$

where Γ_m is the summation of some region widths. It is evident that the following identities hold

$$\sum_m \Gamma_m = \sum_i b_i, \quad \sum_m \frac{\Delta\Theta_m \Gamma_m}{2\pi a} = 1 \quad (5.4)$$

Fig. 5.1(iv) indicates that, for this particular bit, there are 6 distinct angular delays $\Delta\Theta_m$: 2π , $\Delta\Theta_{21}$, $\Delta\Theta_{12}$, $\Delta\Theta_{32}$, $\Delta\Theta_{23}$, and $\Delta\Theta_{13}$; the corresponding Γ_m expressions are $\Gamma_I + \Gamma_V$, $\Gamma_{II} + \Gamma_{III}$, Γ_{II} , $\Gamma_{III} + \Gamma_{IV}$, Γ_{IV} , and Γ_{III} .

Figure 5.1 – Illustration of the bit/rock interaction considering the extended bit geometry (nominal motion).

5.2 Linearized dynamics

Using the partitioning approach described above, we linearize the SD-DDE, by considering a small perturbation to the uniform motion (V_o, Ω_o) of the bit. There are restrictions to the perturbation, however, to ensure that the bit velocity (V, Ω) remains positive. The condition $V > 0$ implies that there is no change in the frictional contact forces, while the condition $\Omega > 0$ prevents stick-slip events and backward rotation of the bit. These restrictions ensure that the bit motion is smooth and therefore differentiable. They also entail that only changes in the cutting components of weight- and torque-on-bit, caused by variation in the cutter engagement area resulting from the perturbations of the bit motion, need to be considered.

The physical meaning of the linearization is best illustrated with the RGD model. In this particular case, the variation of the bit engagement area, $\Delta A(t) := A(t) - A_o$ is simply given by $a(d - d_o)$, with $d(t)$ denoting the combined depth of cut for the n symmetric blades. By expressing d in terms of a difference between the bit axial displacement at a current time and at a previous time, and using (3.12) for d_o , $\Delta A(t)$ can be written as

$$\Delta A(t) = an \left[U(t) - U(t - t_n) - \frac{2\pi V_o}{n\Omega_o} \right] \quad (5.5)$$

with the delay $t_n(t)$ solution of

$$\Phi(t) - \Phi(t - t_n) = \frac{2\pi}{n} \quad (5.6)$$

The above equations can be rewritten in terms of the scaled perturbed displacements $u(\tau)$ and $\varphi(\tau)$ as [27]

$$\Delta \mathcal{A}(\tau) = n [u(\tau) - u(\tau - \tau_n)] + v_o \left(\tau_n - \frac{h}{n} \right) \quad (5.7)$$

$$\varphi(\tau) - \varphi(\tau - \tau_n) = -\omega_o \left(\tau_n - \frac{h}{n} \right) \quad (5.8)$$

where $\mathcal{A} = A/ad_*$, $\tau_n(\tau)$ is the actual delay scaled by t_ω and $h = 2\pi/\omega_o$ is the nominal scaled period of revolution (h/n is the nominal delay). Combining (5.7) and (5.8) yields

$$\Delta \mathcal{A}(\tau) = n [u(\tau) - u(\tau - \tau_n)] - \frac{v_o}{\omega_o} [\varphi(\tau) - \varphi(\tau - \tau_n)] \quad (5.9)$$

Assuming that the perturbations to the uniform motion are small [27], (5.9) can be linearized:

$$\Delta \mathcal{A}(\tau) = n \left[u(\tau) - u\left(\tau - \frac{h}{n}\right) \right] - \frac{v_o}{\omega_o} \left[\varphi(\tau) - \varphi\left(\tau - \frac{h}{n}\right) \right] \quad (5.10)$$

The first term in the above linearized expression captures the variation of $\mathcal{A}$ when the delay is frozen (i.e., when the angular velocity is constant), while the second term takes into account the delay variation $\tau_n - h/n$.

Consider now the class of bits with rectangular cutters, described in Section , which is characterized by M distinct angular delays $\Delta\Theta_m$. The variation of the bit engagement

area is now expressed as $\Delta A(t) = \sum_m (\check{A}_m - \check{A}_{om})$ and thus

$$\Delta A(t) = \sum_m \Gamma_m \left[U(t) - U(t - t_m) - \frac{\Delta \Theta_m V_o}{\Omega_o} \right] \quad (5.11)$$

where t_m is the time delay associated with angular delay $\Delta \Theta_m$. Similarly to the steps followed to derive (5.10), (5.11) can be linearized by assuming the perturbations of the bit motion to be small. Written in terms of scaled quantities, the linearized form of (5.11) reads

$$\Delta \mathcal{A}(\tau) = \sum_m \alpha_m \left[(u(\tau) - u(\tau - k_m h)) - \frac{v_o}{\omega_o} (\varphi(\tau) - \varphi(\tau - k_m h)) \right] \quad (5.12)$$

where the numbers α_m and k_m are defined as

$$\alpha_m = \frac{\Gamma_m}{a}, \quad k_m = \frac{\Delta \Theta_m}{2\pi} \quad (5.13)$$

In fact, $\Delta \mathcal{A}$ can be reinterpreted as $\mathcal{W} - \mathcal{W}_o$, since there is no variation in the contact components of the weight-on-bit.

The variation of the cutting component of torque-on-bit caused by a perturbation in the bit motion can similarly be accounted for. For the considered small perturbations of the bit motion, the radial coordinates of the centroids of the variations $\check{\mathcal{A}}_m - \check{\mathcal{A}}_{om}$ read $\check{\rho}_m$. Furthermore since there is no change in the frictional contact forces, the variation of the torque-on-bit $\mathcal{T} - \mathcal{T}_o$ is given by

$$\mathcal{T} - \mathcal{T}_o = 2 \sum_i (\bar{\mathcal{A}}_i - \bar{\mathcal{A}}_{oi}) \bar{\rho}_i = 2 \sum_m (\check{\mathcal{A}}_m - \check{\mathcal{A}}_{om}) \check{\rho}_m \quad (5.14)$$

where

$$\check{\mathcal{A}}_m = \frac{\check{A}_m}{ad_*}, \quad \check{\rho}_m = \frac{\check{r}_m}{a} \quad (5.15)$$

Hence, after taking into account (5.12), the linearized expression for $\mathcal{T} - \mathcal{T}_o$ is

$$\mathcal{T} - \mathcal{T}_o = \sum_m \beta_m \left[u(\tau) - u(\tau - k_m h) - \frac{v_o}{\omega_o} (\varphi(\tau) - \varphi(\tau - k_m h)) \right] \quad (5.16)$$

with

$$\beta_m = 2\alpha_m\check{\rho}_m \quad (5.17)$$

Introduce η and δ as

$$\eta = \sum_m \alpha_m, \quad \delta = \sum_m \beta_m \quad (5.18)$$

where η can be interpreted as an equivalent number of full blades (not necessarily an integer). Substituting the above expressions for $\mathcal{W} - \mathcal{W}_o$ and $\mathcal{T} - \mathcal{T}_o$ in the equations of motion (3.44), leads to the linearized coupled dynamical equations that govern the evolution of the perturbations $u(\tau)$ and $\varphi(\tau)$

$$\ddot{u} = -\psi \sum_m \alpha_m \left[(u(\tau) - u(\tau - k_m h)) - \frac{v_o}{\omega_o} (\varphi(\tau) - \varphi(\tau - k_m h)) \right] \quad (5.19)$$

$$\ddot{\varphi} + \varphi = - \sum_m \beta_m \left[(u(\tau) - u(\tau - k_m h)) - \frac{v_o}{\omega_o} (\varphi(\tau) - \varphi(\tau - k_m h)) \right] \quad (5.20)$$

The three number sets $\{k_m\}$, $\{\alpha_m\}$ and $\{\beta_m\}$ (each containing M numbers) fully characterize, for the purpose of assessing the bit stability, the design of bits belonging to the class considered here. For the RGD model with n symmetric full blades, $\alpha_1 = n$, $\beta_1 = n$, $k_1 = 1/n$. The following constraints on these numbers exist

$$0 < k_m \leq 1, \quad \sum_m \alpha_m k_m = 1, \quad 0 < \beta_m < 2\alpha_m \quad (5.21)$$

For a core barrel, $\sum_m \alpha_m k_m < 1$.

To derive the characteristic equations, it is advantageous to rewrite the dynamical equations (5.19) and (5.20) using a state space formulation. Let the state variable be defined as $\mathbf{x}(\tau) = \{u(\tau), \dot{u}(\tau), \varphi(\tau), \dot{\varphi}(\tau)\}^T$. Then, (5.19) and (5.20) transform in matrix form as

$$\dot{\mathbf{x}}(\tau) = \mathbf{A}\mathbf{x}(\tau) + \sum_m \mathbf{B}_m \mathbf{x}(\tau - k_m h) \quad (5.22)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\psi\eta & 0 & \frac{v_o}{\omega_o}\psi\eta & 0 \\ 0 & 0 & 0 & 1 \\ -\delta & 0 & -1 + \frac{v_o}{\omega_o}\delta & 0 \end{bmatrix}, \quad \mathbf{B}_m = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \psi\alpha_m & 0 & -\frac{v_o}{\omega_o}\psi\alpha_m & 0 \\ 0 & 0 & 0 & 0 \\ \beta_m & 0 & -\frac{v_o}{\omega_o}\beta_m & 0 \end{bmatrix} \quad (5.23)$$

To determine the characteristic equation of (5.22), assume a perturbation of the form $\mathbf{x} = \mathbf{a}e^{s\tau}$. Hence (5.22) becomes

$$\mathbf{A} + \sum_m e^{-sk_m h} \mathbf{B}_m - s\mathbf{I} = \mathbf{0} \quad (5.24)$$

where null matrix $\mathbf{0}$ and identity matrix $\mathbf{I}$ belong to $\mathbb{R}^{4 \times 4}$. Setting the determinant of (5.24) to zero yields the characteristic equation

$$(s^2 + 1) \left(s^2 + \psi \sum_m \alpha_m (1 - e^{-sk_m h}) \right) - s^2 \frac{v_o}{\omega_o} \sum_m \beta_m (1 - e^{-sk_m h}) = 0 \quad (5.25)$$

It is well-known that, (5.25) has an infinite number of conjugate poles all located in the left half-plane $\alpha \in \mathbb{C} : \Re(\alpha) < \gamma$ where γ is the real part of the right-most root of (5.25). The system is stable in its response to small perturbations if $\gamma < 0$.

5.3 Critical angular velocity

5.3.1 Approach

The objective of the stability analysis is to determine the angular velocity corresponding to the right-most pole crossing the imaginary axis to enter the right half-plane. This is done by tracking the trajectory of the poles in the complex plane as the imposed revolution period h (or equivalently the angular velocity ω_o) varies. In the RGD model, there are two categories of poles [27]: torsional poles in the close vicinity of $\pm j$, which are responsible for the marginally unstable S-regime, and axial poles which control the “fast” unstable U-regime. As shown for the RGD model [27], the critical angular velocity, at which the axial poles cross the imaginary axis and which defines the boundary between stable and unstable axial dynamics, is virtually the same whether computed for the coupled dynamics or for the uncoupled axial dynamics. This characteristic of the problem also holds for the extended model.

It can readily be verified from (5.25) that the pair of conjugate torsional poles located at $\pm j$ when $h = 0$ moves to the right half-plane with increasing h , as

$$\left. \frac{ds}{dh} \right|_{s=\pm j} = \frac{v_o}{2\omega_o} \quad (5.26)$$

For deep drilling, on account that $\frac{v_o}{\omega_o} \ll 1$, the axial poles can be well approximated from the uncoupled axial dynamics with constant delays [27]. They are therefore the solutions of the characteristic equation $P_{\tau_o}(s; h) = 0$, where

$$P_{\tau_o}(s; h) = (s^2 + 1) \left(s^2 + \psi \sum_m \alpha_m (1 - e^{-sk_m h}) \right) \quad (5.27)$$

The pole crossings s and the period of nominal revolution h can be scaled with the axial characteristic time t_v rather than t_ω

$$\mathbf{s} = \frac{s}{\sqrt{\psi}}, \quad \mathbf{h} = \frac{2\pi}{\Omega_o t_v} \quad (5.28)$$

The poles of the axial dynamics are determined by solving the reduced characteristic equation $P_a(\mathbf{s}, \mathbf{h}) = 0$,

$$P_a(\mathbf{s}, \mathbf{h}) = \mathbf{s}^2 + \sum_m \alpha_m (1 - e^{-\mathbf{s}k_m \mathbf{h}}) \quad (5.29)$$

Thus for deep drilling, the influence of bit design on the stability is essentially embedded in the numbers k_m and α_m , $m = 1, 2, \dots, M$, with negligible influence of the numbers β_m . The critical period of revolution $\mathbf{h}_c$ corresponds to a Hopf bifurcation of the decoupled axial dynamics. In the following, we present a general analytical approach to compute the critical angular velocity for any bit design characterized by the set of numbers $\{\alpha_m\}$ and $\{k_m\}$, subject to the constraints (5.21). This approach is based on the D-subdivision method [26, 89].

The infinite number of poles of (5.29) are inside the envelope curve [90]

$$|\mathbf{s}| \leq \alpha + \sum_m \alpha_m e^{-\Re(\mathbf{s})k_m \mathbf{h}} \quad (5.30)$$

Evidently, there exists an upper bound to the real part of the poles. Viewing $\mathbf{h}$ (or equivalently ω_o) as the bifurcation parameter, each pole of (5.29) traces a trajectory

in the complex plane. A loss or acquisition of exponential stability of the decoupled axial dynamics is related to the intersections between those pole trajectories and the imaginary axis.

For $\mathbf{h} = 0$ ($\omega_o = \infty$), there is only one pole located at the origin and the asymptotic stability is maintained as $\mathbf{h} \rightarrow 0^+$ [90]. However, with increasing $\mathbf{h}$, the number n_u of unstable axial poles pair changes at discrete values of $\mathbf{h}$. These changes take place whenever a pair of poles crosses the imaginary axis, with n_u increasing or decreasing depending on the root crossing direction. Thus, the $n_u - \mathbf{h}$ relation can be calculated, with $\mathbf{h}_c$ representing a critical value of $\mathbf{h}$ at which n_u vanishes or originates; each $\mathbf{h}_c$ corresponds to a transition from unstable to stable axial dynamics with increasing $\mathbf{h}$, or vice versa. The critical angular velocities is simply determined by $\omega_c = 2\pi/\mathbf{h}_c$. For the RGD model, there is a unique value of $\mathbf{h}_c$ and the axial dynamics is stable if $\omega > \omega_c$ where

$$\omega_c = \frac{2\pi}{\mathbf{h}_c} = \sqrt{\frac{8}{n}} \quad (5.31)$$

5.3.2 Crossing poles and crossing direction

Denote a crossing pole (in a conjugate pair) as $\mathbf{s}_z = \pm j\varpi$, where $\varpi > 0$ is the frequency of the decoupled axial oscillations. The physical significance of ϖ will be explored later. Define

$$\vartheta = \mathbf{h}\varpi \quad (5.32)$$

Substitute $\mathbf{s}_z$ into (5.29) and separate the real and imaginary parts. The characteristic equation for the axial dynamics $P_a(\mathbf{s}, \mathbf{h}) = 0$ can be rewritten as

$$\begin{cases} -\varpi^2 + \eta - f(\vartheta) = 0 \\ g(\vartheta) = 0 \end{cases} \quad (5.33)$$

where

$$f(\vartheta) = \sum_m \alpha_m \cos(k_m \vartheta), \quad g(\vartheta) = \sum_m \alpha_m \sin(k_m \vartheta) \quad (5.34)$$

Denote ϑ_z as a solution of $g(\vartheta) = 0$, corresponding to a crossing pole. To determine each corresponding root crossing direction, the sign of $\Re\left(\frac{d\mathbf{s}}{d\mathbf{h}}\big|_{\mathbf{s}=j\varpi}\right)$ has to be checked. Taking the total derivative of the axial characteristic equation $P_a(\mathbf{s}, \mathbf{h}) = 0$ yields

$$\frac{d\mathbf{s}}{d\mathbf{h}} = -\frac{\frac{\partial P_a}{\partial \mathbf{h}}}{\frac{\partial P_a}{\partial \mathbf{s}}} \quad (5.35)$$

where

$$\frac{\partial P_a}{\partial \mathbf{h}} = \mathbf{s} \sum_m \alpha_m k_m e^{-k_m \mathbf{s} \mathbf{h}}, \quad \frac{\partial P_a}{\partial \mathbf{s}} = 2\mathbf{s} + \mathbf{h} \sum_m \alpha_m k_m e^{-k_m \mathbf{s} \mathbf{h}} \quad (5.36)$$

Then substituting $\mathbf{s} = j\boldsymbol{\varpi}$ in (5.35) and (5.36) yields

$$\text{sign} \left[\Re \left(\frac{d\mathbf{s}}{d\mathbf{h}} \Big|_{\mathbf{s}=j\boldsymbol{\varpi}} \right) \right] = -\text{sign} [p(\vartheta)] \quad (5.37)$$

where

$$p(\vartheta) := -\frac{dg}{d\vartheta} = \sum_m \alpha_m k_m \cos(k_m \vartheta) \quad (5.38)$$

Define Δn_u as the variation of the number of unstable axial pole pairs with increasing $\mathbf{h}$; the value for Δn_u is related to the sign of function $p(\vartheta_z)$. With increasing $\mathbf{h}$, a right-crossing critical pole indicates the generation of a pair of unstable axial poles; a left-crossing $\mathbf{s}_z$ indicates the vanishing of a pair of unstable axial poles. Note that when a pole trajectory is tangential to the imaginary axis, Δn_u is zero if ϑ_z is a local minimum/maximum of $g(\vartheta)$. The relation between Δn_u and the sign of $p(\vartheta_z)$ reads

$$\Delta n_u(\vartheta_z) = \begin{cases} 1, & -p(\vartheta_z) > 0 \\ -1, & -p(\vartheta_z) < 0 \\ 0, & p(\vartheta_z) = 0, p(\vartheta_z^-) p(\vartheta_z^+) < 0 \\ \Delta n_u(\vartheta_z \pm \varepsilon), & p(\vartheta_z) = 0, p(\vartheta_z^-) p(\vartheta_z^+) > 0 \end{cases} \quad (5.39)$$

where $p(\vartheta_z^\pm) = p(\vartheta_z \pm \varepsilon)$ and $0 < \varepsilon \ll 1$.

5.3.3 Summary

The procedure to determine the critical angular velocities can be summarized as follows.

1. Solve $g(\vartheta_z) = 0$ for $\vartheta_z > 0$.
2. Compute the angular frequency $\boldsymbol{\varpi}$ of the axial oscillations using

$$\boldsymbol{\varpi} = \sqrt{\eta - f(\vartheta_z)} \quad (5.40)$$

and the corresponding time delay from

$$\mathbf{h}_z = \frac{\vartheta_z}{\varpi} \quad (5.41)$$

3. Determine the crossing direction of the corresponding conjugated poles (that are located on the imaginary axis) using (5.37).
4. Establish the relation between the number of unstable axial pole pairs, n_u , and the scaled revolution period $\mathbf{h}$. A critical period $\mathbf{h}_c$ corresponds to a step variation of n_u from 0 to 1 or from 1 to 0. A critical angular velocity Ω_c is then computed according to

$$\Omega_c = \frac{2\pi}{t_v \mathbf{h}_c} \quad (5.42)$$

5.4 Stability analysis for realistic bits

This section presents the formulation of linear stability analysis for a given realistic PDC bit geometry. In essence, the analysis inspects a linearized system which is sufficient to represent the stability of the original system. It will be shown that the formulation requires the knowledge of how each time delay contributes to the perturbation around the nominal weight-on-bit $\mathcal{W}_o$.

5.4.1 Preamble

It can be deduced that the previously presented framework can be extended to realistic PDC bit geometry. A simple proof is that for a given nominal bit penetration d_o , the bit can be viewed as a superposition of many groups of rectangular cutters. It is well established that the bit profile can be discretized into many stripes at discrete r values [91], see Fig.(5.2)(a). In Fig.(5.2)(b), within each r step, the bit can be viewed as several rectangular cutters with identical r -span but different z values, or as a “peculiar” core barrel. For nominal bit motion, the nominal local depths of cut for such a core barrel are piecewise linear functions of the nominal bit penetration per revolution d_o . Separated by some critical values of d_o , there are multiple “regimes”, each characterized by a fixed delay distribution, denoted by Roman numbers in Fig.(5.2)(c). In the context of linear stability analysis, in the neighborhood of a given d_o away from the critical d_o , the core barrel is no different from the bit family studied in [36].

Irrespective of the “striping” approach discussed above, this work incorporates the polygon clipping technique to account for the precise geometric interactions within

Figure 5.2 – A PDC bit can be viewed as a superposition of many groups of rectangular cutters

the analysis. It should be stated that the effect of delay distribution variation on the stability has not been addressed explicitly in this paper, although it is captured in the time simulations. Nonetheless, in the light of the “striping” approach, such effect is only relevant at a discrete array of critical d_o values.

5.4.2 Stability analysis of “real” bits with round cutters

For the linear stability analyses with realistic bit geometry, different from bits with rectangular cutters, $\mathcal{W}_f - \mathcal{W}_{o,f}$ should be considered in addition to $\mathcal{W}_c - \mathcal{W}_{o,c}$ as there is a variation of the wearflat/rock contact area given a perturbation of the depth of cut. Accordingly there are two groups of contributions to the number set $\{\alpha\}$:

$$\{\alpha_p^c\}, \quad \{\alpha_q^f\} \quad (5.43)$$

which correspond to $\mathcal{W}_c - \mathcal{W}_{co}$ and $\mathcal{W}_f - \mathcal{W}_{fo}$ respectively. Note that the dual indices ji of local entries α_{ji}^c and α_{ji}^f at each $\mathcal{S}_i$ are contracted into p or q corresponding to a unique angular delay $\Phi_p = 2k_p\pi$ or $\Phi_q = 2k_q\pi$. When a specific angular delay makes contribution to $\{\alpha_p^c\}$, it may or may not contribute to $\{\alpha_q^f\}$ and vice versa.

When applying a small perturbation to the nominal bit response, the area variation of

$\mathcal{S}_i$ can be associated with α_{ji}^c [36]. Considering the full bit,

$$\|\mathcal{S}\| - \|\mathcal{S}_o\| \approx a \sum_{i=1}^N \sum_{j \in I_i} \alpha_{ji}^c \cdot \left(d_{ji} - \frac{V_o}{\Omega_o} \Phi_{ji} \right) \quad (5.44)$$

where a is the bit radius and $d_{ji} - \frac{V_o}{\Omega_o} \Phi_{ji}$ is infinitesimal. In the bit profile plane, the quantity $a\alpha_{ji}^c$ is the projection length of each arc-like boundary ($\mathcal{L}_{ji}$) on the lower (smaller z) side of $\mathcal{S}_i$. In Fig. 5.3, each arc-like boundary is highlighted by a dotted line; their projections are presented and serve as three entries of $\{\alpha_{ji}^c\}$. The left side of (5.44) is proportional to $\mathcal{W}_c - \mathcal{W}_{o,c}$. After scaling, the following holds

$$\mathcal{W}_c - \mathcal{W}_{o,c} \approx \sum_{i=1}^N \sum_{j \in I_i} \alpha_{ji}^c \cdot \left(\delta_{ji} - \frac{v_o}{\omega_o} \Phi_{ji} \right) \quad (5.45)$$

where the variation $\delta_{ji} - \frac{v_o}{\omega_o} \Phi_{ji}$ can be determined using (3.45). One can contract the dual indices ji into a single one p .

$$\mathcal{W}_c - \mathcal{W}_{o,c} \approx \sum_{p=1} \alpha_p^c \cdot \left(\delta_p - \frac{v_o}{\omega_o} \Phi_p \right) \quad (5.46)$$

Note that $\alpha_{ji}^c < 0$ is possible for few locations, indicating that increasing d_o results in a decrease in $\|\mathcal{S}_i\|$. Physically this is because circular cutters “tunnel” into the rock surface at the inner/outer rims of the bit resulting in leftover rock in contrast to rectangular cutters that ensure a clear removal of rock surface with area ad_o per revolution of uniform bit motion [36]. For simplicity such rare cases of α_{ji}^c are treated as 0. Accordingly the following identity holds

$$\sum_p k_p \alpha_p^c \approx 1 \quad (5.47)$$

Note that “=” can be achieved by bits with rectangular cutters. For realistic PDC cutter layout, $\sum_p k_p \alpha_p^c$ is mostly around 0.95 (the value varies with bit designs and depth of cut d_o). Also

$$\eta = \sum_p \alpha_p^c \quad (5.48)$$

can be used to indicate the effective number of full blades, which will be discussed in more detail later.

Similarly for $b = \sum_{i=1}^N b_i$ (b_i being the length of the projection of $\mathcal{L}_i$ on r -axis), its

variation reads

$$b - b_o \approx \sum_{i=1}^N \sum_{j \in I_i} \alpha_{ji}^f \cdot \left(d_{ji} - \frac{V_o}{\Omega_o} \Phi_{ji} \right) \quad (5.49)$$

Positive entries of $\{\alpha_{ji}^f\}$ indicate that increasing d_o ($d_{ji} = \frac{V_o}{\Omega_o} \Phi_{ji} + \varepsilon$, $0 < \varepsilon \ll 1$) results in an increase in the projection length b . The determination of a local entry α_{ji}^f is described with Fig. 5.3. First sort the end points $\underline{P}$ and $\overline{P}$ of the cutting edge $\mathcal{L}_i$ in a counterclockwise manner. Denote the counterclockwise tangential vector of $\mathcal{L}_i$ and $\mathcal{L}_{ji}$ at point P ¹ as $\ell_i(P) = (\mu_i(P), \nu_i(P))^T$ and $\ell_{ji}(P) = (\mu_{ji}(P), \nu_{ji}(P))^T$ respectively, where $\|\ell_i\| = 1$ and $\|\ell_{ji}\| = 1$. Define

$$\kappa_1(P) = \begin{cases} \nu_i(\underline{P})/\mu_i(\underline{P}), & P = \underline{P} \\ -\nu_i(\overline{P})/\mu_i(\overline{P}), & P = \overline{P} \end{cases} \quad \kappa_2(P) = \begin{cases} \nu_{ji}(\underline{P})/\mu_{ji}(\underline{P}), & P = \underline{P} \\ -\nu_{ji}(\overline{P})/\mu_{ji}(\overline{P}), & P = \overline{P} \end{cases} \quad (5.50)$$

and

$$\chi(P) = \begin{cases} \ell_i(\underline{P}), & P = \underline{P} \\ -\ell_i(\overline{P}), & P = \overline{P} \end{cases} \quad (5.51)$$

Impose a small variation of local shift $\varepsilon = d_{ji} - \frac{V_o}{\Omega_o} \Phi_{ji}$, the objective is to determine the induced variation $b - b_o$. Following Fig. 5.3(b),

$$\alpha_{ji}^f(P) = \frac{\text{sign}(\|\chi(P) - \xi(P)\| - \|\chi(P)\|)}{|\kappa_2(P) - \kappa_1(P)|}, \quad P \in \mathcal{L}_{ji} \cap \mathcal{L}_i \quad (5.52)$$

where $\varepsilon \alpha_{ji}^f(P)$ denotes the contribution to $b_i - b_{o,i}$ at point P ; the vector

$$\xi(P) = \left(\frac{\varepsilon}{\kappa_2(P) - \kappa_1(P)}, \frac{\varepsilon \kappa_1(P)}{\kappa_2(P) - \kappa_1(P)} \right)^T \quad (5.53)$$

solves

$$\begin{cases} z = \kappa_1(P) \cdot r \\ z = \kappa_2(P) \cdot r - \varepsilon \end{cases} \quad (5.54)$$

at end points $\underline{P}$ and $\overline{P}$. For rectangular cutters $\kappa_1(P) = \pm\infty$ and $\kappa_2(P) = 0$ which results in $\alpha_{ji}^f = 0$; when $\kappa_1(P) = 0$, $P \notin \mathcal{L}_i$ and it is assumed that there is no frictional contact stress σ applied after perturbation.

¹The counterclockwise direction is defined by the ascending order of polar angles of points. For $\mathcal{L}_i$ the polar angles are defined with respect to cutter center C_i ; for $\mathcal{L}_{ji}$ the polar angles are defined with respect to shifted cutter center C_j

Alternatively (5.52) can be written in the form of

$$\alpha_{ji}^f = \sin \theta_{1,ji} \cdot \left[\cos \theta_{1,ji} + \frac{\sin \theta_{1,ji}}{\tan(\theta_{2,ji} - \theta_{1,ji})} \right] \quad (5.55)$$

where $\theta_{1,2}$ are defined as follows. At $\underline{P}$, θ_1 is the counterclockwise angle between $\mathbf{e}_z$ and $\ell_i(\underline{P})$ and θ_2 the counterclockwise angle between $\mathbf{e}_z$ and $\ell_{ji}(\underline{P})$. At $\overline{P}$, θ_1 is the clockwise angle between $\mathbf{e}_z$ and $-\ell_i(\overline{P})$ and θ_2 the clockwise angle between $\mathbf{e}_z$ and $-\ell_{ji}(\overline{P})$. The definition alone ensures that $\theta_{1,2} \in [0, 2\pi)$. From observations of bit/rock engagements such as Fig. 5.3(d), the general constraints of $\theta_{1,2}$ for bits with round cutters read

$$0 < \theta_1 < \pi, \quad 0 < \theta_2 - \theta_1 < \frac{\pi}{2} \quad (5.56)$$

For bits with rectangular cutters ($\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \pi$), $\alpha_{ji}^f = 0$. Fig. 5.4 illustrates α_{ji}^f entries (and the corner angles) for Bit A at $d_o = 3$ mm. By viewing various cases, $\alpha_{ji}^f \geq 0$ holds mostly although there are few negative entries of α_{ji}^f close to 0 which takes place close to the outer rim of the bit.

Now (5.49) can be scaled such that

$$\mathcal{W}_f - \mathcal{W}_{o,f} \approx \check{\lambda} \sum_{i=1}^N \sum_{j \in I_i} \alpha_{ji}^f \cdot \left(\delta_{ji} - \frac{v_o}{\omega_o} \Phi_{ji} \right) \quad (5.57)$$

where $\check{\lambda} \propto w/a$ is defined in (3.53) which compares wearflat width to bit radius. Meanwhile α_{ji}^c roughly compares cutter radius with bit radius and entries of α_{ji}^f can be larger than those of α_{ji}^c . Thus the variation (5.57) may be comparable to (5.45) and should not be neglected in the analyses. Similar to (5.46), (5.57) can be written as

$$\mathcal{W}_f - \mathcal{W}_{o,f} \approx \check{\lambda} \sum_q \alpha_q^f \cdot \left(\delta_q - \frac{v_o}{\omega_o} \Phi_q \right) \quad (5.58)$$

It needs to be pointed out that the linear stability analysis is formulated on the basis of some first order approximations (5.44) and (5.49). Strictly, neither $\|\mathcal{S}\| - \|\mathcal{S}_o\|$ nor $b - b_o$ is a linear function of the local shifts $d_{ji} - \frac{\Phi_{ji}}{\Omega_o} V_o$. The linear approximations are acceptable as linear stability predicts "stability in the small" with an infinitesimal perturbation applied to a nominal bit response; also it is assumed that the same order of small perturbation is applied to the contribution of each delay. For bits with rectangular cutters [36], strictly $\alpha_p^c > 0$ and $\alpha_q^f = 0$ for all p and q . For "real" bits,

Figure 5.3 – Definition of the number sets $\{\alpha_{ji}^c\}$ and $\{\alpha_{ji}^f\}$

entries in $\{\alpha_p^c\}$ and $\{\alpha_q^f\}$ are mostly non-negative. This agrees with the observation, as illustrated in Fig. 77, when shifting $\mathcal{L}_{ji}$ in the direction of $-z$, both $\|S_i\|$ and b_i mostly become smaller (physically this means a less aggressive bit/rock engagement). This observation holds true for most cutters on a PDC bit, except for few cases corresponding to cutters close to the outer edge of the bit. Since they contribute to small portion of the total weight-on-bit, the effects are ignored in the stability analysis.

Finally, the characteristic function for the decoupled axial dynamics read

$$P_a(s, \omega_o) = s^2 + \psi \sum_p \alpha_p^c \cdot [1 - \exp(-2\pi k_p s / \omega_o)] + \psi \check{\lambda} \sum_q \alpha_q^f \cdot [1 - \exp(-2\pi k_q s / \omega_o)] \quad (5.59)$$

which is equivalent to

$$P_a(s, \omega_o) = s^2 + \psi \sum_r \alpha_r \cdot [1 - \exp(-2\pi k_r s / \omega_o)] \quad (5.60)$$

where $\alpha_r = \alpha_r^c + \check{\lambda} \alpha_r^f$ and $2\pi k_r$ is the angular delay for which α^c and/or α^f exist. Equation $P_a(s, \omega_o) = 0$ can be solved using the D-subdivision method outlined in [36]; each of the resulting critical rotation speeds ω_c separates fast and slow instability regimes. In axial time scale, the critical rotation speeds ω_c are yielded by solving $P_a(s, \omega) = 0$, where

$$P_a(s, \omega) = s^2 + \sum_r \alpha_r \cdot [1 - \exp(-2\pi k_r s / \omega)] \quad (5.61)$$

Figure 5.4 – Local entries of $\{\alpha_{ji}^f\}$ and corresponding vectors at corner points for Bit A at $d_o = 3$ mm.

Chapter 6

Results

This chapter compiles results obtained with the time simulation algorithm and with the stability analysis. Results for Class I & II Bits are shown sequentially.

6.1 Symmetric and asymmetric full-blade bits

Consider the two bits with three full blades sketched in Fig. 6.1: bit A is symmetric and is thus characterized by a single angular delay so that

$$\alpha_1 = 3, \quad k_1 = \frac{1}{3} \quad (6.1)$$

while bit B is asymmetric and is described by three angular delays; the corresponding sets of numbers $\{\alpha_m\}$ and $\{k_m\}$ therefore read

$$\{\alpha_m\} = \{1, 1, 1\}, \quad \{k_m\} = \left\{ \frac{2}{9}, \frac{3}{9}, \frac{4}{9} \right\} \quad (6.2)$$

Figure 6.1 – Bit A with 3 symmetric full blades and Bit B with as 3 asymmetric full blades.

6.1.1 Critical angular velocity

The critical angular velocities at which the system loses stability can be determined explicitly for both bits, following the general procedure presented in Section 5.3. For Bit A, the general expressions (5.34) for $f(\vartheta)$ and $g(\vartheta)$ simplify to

$$f(\vartheta) = 3 \cos\left(\frac{\vartheta}{3}\right), \quad g(\vartheta) = 3 \sin\left(\frac{\vartheta}{3}\right) \quad (6.3)$$

Thus solving $g(\vartheta_z) = 0$ yields the roots

$$\vartheta_z = 3l\pi, l = 1, 2, \dots \quad (6.4)$$

The crossing position ϖ for the corresponding conjugated poles $\mathbf{s}_z = \pm j\varpi$ is then obtained by substituting (6.4) into (5.40)

$$\varpi = \begin{cases} \sqrt{6}, & \vartheta_z = 3(2l-1)\pi, \\ 0, & \vartheta_z = 6l\pi, \end{cases} \quad (6.5)$$

with $l = 1, 2, 3, \dots$. However, the poles corresponding to $\varpi = 0$ are discarded since they correspond to a vanishing critical angular velocity.

All the poles cross the imaginary axis at the same location, namely $\mathbf{s}_z = \pm j\sqrt{6}$. Furthermore, the crossing direction, determined by the sign of $p(\vartheta_z) = \cos(\vartheta_z)$ is always right going. Thus, the critical angular velocity ω_c (or equivalently the critical period of revolution in axial time scale $\mathbf{h}_c$), corresponding to $\vartheta_z = 3\pi$, represents the stability boundary.

Fig. 6.2 illustrates the procedures for determining $\mathbf{h}_c$. An intersection between $\varpi = \sqrt{6}$ and an hyperbolic curve $\varpi = \vartheta_z/\mathbf{h}$ with $\vartheta_z = 3\pi, 9\pi, 15\pi, \dots$ gives the time delays $\frac{3\pi}{\sqrt{6}}, \frac{9\pi}{\sqrt{6}}, \frac{15\pi}{\sqrt{6}}, \dots$. At each intersection, the number of unstable axial pole pairs increases by one. With n_u increasing monotonously with $\mathbf{h}$ (see the $\mathbf{h} - n_u$ plot in Figure 6.2), the axial dynamics is always unstable for $\mathbf{h} > \frac{3\pi}{\sqrt{6}}$, which corresponds to the first intersection. Hence, there is only one critical angular velocity

$$\omega_c = \sqrt{\frac{8}{3}}, \quad (6.6)$$

which is a particular case for (5.31).

Figure 6.2 – Graphical representation for determining $\mathbf{h}_c$ for Bit A. The arrow at each intersection in the top subplot indicates that the crossing direction of a pole pair is always right-going. The bottom subplot presents the $\mathbf{h} - n_u$ relationship.

Determination of the critical velocities for Bit B follows a similar process. Now

$$f(\vartheta) = \cos\left(\frac{2}{9}\vartheta\right) + \cos\left(\frac{3}{9}\vartheta\right) + \cos\left(\frac{4}{9}\vartheta\right) \quad (6.7)$$

$$g(\vartheta) = \sin\left(\frac{2}{9}\vartheta\right) + \sin\left(\frac{3}{9}\vartheta\right) + \sin\left(\frac{4}{9}\vartheta\right) \quad (6.8)$$

Solving $g(\vartheta_z) = 0$ yields

$$\vartheta_z = 3l\pi, \quad l = 1, 2, 3... \quad (6.9)$$

After substituting each ϑ_z into (5.40), the crossing positions ϖ and the corresponding ϑ_z read

$$\varpi = \begin{cases} \sqrt{5}, & \vartheta_z = 3\pi + 18(l-1)\pi, 15\pi + 18(l-1)\pi \\ \sqrt{3}, & \vartheta_z = 6\pi + 18(l-1)\pi, 12\pi + 18(l-1)\pi \\ \sqrt{2}, & \vartheta_z = 9\pi + 18(l-1)\pi \end{cases} \quad (6.10)$$

with $l = 1, 2, 3, \dots$. The case $\vartheta_z = 18l\pi$ is discarded since it corresponds to $\varpi = 0$. Using (5.37) and (5.39), the variation of unstable pole pairs Δn_u is found to depend

on ϑ_z ,

$$\Delta n_u = \begin{cases} 1, & \vartheta_z = 3\pi + 18(l-1)\pi, 15\pi + 18(l-1)\pi \\ 0, & \vartheta_z = 6\pi + 18(l-1)\pi, 12\pi + 18(l-1)\pi \\ -1, & \vartheta_z = 9\pi + 18(l-1)\pi \end{cases} \quad (6.11)$$

where $l = 1, 2, 3, \dots$. Similar to Fig. 6.2, Fig. 6.3 illustrates the process of finding $\mathbf{h}_z$. Here, the crossing direction depends on the location of the crossing, i.e. on ϖ and thus on ϑ_z , as illustrated by the red marks on Fig. 6.3. Note that ϑ_z has five distinct values with a period of 18π ; within one period, the net increase of the number of unstable axial poles is 1 pair, as indicated by the $\mathbf{h} - n_u$ plot in Fig. 6.2 (right). Thus, except for the first period of ϑ_z where there are two windows of stable axial dynamics ($n_u = 0$) corresponding to $0 < \mathbf{h} < \frac{3\pi}{\sqrt{5}}$, and $\frac{9\pi}{\sqrt{2}} < \mathbf{h} < \frac{15\pi}{\sqrt{5}}$, the dynamical system is in the fast unstable regime. The critical angular velocities for Bit B read

$$\omega_c = \left\{ \frac{2}{15}\sqrt{5}, \frac{2}{9}\sqrt{2}, \frac{2}{3}\sqrt{5} \right\} \quad (6.12)$$

This results in four intervals, two U-regimes $\omega_o \in (0, \frac{2}{15}\sqrt{5})$, $\omega_o \in (\frac{2}{9}\sqrt{2}, \frac{2}{3}\sqrt{5})$ and two S-regimes $\omega_o \in (\frac{2}{15}\sqrt{5}, \frac{2}{9}\sqrt{2})$, $\omega_o \in (\frac{2}{3}\sqrt{5}, \infty)$.

Figure 6.3 – Graphical representation for determining $\mathbf{h}_c$ for Bit B. The arrow at each intersection in the top subplot indicates the crossing direction of the critical pole. Note that the vertical dash means that the pole trajectory is tangential to the imaginary axis, and for this case particularly, the tangent point is a local minimum/maximum, thus resulting in $\Delta n_u = 0$ at those critical poles. The bottom subplot presents the $\mathbf{h} - n_u$ relation.

Figure 6.4 – Time simulation results for Bit A: (i) evolution of axial velocity (blue curve) and angular velocity (red curve) for $\omega_o = 8$ (ii) evolutions of axial velocity (blue curve) and angular velocity (red curve) for $\omega_o = 14$.

6.1.2 Time-dependent response

The critical angular velocities for Bit A and Bit B, in (6.6) and (6.12), are verified by conducting time simulations of the full dynamics. The description of the algorithm can be found in [3]. The calculations are performed for the following configuration: $\psi = 50$, $\beta = 0.3$, $\zeta = 0.5$, $\mathcal{W}_{of} = 5$ and $\mathcal{W}_{oc} = 2$. To force departure of the system from its equilibrium point, a small resisting torque $\mathcal{T}_p = 0.1$ with a duration of $\Delta\tau_p = 2$ is added to the torque-on-bit.

For Bit A, there is a single critical angular velocity $\omega_c = \sqrt{8\psi/3} \simeq 11.55$, which delimits the boundary between the S- ($\omega_o > \omega_c$) and the U-regime ($\omega_o < \omega_c$). The evolution of axial and angular velocities are shown in Fig. 6.4. It can be observed that for $\omega_o = 8 < \omega_c$, the applied small perturbation grows until axial stick-slip oscillations are observed, with no torsional stick-slip oscillations; meanwhile for $\omega_o = 14 > \omega_c$, a quasi-periodic motion is observed.

For Bit B, the critical angular velocities read $\omega_c = \{\frac{2}{15}\sqrt{5\psi}, \frac{2}{9}\sqrt{2\psi}, \frac{2}{3}\sqrt{5\psi}\} = \{2.11, 2.22, 10.54\}$, with the two S-regimes corresponding to $\omega_o > 10.54$ and $2.11 < \omega_o < 2.22$, and the two U-regimes to $0 < \omega_o < 2.11$ and $2.22 < \omega_o < 10.54$. Figure 6.5 illustrates the time responses for 4 cases: (i) $\omega_o = 1.5$, (ii) $\omega_o = 2.15$, (iii) $\omega_o = 6$, and (iv) $\omega_o = 11$. Cases (ii) and (iv) in the S-regime display an early-time quasi-periodic response, noting that the magnitude of the oscillations actually depend on the initial perturbations. On the other hand, the time-response for the U-cases (i) and (iii) are seen to converge towards a limit cycle. There are, however, differences in the early-time transient and stationary responses between these two U-cases, as can be observed in Fig. 6.5. In case (i), the generation of axial stick-slip oscillations eventually lead to a limit cycle

without torsional stick-slip, while in case (iii) the limit cycle is characterized by both axial and torsional stick-slip oscillations. The bit response in Fig. 6.5-(i') is extended for a longer duration, but the occurrence of torsional stick-slip is still not detected after 1000 bit revolutions. Phase portraits of the time responses for the cases $\omega_o = 1.5$ and $\omega_o = 6$ are shown in Fig. 6.6.

Figure 6.5 – Time simulation results for Bit B: (i) and (i') show the early and developed evolution of velocities for $\omega_o = 1.5$ (U-regime). A steady-state limit cycle can be observed with axial stick-slips but no torsional stick-slip. For $\omega_o = 2.15$ (S-regime), (ii) shows a quasi-periodic response. (iii) and (iii') show the early and developed evolutions of velocities for $\omega_o = 6$ (U-regime). A steady-state limit cycle corresponding to both axial and torsional stick-slip is observed. (iv) Corresponds to $\omega_o = 11$, similar to (ii), a quasi-periodic response is observed (S-regime).

To this point, viewing the various responses of bits with different geometry under identical configuration, it is evident that the bit design affects the stability of the bit.

Figure 6.6 – Limit cycles for the time responses: (i) $\omega_o = 1.5$, limit cycle without torsional stick-slip ; (ii) $\omega_o = 6$, limit cycle with stick-slip.

The torsional resonance frequency $f = 1/2\pi$ (scaled by t_ω) can be clearly observed in both the angular and axial limit cycles. While the angular oscillations are virtually monochromatic, the higher frequency axial oscillations are modulated by $f = 1/2\pi$. This is verified by a frequency spectrum analysis of $v(\tau)$, once the limit cycle is reached, conducted for various angular velocities in the U-regime (see Fig. 6.7). This figure shows that the uncoupled axial dynamics features, in all cases, a single unstable pole denoted as s_a such that $\sqrt{2\psi}/2\pi < \Im(s_a) < \sqrt{5\psi}/2\pi$. It can be confirmed by an examination of Fig. 6.7 that the dominant axial frequencies, besides $1/2\pi$, are indeed related to $\Im(s_a)$. The saturation condition due to unilateral contact, which is responsible for stabilizing the unstable system to a limit cycle, results in the addition of multiple harmonics around the dominant component of the axial dynamics.

6.2 Fishtail bits

Here we construct the stability maps for fishtail bits, which consist of full blades with two distinct angular offsets (Fig. 6.8). Define the number of blades as 2ν ; thus the summation of the two distinct angular offsets is $2\pi/\nu$. Define one of the offsets as $2\pi\chi/\nu$, then the other one is $2\pi(1-\chi)/\nu$. Due to symmetry it is required that $0 \leq \frac{2\pi\chi}{\nu} \leq \frac{\pi}{\nu}$; thus $\chi \in [0, 1/2]$. Hence, this class of bits is completely described by ν and χ , and the two set of numbers $\{k_m\} = \{\chi/\nu, (1-\chi)/\nu\}$ and $\{\alpha_m\} = \{\nu, \nu\}$.

The case $\chi = 0$ depicts a bit with ν symmetric blades ($\nu \geq 2$), while $\chi = 1$ corresponds to a bit with 2ν symmetric blades.

Figure 6.7 – Frequency spectra of $v(\tau)$ within axial limit cycles and the root-locus plots for several ω_o in the U-regime (the red circles represent the poles corresponding to a given ω_o). The resonance frequency $f = 1/2\pi$ can clearly be observed. In all cases, there is a single unstable pole s_a and the high frequency components of $v(\tau)$ are related to $\Im(s_a)$.

Figure 6.8 – General fishtail bits under consideration: each bit can be characterized by two set of numbers $\{k_m\} = \{\chi/\nu, (1 - \chi)/\nu\}$ and $\{\alpha_m\} = \{\nu, \nu\}$. Note that the range of χ reads $\chi \in [0, \frac{1}{2\nu}]$.

6.2.1 Critical frequencies

For these bits, expressions (5.34) and (5.38) for functions $f(\vartheta)$, $g(\vartheta)$ and $p(\vartheta)$ simplify to

$$g(\vartheta) = 2\nu \sin\left(\frac{1}{2\nu}\vartheta\right) \cos\left(\frac{2\chi - 1}{2\nu}\vartheta\right) \quad (6.13)$$

$$f(\vartheta) = 2\nu \cos\left(\frac{1}{2\nu}\vartheta\right) \cos\left(\frac{2\chi - 1}{2\nu}\vartheta\right) \quad (6.14)$$

$$p(\vartheta) = \chi \cos\left(\frac{\chi}{\nu}\vartheta\right) + (1 - \chi) \cos\left(\frac{1 - \chi}{\nu}\vartheta\right) \quad (6.15)$$

Solving $g(\vartheta_z) = 0$ for ϑ_z , and compute ϖ using (5.40) yields two groups of solutions:

$$\begin{cases} \vartheta_z(m, \nu, \chi) = 2m\nu\pi \\ \varpi(m, \nu, \chi) = \sqrt{2\nu - 2\nu \cos(2m\pi\chi)} \end{cases} \quad m = 1, 2, 3, \dots \quad (6.16)$$

$$\begin{cases} \vartheta_z(m, \nu, \chi) = \frac{(1-2m)\nu\pi}{2\chi-1} \\ \varpi(m, \nu, \chi) = \sqrt{2\nu} \end{cases} \quad m = 1, 2, 3, \dots \quad (6.17)$$

The values of $\omega = 2\pi/h$ corresponding to pole-crossing read

$$\omega(m, \nu, \chi) = \begin{cases} \omega_1 = \frac{\sqrt{2-2\cos(2m\pi\chi)}}{m\sqrt{\nu}} \\ \omega_2 = \frac{2\sqrt{2}(2\chi-1)}{(-2m+1)\sqrt{\nu}} \end{cases} \quad m = 1, 2, 3, \dots \quad (6.18)$$

Combining (6.15) and (6.18) yields sub-regions in the $\omega_o - \chi$ quarter-plane ($\omega > 0$, $\chi > 0$), whose boundaries are defined by the intersecting curves in (6.18); also the boundary curves give information on the pole crossing direction according to the sign of (6.15).

Such sub-regions are shown in Fig. 6.9a for a fishtail bit with 4 blades ($\nu = 2$). It is shown that $\omega_1(m, \nu, \chi)$ is periodic in χ , with a period of $1/m$; each period of $\omega_1(m, \nu, \chi)$ is symmetric with respect to χ ; meanwhile, $\omega_2(m, \nu, \chi)$ is linear with respect to χ , with slopes varying with m . In Fig. 6.9a, the segments in blue corresponds to the generation of a pair of conjugate unstable poles, while the red segments relate to the vanishing of a pair of conjugate unstable poles.

6.2.2 Stability map for fishtail bits

The final stability boundaries for the fishtail bits with 4 blades ($\nu = 2$) are shown in Fig. 6.9b. The map also indicate the regions corresponding to the S- and the U-regime. Similar to the bit with 3 asymmetric blades, S-regions can be reached with relatively smaller nominal rotation speeds. The $h - n_u$ relationship for a fishtail bit characterized by $\nu = 2$ and a given χ can be deduced from this map and thus the critical rotation speed(s) ω_c . From (6.18), it is also noted that $\omega_c \propto \frac{1}{\sqrt{\nu}}$, so the stability map for $\nu = 2$ can be interpreted as the stability map for fishtail bits with a different ν , through a simple transformation.

A few salient features can be observed from the map in Fig. 6.9b. The boundaries with $\omega_c > \sqrt{2/\nu}$ are simply defined by $\omega_1(1, \nu, \chi)$ and $\omega_2(1, \nu, \chi)$. The points $(\sqrt{4/\nu}, 1/2)$ and $(\sqrt{8/\nu}, 0)$ correspond to symmetric bits with 2ν and ν full blades, respectively. The point $(\sqrt{2/\nu}, 1/4)$ corresponds to $\Delta n_u = 0$, which implies that two S-regions are connected at this point, and so are two U-regions. Note that there is a triangle-like S-region bounded by $\omega_1(1, \nu, \chi)$, $\omega_1(2, \nu, \chi)$, $\omega_2(1, \nu, \chi)$ and $\omega_2(2, \nu, \chi)$. This is a large “sweet-spot” region where axial stability is possible for smaller nominal angular velocity. If one evaluates the stability of the bits using the maximum ω_c (noting that smaller ω_c corresponds to a better design in regard to stability), this S-region is also interesting since it includes the point $(\sqrt{1/\nu}, 1/4)$, which corresponds the minimum $\max(\omega_c)$ for any χ . Also, the intersection between $\omega_c = \sqrt{4/\nu}$ and $\omega_2(1, \nu, \chi)$ gives $\chi_{\text{opt}} \simeq 0.1464$, meaning that with $\chi \in (\chi_{\text{opt}}, 1/2)$, the stability of a fishtail bit is better than its symmetric counterpart.

In theory, there also exist infinitely many S-regions with smaller size, as shown in Fig. 6.9c. Particularly, there are two groups of such regions: one is bounded by

$$\omega_1(p, \nu, \chi), \omega_2(p+1, \nu, \chi), \omega_1(p+1, \nu, \chi), \omega_2(p, \nu, \chi), p = 1, 2, 3, \dots \quad (6.19)$$

while the other is bounded by

$$\omega_2(1, \nu, \chi), \omega_1(q, \nu, \chi), \omega_1(q+1, \nu, \chi), \omega_1(q+2, \nu, \chi), q = 2, 4, 6, \dots \quad (6.20)$$

As $\chi \rightarrow 0$, the boundaries of a region defined by (6.19) converge to $\omega_1(1, \nu, \chi)$; as $\chi \rightarrow \frac{1}{2}$, the boundaries of regions defined by (6.20) converge to $\omega_2(1, \nu, \chi)$.

Figure 6.9 – Stability map for fishtail bits with 4 blades ($\nu = 2$): a. the intersecting $\omega(m, \nu, \chi)$ curves and the critical pole-crossing direction (blue: right-going as $\mathbf{h}$ increases or ω decreases; red: left-going as $\mathbf{h}$ decreases or ω increases). b. The final stability map and some highlighted features. c. Determinations of the infinitely many smaller S-regions.

6.2.3 Frequency of the decoupled axial oscillations

Numerical simulation is again carried out to verify the stability map shown in Fig. 6.9b. The expected responses are confirmed for the various S- and U-regimes, which are not included here for the sake of brevity. The frequency spectrum of the axial response in the U-regimes is shown in Fig. (6.10), where a resonance frequency at $f = 1/2\pi$ (scaled by t_ω) is observed. As determined above, there exists a minimum

$\max(\omega_c)$ for the fishtail bits that corresponds to the point $(\omega_c, \chi) = \left(\sqrt{\frac{1}{\nu}}, \frac{1}{4}\right)$ in the stability map; for this particular case, the frequency spectra of the axial stick-slip response should distribute in the vicinity of $f = \sqrt{2\nu\psi}/2\pi \propto \sqrt{\nu}$. Considering an initial nominal motion $\omega_o = 0.9\sqrt{\frac{\psi}{\nu}}$ (with $\omega_c = \sqrt{\frac{\psi}{\nu}}$), the frequency spectra of $v(\tau)$ for various values of ν are shown in Fig. 6.10. For different ν , the poles travel in the complex plane in the same manner; evidently the relative position is constant between $f = \sqrt{2\nu\psi}/2\pi$ and the center of the high frequency lobe. Therefore, scaling the frequency spectra with $\sqrt{\nu}$ results in an overlapping in frequency distribution, although the magnitudes of the frequency components are different.

Figure 6.10 – Time simulations are conducted for fishtail bits with $\chi = \frac{1}{4}$. For different bits, the same configuration ($\psi = 50$, $\beta = 0.3$, $\zeta = 0.5$, $\mathcal{W}_{fo} = 5$, $\mathcal{W}_{co} = 2$) is considered together with a nominal rotation speed $0.9\sqrt{\frac{\psi}{\nu}}$. Once a steady-state limit cycle is reached, a frequency spectrum is generated from the axial motion for each bit. For various ν the frequency spectra are shown in subplot (i). It is verified in (ii) that the dominant higher frequency of the axial oscillations is proportional to $\sqrt{\nu}$.

6.3 Bits with partial blades

Here we analyze the stability of a class of symmetric bits characterized by the same number of full and partial blades. The two kind of blades are arranged alternately around the bit axis as illustrated in Fig. 6.11. The bit design is thus described by

two dimensionless parameters: ξ ($0 < \xi < 1$), the number defining the ratio of the inner radius of the partial blades over the bit radius a and $v > 1$ the number of full or partial blades. Such a bit can also be seen as a combination of two symmetric bits: a bit of radius ξa consisting of v blades, and a core barrel made of $2v$ identical blades with inner radius $a\xi$ and outer radius a . This class of bits is thus characterized by two sets $\{k_m\} = \{1/v, 1/2v\}$ and $\{\alpha_m\} = \{v\xi, 2v(1-\xi)\}$, with $\xi \in [0, 1]$. The limits $\xi = 0$ and $\xi = 1$ corresponds to symmetric bits with $2v$ and v full blades, respectively.

Figure 6.11 – Bits with v full blades and v partial blades: each bit can also be viewed as a combination of two symmetric bits. The blades inside inner circle of radius of $a\xi$ corresponds to a symmetric bit with v blades; The blades outside the circle of radius of $a\xi$ to a symmetric core barrel with $2v$ blades.

6.3.1 Critical frequencies

For this class of bits, the functions $f(\vartheta)$, $g(\vartheta)$ and $p(\vartheta)$, defined in (5.34) and (5.38), become

$$f(\vartheta) = v\xi \cos\left(\frac{\vartheta}{v}\right) + 2v(1-\xi) \cos\left(\frac{\vartheta}{2v}\right) \quad (6.21)$$

$$g(\vartheta) = \sin\left(\frac{\vartheta}{2v}\right) \left[2v\xi \cos\left(\frac{\vartheta}{2v}\right) + 2v(1-\xi) \right] \quad (6.22)$$

$$p(\vartheta) = \xi \cos\left(\frac{\vartheta}{v}\right) + (1-\xi) \cos\left(\frac{\vartheta}{2v}\right) \quad (6.23)$$

As done previously, $\mathbf{h}$, $\boldsymbol{\varpi}$ and Δn_u can be determined in closed-form. There are 3 groups of solutions:

- $\mathcal{G}_1$ with $0 \leq \xi < \frac{1}{2}$,

$$\frac{\mathbf{h}}{\sqrt{v}} = \frac{(2m-1)\pi}{\sqrt{(1-\xi)}}, \quad m = 1, 2, 3, \dots, \quad \frac{\boldsymbol{\varpi}}{\sqrt{v}} = 2\sqrt{(1-\xi)}, \quad -p(\vartheta_z) = 1-2\xi > 0$$

- $\mathcal{G}_2$ with $0 \leq \xi < \frac{1}{2}$,

$$\frac{\mathbf{h}}{\sqrt{v}} = \frac{2\varrho}{\sqrt{2}} \text{ or } \frac{\mathbf{h}}{\sqrt{v}} = \frac{\pm 2\varrho + 4m\pi}{\sqrt{2}}, \quad m = 1, 2, 3, \dots, \quad \frac{\varpi}{\sqrt{v}} = \sqrt{2v}, \quad -p(\vartheta_z) = 2 - \frac{1}{\xi} \geq 0$$

with $\varrho \in [0, \pi]$ defined as $\varrho = \arccos((\xi - 1)/\xi)$

- $\mathcal{G}_3$ with $\frac{1}{2} \leq \xi \leq 1$,

$$\frac{\mathbf{h}}{\sqrt{v}} = \frac{(2m-1)\pi}{\sqrt{(1-\xi)}}, \quad m = 1, 2, 3, \dots, \quad \frac{\varpi}{\sqrt{v}} = 2\sqrt{(1-\xi)}, \quad -p(\vartheta_z) = 1 - 2\xi \leq 0$$

The third group of solutions is similar to the first group $\mathcal{G}_1$, but the crossing directions is opposite. Using (5.39), one can create the $\mathbf{h} - n_u$ plot and determine the critical angular velocity ω_c . Similar to fishtail bits, it is observed that given a constant value of ξ , symmetric bits with an equal number ν of partial and full blades satisfy the following

$$\omega \propto \frac{1}{\sqrt{v}} \quad \varpi \propto \sqrt{v} \quad (6.24)$$

In other words, stability maps in terms of $\omega_c - \xi$ for bits with different v maintains a similar shape.

6.3.2 Stability map

The boundary curves for the stability map given by

$$\omega_c(\nu, \xi) = \begin{cases} \omega_1 = 2\sqrt{\frac{1-\xi}{v}}, & \xi \in (0, \frac{1}{2}) \\ \omega_2 = 2\sqrt{\frac{1-\xi}{v}}, & \xi \in (\frac{1}{2}, \xi_c) \\ \omega_3 = \frac{\pi}{\arccos(\frac{\xi-1}{\xi})} \sqrt{\frac{2}{v}}, & \xi \in (\frac{1}{2}, \xi_c) \\ \omega_4 = \frac{\pi}{-\arccos(\frac{\xi-1}{\xi}) + 2\pi} \sqrt{\frac{2}{v}}, & \xi \in (\frac{1}{2}, 1) \end{cases} \quad (6.25)$$

where $\xi_c \simeq 0.7395$ is the root of

$$2\sqrt{1-\xi_c} = \frac{\pi\sqrt{2}}{\arccos\left(\frac{\xi_c-1}{\xi_c}\right)}, \quad \frac{1}{2} < \xi_c < 1 \quad (6.26)$$

As an example, the stability map $\omega_c - \xi$ for the class of bits characterized by $v = 3$ is shown in Fig. 6.12. Similar to Fig. 6.9, the map indicates that for certain ξ , there

Figure 6.13 – Time simulation results for case U_I , $\omega_o = 0.6241$: (i) (ii) show the early and developed evolution of velocities, (iii) shows a steady-state limit cycle with torsional stick-slip

exist multiple S- and U-regions. Some special points can also be recognized in the stability map. Points $(\sqrt{8/v}, 1)$ and $(\sqrt{4/v}, 0)$ are related to symmetric bits with v and $2v$ full blades. The intersections between curves $\omega_2(v, \xi)$ and $\omega_3(v, \xi)$ read $(2\sqrt{(1 - \xi_c)v}, \xi_c)$ and $(\sqrt{2/v}, 1/2)$. Note that at $\xi = \frac{1}{2}$, there is only one critical angular velocity $\sqrt{2/v}$. Increasing the relative length of the partial blades from 0 to $a/2$ (decreasing ξ from 1 to $1/2$), the stability improves progressively. As the partial blades grows from $a/2$ to a (decreasing ξ from $1/2$ to 0), the stability deteriorates, although it is still better compared to a symmetric bit with $2v$ full blades; in fact, the stability of the bits with partial blades is better, given that $0 < \xi < \xi_{\text{opt}}$, where $\xi_{\text{opt}} \simeq 0.6230$. It is also observed in the map that for $\frac{1}{2} < \xi < \xi_c$, there exist multiple S- and U-regions, including the S-regimes possible for relatively smaller ω_o .

Figure 6.12 – $\omega_c - \xi$ plot for bits with 3 symmetric full blades and 3 symmetric partial blades ($v = 3$)

Figure 6.14 – Time simulation results for case U_{II} , $\omega_o = 0.7503$: (i) (ii) show the early and developed evolution of velocities, (iii) shows a steady-state limit cycle with no torsional stick-slip (but with axial stick-slip)

Figure 6.15 – Time simulation results: (i) case S_I , $\omega_o = 0.6841$ and (ii) S_{II} , $\omega_o = 1.3150$. For each case, a quasi-periodic response is observed.

Parameter	Symbol	Value	Unit
Drill pipe torsional stiffness	C	470	N·m
Drill pipe torsional inertia	I	103.8	kg·m ²
Drill pipe axial inertia	M	31350	kg
Rock specific strength	ϵ	60	MPa
Cutting force inclination	ζ	0.7	
Bit-rock friction coefficient	μ	0.6	
Bit-rock contact stress	σ	60	MPa

Table 6.1 – List of some parameters considered in the model

6.3.3 Verification with time simulations

Time simulations are performed with a similar blade configuration. As an external perturbation to the nominal bit motion, a small resisting torque $\mathcal{T}_p = 0.3$ with a duration of $\Delta\tau_p = 2$ is added to the torque-on-bit. We choose to show some results for the bit with $\xi = 0.6$ and $\nu = 3$. From Figure. 6.12 and (6.25), there exist three critical angular velocities

$$\begin{aligned}\omega_c(v = 3, \xi = 0.6) &= \{\omega_3, \omega_2, \omega_4\} \\ &\simeq \{0.6441, 0.7303, 1.1150\}\end{aligned}\tag{6.27}$$

Thus there are two U-regimes corresponding to $0 < \omega_o < 0.6441$, $0.7303 < \omega_o < 1.1150$ and two S-regimes corresponding to $0.6441 < \omega_o < 0.7303$, $\omega_o > 1.1150$.

Figures 6.13-6.15 illustrates the time responses for 4 cases: $\omega_o = 0.6241$ (U_I), $\omega_o = 0.7503$ (U_{II}), $\omega_o = 0.6841$ (S_I) and $\omega_o = 1.3150$ (S_{II}). The time responses for the U-cases both converge to a limit cycle, but differences are observed in the early and more developed responses. In case U_I , the occurrence of axial stick-slip leads to a limit cycle corresponding to both axial and torsional stick-slip oscillations, while in case U_{II} , the limit cycle does not correspond to the torsional stick-slip. For the S-cases, a quasi-periodic motion is observed upon the application of external perturbation.

6.4 Results for Bit A (realistic bit)

This Section presents results achieved with the presented methods for Bit A. Fig. 6.16 presents the top view of the bit which is composed of 32 round cutters placed on 3 full blades and 3 partial blades. The wearflat width per cutter is assumed as $w = 0.46$ mm. The other parameters are listed in Table 6.1.

Figure 6.16 – Top view of Bit A (looking toward $-Z$ direction)

6.4.1 Geometric properties

Here some geometric properties of Bit A are inspected. These properties reflect the cutter layout information that is imbedded in the bit/rock interaction under nominal motion. The nominal motion is characterized by a constant bit penetration per revolution d_o . Fig. 6.17 considers Bit A given d_o within the range of $[0, 10 \text{ mm}]$.

Figure 6.17(a) and (b) plot the errors

$$e_1 = \frac{A/a - d_o}{d_o}, \quad e_2 = \frac{A\rho_c - \frac{1}{2}a^2d_o}{\frac{1}{2}a^2d_o} \quad (6.28)$$

with respect to d_o . For Bit A, $|e_1|$ and $|e_2|$ are around $1\% \sim 2\%$ for the array of d_o considered. In the light of the earlier discussion on (5.47), it is verified that A/a is slightly smaller than d_o due to the leftover rock at the rim of the bit. Thus the following identity holds with a small error

$$\frac{A}{a} \simeq d_o, \quad A\rho_c \simeq \frac{1}{2}a^2d_o \quad (6.29)$$

where A is the total area of all the local engagement surfaces $\mathcal{S}_i$, a is the bit radius and ρ_c corresponds to the centroid of the full collection of all $\mathcal{S}_i$.

In Fig. 6.17(c), the quantity η is defined by

$$\eta = \frac{b}{a} \quad (6.30)$$

where b is the total length of all the radial projections of $\mathcal{L}_i$ on r -axis. The number η can be related to $\{\alpha_p^c\}$ in the stability analysis via (5.48). This quantity can be interpreted as the effective number of full blades on the bit, which is in proportion to $W_{o,f}$ assuming a constant wearflat width. From (6.16), if all blades are treated as continuous blades while assuming the boundary between the inner and outer rings is at $r = a/2$, we have $\eta' = 4.5$. The actual η value not only varies with d_o but also are smaller than the estimate, which illustrates some limitation of the assumption of continuous blades.

In Fig. 6.17(d), the quantity plotted reads

$$L\rho_f \quad (6.31)$$

where L is the total arc length of all Γ_i and ρ_f is related to the centroid of the collection. Readily $T_{o,f}$ is proportional to $L\rho_f$.

Fig. 6.17(e) and (f) present two quantities

$$\kappa_W = \frac{A}{wb}, \quad \kappa_T = \frac{A\rho_c}{wL\rho_f} \quad (6.32)$$

where κ_W and κ_T can be related to the ratios W_c/W_f and T_c/T_f respectively through

$$\frac{W_c}{W_f} = \kappa_W \frac{\epsilon\zeta}{\sigma}, \quad \frac{T_c}{T_f} = \kappa_T \frac{\epsilon}{\mu\sigma} \quad (6.33)$$

The similar trends of $\kappa_W(d_o)$ and $\kappa_T(d_o)$ indicates that as d_o increases, the cutting component is taking an increasingly larger portion of the applied weight-on-bit W_o and torque-on-bit T_o . In [3], the effect of the ratios (6.33) in the model was mentioned: as more energy is diverted to the cutting process, the regenerative cutting effect is becoming more dominant, rendering an earlier occurrence of the axial-torsional stick (for fast instability regimes). This effect has not been consistently verified with “real” bits, as the frictional component of the drilling force is also subject to the regenerative drilling process. With the same model configuration and identical W_o , a larger κ_W corresponds to a more aggressive bit/rock interaction characterized by a larger depth of cut.

Fig. 6.17(g) shows the $d_o - \gamma$ relation. It is verified with “real” bits that configurations

Figure 6.17 – Geometric properties of Bit A

with $\beta > 1$ can either delay or even eliminate the occurrence of torsional stick-slip vibrations. Therefore a larger γ value means it is more likely for axial stick-slips to damp the torsional vibrations.

Finally Fig. 6.17(h) shows the average depth of cut per cutter which is defined by

$$\bar{d} = \frac{A}{a\eta} \quad (6.34)$$

6.4.2 Stability map for the decoupled axial dynamics

Following the formulation in section 5.4.2, stability analysis is performed for Bit A. First, the number sets $\{k_r\}$ and $\{\alpha_r\}$ are assembled for each entry of the d_o array. Typically, the number of k_r (angular delay) for a PDC bit is in the order of $O(10 \sim 100)$.

Fig. 6.18 shows the stability map of the decoupled axial dynamics for Bit A. Fig. 6.18(a) is the base case where the wearflat width w takes a realistic value of 0.4572 mm, while (b) assumes that there is no variation of W_f and (c) doubles the w value which exacerbates the effect of frictional component imposed on the decoupled axial

dynamics.

For a given d_o , there always exist an odd number of ω_c , resulting in a guaranteed U -regime for small ω and an S -regime for large ω . Notably, the boundary in (b) appears to be more smooth as the cutting components of $\{\alpha_r\}$ vary gradually with respect to d_o . In contrast, the frictional components of $\{\alpha_r\}$ vary in a more volatile manner. On top of the stability maps, two instances of ω are plotted. The critical angular velocity $\omega = \sqrt{8}$ corresponds to a bit with a single blade (i.e. there is only one angular delay 2π). This can be understood with the theoretical bit in Fig.5.2: for a small depth of cut there is only one cutter cutting rock. Meanwhile if Bit A is treated as a bit with continuous full/partial blades that are evenly spaced, with

$$\{k\} = \left\{ \frac{1}{6}, \frac{1}{3} \right\}, \quad \{\alpha\} = \left\{ 3, \frac{3}{2} \right\} \quad (6.35)$$

which results in $\omega_c = \sqrt{\frac{2}{3}}$ [36]. Interestingly for $d_o \approx 7$ mm, the stability analysis of the “real” bit gives a ω_c that seemingly agrees with the value $\sqrt{\frac{2}{3}}$. This agreement should be viewed with caution. For larger d_o , similar asymptotes of $\omega_c = 0.5 \sim 1$ can be observed with some other PDC bits equipped with round cutters. The explanation remains unclear within the limitation of this work.

It must be pointed out that the stability map alone is insufficient in explaining the common practice of mitigating stick-slip drillstring vibrations through reasonably reducing the depth of cut. The stability map in Fig. 6.18 clearly shows that the stable region for the decoupled axial motion increases with growing depth of cut d_o . This disagreement may be caused by the simple lumped drillstring model. Nonetheless, the work reported in this thesis is self-contained and can be viewed as an extreme case of a more accurate model.

6.4.3 Time simulation verification

In the following, we examine the stability map with time simulations achieved from the time simulation code outlined in a previous chapter.

6.4.3.1 Verification of the stability map

The stability map for the decoupled axial dynamics has been verified. The following presents the verification of the stability map in Fig. 6.18(a) at $d_o = 2.5$ mm. For this

Figure 6.18 – Stability map for Bit A (shaded area represents U -regime in contrast to the S -regime): (a) wearflat width $w = 0.46$ mm (b) with no variation of W_f (c) wearflat width $w = 0.92$ mm (d) stacked stability boundaries

case, the number sets $\{\alpha_r\}$ and $\{k_r\}$ of the characteristic function (5.60) or (5.61) are shown in Fig. (6.19), which leads to the corresponding array of critical rotation speeds ω_c

$$\omega_c = \{1.918, 1.237, 0.588, 0.512, 0.483\} \quad (6.36)$$

In the top subplot of Fig. 6.20, the black curve represents the stability boundaries that separate various U - and S -regimes; the green dashed line corresponds to the boundary

Figure 6.19 – Number sets $\{\alpha_r\}$ and $\{k_r\}$ for the characteristic function P_a of Bit A at $d_o = 2.5$ mm

corresponding to invariant W_f . Cases corresponding to various ω are investigated in detail.

Cases with $\omega = \{2.4, 1.15, 0.8, 0.495\}$ correspond to the S -regimes where quasi-periodic responses are observed for the axial and torsional dynamics, see Fig. 6.20(a)(c)(d)(f). Note that cases (c) and (f) correspond to U -regimes if the variation of W_f is ignored. The early response of axial velocity v in (f) sees a high frequency component that gradually disappears as a quasi axial limit cycle develops. Due to absence of any axial pole, the dynamics eventually settles in a quasi-periodic motion as expected. Cases with $\omega = \{1.6, 0.55, 0.4\}$ are related to the U -regimes where axial velocity with some high frequency oscillatory components quickly evolves into axial stick-slips, see Fig. 6.20(b)(e)(g). Case (e) corresponds to an S -regime if the variation of W_f is neglected. A common feature observed in the torsional dynamics for U -regimes is that the angular velocity profile exhibits a high-frequency oscillatory components only after an axial limit cycle corresponding to axial stick-slips has been reached.

Nonetheless, for both regimes of instability, an eventual limit cycle should be reached that correspond to a torsional stick-slip. Such a torsional limit cycle can be reached more rapidly with stick-slip vibrations in the axial mode.

6.4.3.2 Transient axial response

A closer look is taken for the transient axial response before the initiation of the axial stick-slip phase. It is claimed that this transient behavior is influenced by the positions of the unstable axial poles s in regard to (5.61).

It is stated in [36] that such a characteristic equation results in many pole trajectories evolving in the complex plane as ω (or equivalently h , the nominal bit revolution period in axial time scale t_v) changes. The work [92] enables the computation for the pole coordinates in the complex plane given a fixed ω . Accordingly, Fig. 6.22 show the pole (the one with positive imaginary part in the conjugate pair) trajectory evolutions for the system under consideration with decreasing ω . The red curves represent the pole trajectories appearing within the current ω interval, while the blue curves represent the counterparts in all previous ω intervals.

Subplots (a)(b)(e)(f)(h) show the crossing of a certain pole trajectory that crosses the imaginary axis, which correspond to ω_c . Alternatively, the crossing (finite in the

Figure 6.20 – Verification of the stability map of Bit A. Top: stability boundaries of Bit A around $d_o = 2.5$ mm and the cases being showcased: (a) $\omega = 2.4$, (b1)-(b2) $\omega = 1.6$, (c) $\omega = 1.15$, (d) $\omega = 0.8$

Figure 6.21 – Verification of the stability map of Bit A (continued): (e1)-(e2) $\omega = 0.55$, (f1)-(f2) $\omega = 0.495$ and (g1)-(g2) $\omega = 0.4$

amount) and the corresponding ω_c can be solved using the D-subdivision methods outlined in [36] once the sets $\{\alpha_r\}$ and $\{k_r\}$ are determined. Subplots (c)(d)(g) captures the generation of a new pole branch. Note that number of trajectories increases as ω decreases from a relatively large value. The pole branch responsible for the minimum $\omega_c = 0.483$ provides the the unstable poles for $0.163 < \omega < 0.483$ until it is taken over by another trajectory at $\omega = 0.163$, shown in subplot (i), which is not a critical rotation speed as there is no elimination of unstable axial poles. Finally, subplot (j) shows the all the existing trajectories at $\omega = 0.01$, where many more unstable axial poles come to place. Note that all plots in Fig. 6.22 are composed of scattered points with respect to an ω array with an increment of 0.01. It seems that the propagation speed of $\mathbf{s}(\omega)$ depends on the value of ω , but this is irrelevant to this study.

Fig. 6.23 simply presents the results in Fig. 6.22 in a more pleasant manner, where the evolutions of the real part of rightmost pole and the number of unstable pole pairs n_u are respectively shown. In retrospect of the time simulation results in Section (6.4.3.1), cases (b)(e)(g) in Fig. 6.20 and Fig. 6.21 all have a single unstable axial pole, see Fig. 6.23(b). From 6.23(b),

$$\Re(\mathbf{s}_g) > \Re(\mathbf{s}_b) > \Re(\mathbf{s}_e) \quad (6.37)$$

where

$$\Re(\mathbf{s}_g) = 0.0983, \quad \Re(\mathbf{s}_b) = 0.0240, \quad \Re(\mathbf{s}_e) = 0.0087 \quad (6.38)$$

Fig. 6.24 presents the local minima/maxima of the axial velocity perturbation $\dot{u}$. Indeed, case with larger $\Re(\mathbf{s})$ value sees a more rapid growth in the size of the $\dot{u}$ envelope. It should be pointed out that the event time for the first occurrence of axial stick is a result of the interplay between the just mentioned axial pole location and the v_o value. This can be shown with cases (b) and (e), the first axial stick is reached slightly earlier for case (e) due to a smaller v_o .

6.4.4 Parameter β and the torsional response

It was shown that the bit design factor γ , which depicts the wearflat distribution [21], is larger than the assumed value 1 in Richard et al. [3]. Accordingly, when taking larger values for μ and ζ in contrast to the base case in Table 6.1, the resulting β value can surpass 1. Fig. 6.25 compares time responses of Bit A for $\beta < 1$ (base

Figure 6.22 – Evolution of pole trajectories for Bit A at $d_o = 2.5$ mm. Snapshots of the trajectories are taken with decreasing ω (or increasing h , nominal bit revolution period in axial time scale t_v): (a) $\omega = 1.9180$ (b) $\omega = 1.2370$ (c) $\omega = 1.0150$ (d) $\omega = 0.6700$ (e) $\omega = 0.5880$ (f) $\omega = 0.5210$ (g) $\omega = 0.4610$ (h) $\omega = 0.4830$ (i) $\omega = 0.1630$ (j) $\omega = 0.0100$. The red curves represent the pole trajectories appearing within the current ω interval, while the blue curves represent the counterparts in all previous ω intervals. In (c)(d)(g), the generation of a new branch is highlighted in a green circle.

Figure 6.23 – For Bit A at $d_o = 2.5$ mm, with respect to ω , (a) the evolution of the real part of rightmost pole (b) the evolution of the number of unstable pole pairs n_u

Figure 6.24 – Influence of pole location on the transient response of the coupled axial dynamics

Figure 6.25 – Comparison of time responses of Bit A ($d_o = 2.5$ mm, $\omega = 0.319$): (a) $\beta < 1$ and (b) $\beta > 1$

case) and $\beta > 1$ ($\mu = 1$ and $\zeta = 0.75$)¹. The very similarity in the early responses is governed by the identical pole positions of the decoupled axial dynamics, but the further developments are evidently influenced by the parameter β . Even the decoupled axial dynamics is subject to the fast instability regime, the perturbations in torsional velocity is contained for $\beta > 1$.

6.4.5 Parameter κ_W and the torsional response

It was established in Richard et al. [3] that given the same configuration, a larger κ_W , defined in 6.32, results in an earlier onset of axial-torsional stick-slip regime, if it is achievable. Figure 6.26 presents a detailed parametric study showcasing this trend. One interesting regime is the so-called “late axial-torsional stick slips” where the decoupled axial dynamics is unstable (U -regime) with the torsional motion evolving rather slowly (up to hundreds of bit revolutions) towards torsional stick slips; this regime is drastically different from its fast counterpart. Indeed, a larger initial ω_o may

¹The parameters γ and accordingly β are subject to the dynamic bit rock interactions. Here β is evaluated with the nominal responses.

Figure 6.26 – Parametric study of a reference RGD model ($n = 6$) with various magnitudes of external perturbation, parameters chosen follow Figure 5 of Richard et al. [3]. The bit responses are achieved by applying an external perturbing torque to a prescribed nominal motion, the duration for applying $\mathcal{T}_p$ is $\Delta\tau_p = 1$.

be a contributing factor for later onset of axial-torsional stick slips, it is also evident that the regime is vanishing with increasing δ_o value, equivalent to larger κ_W as the frictional WOB is constant in the RGD model. A possible explanation is that a larger cutting component in the bit/rock interaction exacerbates the regenerative process.

Following this, it can be speculated that for realistic bits, κ_W may exhibit a similar effect on torsional bit response so long as the variation of W_f is more contained in comparison to that of W_c . Although for realistic bits, $\max W_f$ is also subjective to the regenerative bit/rock interaction, such a condition often holds. Figure 6.27 compares responses of Bit A at different κ_W values. Indeed a larger d_o value, corresponding to a larger κ_W , results in an earlier onset of axial-torsional stick.

6.5 Evaluation of stability properties of existing designs

Two bits with round cutters are compared: bit 4 and bit 20. The cutter layout is presented in Fig. 6.28 (view in the direction opposite to bit axial advance). For Bit 4 there are 3 blades in the inner ring and 5 blades in the outer ring. For Bit 20 there are 3 blades in the inner ring and 7 blades in the outer ring. Fig. 6.29 shows the same two bits equipped backup cutters.

6.5.1 Comparison of nominal responses

Here we inspect some geometric properties of the bit. In the presented results, we look at d_o within the range of $[0, 10 \text{ mm}]$. Fig. 6.30 compares some geometric quantities

Figure 6.27 – Comparison of time responses of Bit A ($\omega = 0.319$): (a) $d_o = 2.5$ mm and (b) $d_o = 3$ mm

Figure 6.28 – Bottom views of Bit 4 and Bit 20.

Figure 6.29 – Bottom views of Bit 4 and Bit 20.

that are extracted from the nominal bit/rock interactions.

According to Fig. 6.30(a), treating all blades as continuous cutter leads to approximated η values of 4 and 5 for bit 4 and 20. The actual η values not only vary with d_o but also are smaller than the estimate. From Fig. 6.30(b), bit 20 has larger frictional contact compared to bit 4; also the contacts increase with d_o for both bits. In Fig. 6.30(c), with the same model configuration and identical W_o , a larger κ_W corresponds to a more aggressive bit/rock interaction. Time simulations have shown that for similar configuration in the U -regime, Bit 4 tends to achieve axial-torsional stick-slips earlier compared to Bit 20. Fig. 6.30(d) shows the $d_o - \gamma$ relation. Finally Fig. 6.30(f) shows the average depth of cut per cutter. It can be concluded that given the same d_o , Bit 4 is more “aggressive” compared to Bit 20.

Figure 6.31 shows the results for the same set of bits equipped with backup cutters (Fig. 6.29). Figure 6.31(a) and (b) indicate that the backup cutters substantially increases the frictional contact for Bit 20 once d_o reaches a critical value of 1.2 mm; the backup cutters for Bit 4, however, only poses a slight influence on the frictional contact. Figure 6.31(c) and (d) are evidently results of reduced ratio of the cutting component in W_o and T_o , theoretically delaying/eliminating stick-slips in a better way compared to the respective cases without backup cutters. Figure 6.31(e) shows that adding backup cutters increases γ for both bits; it has a more substantial effect on Bit 20, in accordance with results in Fig. 6.31(a) and (b). Finally, Fig. 6.31(f) indicates

Figure 6.30 – Comparison of geometric properties of Bit 4 (blue) and Bit 20 (red)

that adding backup cutters enables a less aggressive bit/rock interaction for a PDC bit.

6.5.2 Comparison of linear stability

The stability maps for Bit 4 and Bit 20 are shown in Fig. 6.32. For smaller depth of cut ($d_o < 4$ mm), Bit 20 has a larger region in the $RPM - d_o$ space that corresponds to the slow instability regimes where the bit has a quasi-periodic responses in both axial and torsional motions. For larger depth of cut ($d_o > 4$ mm), there exists a single critical rotation speed ω_c that varies slightly with d_o , and the linear stability boundaries are similar for both bits. Fig. 6.32 also shows results for the same bits with corresponding backup cutters. It is shown that the backup cutters do not affect much the linear stability.

6.5.3 Bit responses

First, two different instability regimes predicted from the linear stability maps can be well verified using the time simulation code. Indeed, for slow instability regimes, the bit experiences quasi-periodic responses. For fast instability regimes, some typical bit responses are shown in Fig. 6.33. Subplots (a1)-(a2) show the early/late response of a

Figure 6.31 – Comparison of geometric properties of Bit 4 with backup cutters (blue) and Bit 20 with backup cutters (red). The dashed lines correspond to the results without considering backup cutters, identical to the plots in Fig. 6.30.

Figure 6.32 – Stability maps for Bit 4 (with or without backup cutters), Bit 20 (with or without backup cutters)

Figure 6.33 – Time simulation results for bit 4

fast instability; axial-torsional stick-slips develop rather quickly. Subplots (b1)-(b2) also correspond to a fast instability, but the transient response of the torsional motion sees a slow growth in the magnitude of RPM. Differences between (a) and (b) cannot be interpreted by the linear stability. This effect seems to correspond to the value of κ_W (or κ_T). For this specific case indeed $\kappa_{Wa} > \kappa_{Wb}$. Subplots (c1) and (c2) correspond to the same model configuration but with various β value. For $\beta > 1$, the axial oscillations have a damping effect to the torsional motion, which prevent the establishment of a torsional limit cycle.

6.6 Geometric extension of the model

An useful extension of the work described is a “geometric” one that alters the cutter shape with no need for other changes. This thesis presents two examples of this extension: one deals with scribe cutters and one applies wear to any existing bit geometry with any cutter shapes.

6.6.1 Scribe cutters

Geometrically, a scribe cutter face can be seen as a circle clipped by an isosceles triangle that is offset in the direction of bit axial advance. Numerically, the cutter shape is defined as a 2-D polygon in the plane orthogonal to the cutter normal. The cutter shape in this plane is determined by the set $\{\Phi_{sc}, \Delta_{sc}, R_{sc}\}$ (added subscript sc denotes scribe cutter) in Fig. 6.34, where R_{sc} is the round cutter radius prior to the “clipping”, Δ_{sc} is the positive shift of the clipping isosceles triangle and Φ_{sc} pinpoints the shape of that triangle. In the bit design file provided, an angle γ_{sc} is given to depict the in-plane rotation of the cutter projection in the bit profile. This is an inconvenience since the bit profile plane is not parallel to the cutter plane. In other words, after the side and back rake angles are applied, the in-plane cutter rotation - in the plane perpendicular to cutter normal - has to be computed from the orientation of the scribe cutter in the bit profile. In this way, the cutter projection of a scribe cutter can be properly finalized. The details of the algorithm is not included here; the main idea is to project two intersecting lines defining the angle from one plane to another.

We briefly presents some results for two bits equipped with scribe cutters (Fig. 6.35). Bit B is equipped with a majority of scribe cutters with 5 round cutters at the edge; Bit C has main cutters with round shape and some backup scribe cutters. The geometric properties are shown in Fig. 6.36 to give a comparison between Bit B, Bit C and Bit 20. The stability maps are shown in Fig. 6.37.

6.6.2 Worn cutters

The change of cutter geometry due to wear is implemented in a simple manner, as illustrated in Fig. 6.38. First, for a given d_o , the nominal cutter/rock engagement $\mathcal{S}_i(d_o)$ is superimposed on the cutter projection. Then determine the line passing the left/rightmost points of the cutting edge Γ_i . Perpendicular to the line is the unit vector $\mathbf{e}_{\text{wear}}$. Last, shift the line by $y\mathbf{e}_{\text{wear}}$, where $y > 0$ indicating less remaining cutter

Figure 6.34 – Cutter face of a scribe cutter (upward is the positive direction for bit axial advance)

Figure 6.35 – Bottom views and bit profile of Bit B and Bit C.

Figure 6.36 – Comparison of geometric properties of Bit B, Bit C and Bit 20

Figure 6.37 – Stability maps for Bit 20, Bit B and Bit C

Figure 6.38 – Modeling of cutter wear: examples are shown for both round and scribe cutters

face area, and vice versa. Using the shifted line to clip the original projected cutter face gives the remained cutter face under wear condition. They are represented by red polygons in Fig. 6.38.

Figure 6.39 investigates the effect of cutter wear on the geometric properties discussed previously. Two cases of worn bits are considered, a smaller wear defined by ($d_o = 0.5$ mm, $y = 0.5$ mm) and a larger one ($d_o = 0.5$ mm, $y = 1.5$ mm). From Fig. 6.39(a) (b), cutter wear increases the frictional contact as increasing y is effectively changing a round cutter to be closer to a rectangular cutter. Accordingly in Fig. 6.39(c) (d), the cutting component takes up a smaller portion out of W_o and T_o . Figure 6.39(d) shows that increasing y results in a larger γ value and finally shown in Fig. 6.39(f) the bit is less aggressive with increasing y . Figure 6.40 shows the stability maps for Bit 4 at various degrees of wearing. Interestingly, for smaller depth of cut, cutter wearing also improves the linear stability of the bit. In summary, wearing of round cutters is able to improve the stability property of a bit within the modeling framework.

Figure 6.39 – Comparison of geometric properties of Bit 4 (intact, two different wearing degrees)

Figure 6.40 – Stability maps for Bit 4 with various degrees of wearing

Chapter 7

Conclusion and Future Work

7.1 Contributions

This thesis extends the RGD model [3], by considering drag bits with more realistic representations (Classes I and II) of the regenerative process. Such bit designs introduce multiple time delays, in contrast to the single time delay of the RGD model. Formulation of the bit/rock interaction for these bits is the stepping stone to the linear stability analysis and to the algorithm developed to simulate the self-excited vibrations of a rotary drilling system.

Specifically, two types of Class I bits are considered: fishtail bits with varying angular delays and symmetric bits with equal number of full and partial blades. The geometry of these two groups of bits are quantified by two numbers, one related to the number of blades, the other to the distribution of angular delays for the fishtail bit or the length of the partial blades for the symmetric bits. For some limits of this second number, both bits degenerate into symmetric bits with identical blades, like those considered in the RGD model. Stability maps for the two groups of bits are constructed analytically through an application of the D-subdivision method [89]. The stability maps indicate that both groups of bits have better stability property compared to the RGD counterparts; also optimal designs for both groups are determined. Furthermore, multiple critical rotation speeds exist for certain bit geometries, in contrast to the RGD bit which has a single stability boundary. The results of the linear stability analysis were confirmed by conducting time simulations. In all the cases considered, the computed time response for the predicted unstable configurations evolve towards limit cycles characterized by stick-slip events.

On the basis of Class I bits, this work further extends the RGD model by considering realistic cutter shape and layout of a given drag bit (rake angles and positions). An elaborate modeling approach for the bit/rock interaction has been presented, which

is implemented by incorporating the polygon clipping technique. For realistic bit geometries, the formulation for the linear stability analysis has been constructed based on the geometric variations of bit/rock engagement surfaces. The resulting stability boundary (ω_c, d_o) for the decoupled axial dynamics has been well verified with a modified inverse time simulation algorithm first given in [81]. In conjunction with the exact location of unstable axial poles, the stability boundaries can be used to predict the occurrence of axial stick-slip oscillations in the coupled dynamics, which is crucial for simulating the axial-torsional stick-slip oscillations. The bit design factor γ is shown to control the torsional behavior of the bit response. With $\beta = \mu\zeta\gamma > 1$, the axial stick-slip oscillations can curb the torsional oscillations.

In the current model, the drillstring is represented by a simple lumped mass as in [3]. Nonetheless, the model predicts that the axial/torsional velocity profiles can fully develop into certain stationary patterns (limit cycles or quasi-periodic responses¹), as shown in Fig. 7.1:

- (A) “Pseudo” limit cycles are formed for both modes of motion, axial stick-slips take place without torsional stick-slips. Given longer time, the real limit cycles correspond to axial-torsional stick-slips.
- (B) Limit cycles are formed for both modes of motion, axial stick-slips accompany torsional stick-slips.
- (C) Quasi-periodic responses are observed for both modes of motion, where velocities have a very slow growth.
- (D) Axial stick-slips have a damping effect on the torsional oscillation.

7.2 Engineering applications

Whenever torsional stick-slip takes place, a faster axial stick-slip oscillation accompanies; it should be addressed that the torsional stick-slip is damaging to the drilling equipments and should be mitigated. It seems that the torsional stick-slips can be prevented through the following:

¹When axial stick-slip oscillations are absent, the developed bit response (quasi-periodic as in Fig. 7.1C) is transient with a very slow growth in the time scale of drilling operations. Thus technically this is not a limit cycle. For such configurations, with the addition of damping, stationary axial/torsional vibrations or even non-oscillatory motion can be observed.

- Apply the proper combination of applied RPM/ROP at the rig such that the nominal bit motion ensures the regime of “slow” instability [27] where quasi-periodic responses are achieved, as depicted in Fig. 7.1C. For a specific bit design, linear stability analysis for the decoupled axial dynamics can be conducted to yield stability maps in terms of applied RPM/ROP to predict the desired “slow” instability regime [93].
- Choose the nominal bit motion such that limit cycles like Fig. 7.1A can be achieved. At this stage the desired rig RPM/ROP for this goal can only be predicted from case-by-case time simulations. A systematic determining process requires a better understanding of this type of limit cycles.
- Adjust the wearflat distribution of the bit such that an *apparent* “velocity-strengthening” reaction torque-on-bit can be achieved. Bits with such designs are likely to develop responses in Fig. 7.1D, where the torsional oscillation is damped irrespective of the axial stick-slips.

7.3 Future work

For engineering applications, the insights drawn from the current model need to be refined with the support of laboratory/field data. The scopes for the proposed research are listed as follows:

- Calibrate the existing bit/rock interaction model using laboratory data. In [58], backed by laboratory measurements of nominal bit responses, an empirical relationship has been set up between bit/rock engagement geometry and the drilling force while considering the effect of rock chips in front of cutters. Combining both models should yield a more reliable depiction of the bit/rock interaction when the system experiences self-excited vibrations.
- Analyze field data to verify the existence of various types of stationary pattern developed in bit responses. After verification, with the help of the modeling, try to understand the mechanism for the geneses of various types of the patterns. Finally, bridge the gap between the drilling configurations (operation parameters such as ROP/RPM, bit/drillstring design factors, rock properties etc.) and the patterns.
- Improve the current model. Although the discrete lumped model of the drillstring yields limited results that are meaningful when comparing different bit designs,

for more detailed parameter planning by modeling and simulation, a multiple degrees-of-freedom drillstring model is necessary. A preliminary code has been developed for a more realistic drillstring model [94], which is based on an angle-based “inverse” algorithm. The author proposes that this algorithm will be improved by incorporating with the realistic bit/rock interaction model mentioned earlier.

- Design field tests that are necessary to verify the theoretical results derived from the modeling.

The research scopes listed above aim to address two fundamental engineering challenges:

- For existing bit designs, which one is better in terms of torsional stick-slip mitigations?
- What are the key design factors of an optimal bit that yields better stability?

Figure 7.1 – Different types of “limit cycles” observed in the current stick-slip model: (A) axial stick-slips without torsional stick-slips, (B) axial stick-slips with torsional stick-slips, (C) a quasi-periodic motion and (D) axial stick-slips with damped torsional oscillation. For the responses (C) and (D), with the addition of damping, one can observe stationary torsional oscillations.

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