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Sensor Placement and Selection for Bearing Sensors with Bounded
Uncertainty

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(Extended Abstract)

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Abstract

We study the problem of placing sensors so as to accurately estimate the location of a target in a given environment. We focus on bearing sensors (such as cameras, microphone arrays) which are commonly used for target localization. We seek to discover and exploit the geometric structure associated with both the sensing model and the estimation process in order to obtain sensor placement schemes with performance guarantees.

We use a geometric model to represent the sensing uncertainty: Each measurement yields an unbounded 2D wedge with angular width 2α around the measurement (α is an input parameter representing the maximum sensing noise). The wedge is guaranteed to contain the true location of the target. The target is localized by intersecting the wedges obtained from *all* sensors. The quality of the placement is given by the area or diameter of the intersection in the worst-case (i.e. regardless of the target's location). We study the bi-criteria optimization problem of placing a small number of sensors while guaranteeing a worst-case bound on the estimation error.

Our main result is a constant-factor approximation for this problem: Let U_D^* and U_A^* be the diameter and area uncertainty achieved by an optimal algorithm using n^* sensors. We show that at most $9n^*$ sensors placed on a triangular grid has diameter uncertainty of at most $5.88U_D^*$ and area uncertainty of at most $7.76U_A^*$, when the sensing noise $0 < 2\alpha \leq \frac{\pi}{2}$. For $\frac{\pi}{2} < 2\alpha < \pi$, we bound the number sensors sufficient to have a bounded target estimate. When the maximum sensing noise $2\alpha \geq \pi$, we show that there always exists a set of measurements, for any placement of n sensors, such that the intersection of all wedges is unbounded. In obtaining these results, we provide some interesting structural properties which may be of independent interest. First, we study general placements and obtain a lower bound on the best uncertainty that can be achieved when placing n sensors on the plane. We then focus on the grid-based placement scheme and show that a placement of n sensors placed on a triangular grid does not deviate much from this lower-bound. This latter result also yields a sensor selection mechanism: We show that in the triangular grid, only a constant number of sensors need to be activated to achieve the upper-bound.

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1. Introduction

Sensor placement for target localization is a fundamental problem in robotics and sensor networks. The goal is to place sensors so as to accurately estimate the unknown location of a target in a given environment. A good sensor placement scheme can improve the performance of a sensor network in applications such as detecting an intruder in surveillance, robot navigation using exteroceptive measurements, and providing location information to mobile devices for location-aware computing.

Most sensors employed for localization have a strong structure associated with their sensing and measurement model. This structure plays a critical role in how information from multiple sensors can be combined. In this paper, we study the sensor placement problem for bearing sensors. Bearing sensors are commonly used in robotic and networked sensor systems: Monocular cameras, microphone/acoustic arrays, directional radio antennas, passive infrared receivers, etc. all measure bearing towards a target. We consider the scenario when the output of each sensor is a bearing corrupted with unknown but bounded noise. Bounded noise models provide a useful alternative to probabilistic models especially when a precise device model is not available (perhaps due to the difficulty of calibration or changing device parameters).

In the bounded uncertainty model, the true bearing to the target is guaranteed to lie in a 2D wedge which is centered at the measured bearing and has an apex angle equal to the maximum sensing noise. Measurements from multiple sensors are combined by intersecting the corresponding sensing wedges. The intersection yields an estimate for the target’s location. The uncertainty in the estimate is usually taken to be the diameter or the area of the intersection. We seek worst-case quality guarantees for our estimate: Given the true location of the target, imagine an adversary choosing the noise values for sensors. The true location can be anywhere in the intersection and we would like this set to be “small” no matter where the target is.

Figure 1: An example of the target estimate obtained by combining sensing wedges from all sensors (marked as circles) when the maximum sensing noise is $\frac{\pi}{25}$, $\frac{\pi}{15}$ and $\frac{\pi}{5}$. The true target is marked by a cross.

We consider a simple workspace for the problem: The target can lie anywhere within a square without any obstacles or visibility constraints. Even in this basic setting, devising a sensor placement scheme is tricky. It is intuitively clear that the optimal placement should be some kind of a uniform grid. However it is not clear if the grid should be square, triangular or some other shape. Further, optimizing parameters of the grid (e.g. resolution) is not straightforward because as illustrated in Figure 1, the estimate is obtained by combining measurements from *all* sensors. This

makes it difficult to express its area or diameter in closed-form in order to optimize grid parameters.

Our contributions: We study the bi-criteria problem of minimizing the number of sensors and finding a placement in the workspace to guarantee a desired level of worst case uncertainty. First, we relate the worst-case uncertainty of a placement to the radius of the largest empty circle in the workspace (Lemma 2). Based on this relationship, we present lower bounds on the number of sensors required to achieve a desired uncertainty (Corollary 1 & Corollary 2). Next, we study simple sensor configurations and bound the maximum uncertainty when the target lies inside an equilateral triangle or regular hexagon with sensors placed on the vertices, as a function of the side length (Lemma 3 & Lemma 4). We then use these results to study the performance of a triangular placement (Lemma 6). This analysis provides a natural placement scheme with constant-factor approximation guarantees for placing sensors. Specifically, in a square workspace, when the maximum sensing noise is less than $\frac{\pi}{4}$ we get a constant-factor approximation (Theorem 1). For sensing noise between $\frac{\pi}{4}$ and $\frac{\pi}{2}$, we bound the number of sensors sufficient to ensure bounded intersection of sensing wedges (Lemma 5). When the maximum sensing noise is greater than $\frac{\pi}{2}$, we show there always exists a set of measurements for any placement of n sensors with an unbounded intersection of sensing wedges (Lemma 1). Therefore, in this last case, no placement can guarantee bounded error. In obtaining these results, we also show that one can obtain good estimates by activating only a constant number of sensors (Corollary 3). This result is important for sensor-network applications with energy or bandwidth constraints.

We start with an overview of related work.

2. Related Work

The problem of optimizing the placement of sensor nodes in sensor networks research has been well studied [8]. A large amount of research has focused on self-localization of stationary sensor networks. For example, Agarwal et al. [1] studied the problems of localization, clustering and sensor suspension by refining the self-localization of a network of sensors with noisy but bounded uncertainty and correlated measurements of some environmental characteristic. In our present work, we assume that the locations of the sensors themselves are accurately known and focus on the complementary problem of placing sensors so as to localize targets.

The Art Gallery Problem [6] is a classical problem of placing the minimum number of guards (or sensors like omnidirectional cameras), usually in polygonal environments, such that a target present anywhere within the polygon is visible from at least one guard. The original version of the problem only considers *detection* (or seeing) the target with a single guard. There is no notion of *localizing* the target, which could require combining measurements from multiple guards. Recently, Eppstein et al. [3] introduced a variant of placing a minimum number of “angle guards” in a polygon, to decide whether a target lies in the interior of the polygon. Each angle guard detects if a target lies in the guard’s unbounded 2D wedge (whose apex angle is free to be set) without any visibility constraints. The target location is said to be in the interior of the polygon if and only if the location lies in the intersection of wedges from two or more guards whose intersection is known to lie in the interior of the polygon. The paper presents bounds on the number of guards for various classes of polygons, as well as a 2-approximation for the general case. While this sensing model considers uncertainty in the sense that the measurement is a wedge, there is no notion of measurement noise or quality of localization (the goal is to decide if the target is inside or not). In this paper, we

specifically consider the uncertainty due to bounded noise in bearing sensors.

The uncertainty in the target estimate obtained by combining bearing measurements depends on the relative position of the sensors and the target. For example, if the sensors are nearly collinear with the target, the resulting estimate will have significant uncertainty. With this as motivation, Efrat et al. [2] studied the following two variants of the Art Gallery Problem: minimize the number of guards such that (i) each point in the polygon is visible from at least two sensors with relative angle between them constrained to a desired interval, or (ii) each point is visible from at least three sensors enclosing the point. For both problems, the authors presented a log factor approximation, up to a fine discretization parameter.

In addition to the relative angles, the uncertainty in estimation is also affected by the distance of the target from the sensors. The Geometric Dilution of Precision (GDOP) is one such measure giving in closed form, the relation of uncertainty with distance to target and relative angles. Tekdas and Isler [7] studied the problem of placing a minimum number of bearing sensors using GDOP as the objective function. They presented a placement scheme which guarantees that for any target location there are always two sensors whose GDOP is a constant factor of the GDOP achieved by *two* sensors from an optimal placement. In our work, we do not restrict the estimation process to use only two sensors. Instead, we compare our performance against an optimal placement combining measurements from possibly all sensors. Ercan et al. [4] studied the problem of placing horizontal scan-line cameras along the boundary of a circular room to minimize least-squares localization error for a target with a given prior. Their placement result shows that a uniform placement along the boundary is optimal. In our work, we allow sensors to be placed anywhere within a square workspace, without assuming any prior for the target’s location.

Isler and Magdon-Ismael [5] considered the problem of selecting a small subset of sensors from a given placement; each sensor’s output is a convex subset of the plane. They proved that irrespective of the total number of sensors, there is always a subset of four sensors that can be selected, whose measurements when combined yield an intersection area at most twice of that obtained by intersecting all measurements. In their problem, the placement of the sensors and the actual sensor measurements are already given. For the same placement of sensors, this subset would change if the measurement changes. This poses an interesting question whether there is some placement of sensors for which the same subset can be used to approximate the uncertainty region for different (but perhaps “nearby”) measurements. In this paper, we present a result in this direction for bearing measurements with bounded noise.

3. Problem Formulation

In this section, we first describe the notation used in this paper, define the sensing and estimation models, and use them to formulate the problem studied in this paper.

3.1. Notation and Sensing Model

The workspace is a $d \times d$ square denoted by $\mathcal{A}$. The target can be anywhere within $\mathcal{A}$. The target’s true location is denoted by x . Consider the sensor placement $S = \{s_1, \dots, s_n\}$ where each s_i denotes the sensor location. All sensors in S lie within $\mathcal{A}$. Each sensor measures the bearing towards the target as $\theta_i^m = \theta_i^t + n_i$, where $\theta_i^t \in [0, 2\pi)$ is the true bearing (Figure 2(a)). $n_i \in [-\alpha, +\alpha]$ is the

Figure 2: (a) Sensing model: The target is located at x . The true bearing from sensor s_i is given by θ_i^t . The actual measurement θ_i^m lies anywhere between $\theta_i^t \pm \alpha$. The wedge for a given measurement θ_i^m , (given by h_i^l and h_i^r), is guaranteed to contain x . (b)&(c) Two estimates for the same target location and placement of sensors, but different set of actual measurements resulting in different uncertainty. We use the worst-case intersection area and diameter as the uncertainty measure.

bounded sensor noise. α is the bound on the absolute noise in the sensor. The true bearing θ_i^t takes values from 0 to 2π , hence its preimage is a ray which extends in one direction as opposed to a line which extends in both directions. The preimage of a measurement θ_i^m is a 2D wedge (denoted by $W(s_i, \theta_i^m)$) extending from $(\theta_i^m - \alpha)$ to $(\theta_i^m + \alpha)$ as shown in Figure 2(a).

Definition 1 (Sensing Wedge $W(s_i, \theta_i^m)$) For sensor at s_i yielding a bearing measurement θ_i^m , a sensing wedge is defined as the intersection of two half-planes: (1) h_i^r passing through s_i with an angle $\theta_i^m - \alpha$, and (2) h_i^l passing through s_i with an angle $\theta_i^m + \alpha$, such that all points in the direction of θ_i^m are included for each half-plane (h_i^l and h_i^r form the “left” and “right” boundaries).

Remark 1 The wedge defined above is not the same as a fixed field-of-view sensor; for the same target location, the sensor can receive any sensing wedge of angular width 2α so long as it contains the true target location (Figure 2(a)).

The sensing wedge corresponding to a measured bearing, represents the smallest set (in terms of angular width) guaranteed to contain the true target location. Measurements from multiple sensors can be combined, to yield a smaller subset of $\mathcal{A}$ guaranteed to contain x . The estimation process consists of intersecting the sensing wedges from all the sensors, as defined next.

Definition 2 (Target Estimate) The target estimate obtained by combining a set of measurements $\theta^m = [\theta_1^m, \dots, \theta_n^m]^T$ obtained from n sensors, is defined as the intersection of the n sensing wedges $W(s_i, \theta_i^m)$. That is, $\hat{P}(S, \theta^m) \triangleq \bigcap_{i=1}^n W(s_i, \theta_i^m)$.

Here $\hat{P}$ is a convex polygonal region which can possibly be unbounded.

3.2. Adversarial Formulation of Uncertainty

The quality of the target estimate depends on the size of $\hat{P}$. $\hat{P}$ itself depends on the actual measurements. Figure 2(b) and Figure 2(c) show two instances where the size of $\hat{P}$ differs significantly

for different measurements obtained from the same placement of sensors. The actual measurements obtained by the sensors cannot be controlled by the user. However, we will show that by carefully placing the sensors one can guarantee there always exists a good set of valid measurements.

We model the objective using an adversarial process: Given a placement of sensors, an adversary selects a target location within the square and a corresponding set of measurements to maximize the uncertainty in the target estimate. We use two measures (area and diameter of $\hat{P}$) to define the uncertainty in target's estimate.

Definition 3 (Diameter Uncertainty) The diameter uncertainty of a placement of sensors S in a workspace $\mathcal{A}$, is defined as the diameter of the target estimate polygon $\hat{P}$:

$$U_D(S) \triangleq \max_{x \in \mathcal{A}} \max_{\theta^m \in \theta(x)} \text{diameter}(\hat{P}(S, \theta^m)),$$

where $\theta(x)$ is the set of valid measurements that can be obtained from S for a target location x .

Here, the first max is over the set of possible target locations and the second max is over the set of corresponding valid measurements the adversary can choose, given a placement S of sensors. The area uncertainty can be similarly defined.

3.3. Objective

Broadly, there are two factors that affect the worst-case uncertainty: (i) the number of sensors, and (ii) the location of placed sensors. In this work, we take the approach that the user specifies a desired uncertainty and the objective is to minimize the number of sensors and find the corresponding placement to guarantee that the worst-case uncertainty is below the user-specified value. In particular, we address the following problem.

Find the minimum number of sensors required and the corresponding placement to achieve a desired diameter uncertainty U_D^ (or desired area uncertainty U_A^*).*

Our main result shows that by placing sensors on a triangular grid-like placement, 9 times as many sensors as an optimal algorithm are sufficient to guarantee 5.88 times the desired diameter uncertainty (respectively, 7.76 times the desired area uncertainty) when the maximum sensing noise is less than $\frac{\pi}{4}$.

Theorem 1 *Let the maximum absolute noise for bearing sensors be $0 < \alpha \leq \frac{\pi}{4}$. Let the desired diameter uncertainty for a $d \times d$ square environment be $U_D^* < \frac{d}{7 \sin \alpha}$ (respectively, area uncertainty be $U_A^* < \frac{\pi \sin^2 \alpha}{196} d^2$). If an optimal placement algorithm achieves U_D^* (respectively, U_A^*) with n^* sensors, then a triangular grid-like placement achieves at most $5.88U_D^*$ (respectively, at most $7.76U_A^*$) with at most $9n^*$ sensors.*

The analysis for [Theorem 1](#) is based covering the $d \times d$ square with equilateral triangles of sensors. We use the term “grid-like” to indicate that in addition to placing sensors on a triangular grid of side r , we place additional sensors near the boundary of $\mathcal{A}$ to cover points which may not be enclosed by equilateral triangles. The overhead of additional sensors decreases as the desired uncertainty becomes smaller. When the desired uncertainty is higher than the restriction in [Theorem 1](#) and

comparable to the size of $\mathcal{A}$, an optimal placement may use very few sensors. Nevertheless, even for that case the total number of sensors for the grid-like placement is bounded (given by [Lemma 6](#)).

In the following sections, we analyze the number of sensors required for an optimal algorithm and for a triangular grid-like placement. In this extended abstract, we state the key lemmas and sketch the proofs. The full proofs are included in the appendix.

4. Lower Bounds for Optimal Placement

In this section, we first present lower bounds on the uncertainty achieved by any placement of sensors in the plane. We apply this to bound the number of sensors placed within $\mathcal{A}$ by an optimal algorithm.

First consider the case when the maximum sensing noise $\alpha \geq \frac{\pi}{2}$, i.e., the sensing wedges are at least half-planes. We show that the adversary can always choose a valid measurement set for any placement of n sensors, such that the sensing wedges have an unbounded intersection.

Lemma 1 (Unbounded Estimate when $\alpha \geq \frac{\pi}{2}$) *For any placement S of n bearing sensors with maximum absolute noise $\alpha \geq \frac{\pi}{2}$, there exists a measurement set θ^m such that the intersection of the wedges $(\bigcap_{i=1}^n W(s_i, \theta_i^m))$ is unbounded.*

[Lemma 1](#) implies that when $\alpha \geq \frac{\pi}{2}$ the uncertainty can be as large as $\mathcal{A}$, i.e., $U_A(S) = \Theta(d^2)$ and $U_D(S) = \Theta(d)$ for any placement of sensors, including the optimal. The proof is based on constructing a simple instance when $\alpha = \frac{\pi}{2}$, i.e. the sensing wedges are half-planes. We create a measurement set where the lines corresponding to all half-planes pass through the target location. We assign directions to each half-planes to ensure that their intersection is unbounded.

[Lemma 1](#) is not surprising, since $\alpha \geq \frac{\pi}{2}$ corresponds to a very noisy sensor. Practical bearing sensors often are much more accurate. For the rest of the paper, we only focus on the case when the maximum sensing noise $\alpha < \frac{\pi}{2}$.

In the following, we will lower bound the uncertainty for any placement parametrized by the distance of the target to the closest sensor. Recall from [Definition 3](#), the uncertainty is defined as the max over all possible target locations, and all valid measurements. Hence, to lower bound the uncertainty it is sufficient to consider a particular target location and valid measurement set. For [Lemma 2](#), we consider the case when the target is located at the center of a circle with no sensors present in its interior. The lower bound is parametrized by the radius of the circle.

Lemma 2 (General Lower Bound) *Let S be a placement of n bearing sensors on the plane, with maximum absolute noise α . If there exists a circle C with no sensor in its interior, then the diameter uncertainty is bounded from below as $U_D(S) \geq 2r \sin \alpha$ (respectively, $U_A(S) \geq \pi r^2 \sin^2 \alpha$), where r is the radius of C .*

To prove [Lemma 2](#), we show that when the target lies at the center of C and each sensor receives a measurement equal to the true bearing, we can draw a circle of radius $r \sin \alpha$ centered at the target which lies completely within the intersection of all sensing wedges. This instance allows us to lower bound the worst-case uncertainty.

When a desired uncertainty is given, we can apply [Lemma 2](#) to find the radius of the largest such circle lying in the workspace $\mathcal{A}$ and not containing any sensor.

Corollary 1 *Let S^* be an optimal placement of bearing sensors with maximum absolute noise α , achieving a desired diameter uncertainty U_D^* (respectively, area uncertainty U_A^*) in a square workspace of side d . If r^* is the radius of the largest circle lying completely within $\mathcal{A}$ and not containing any sensor in its interior, then $r^* \leq \frac{U_D^*}{2 \sin \alpha}$ (respectively, $r^* \leq \sqrt{\frac{U_A^*}{\pi}} \frac{1}{\sin \alpha}$).*

Corollary 1 implies an upper bound on how far each point in $\mathcal{A}$ can be from any sensor or the boundary of $\mathcal{A}$. This allows us to bound the number of sensors required for an optimal algorithm as a function of r^* . **Corollary 2** states that $\Omega(\frac{d^2}{r^{*2}})$ sensors are needed to guarantee coverage of a $d \times d$ area.

Corollary 2 *Given an optimal placement inside a square of side length d , let r^* be the radius of the largest circle within the square, not containing any sensor from an optimal placement in its interior. If the desired diameter uncertainty is $U_D^* < d \sin \alpha$ (respectively, area uncertainty $U_A^* < d^2 \frac{\pi \sin^2 \alpha}{4}$) then the number of sensors for an optimal algorithm $n^* \geq \frac{(d-2r^*)^2}{\pi r^{*2}}$.*

The bound on the uncertainty implies that $d > 2r^*$: If $d > 2r^*$, then there is a smaller square within $\mathcal{A}$ where all points are more than r^* away from the boundary and hence require at least one sensor within r^* . We can show that the set of circles of radii r^* drawn about each sensor in the optimal placement, should form a cover of this smaller square, yielding the bound. When $d \leq 2r^*$, the desired uncertainty is comparable to $\mathcal{A}$, and the optimal algorithm would place very few sensors, yielding a trivial lower bound.

5. Performance of the Triangular Grid

In this section, we analyze the number of sensors required and the uncertainty achieved by placing the sensors on a triangular grid-like placement.

Figure 3: We consider two configurations of sensors to upper bound the uncertainty. For $0 < \alpha < \frac{\pi}{6}$, the target can be anywhere within an equilateral triangle $\triangle s_1 s_2 s_3$, with sensors on its vertices. For $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{4}$, the target can be anywhere within a circle of radius $\frac{r}{\sqrt{3}}$ within a regular hexagon of side r , with sensors on its vertices.

5.1. Uncertainty with Triangular Grid

Before the main analysis, first consider two special configurations of sensors: (i) three sensors placed on the vertices of an equilateral triangle when $0 < \alpha < \frac{\pi}{6}$, and (ii) six sensors placed on the vertices of a regular hexagon when $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{4}$. For case (i), let $\triangle s_1 s_2 s_3$ be an equilateral triangle of side length r with one sensor placed at s_1 , s_2 and s_3 each (Figure 3). The target can lie anywhere

within $\triangle s_1 s_2 s_3$. While for lower bounds on uncertainty it sufficed to consider a specific instance, the upper bounds require considering all possible target locations and valid set of measurements. We further divide the analysis into intervals based on α , given next.

Lemma 3 (Upper bound on uncertainty) *Let $\triangle s_1 s_2 s_3$ be an equilateral triangle of side r with a bearing sensor placed at each vertex. If the target lies inside an equilateral triangle then the diameter uncertainty is bounded by*

$$U_D(\{s_1, s_2, s_3\}) \leq \begin{cases} 11.35r \sin \alpha & 0 < \alpha < \frac{\pi}{18}, \\ 2.04r & \frac{\pi}{18} \leq \alpha < \frac{\pi}{12}, \\ \left(1 + \frac{1}{\sqrt{3}}\right) r & \frac{\pi}{12} \leq \alpha < \frac{\pi}{6} \end{cases}$$

and the area uncertainty is bounded by

$$U_A(\{s_1, s_2, s_3\}) \leq \begin{cases} 23.46r^2 \sin^2 \alpha & 0 < \alpha < \frac{\pi}{18}, \\ \frac{\sqrt{3}r^2}{4} + 10.001r^2 \sin^2 \alpha & \frac{\pi}{18} \leq \alpha < \frac{\pi}{12}, \\ \frac{3\sqrt{3}r^2}{4} & \frac{\pi}{12} \leq \alpha < \frac{\pi}{6}. \end{cases}$$

The proof partitions the triangle into three regions, and assigns sensors for each region such that *any* valid set of measurements results in bounded intersection. The sensing wedges corresponding to each partition are over-approximated and their intersection diameter and area is bounded. The full proof is included in the appendix. Note that we do not restrict the intersection to be contained within $\triangle s_1 s_2 s_3$, allowing us to use this lemma for the general case.

When $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{4}$, the sensing wedges become too large to result in bounded intersection with three sensors. Instead we use six sensors, placed on a regular hexagon with center o and side r , to bound their intersection. The target can lie anywhere within a circle of radius $\frac{r}{\sqrt{3}}$ centered at o . We find an upper bound to the uncertainty, by finding the intersection of the union of all sensing wedges for each sensor, corresponding to all target locations within the circle.

Lemma 4 (Upper bound on uncertainty) *Let $s_1 s_2 s_3 s_4 s_5 s_6$ be a regular hexagon of side r and center o with a bearing sensor placed at each vertex, and maximum absolute noise $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{4}$. If the target lies inside a circle of radius $\frac{r}{\sqrt{3}}$ centered at o then the diameter uncertainty is bounded by*

$$U_D(\{s_1, \dots, s_6\}) \leq r (\sin^2 \alpha + 3.76 \sin \alpha + 1.232)$$

and the area uncertainty is bounded by

$$U_A(\{s_1, \dots, s_6\}) \leq 1.5r^2 (\sin^2 \alpha + 3.76 \sin \alpha + 1.232)$$

[Lemma 4](#) bounds the intersection when the sensing wedges are at most a quadrant ($\alpha \leq \frac{\pi}{4}$), and the target lies within a circle of radius $\frac{r}{\sqrt{3}}$. [Lemma 1](#) shows that when sensing wedges are at least a half-plane ($\alpha \geq \frac{\pi}{2}$), the resulting intersection can be unbounded in the worst-case. We can extend the result in [Lemma 4](#) for $\alpha = \frac{\pi}{2} - \epsilon$ with $0 < \epsilon$, to bound the number of sensors placed on a triangular grid, sufficient to guarantee that the intersection of all wedges is bounded.

Figure 4: (a) Placement of sensors on a triangular grid of side r within a $d \times d$ square environment $\mathcal{A}$. (b) To apply [Lemma 3](#) we pad the three regions on the boundary with additional sensors to ensure any point in $\mathcal{A}$ is enclosed by an equilateral triangle of sensors. The sensors in odd rows and columns are marked white and those in even rows and columns are marked black.

Lemma 5 (Bounded Intersection for $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$) *Let the maximum absolute noise for the sensors be $\alpha = \frac{\pi}{2} - \epsilon$ with $0 < \epsilon$. If $\sin^{-1}\left(\frac{1}{\sqrt{3}k}\right) + \frac{\pi}{6(k-1)} < 2\epsilon$ then $\mathcal{O}(k^2)$ sensors placed on a triangular grid are sufficient for ensuring bounded intersection of sensing wedges when the target lies within a circle of radius $\frac{r}{\sqrt{3}}$.*

5.2. Number of Sensors with Triangular Grid

[Lemma 3](#) gives an upper bound on the uncertainty for a placement of sensors in $\mathcal{A}$, if there exists an equilateral triangle of sensors enclosing any point in $\mathcal{A}$. Consider a placement of sensors on an equilateral triangular grid of side r ([Figure 4\(a\)](#)) within $\mathcal{A}$. Since the sensors cannot be placed outside of $\mathcal{A}$, regions near the boundary of $\mathcal{A}$ may not have an enclosing equilateral triangle. The three regions where this occurs for the placement in [Figure 4\(a\)](#) are marked $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ in [Figure 4\(b\)](#). We place additional sensors within these regions to ensure that any point in $\mathcal{A}$ is enclosed by an equilateral triangle of sensors. We mark the sensors in odd rows and columns by white and those in even rows and columns as black. The total number of sensors required can be given as a function of the side length r of the grid. [Lemma 6](#) states that $\mathcal{O}(\frac{d^2}{r^2})$ sensors are sufficient to cover a square of area $d \times d$.

Lemma 6 (Upper Bound on Number of Sensors) *If w_r, w_c, b_r, b_c are the number of sensors in odd and even rows and columns, respectively of a triangular grid of sensors with side r (marked white and black respectively in [Figure 4\(b\)](#)) in a square of side d , such that*

$$w_r = \left\lfloor \frac{d}{r} \right\rfloor + 1, \quad w_c = \left\lfloor \frac{d}{\sqrt{3}r} \right\rfloor + 1, \quad b_r = \left\lfloor \frac{d - \frac{r}{2}}{r} \right\rfloor, \quad b_c = \left\lfloor \frac{d - \frac{\sqrt{3}r}{2}}{\sqrt{3}r} \right\rfloor + 1.$$

then $w_r w_c + b_r b_c + 3(2w_r + b_r) + 8$ sensors are sufficient to cover the square with equilateral triangles of side r .

The lower and upper bounds obtained can be applied to get the main result of this paper: Recall from [Corollary 1](#), that an optimal algorithm has to place a sensor within distance $r^* \leq \frac{U_D^*}{2 \sin \alpha}$

(equivalently, $r^* \leq \sqrt{\frac{U_A^*}{\pi}} \frac{1}{\sin \alpha}$) of every point $\mathcal{A}$ to ensure the desired uncertainty. For the triangular grid placement, set the grid length as $r = \frac{U_D^*}{2 \sin \alpha}$ (respectively, $r = \sqrt{\frac{U_A^*}{\pi}} \frac{1}{\sin \alpha}$). Hence, $r^* \leq r$. [Corollary 2](#) gives a lower bound on the number of sensors required for an optimal algorithm in terms of r^* , and [Lemma 6](#) gives an upper bound for the grid-like placement in terms of r . [Lemma 3](#) and [Lemma 4](#) bound the uncertainty of the grid-like placement in terms of r . Using $r^* \leq r$ and substituting the value of r , the result in [Theorem 1](#) can be obtained.

The upper bounds from [Lemma 3](#) and [Lemma 4](#) reveal that only a small number of sensors in our placement suffice to achieve an uncertainty comparable to that obtained by combining all measurements from an unknown optimal placement. This is useful when there is prior knowledge about the target location (e.g. a subset of $\mathcal{A}$) and only those sensors corresponding to the enclosing triangle or hexagon need be queried for their measurements.

Corollary 3 *Given a target location x within a square and desired diameter uncertainty U_D^* (respectively, area uncertainty U_A^*), if sensors are placed on a triangular grid with side $r = \frac{U_D^*}{2 \sin \alpha}$ (respectively, $r = \sqrt{\frac{U_A^*}{\pi}} \frac{1}{\sin \alpha}$), three sensors are sufficient when $0 < \alpha < \frac{\pi}{6}$ and six sensors are sufficient when $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{4}$ to ensure diameter uncertainty at most $5.88U_D^*$ (respectively, $7.76U_A^*$).*

[Corollary 3](#) provides a sensor selection method which may be useful in sensor network applications with energy or bandwidth constraints that require activating only a small number of sensors.

6. Concluding Remarks

In this paper, we studied a basic sensor placement problem for bearing sensors which are commonly used in sensor network applications. We used a bounded uncertainty formulation which allowed us to represent each measurement as a wedge. The measurements from multiple sensors were merged by intersecting the corresponding wedges. The intersection of these wedges provided an estimate of the target's location. The quality of the estimate was quantified by the diameter or the area of the intersection. In this setting, a fundamental question that arises is: What is the minimum number and placement of sensors that guarantee that no matter where the target is, or what the actual measurements are, the uncertainty in the estimate is below a given desired uncertainty level?

This basic question turned out to be surprisingly hard due to the fact that the quality of the estimation depends on the locations of all sensors as well as the actual measurements. Therefore, for a simple environment such as a square, while it is intuitively clear that the optimal sensor placement should be a uniform one, the shape and parameters of the placement were unknown. Our results provided insights about the structure of this sensor placement problem and yielded a placement scheme with constant-factor approximation guarantees. In particular, we showed that unless the sensor noise is too large, a placement of sensors on a triangular grid yields a good performance. Further, (excluding some extreme cases) we showed that the triangular placement has the nice property that if a rough estimate of the target location is available, one can obtain a good estimate by querying only a fixed number of sensors. This latter sensor selection scheme makes the triangular placement particularly appealing for resource constrained sensor-network applications.

Can this result be improved, perhaps by showing that the triangular placement is optimal? This is a question we pursue in our immediate future work. Future work also includes addressing sensing limitations such as visibility constraints and placement in more complex environments.

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A. Lower Bounds

In this appendix, we provide the proofs for the results presented in the accompanying extended abstract.

A.1. Proof for Lemma 1

PROOF Consider the following construction for $\alpha = \frac{\pi}{2}$: Draw a line l passing through any sensor s_i and any point in $\mathcal{A}$ (denoted by x). For all sensors s_b such that vector product $\overrightarrow{xs_i} \times \overrightarrow{xs_b}$ is zero or negative, let θ^m be the direction obtained by rotating $\overrightarrow{s_bx}$ clockwise by $\frac{\pi}{2}$ and for all sensors s_u such that $\overrightarrow{xs_i} \times \overrightarrow{xs_u}$ is positive, let θ^m be obtained by rotating $\overrightarrow{s_ux}$ counter-clockwise by $\frac{\pi}{2}$. For all sensors, the sensing wedges are half-planes described by lines which pass through x . Further each sensing wedge contains the half-line starting from x and passing through s_i . Hence, the intersection of all sensing wedges is unbounded. For all $\alpha > \frac{\pi}{2}$, the sensing wedges are a superset of that obtained with $\alpha = \frac{\pi}{2}$. Hence, the proof holds.

A.2. Proof for Lemma 2

Figure 5: Five sensors placed uniformly on a circle with radius r . The true target is located at the center of the circle. Each sensor receives a measurement with no noise.

PROOF Before we prove the lower bound for *any* placement, we first consider the case of $n \geq 3$ sensors placed uniformly on the boundary of C (Figure 5). The true target location x is at the center o and all sensors receive measurements θ_i^m without any noise. Hence, for any point q on any sensing wedge, we have $oq \geq r \sin \alpha$, since $r \sin \alpha$ is the perpendicular distance of o to any of the bounding half-planes.

Recall that $\hat{P}$ denotes the possibly unbounded convex polygonal region of intersection of sensing wedges. We claim that the area of $\hat{P}$ in this case is lower bounded by $\pi r^2 \sin^2 \alpha$. Since o is in the intersection, if we show that there is a circle C centered at o with radius $r \sin \alpha$ such that it lies in the interior or shares a boundary point with $\hat{P}$ then the claim holds. Suppose there is a point p in the interior of C such that $p \notin \hat{P}$. Let $p' \neq p$ be the point of intersection of the segment op with the boundary of $\hat{P}$. We have $op' < op < r \sin \alpha$. Since p' lies on the boundary of $\hat{P}$, it must also lie on one of the bounding half-planes of some sensing wedge. Hence, $op' \geq r \sin \alpha$ which is a contradiction. Since n here was arbitrary, the result holds for the case when the sensors are everywhere on the boundary of C .

Now consider any placement of sensors S . From [Definition 3](#), we know that $U(S)$ is defined as the maximum over all possible true target locations. Hence, $U(S) \geq U(S|x = o)$ which is the uncertainty when the target is located at o . Assume that all sensors receive a measurement with zero noise, i.e., $\theta_i^m = \theta_i^t$ for all $i = 1, \dots, n$. Hence, $U_D(S)$ and $U_A(S)$ are lower bounded by the diameter and area of intersection of all such sensing wedges.

We further lower bound this by the following construction: Replace each sensor s_i with the point of intersection of the boundary of C with the segment joining o and s_i . Denote this point of intersection by s'_i . Note $W(s_i, \theta_i^t) \supseteq W(s'_i, \theta_i^t)$ and hence the intersection area and diameter formed by $W(s'_i, \theta_i^t)$ is a lower bound on $U(S)$. Hence,

$$U_A(S) \geq \text{Area} \left(\bigcap_{i=1}^n W(s_i, \theta_i^t) \right) \geq \text{Area} \left(\bigcap_{i=1}^n W(s'_i, \theta_i^t) \right) \geq \pi r^2 \sin^2(\alpha)$$

since the last step covers the case that the sensors are everywhere on the circle including all s'_i locations. Similarly, $U_D(S) \geq 2r \sin \alpha$.

A.3. Proof for [Corollary 1](#)

PROOF Suppose the corollary does not hold. Then there must exist a point, say p , with distance $r^* > \frac{U_D^*}{2 \sin \alpha}$ (or $r^* > \sqrt{\frac{U_A^*}{\pi}} \frac{1}{\sin \alpha}$) to the boundary and any sensor in S^* . We can draw a circle lying completely inside $\mathcal{A}$ with radius r^* centered at p , not containing any sensor in its interior. By [Lemma 2](#), $U_D^* \geq 2r^* \sin \alpha$ or $r^* \leq \frac{U_D^*}{2 \sin \alpha}$ (equivalently $U_A^* \geq \pi r^{*2} \sin^2 \alpha$ or $r^* \leq \sqrt{\frac{U_A^*}{\pi}} \frac{1}{\sin \alpha}$), which is a contradiction.

A.4. Proof for [Corollary 2](#)

PROOF We first show that $U_D^* < \sin \alpha \cdot d$ implies $d > 2r^*$. Suppose not, i.e., $d \leq 2r^*$. Hence we have, $U_D^* < 2r^* \sin \alpha$. However, from [Corollary 1](#) we have $U_D^* \geq 2r^* \sin \alpha$, which is a contradiction. Similarly we can show $U_A^* < \frac{\pi \sin^2 \alpha}{4} \cdot d^2$ implies $d > 2r^*$.

Consider a square $\mathcal{A}'$ centered at the center of $\mathcal{A}$ but with side length $d - 2r^*$. From the definition of r^* , we observe that any point within $\mathcal{A}'$ will be at most r^* away from a sensor in S^* . This implies that the set of circles of radii r^* centered at each sensor in S^* cover $\mathcal{A}'$. Hence,

$$\begin{aligned} n^* \pi r^{*2} &\geq (d - 2r^*)^2, \\ \therefore n^* &\geq \frac{(d - 2r^*)^2}{\pi r^{*2}}. \end{aligned}$$

This completes the proof.

B. Upper Bounds

B.1. Proof for Lemma 3

We break down the proof for Lemma 3 into the following three lemmas for each of the following cases.

First consider the case of the area of intersection of wedges from two sensors. Let $S_{ALG} = \{s_i, s_j, s_k\}$ be three sensors forming an equilateral triangle of side r (Figure 6(a)). We divide $\triangle s_i s_j s_k$ into three equal regions R_{ij}, R_{jk} and R_{ki} as shown in Figure 6(a) using three perpendicular bisectors. Suppose the true target $x \in R_{jk}$. We begin by bounding $\text{Area}(W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m))$, where θ_j^m, θ_k^m are any two valid measurements from s_j and s_k respectively. Let $\hat{x}$ be the intersection of rays along θ_j^m and θ_k^m . Note that $\hat{x}$ is not necessarily coincident with x but $\hat{x}, x \in W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)$. Then we have the following:

Figure 6: (a) We divide the interior of the $\triangle s_i s_j s_k$ with perpendicular bisectors into R_{ij}, R_{jk}, R_{ki} . For $x \in R_{jk}$, we show that the area of intersection of any valid sensing wedges from s_j and s_k is bounded. (b) Intersection of sensing wedges from s'_j and s'_k is a kite. s'_j and s'_k are obtained by extending s_j and s_k along lines θ_j^m and θ_k^m with $\hat{x}s'_j = \hat{x}s'_k = r'$. This kite bounds the intersection of the original sensing wedges.

Lemma 7 (Intersection of two wedges) When $0 < \alpha < \frac{\pi}{18}$,

$$\max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m} \text{Diameter}(W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) \leq 11.35r \sin \alpha$$

and,

$$\max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m} \text{Area}(W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) \leq 23.46r^2 \sin^2(\alpha)$$

PROOF Consider Figure 6(a). We have $\frac{\pi}{3} \leq \angle s_j x s_k \leq \frac{2\pi}{3}$ and $d(s_j, x), d(s_k, x) \leq r$, where $d(\cdot, \cdot)$ gives the distance between two points. We now bound $d(s_j, \hat{x})$, $d(s_k, \hat{x})$ and $\angle s_j \hat{x} s_k$. We have: $\angle s_j \hat{x} s_k = \pi - \angle \hat{x} s_j s_k - \angle \hat{x} s_k s_j$.

$$\therefore \min \angle s_j \hat{x} s_k = \pi - \max \angle \hat{x} s_j s_k - \max \angle \hat{x} s_k s_j \therefore \min \angle s_j \hat{x} s_k = \pi - 2\left(\frac{\pi}{3} + \alpha\right) = \left(\frac{\pi}{3} - 2\alpha\right).$$

Similarly $\max \angle s_j \hat{x} s_k = \pi - \min \angle \hat{x} s_j s_k - \min \angle \hat{x} s_k s_j = (\frac{2\pi}{3} + 2\alpha)$. Hence, we have $(\frac{\pi}{3} - 2\alpha) \leq \angle s_j \hat{x} s_k \leq (\frac{2\pi}{3} + 2\alpha)$. Since both $\angle \hat{x} s_j s_k$ and $\angle \hat{x} s_k s_j$ are acute, $s_j s_k = s_j y + s_k y$. Consider $\triangle \hat{x} y s_k$. By law of sines we have,

$$\frac{\sin(\angle s_j \hat{x} s_k)}{s_j s_k} = \frac{\sin(\angle \hat{x} s_j s_k)}{\hat{x} s_k}$$

$$\hat{x} s_k = \frac{\sin(\angle \hat{x} s_j s_k)}{\sin(\angle s_j \hat{x} s_k)} s_j s_k = \frac{\sin(\angle \hat{x} s_j s_k)}{\sin(\angle s_j \hat{x} s_k)} r \leq \frac{\sin(\frac{\pi}{3} + \alpha)}{\sin(\angle s_j \hat{x} s_k)} r$$

Let $\hat{\theta} \triangleq \angle s_j \hat{x} s_k$ and $r' \triangleq \frac{\sin(\frac{\pi}{3} + \alpha)}{\sin(\hat{\theta})} r$. By symmetry we have $d(s_j, \hat{x}), d(s_k, \hat{x}) \leq r'$. The actual distance varies depending on θ_j^m, θ_k^m . In order to bound $\text{Area}(W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m))$ for all possible θ_j^m, θ_k^m , we construct the following instance:

- For any θ_j^m, θ_k^m draw a circle C , centered at $\hat{x}$ with radius r' .
- Extend segment from $\hat{x}$ towards s_j and s_k . Let s'_j and s'_k be its points of intersection with C . Note that $s_j, s_k \in C$, $d(s'_j, \hat{x}) = d(s'_k, \hat{x}) = r'$, and $W(s_j, \theta_j^m) \subseteq W(s'_j, \theta_j^m)$ and $W(s_k, \theta_k^m) \subseteq W(s'_k, \theta_k^m)$.

We have, $\text{Area}(W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) \leq \text{Area}(W(s'_j, \theta_j^m) \cap W(s'_k, \theta_k^m))$. We first show that the intersection of $h_{j'}^l, h_{j'}^r, h_{k'}^l, h_{k'}^r$ is a bounded quadrilateral $acbd$ (Figure 6(b)). First consider the intersection of $h_{j'}^r$ with $h_{k'}^l$ denoted by a . We have,

$$\angle s'_j a s'_k = \pi - 2(\angle a s'_j s'_k) = \pi - 2\left(\frac{\pi}{2} - \frac{\hat{\theta}}{2} - \alpha\right)$$

$$= \hat{\theta} + 2\alpha \leq \frac{2\pi}{3} + 4\alpha < \pi.$$

Hence a lies on the same side of $s'_j s'_k$ as $\hat{x}$. Further this implies that the intersection of $h_{j'}^l$ with $h_{j'}^r$ (i.e., s'_j) and $h_{k'}^l$ with $h_{k'}^r$ (i.e., s'_k) does not lie within or on the intersection region of the sensing wedges. Next consider the intersection of $h_{j'}^l$ with $h_{k'}^r$ denoted by b . Suppose that b does not exist. Let $b_j \neq s'_j$ and $b_k \neq s'_k$ be any points on $h_{j'}^l$ and $h_{k'}^r$ respectively. Since $h_{j'}^l$ and $h_{k'}^r$ do not intersect we have,

$$\angle b_j s'_j s'_k = \angle b_j s'_j \hat{x} + \angle \hat{x} s'_j s'_k \geq \frac{\pi}{2}$$

$$\therefore \alpha + \frac{\pi}{2} - \frac{\hat{\theta}}{2} \geq \frac{\pi}{2} \implies \alpha \geq \frac{\pi}{12}$$

which is a contradiction since $0 < \alpha < \frac{\pi}{18}$. Since a and b exist the intersection of $h_{j'}^r$ with $h_{k'}^r$ (denoted by c) and $h_{j'}^l$ with $h_{k'}^l$ (denoted by d) also exist. By symmetry, it is easy to show that $\triangle bcs'_j \cong \triangle bds'_k$ and $\triangle das'_j \cong \triangle cas'_k$ so that $bc = bd$ and $ac = ad$. Thus, $acbd$ is a kite. Furthermore, $\hat{x}$ is collinear with a and b since $\triangle s'_j \hat{x} s'_k$ and $\triangle s'_j a s'_k$ are both isosceles.

By property of a kite, $ab \perp cd$ and $\text{Area}(acbd) = 0.5ab \times cd$. Next, we compute the length of the two diagonals ab and cd . First we find $ab = a\hat{x} + \hat{x}b$.

We have $\triangle s'_j \hat{x} a \cong \triangle s'_k \hat{x} a$. Hence, $\angle s'_j \hat{x} a = \frac{\angle s'_j \hat{x} s'_k}{2}$. Thus, $\angle s'_j \hat{x} a = \frac{\hat{\theta}}{2}$. Further $\angle s'_j a \hat{x} = \pi - \left(\frac{\hat{\theta}}{2} + \alpha\right)$ yielding $\angle s'_j b \hat{x} = \angle s'_j b a = \left(\frac{\hat{\theta}}{2} - \alpha\right)$. By law of sines in $\triangle s'_j b \hat{x}$,

$$\hat{x}b = \frac{\sin(\angle b s'_j \hat{x})}{\sin(\angle s'_j b \hat{x})} s'_j \hat{x} = \frac{\sin(\alpha)}{\sin\left(\frac{\hat{\theta}}{2} - \alpha\right)} r'.$$

Similarly applying law of sines in $\triangle s'_j \hat{x} a$ we get $a\hat{x} = \frac{\sin(\alpha)}{\sin\left(\frac{\hat{\theta}}{2} + \alpha\right)} r'$. Hence,

$$ab = a\hat{x} + \hat{x}b = r' \sin \alpha \left[\sin^{-1}\left(\frac{\hat{\theta}}{2} + \alpha\right) + \sin^{-1}\left(\frac{\hat{\theta}}{2} - \alpha\right) \right] \quad (1)$$

If we extend segment from b to a onto $s'_j s'_k$, by symmetry we can show that the segment is a perpendicular bisector of $s'_j s'_k$. Since $cd \perp ab$, quadrilateral $abcd$ forms a trapezoid. We have,

$$\angle cs'_k s'_j = \angle cs'_k \hat{x} + \angle \hat{x} s'_k s'_j = \alpha + \left(\frac{\pi}{2} - \frac{\hat{\theta}}{2}\right) = \frac{\pi}{2} - \left(\frac{\hat{\theta}}{2} - \alpha\right).$$

Hence $\angle dcs'_k = \pi - \angle cs'_k s'_j = \frac{\pi}{2} + \left(\frac{\hat{\theta}}{2} - \alpha\right)$ and $\angle ds'_k s'_j = \frac{\pi}{2} - \left(\frac{\hat{\theta}}{2} + \alpha\right)$.

Drop a perpendicular from d onto $s'_j s'_k$ and let y be its point of intersection. Since $\angle bs'_k s'_j, bs'_j s'_k < \frac{\pi}{2}$, y lies between s'_j and s'_k . Now we get,

$$ys'_k = ds'_k \cos\left(\frac{\pi}{2} - \left(\frac{\hat{\theta}}{2} + \alpha\right)\right) = ds'_k \sin\left(\frac{\hat{\theta}}{2} + \alpha\right)$$

and

$$dy = ds'_k \sin\left(\frac{\pi}{2} - \left(\frac{\hat{\theta}}{2} + \alpha\right)\right) = ds'_k \cos\left(\frac{\hat{\theta}}{2} + \alpha\right)$$

Further,

$$\angle ds'_j s'_k = \angle bs'_j s'_k = \angle bs'_j \hat{x} + \angle \hat{x} s'_j s'_k = \alpha + \frac{\pi}{2} - \frac{\hat{\theta}}{2} = \frac{\pi}{2} - \left(\frac{\hat{\theta}}{2} - \alpha\right).$$

Therefore,

$$s'_j y = dy \cot(\angle ds'_j y) = dy \tan\left(\frac{\hat{\theta}}{2} - \alpha\right) = ds'_k \cos\left(\frac{\hat{\theta}}{2} + \alpha\right) \tan\left(\frac{\hat{\theta}}{2} - \alpha\right)$$

We can now solve for ds'_k ,

$$\begin{aligned} s'_j s'_k &= s'_j y + ys'_k \\ \therefore 2r' \sin\left(\frac{\hat{\theta}}{2}\right) &= ds'_k \cos\left(\frac{\hat{\theta}}{2} + \alpha\right) \tan\left(\frac{\hat{\theta}}{2} - \alpha\right) + ds'_k \sin\left(\frac{\hat{\theta}}{2} + \alpha\right) \\ \therefore ds'_k &= \frac{2r' \sin\left(\frac{\hat{\theta}}{2}\right)}{\sin\left(\frac{\hat{\theta}}{2} + \alpha\right) + \cos\left(\frac{\hat{\theta}}{2} + \alpha\right) \tan\left(\frac{\hat{\theta}}{2} - \alpha\right)} \end{aligned}$$

By law of sines in $\triangle dcs'_k$,

$$\begin{aligned}
\frac{cd}{\sin(\angle cs'_k d)} &= \frac{ds'_k}{\sin(\angle dcs'_k)} \\
\therefore cd &= \frac{2r' \sin(2\alpha) \sin\left(\frac{\hat{\theta}}{2}\right)}{\sin\left(\frac{\pi}{2} + \left(\frac{\hat{\theta}}{2} - \alpha\right)\right) \left[\sin\left(\frac{\hat{\theta}}{2} + \alpha\right) + \cos\left(\frac{\hat{\theta}}{2} + \alpha\right) \tan\left(\frac{\hat{\theta}}{2} - \alpha\right)\right]} \\
&= \frac{4r' \sin(\alpha) \cos(\alpha) \sin\left(\frac{\hat{\theta}}{2}\right)}{\cos\left(\frac{\hat{\theta}}{2} - \alpha\right) \left[\sin\left(\frac{\hat{\theta}}{2} + \alpha\right) + \cos\left(\frac{\hat{\theta}}{2} + \alpha\right) \tan\left(\frac{\hat{\theta}}{2} - \alpha\right)\right]} \\
&= \frac{4r' \sin(\alpha) \cos(\alpha) \sin\left(\frac{\hat{\theta}}{2}\right)}{\sin(\hat{\theta})} = \frac{2r' \sin(\alpha) \cos(\alpha)}{\cos\left(\frac{\hat{\theta}}{2}\right)} \tag{2}
\end{aligned}$$

Using Equations 1 and 2 we get,

$$\begin{aligned}
\text{Area}(acbd) &= 0.5ab \times cd \\
&= \frac{0.5 \times 2r'^2 \sin^2(\alpha) \cos(\alpha) \left(\sin^{-1}\left(\frac{\hat{\theta}}{2} + \alpha\right) + \sin^{-1}\left(\frac{\hat{\theta}}{2} - \alpha\right)\right)}{\cos\left(\frac{\hat{\theta}}{2}\right)}
\end{aligned}$$

Now, $r' = r \frac{\sin\left(\frac{\pi}{3} + \alpha\right)}{\sin(\hat{\theta})} \leq r \frac{\sin\left(\frac{\pi}{3} + \frac{\pi}{18}\right)}{\sin(\hat{\theta})} = \frac{0.9397r}{\sin(\hat{\theta})}$.

Substituting,

$$\text{Area}(acbd) \leq \frac{0.9397^2 r^2 \sin^2(\alpha) \cos(\alpha) \left(\sin^{-1}\left(\frac{\hat{\theta}}{2} + \alpha\right) + \sin^{-1}\left(\frac{\hat{\theta}}{2} - \alpha\right)\right)}{\sin^2(\hat{\theta}) \cos\left(\frac{\hat{\theta}}{2}\right)}$$

Recall that we have $\left(\frac{\pi}{3} - 2\alpha\right) \leq \hat{\theta} \leq \left(\frac{2\pi}{3} + 2\alpha\right)$ and $0 < \alpha < \frac{\pi}{18}$. We split this into the following:

(a) $\left(\frac{\pi}{3} - 2\alpha \leq \hat{\theta} \leq \frac{\pi}{2}\right)$ and (b) $\left(\frac{\pi}{2} < \hat{\theta} \leq \frac{2\pi}{3} + 2\alpha\right)$.

For (a) we get,

$$\text{Area}(acbd) \leq \frac{0.9397^2 r^2 \sin^2(\alpha) \left(\sin^{-1}\left(\frac{\pi}{6}\right) + \sin^{-1}\left(\frac{\pi}{18}\right)\right)}{\sin^2\left(\frac{2\pi}{9}\right) \cos\left(\frac{\pi}{4}\right)} \leq 23.46r^2 \sin^2(\alpha).$$

Similarly for (b) we get,

$$\text{Area}(acbd) \leq \frac{0.9397^2 r^2 \sin^2(\alpha) \left(\sin^{-1}\left(\frac{\pi}{4}\right) + \sin^{-1}\left(\frac{7\pi}{36}\right)\right)}{\sin^2\left(\frac{2\pi}{9}\right) \cos\left(\frac{7\pi}{18}\right)} \leq 19.74r^2 \sin^2(\alpha).$$

The diameter for the intersection of wedges is bounded by the diameter for $acbd$. We have,

$$\begin{aligned}
\max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m} \text{Diameter}(W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) &\leq \text{Diameter}(acbd) \\
&= \max\{ab, cd\} \\
&\leq 11.35r \sin \alpha,
\end{aligned}$$

where the last step is obtained by substituting the expressions for ab and cd using Equation 1 and 2 and finding the bound similar to the area case.

For other cases since $\alpha \geq \frac{\pi}{18}$, we know from Lemma 2 that $U(S^*)$ is lower bounded by $\pi r^{*2} \sin^2(\frac{\pi}{18}) \approx 0.095r^{*2}$ unlike in Lemma 7 where the lower bound could be arbitrarily small. However since $\alpha \geq \frac{\pi}{18}$ we cannot use just two sensors to get a good approximation especially when the angle between the sensors and the target is close to $\frac{\pi}{3}$. Instead we approximate the sensing wedges from three neighboring sensors in the next lemma.

Lemma 8 (Intersection of three sensing wedges) When $\frac{\pi}{18} \leq \alpha < \frac{\pi}{12}$,

$$\max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m, \theta_k^m} \text{Diameter}(W(s_i, \theta_i^m) \cap W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) \leq 2.04r$$

and

$$\max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m, \theta_k^m} \text{Area}(W(s_i, \theta_i^m) \cap W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) \leq \frac{\sqrt{3}r^2}{4} + 10.001r^2 \sin^2 \alpha.$$

Figure 7: Instead of projecting the sensors back, as in Figure 6, we approximate each sensor by taking the union of all feasible sensing wedges when $x \in R_{jk}$. For (a) $\frac{\pi}{18} \leq \alpha < \frac{\pi}{12}$, and (b) $\frac{\pi}{12} \leq \alpha < \frac{\pi}{6}$. Note for R_{jk} is different for (a) and (b).

PROOF Consider Figure 7(a) where s_i, s_j, s_k are three neighboring sensors. $x \in R_{jk}$ where R_{jk} is defined as in Lemma 7. We have $\frac{\pi}{3} \leq \angle xs_js_k, \angle xs_k s_j \leq \frac{2\pi}{3}$ and $0 \leq \angle xs_i s_j \leq \frac{\pi}{3}$. For each sensor we are going to build a larger sensing wedge with the union of sensing wedges $\forall x \in R_{jk}$. Denote such a sensing wedge by $\hat{W}(s_t, R_{jk})$ where $t = i, j, k$. $\hat{W}$ is an approximation to the actual sensing wedges: $W(s_t, \theta_t^m) \subset \hat{W}(s_t, R_{jk}), \forall x \in R_{jk}$. We have,

$$\max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m, \theta_k^m} \text{Area} \left(\bigcap_{t=i,j,k} W(s_t, \theta_t^m) \right) \leq \text{Area} \left(\bigcap_{t=i,j,k} \hat{W}(s_t, R_{jk}) \right)$$

$\hat{W}(s_j, R_{jk})$ and $\hat{W}(s_k, R_{jk})$ have an angular width of $\frac{\pi}{6} + 4\alpha$ whereas $\hat{W}(s_i, R_{jk})$ has $\frac{\pi}{3} + 4\alpha$. Let $acebfd$ be the six-sided convex polygon formed by the intersection of three $\hat{W}$'s as shown in Figure 7(a). Since there are six bounding half-lines, we have fifteen cases to consider.

We denote the bounding half-planes of $\hat{W}(s_t, R_{jk})$ by $\hat{h}_t^l$ and $\hat{h}_t^r$. Consider point of intersection of $\hat{h}_j^r$ and $\hat{h}_k^l$, denoted by a . In $\triangle s_j a s_k$ we have,

$$\angle s_j a s_k = \pi - 2 \left(\frac{\pi}{6} - 2\alpha \right) = \frac{2\pi}{3} + 4\alpha \leq \pi.$$

Hence a lies to the same side of $s_j s_k$ as s_i and thus s_j (intersection of $\hat{h}_j^l$ and $\hat{h}_j^r$) and s_k (intersection of $\hat{h}_k^l$ and $\hat{h}_k^r$) do not lie on or within the intersecting polygon. Now consider the intersection of $\hat{h}_j^l$ and $\hat{h}_i^r$, denoted by f . In $\triangle s_i f s_j$ we have,

$$\angle s_i f s_j = \pi - 4\alpha.$$

Since $\frac{\pi}{18} \leq \alpha < \frac{\pi}{12}$,

$$\frac{2\pi}{3} < \angle s_i f s_j \leq \frac{7\pi}{9}.$$

Denote by d the point of intersection of $\hat{h}_j^l$ and $\hat{h}_k^l$. We have

$$\angle s_j d s_k = \pi - \left(\frac{\pi}{3} + 2\alpha + \frac{\pi}{6} - 2\alpha \right) = \frac{\pi}{2}.$$

The considerations for e and c (as shown in Figure 7(a)) are symmetric to f and d respectively. Vertex b is coincident with s_i , hence the points of intersection of $\hat{h}_j^r$ with $\hat{h}_i^r$, $\hat{h}_j^l$ with $\hat{h}_i^l$, $\hat{h}_k^r$ with $\hat{h}_i^r$, and $\hat{h}_k^l$ with $\hat{h}_i^l$ do not lie within or on the intersecting polygon. Similarly, the point of intersection of $\hat{h}_j^l$ with $\hat{h}_k^r$ is at a height of $\left(\frac{r}{2} \tan \left(\frac{\pi}{3} + 2\alpha \right) > \frac{\sqrt{3}}{2} r \right)$ perpendicular to $s_j s_k$ and hence outside $\hat{W}(s_i, R_{jk})$. Finally consider g and h , the points of intersection of $\hat{h}_j^r$ with $\hat{h}_i^l$ and $\hat{h}_k^l$ with $\hat{h}_i^r$ respectively. Since $\angle s_j d s_k = \frac{\pi}{2}$ i.e., $\hat{h}_j^l \perp \hat{h}_k^l$ and $\hat{h}_i^r$ intersects $\hat{h}_j^l$, then s_i and h cannot lie on the same side of $\hat{h}_j^l$. Since $s_i = b$ lies on the intersecting polygon, hence h does not. The case for g is symmetric.

We can verify that the vertices appear cyclically on the intersection polygon in the order $acebfd$. Hence, the intersecting polygon of the three $\hat{W}$ wedges is the six-sided convex polygon $acebfd$.

Now we compute and bound the area of the polygon:

$$\text{Area}(acebfd) = \text{Area}(agbh) - 2\text{Area}(\triangle dfh) = 0.5ab \times gh - df \times hd. \quad (3)$$

The above equation follows from the properties that $\triangle dfh \cong \triangle ceg$ and quadrilateral $agbh$ is a kite due to symmetry. Next we compute each of the lengths in turn. Drop a perpendicular from a onto $s_j s_k$ and denote its point of intersection with y . Since $\triangle s_j a s_k$ is isosceles, y, a and b are collinear. We have,

$$\begin{aligned} ab &= by - ay \\ &= \frac{\sqrt{3}r}{2} - \frac{r}{2} \tan \left(\frac{\pi}{6} - 2\alpha \right) \\ &= \frac{r}{2} \left(\sqrt{3} - \tan \left(\frac{\pi}{6} - 2\alpha \right) \right) \end{aligned} \quad (4)$$

Similarly f , w and s_k are collinear. Thus,

$$\begin{aligned} s_k f &= s_k w + w f \\ &= \frac{\sqrt{3}r}{2} + \frac{r \tan(2\alpha)}{2} = \frac{r}{2} \left(\sqrt{3} + \tan(2\alpha) \right). \\ df &= s_k f \sin(2\alpha) = \frac{r \sin(2\alpha)}{2} \left(\sqrt{3} + \tan(2\alpha) \right). \\ ds_k &= s_k f \cos(2\alpha) = \frac{r \cos(2\alpha)}{2} \left(\sqrt{3} + \tan(2\alpha) \right). \end{aligned}$$

Consider $\triangle s_j ds_k$,

$$ds_k = s_j s_k \sin\left(\frac{\pi}{3} + 2\alpha\right) = r \sin\left(\frac{\pi}{3} + 2\alpha\right)$$

Now in $\triangle df s_k$ we get,

$$df = ds_k \tan(2\alpha) = r \sin\left(\frac{\pi}{3} + 2\alpha\right) \tan(2\alpha) \quad (5)$$

In $\triangle dfh$ we have $\angle hdf = \frac{\pi}{2}$ and

$$\angle hfd = \pi - \angle s_j f s_i = \pi - (\pi - 4\alpha) = 4\alpha.$$

Therefore,

$$hd = df \tan(\angle hfd) = r \sin\left(\frac{\pi}{3} + 2\alpha\right) \tan(2\alpha) \tan(4\alpha) \quad (6)$$

Finally since $agbh$ is a kite, ab is a perpendicular bisector of gh giving

$$gh = 2ah \sin(\angle hab) = 2ah \sin\left(\frac{\angle s_j a s_k}{2}\right) = 2ah \sin\left(\frac{\pi}{3} + 2\alpha\right)$$

where,

$$\begin{aligned} ah &= ds_k + hd - s_k a \\ &= r \sin\left(\frac{\pi}{3} + 2\alpha\right) + r \sin\left(\frac{\pi}{3} + 2\alpha\right) \tan(2\alpha) \tan(4\alpha) - \frac{r}{2 \sin\left(\frac{\pi}{3} + 2\alpha\right)}. \end{aligned}$$

Substituting we get,

$$gh = 2r \sin^2\left(\frac{\pi}{3} + 2\alpha\right) [1 + \tan(2\alpha) \tan(4\alpha)] - r. \quad (7)$$

Now since $\frac{\pi}{18} \leq \alpha < \frac{\pi}{12}$ from Equations 4 and 7 we get,

$$\begin{aligned}
ab \times gh &= \frac{r}{2} \left[\sqrt{3} - \tan \left(\frac{\pi}{6} - 2\alpha \right) \right] \times \left[2r \sin^2 \left(\frac{\pi}{3} + 2\alpha \right) [1 + \tan(2\alpha) \tan(4\alpha)] - r \right] \\
&\leq \frac{\sqrt{3}r}{2} \times \left[2r \sin^2 \left(\frac{\pi}{3} + 2\alpha \right) [1 + \tan(2\alpha) \tan(4\alpha)] - r \right] \\
&\leq \sqrt{3}r^2 (1 + \tan 2\alpha \tan 4\alpha) - \frac{\sqrt{3}r^2}{2} \\
&= \frac{\sqrt{3}r^2}{2} + \frac{8\sqrt{3}r^2 \sin^2 \alpha \cos^2 \alpha}{\cos 4\alpha} \\
&\leq \frac{\sqrt{3}r^2}{2} + \frac{8\sqrt{3}r^2 \sin^2 \alpha \cos^2 \left(\frac{\pi}{18} \right)}{\cos \left(\frac{\pi}{3} \right)} \\
&\leq \frac{\sqrt{3}r^2}{2} + 26.88r^2 \sin^2 \alpha.
\end{aligned}$$

Also from Equations 5 and 6,

$$\begin{aligned}
df \times hd &= r^2 \sin^2 \left(\frac{\pi}{3} + 2\alpha \right) \tan^2 2\alpha \tan 4\alpha \\
&\geq r^2 \sin^2 \left(\frac{\pi}{3} + \frac{\pi}{9} \right) \tan 2\alpha (\tan 2\alpha \tan 4\alpha) \\
&\geq r^2 \sin^2 \left(\frac{4\pi}{9} \right) \tan 2\alpha \left(\frac{8 \sin^2 \alpha \cos^2 \alpha}{\cos 4\alpha} \right) \\
&\geq 8r^2 \sin^2 \alpha \sin^2 \left(\frac{4\pi}{9} \right) \tan \left(\frac{\pi}{9} \right) \left(\frac{\cos^2 \left(\frac{\pi}{12} \right)}{\cos \left(\frac{2\pi}{9} \right)} \right) \\
&\geq 3.439r^2 \sin^2 \alpha.
\end{aligned}$$

Finally substituting in Equation 3,

$$\begin{aligned}
\text{Area}(acebfd) &\leq \frac{\sqrt{3}r^2}{4} + 13.44r^2 \sin^2 \alpha - 3.439r^2 \sin^2 \alpha \\
&= \frac{\sqrt{3}r^2}{4} + 10.001r^2 \sin^2 \alpha.
\end{aligned}$$

The bound for diameter follows immediately,

$$\begin{aligned}
\max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m, \theta_k^m} \text{Diameter}(W(s_i, \theta_i^m) \cap W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) &\leq \text{Diameter}(acebfd) \\
&\leq \text{Diameter}(agbh) \\
&= \max\{ab, gh\} \\
&\leq 2.04r \sin \alpha
\end{aligned}$$

where the last step is obtained by substituting the values of ab and gh .

Next we consider $\frac{\pi}{12} \leq \alpha < \frac{\pi}{6}$ in the following lemma. Instead of using R_{jk} as in Figure 7(a) we now use the partition as shown in Figure 7(b). Similar to the previous lemma, we consider for each sensor the union of all possible sensing wedges, and show that the intersection of such wedges for the three sensors is bounded.

Lemma 9 (Intersection of three sensing wedges) When $\frac{\pi}{12} \leq \alpha < \frac{\pi}{6}$,

$$\max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m, \theta_k^m} \text{Diameter}(W(s_i, \theta_i^m) \cap W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) \leq \left(1 + \frac{1}{\sqrt{3}}\right) r.$$

and

$$\max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m, \theta_k^m} \text{Area}(W(s_i, \theta_i^m) \cap W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) \leq \frac{3\sqrt{3}}{4} r^2.$$

PROOF We use a procedure similar to the previous lemma, except that the assigned regions $x \in R_{jk}$ is changed (Figure 7(b)). We have $\frac{2\pi}{3} \leq \angle s_j x s_k \leq \pi$ and $0 \leq \angle s_j x s_i \leq \frac{\pi}{3}$. For sensors s_j and s_k , we draw sensing wedges $\hat{W}(s_j, R_{jk})$ and $\hat{W}(s_k, R_{jk})$ that approximate any valid sensing wedges. As before, we denote the bounding half-lines of $\hat{W}(s_j, R_{jk})$ ($\hat{W}(s_k, R_{jk})$) by $\hat{h}_j^l$ and $\hat{h}_j^r$ ($\hat{h}_k^l$ and $\hat{h}_k^r$ respectively). We consider the intersection of all the bounding half-lines and bound the area using these points.

Since $\alpha < \frac{\pi}{6}$, we draw $\hat{h}_j^r$ by rotating line through $s_j s_k$ clockwise by $\frac{\pi}{3}$ about s_j . Similarly $\hat{h}_j^l$ is drawn by rotating the line through s_j and perpendicular to $s_i s_k$ counter-clockwise by $\frac{\pi}{3}$ about s_j . $\hat{h}_k^l$ and $\hat{h}_k^r$ are drawn symmetrically. $\hat{h}_i^l$ ($\hat{h}_i^r$) are drawn by rotating line $s_i s_k$ ($s_i s_j$) counter-clockwise (clockwise) by $\frac{\pi}{3}$ about s_i . Note that for all $x \in R_{jk}$, for any valid θ_j^m and θ_k^m we have $\hat{W}(s_j, R_{jk}) \supseteq W(s_j, \theta_j^m)$ and $\hat{W}(s_k, R_{jk}) \supseteq W(s_k, \theta_k^m)$.

Denote by d the intersection of $\hat{h}_i^r$ with $\hat{h}_j^l$, by c the intersection of $\hat{h}_i^l$ with $\hat{h}_k^r$, and by a the intersection of $\hat{h}_j^r$ with $\hat{h}_k^l$. We can see that $dc s_k s_j$ is a rectangle with sides $s_j s_k = dc = r$ and $s_j d = s_k c = \frac{\sqrt{3}r}{3}$. Further $\triangle s_j s_k a$ is an equilateral triangle with side r . Hence,

$$\begin{aligned} & \max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m, \theta_k^m} \text{Area}(W(s_i, \theta_i^m) \cap W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) \\ & \leq \text{Area}(s_j s_k c d) + \text{Area}(s_j s_k a) = \frac{\sqrt{3}}{2} r^2 + \frac{\sqrt{3}}{4} r^2 = \frac{3\sqrt{3}}{4} r^2. \end{aligned}$$

The bound on the diameter follows:

$$\begin{aligned} \max_{x \in R_{jk}} \max_{\theta_i^m, \theta_j^m, \theta_k^m} \text{Diameter}(W(s_i, \theta_i^m) \cap W(s_j, \theta_j^m) \cap W(s_k, \theta_k^m)) & \leq \text{Diameter}(a s_j d c s_k) \\ & = ad = \left(1 + \frac{1}{\sqrt{3}}\right) r \end{aligned}$$

PROOF (LEMMA 3) Using Lemmas 7, 8, and 9.

B.2. Proof for Lemma 4

PROOF Consider Figure 8(a). Sensors $s_i, \dots, s_o$ are seven neighboring sensors in S_{ALG} : s_k is the center of a regular hexagon of side length r formed by $s_i, s_j, s_l, s_m, s_n, s_o$. Instead of considering that the target lies in one of the triangular faces, we instead draw a circle of radius $r' = \frac{r}{\sqrt{3}}$ centered at each sensor and bound the uncertainty when the true target lies in one of such circles. The union of all such circles covers $\mathcal{A}$ and hence, the uncertainty bound holds for any true target location. Denote by C_k the circle corresponding to sensor s_k as shown in Figure 8(a), and let $x \in C_k$.

Figure 8: Bounding the intersecting area with 6 sensors $s_i, s_j, s_l, s_m, s_n, s_o$. The target can be anywhere within a circle C of radius $\frac{r}{\sqrt{3}}$ centered at sensor s_k . (a) We ignore s_k and approximate each other sensor by taking the union of all valid sensing wedges when $x \in C$. The resulting shape is a star with six petals. (b) An instance of execution with randomly generated target (blue square) and measurements with $\pm\alpha = \frac{\pi}{4}$.

As in the previous lemmas, we approximate the sensing wedges from each sensor and then bound their intersection area. Consider sensor s_i in Figure 8(a). When $x \in C_k$, θ_i^t is bounded between the two tangents from s_i to C_k i.e., between within $\angle as_ie$. From $\triangle s_i as_k$ we have, $\angle as_i s_k = \sin^{-1}\left(\frac{as_k}{s_i s_k}\right) = \sin^{-1}\left(\frac{r}{\sqrt{3}r}\right) = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$. Define $\theta \triangleq \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$. By symmetry, $\angle ds_i s_k = \angle cs_j s_k = \theta$. Recall that the sensing wedge is defined as the intersection of two bounding half-planes.

We now approximate *all* sensing wedges from s_i (when $x \in C_k$) by a larger wedge $\hat{W}(s_i, C_k)$ using the following operations: (a) Draw $\hat{h}_i^r$ by rotating line $s_i a$ pivoted at s_i , clockwise by an angle of 2α . (b) Draw $\hat{h}_i^l$ by rotating line $s_i b$ pivoted at s_i , counter-clockwise by an angle of 2α . (c) $\hat{W}(s_i, C_k)$ is defined as the intersection of the two half-planes given by $\hat{h}_i^l$ and $\hat{h}_i^r$. We can see that $\hat{W}(s_i, C_k)$ as defined above ensures that $W(s_i, \theta_i^m) \subset \hat{W}(s_i, C_k)$ for all θ_i^m . We can similarly define $\hat{W}$ for all other sensors. The intersection of any combination of $W(s_t, \theta_t^m)$ sensing wedges lie completely inside the intersection of $\hat{W}$ sensing wedges. Hence, we compute the area of intersection of all $\hat{W}$ sensing wedges next.

Consider two adjacent sensors s_i and s_j as shown in Figure 8(a). We first show that $\hat{h}_i^r$ and $\hat{h}_j^l$ intersect at a point b such that $s_k s_j b s_i$ forms a kite. Suppose not. Let b_i and b_j be any points on $\hat{h}_i^r$ and $\hat{h}_j^l$ lying on opposite sides of the line $s_i s_j$ as s_k . Hence we must have $\angle b_i s_i s_j, \angle b_j s_j s_i \geq \frac{\pi}{2}$. Since $\angle b_i s_i s_k = \theta + 2\alpha$, we get $2\alpha + \theta - \frac{\pi}{3} \geq \frac{\pi}{2}$. Therefore $2\alpha \geq \frac{5\pi}{6} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \implies \alpha \geq \frac{\pi}{3.313}$. But $\alpha \leq \frac{\pi}{4}$, hence its a contradiction and $\hat{h}_i^r$ and $\hat{h}_j^l$ must intersect at a point b . Furthermore by symmetry, $s_k s_j b s_i$ is a kite and bs_k is a perpendicular bisector of $s_i s_j$. We compute the area of the kite next, and use that to bound the total area.

We have,

$$\begin{aligned}
bs_k &= bd + ds_k \\
&= \frac{r}{2} \tan(\angle bs_j s_i) + \frac{\sqrt{3}}{2} r \\
&= \frac{r}{2} \left[\tan\left(\theta + 2\alpha - \frac{\pi}{3}\right) + \sqrt{3} \right]
\end{aligned}$$

By the property of kite,

$$\begin{aligned}
\text{Area}(s_k s_j bs_i) &= 0.5 bs_k \times s_i s_j \\
&= \frac{r^2}{4} \left[\tan\left(2\alpha - \left(\frac{\pi}{3} - \theta\right)\right) + \sqrt{3} \right] \\
&= \frac{r^2}{4} \left[\frac{\sin 2\alpha \cos\left(\frac{\pi}{3} - \theta\right)}{\cos\left(2\alpha - \left(\frac{\pi}{3} - \theta\right)\right)} - \frac{\cos 2\alpha \sin\left(\frac{\pi}{3} - \theta\right)}{\cos\left(2\alpha - \left(\frac{\pi}{3} - \theta\right)\right)} + \sqrt{3} \right] \\
&= \frac{r^2}{4} \left[\frac{2 \sin \alpha \cos \alpha \cos\left(\frac{\pi}{3} - \theta\right) - \cos^2 \alpha \sin\left(\frac{\pi}{3} - \theta\right) + \sin^2 \alpha \sin\left(\frac{\pi}{3} - \theta\right)}{\cos\left(2\alpha - \left(\frac{\pi}{3} - \theta\right)\right)} + \sqrt{3} \right] \\
&\leq \frac{r^2}{4} \left[\frac{2 \sin \alpha \cos \frac{\pi}{6} \cos\left(\frac{\pi}{3} - \theta\right) - \cos^2 \frac{\pi}{4} \sin\left(\frac{\pi}{3} - \theta\right) + \sin^2 \alpha \sin\left(\frac{\pi}{3} - \theta\right)}{\cos\left(\frac{\pi}{2} - \left(\frac{\pi}{3} - \theta\right)\right)} + \sqrt{3} \right] \\
&= \frac{r^2}{4} (\sin^2 \alpha + 3.76 \sin \alpha + 1.232).
\end{aligned}$$

Now the complete intersection polygon is made up of the union of six such kites (one per adjacent sensor pair). Hence, the uncertainty when $x \in C_k$ is bounded by $\frac{6r^2}{4} (\sin^2 \alpha + 3.76 \sin \alpha + 1.232)$.

The diameter can be bounded as,

$$U_D(\{s_1, \dots, s_6\}) \leq 2bs_k \leq r (\sin^2 \alpha + 3.76 \sin \alpha + 1.232)$$

B.3. Proof for Lemma 6

PROOF Consider Figure 9. Regions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are not covered by equilateral triangles. The width of each of these regions is d and the height is at most r . We place additional sensors in each of these regions as shown in Figure 9(b). Each region will require at most $2w_r + b_r$ of such sensors. Observe that, these sensors cover the three regions ($\mathcal{A}_{1,2,3}$) everywhere except the two ends. We will add additional sensors to ensure that $\mathcal{A}$ is covered by equilateral triangles.

Note that amongst the newly added sensors, one sensor is placed at one of the four corners (marked O in Figure 9(b)). We align this corner with vertex B, C and D for regions $\mathcal{A}_3, \mathcal{A}_2$ and $\mathcal{A}_1$ respectively. The uncovered portions of $\mathcal{A}$ are now four regions near each of the corners of sides at most r . For each of the corners we have a situation as shown in Figure 9(c), where already placed sensors are marked as white and black circles. We require two additional sensors per corner (marked as squares) to cover the remaining portion of $\mathcal{A}$. Hence, the total number of sensors required are,

$$n_{ALG} \leq (w_r w_c + b_r b_c) + 3(2w_r + b_r) + 4 \cdot 2.$$

This completes the proof.

Figure 9: (a) Regions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are not covered by equilateral triangles by sensors placed on a triangular grid. (b) We place additional sensors in these three regions to cover all portions except near the ends. (c) We place two additional sensors per corner (shown in square) to cover the rest of $\mathcal{A}$.

B.4. Proof for Lemma 5

Figure 10: The target can lie anywhere within C . All sensors inside a circle of radius kr are projected on to its boundary. The resulting set of sensors has bounded intersection for any $\alpha = \frac{\pi}{2} - \epsilon$ if $\sin^{-1}\left(\frac{1}{\sqrt{3k}}\right) + \frac{\pi}{6(k-1)} < 2\epsilon$.

PROOF The construction is similar to that in Lemma 4. Let o be the center of the circle C of radius $\frac{r}{\sqrt{3}}$ that contains the target. Let s_i be a sensor on the triangular grid at a distance $r' = kr$ from o (let o lie on a grid location). Draw two tangents from s_i onto C . Let a and e be their point of intersection with C (see Figure 10). Draw $\hat{h}_i^l$ and $\hat{h}_i^r$ by rotating $s_i a$ and $s_i b$ about s_i away from $s_i s_k$ by an angle of $2\alpha < \frac{\pi}{2}$. $\hat{h}_i^l$ and $\hat{h}_i^r$ enclose any valid sensing wedge for s_i when the target lies within C .

Let b_l and b_r be a point on the half-line $\hat{h}_i^l$ and $\hat{h}_i^r$ (starting at s_i) respectively. We have,

$$\begin{aligned}\angle b_l s_i o &= \angle b_l s_i a + \angle a s_i o = 2\alpha + \sin^{-1} \left(\frac{1}{\sqrt{3k}} \right) < \pi \\ \angle b_r s_i o &= \angle b_r s_i e + \angle e s_i o = 2\alpha + \sin^{-1} \left(\frac{1}{\sqrt{3k}} \right) < \pi.\end{aligned}$$

Draw a circle C' of radius kr centered at o . We project all sensors lying in C' on to its boundary, along the line passing through the origin and the sensor. $\hat{h}_j^l$ and $\hat{h}_j^r$ drawn about the projected sensor location enclose the original sensing wedges for every s_j . Observe that for every integer k , there exist six sensors in the triangular grid that enclose a regular hexagon of side kr . Further there are $k - 1$ sensors that lie along each side of this hexagon, not counting the sensors on the vertices. Hence, when the sensors are projected on to a circle of radius kr , it must enclose a hexagon of side at least $k - 1$. Further, if s'_i and s'_j are adjacent projected sensors along the boundary, then $\angle s'_i o s'_j \leq \frac{\pi}{3(k-1)}$.

The area of C' is $\pi k^2 r^2$, where r is the grid side. Each triangle in the grid covers an area $\frac{\sqrt{3}}{4} r^2$. Hence, C' contains $\mathcal{O}(k^2)$ sensors.

Now for bounded intersection, we require $\hat{h}_i^l$ to intersect with $\hat{h}_j^r$ (or vice-versa) for every adjacent pair of projected sensors. Suppose the intersection is unbounded for some pair s'_i and s'_j . Then, $\hat{h}_i^l$ does not intersect with $\hat{h}_j^r$. Hence, $\angle b_i s'_i s'_j \geq \frac{\pi}{2}$ where b_i is any point on h_i^l . But $\angle b_i s'_i s'_j = \angle b_i s'_i o - \angle o s'_i s'_j$. Therefore $2\alpha + \sin^{-1} \left(\frac{1}{\sqrt{3k}} \right) - \left(\frac{\pi}{2} - \frac{\pi}{6k} \right) \geq \frac{\pi}{2}$. That is, $\sin^{-1} \left(\frac{1}{\sqrt{3k}} \right) + \frac{\pi}{6(k-1)} \geq 2\epsilon$. This is a contradiction. Hence, the intersection of all valid sensing wedges is bounded.

C. Proof for Theorem 1

PROOF First consider the number of sensors. Let n_{ALG} be the number of sensors placed by our algorithm for covering $\mathcal{A}$ with equilateral triangles. From Lemma 6,

$$\begin{aligned}
n_{ALG} &\leq w_r w_c + b_r b_c + 6w_r + 3b_r + 8, \\
&\leq 2w_r w_c + 9w_r + 8, \\
&\leq 2 \left(\frac{d}{r} + 1 \right) \left(\frac{d}{\sqrt{3}r} + 1 \right) + 9 \left(\frac{d}{r} + 1 \right) + 8, \\
&= \frac{2}{\sqrt{3}} \frac{d^2}{r^2} + \left(11 + \frac{2}{\sqrt{3}} \right) \frac{d}{r} + 19.
\end{aligned} \tag{8}$$

From Corollary 2, we now the number of sensors used by an optimal algorithm is bounded by,

$$\begin{aligned}
n^* &\geq \frac{(d - 2r^*)^2}{\pi r^{*2}}, \\
&= \frac{1}{\pi} \frac{d^2}{r^{*2}} - \frac{4}{\pi} \frac{d}{r^*} + \frac{4}{\pi}.
\end{aligned}$$

Recall from Corollary 1, $r^* \leq \frac{U_D^*}{2 \sin \alpha}$ and $r^* \leq \sqrt{\frac{U_A^*}{\pi}} \frac{1}{\sin \alpha}$. For the triangular grid placement, we set the grid length $r = \frac{U_D^*}{2 \sin \alpha}$ and $r = \sqrt{\frac{U_A^*}{\pi}} \frac{1}{\sin \alpha}$, for the diameter and area uncertainty problems respectively. Hence $r^* \leq r$ which gives,

$$n^* \leq \frac{1}{\pi} \frac{d^2}{r^2} - \frac{4}{\pi} \frac{d}{r} + \frac{4}{\pi}. \tag{9}$$

When $d \leq 14r$, we have $d \leq \frac{14U_D^*}{2 \sin \alpha}$ which implies $U_D^* \geq \frac{d}{7 \sin \alpha}$ (equivalently $d \leq \frac{14}{\sin \alpha} \sqrt{\frac{U_A^*}{\pi}}$ which implies $U_A^* \geq \frac{\pi d^2 \sin^2 \alpha}{196}$). This refers to the case when the desired uncertainty is greater than a constant times the diameter or area of $\mathcal{A}$. In this case, since we know $\frac{d}{r} \leq 14$, we only require a constant number of sensors (n_{ALG}) as given by Equation 8.

Now consider $d > 14r$. Then we have,

$$\begin{aligned}
\frac{2}{\sqrt{3}} \frac{d^2}{r^2} + \left(11 + \frac{2}{\sqrt{3}} \right) \frac{d}{r} + 19 &\leq 9 \left(\frac{1}{\pi} \frac{d^2}{r^2} - \frac{4}{\pi} \frac{d}{r} + \frac{4}{\pi} \right) \\
\therefore n_{ALG} &\leq 9n^*.
\end{aligned}$$

Next, we bound the uncertainty achieved by our placement with that by an optimal. We consider each of the four cases defined in Lemma 3 and Lemma 4 in turn. For the first three cases, we require the target to lie within an equilateral triangle. From Lemma 6 we know that our placement strategy ensures a cover of $\mathcal{A}$ with equilateral triangles. Recall that we set $r = \frac{U_D^*}{2 \sin \alpha}$ and $r = \sqrt{\frac{U_A^*}{\pi}} \frac{1}{\sin \alpha}$, for the diameter and area uncertainty problems respectively.

CASE (a) $0 < \alpha < \frac{\pi}{18}$:

For diameter uncertainty,

$$\begin{aligned}
U_D(S_{ALG}) &\leq 11.35r \sin \alpha, \\
&= 11.35 \frac{U_D^* \sin \alpha}{2 \sin \alpha} \\
\therefore U_D(S_{ALG}) &\leq 5.675U_D^*.
\end{aligned}$$

For area uncertainty,

$$\begin{aligned}
U_A(S_{ALG}) &\leq 23.46r^2 \sin^2 \alpha \\
&= 23.46 \frac{U_A^* \sin^2 \alpha}{\pi \sin^2 \alpha} \\
\therefore U_A(S_{ALG}) &\leq 7.47U_A^*.
\end{aligned}$$

CASE (b) $\frac{\pi}{18} \leq \alpha < \frac{\pi}{12}$:

For diameter uncertainty,

$$\begin{aligned}
U_D(S_{ALG}) &\leq 2.04r \\
&= 2.04 \frac{U_D^*}{2 \sin \alpha} \\
&\leq 2.04 \frac{U_D^*}{2 \sin \frac{\pi}{18}} \\
\therefore U_D(S_{ALG}) &\leq 5.88U_D^*.
\end{aligned}$$

For area uncertainty,

$$\begin{aligned}
U_A(S_{ALG}) &\leq \frac{\sqrt{3}r^2}{4} + 10.001r^2 \sin^2 \alpha \\
&= \frac{\sqrt{3}U_A^*}{4\pi \sin^2 \alpha} + 10.001 \frac{U_A^* \sin^2 \alpha}{\pi \sin^2 \alpha} \\
&\leq \frac{\sqrt{3}U_A^*}{4\pi \sin^2 \frac{\pi}{18}} + 10.001 \frac{U_A^*}{\pi} \\
\therefore U_A(S_{ALG}) &\leq 7.76U_A^*.
\end{aligned}$$

CASE (c) $\frac{\pi}{12} \leq \alpha < \frac{\pi}{6}$:

For diameter uncertainty,

$$\begin{aligned}
U_D(S_{ALG}) &\leq \left(1 + \frac{1}{\sqrt{3}}\right) r \\
&= \left(1 + \frac{1}{\sqrt{3}}\right) \frac{U_D^*}{2 \sin \alpha} \\
&\leq \left(1 + \frac{1}{\sqrt{3}}\right) \frac{U_D^*}{2 \sin \frac{\pi}{12}} \\
\therefore U_D(S_{ALG}) &\leq 3.05U_D^*.
\end{aligned}$$

For area uncertainty,

$$\begin{aligned}
U_A(S_{ALG}) &\leq \frac{3\sqrt{3}}{4}r^2 \\
&= \frac{3\sqrt{3}U_A^*}{4\pi \sin^2 \alpha} \\
&\leq \frac{3\sqrt{3}U_A^*}{4\pi \sin^2 \frac{\pi}{12}} \\
\therefore U_A(S_{ALG}) &\leq 6.17U_A^*.
\end{aligned}$$

CASE (d) $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{4}$:

We apply Lemma 4 to bound this uncertainty. In Lemma 4, we constructed an over approximation to the union of all possible sensing wedges from the sensors placed on the hexagon. Without loss of generality let $x \in C_k$. The union of all C_k circles covers $\mathcal{A}$. When x is close to the boundary of $\mathcal{A}$, we may find only a subset of the sensors $i, \dots, o$, as shown in Figure 8(a), which are required by Lemma 4. However, the intersection of the available sensing wedges from $t = i, \dots, o$ with the boundary of $\mathcal{A}$ is a subset of the over approximation used in Lemma 4.

Hence for diameter uncertainty,

$$\begin{aligned}
U_D(S_{ALG}) &\leq r (\sin^2 \alpha + 3.76 \sin \alpha + 1.232) \\
&= \frac{U_D^*}{2 \sin \alpha} (\sin^2 \alpha + 3.76 \sin \alpha + 1.232) \\
&\leq \frac{U_D^*}{2} \left(\sin \alpha + 3.76 + \frac{1.232}{\sin \alpha} \right) \\
\therefore U_D(S_{ALG}) &\leq 3.47U_D^*.
\end{aligned}$$

For area uncertainty,

$$\begin{aligned}
U_A(S_{ALG}) &\leq 1.5r^2 (\sin^2 \alpha + 3.76 \sin \alpha + 1.232) \\
&= 1.5 \frac{U_A^*}{\pi \sin^2 \alpha} (\sin^2 \alpha + 3.76 \sin \alpha + 1.232) \\
&= 1.5 \frac{U_A^*}{\pi} \left(1 + \frac{3.76}{\sin \alpha} + \frac{1.232}{\sin^2 \alpha} \right) \\
&\leq 1.5 \frac{U_A^*}{\pi} \left(1 + \frac{3.76}{\sin \frac{\pi}{6}} + \frac{1.232}{\sin^2 \frac{\pi}{6}} \right) \\
\therefore U_A(S_{ALG}) &\leq 6.42U_A^*.
\end{aligned}$$

This completes the proof.