Technical Report

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TR 10-002

Maintaining Connectivity in Environments with Obstacles

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January 29, 2010

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Abstract—Robotic routers (mobile robots with wireless communication capabilities) can create an adaptive wireless network and provide communication services for mobile users on-demand. Robotic routers are especially appealing for applications in which there is a single mobile user whose connectivity to a base station must be maintained in an environment that is large compared to the wireless range.

In this paper, we study the problem of computing motion strategies for robotic routers in such scenarios, as well as the minimum number of robotic routers necessary to enact our motion strategies. Assuming that the routers are as fast as the user, we present an optimal solution for cases where the environment is a simply-connected polygon, a constant factor approximation for cases where the environment has a single obstacle, and an $O(h)$ approximation for cases where the environment has h circular obstacles. The $O(h)$ approximation also holds for cases where the environment has h arbitrary polygonal obstacles, provided they satisfy certain geometric constraints – e.g. when the set of their minimum bounding circles is disjoint.

I. INTRODUCTION

Suppose a mobile user operating in a large environment needs network connectivity to a base station. The user may be a tele-operated robot and the base station may be a gateway to the Internet. Without a communication infrastructure, the mobility of the user would be restricted by the communication radius of the base-station. The traditional solution for providing long-range network connectivity is to deploy a network of static wireless routers which cover the entire area of interest. However when the environment is large, covering it can be costly. Moreover, in some scenarios (such as natural disasters or hazardous conditions) it might be impossible to manually deploy this network in advance.

On the other hand, we can deploy a small number of robots to act as mobile routers. These robots can autonomously relocate themselves according to the movement of the user and maintain their connectivity with the base station. In order to demonstrate the potential gain attained by using robots, let us consider the scenario shown in Figure 1. In this scenario, a user u navigates inside a circular arena, and wishes to remain connected to the base station at c at all times. Suppose the communication range of all devices is σ . If we deploy a stationary network to cover the arena, the number of necessary routers is $\Theta(R^2/\sigma^2)$. Instead, $\Theta(R/\sigma)$ robotic routers can maintain the user's connectivity by staying on the line segment $[cu]$ (the details of a generalization of this strategy are given in Section III). Hence, the number of

routers used can be drastically reduced by using the mobility of robots.

Fig. 1. An example which demonstrates the potential gain of using robotic routers. We need $\Theta(R^2/\sigma^2)$ static routers whereas $\Theta(R/\sigma)$ robots are sufficient to keep the user connected. When the user moves from u to u' , robotic routers move with the same angular velocity to keep the user connected.

In this paper, we study the problem of computing the minimum number of robotic routers (and their motion strategies) in order to maintain the connectivity of a single user to a base station. After an overview of related work, we formalize the robotic routers problem in Section II. In Section III, we present an optimal solution for simply-connected polygons. In Section IV, we present a constant-factor approximation for a polygonal environment with a single obstacle. In Section V, we present an $O(h)$ -approximation algorithm for polygons with multiple obstacles.

A. Related work

Exploiting the controlled mobility of robots to improve connectivity has received significant attention. Related problems include establishing connectivity [8], [15], building a bridge between two points [20], network coverage [6], [16], repairing a network's connectivity [5], [1], and improving it [4], [10]. In sensor networks literature, robots are used as mobile sensors to extend sensing regions [7], [11] or to collect data from stationary sensors [21], [17]. Moreover, a network of robots can be used for improving the communication in search and rescue tasks [12], [13].

In our recent work, we studied the problem of maintaining connectivity between a mobile user and a base station for two different user motion models [19]. We presented optimal algorithms to compute the minimum number of routers to maintain connectivity for both models. While the algorithms can incorporate arbitrary communication link models, their running times are exponential in the number of robots. In this work, we focus on geometric domains and geodesic distance based connectivity models, and present polynomial time approximation algorithms.

In other related work, Stump et al propose two metrics for characterizing connectivity and present a framework which

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chooses the best local decision to maintain the connectivity of an independently moving target [18]. Here, we are able to give global guarantees by controlling the number of robots. In a related paper [9], Dixon et al study the problem of forming a chain of robotic relays and present an algorithm to control robots along the chain to improve the signal-to-noise ratio. A similar problem is considered by Kutylowski et. al [14]. They present a global strategy for using a chain of robots to create a communication bridge between a stationary camp and a mobile explorer. In [2], they extend this strategy to local (distributed) strategies. However, in complex environments with obstacles, topologies more general than chains must be used. This is the main focus of the present work.

II. PROBLEM FORMULATION AND NOTATION

In this section, we present the terminology and notation used throughout the paper, and formalize the robotic router problem.

A *robotic router* is a mobile robot which can communicate wirelessly. Robotic routers are subject to communication and motion constraints such as limited communication range and a bounded maximum speed. The *base station* is a static entity to which the *user* wishes to establish connection. All entities are contained in a shared environment denoted by $\mathcal{P}$. The user is connected to the base station through a communicating bridge of robotic routers.

Throughout the paper, we assume that the time domain is continuous. We denote the position of the user at time t as $u(t)$, and that of the i^{th} robotic router as $r_i(t)$. We assume that both the robotic routers and the user have the same maximum speed. We call this requirement as *the motion constraint*. We will prove the correctness of our strategies by showing that the speed of each robotic router at time t is less than or equal to the speed of the user at time t . In other words, let $|\dot{u}(t)|$ and $|\dot{r}_i(t)|$ be their respective speeds; we present strategies in which $|\dot{r}_i(t)| \leq |\dot{u}(t)|$ always holds.

We measure the distance between any two points $x, y \in \mathcal{P}$ by the length of the geodesic path from x to y , i.e. the shortest path from x to y that lies inside $\mathcal{P}$ and does not cut through any obstacles. For any time t , we denote the geodesic shortest path from the base station b to the user $u(t)$ as $SP(t)$. The shortest geodesic distance between x and y is denoted by $d(x, y)$.

Various models for radio propagation are studied in the literature. Due to various environment dependent effects (such as multi-path, fading, occlusion, etc.), it is difficult to provide a generic model which incorporates all these effects. In this work, we assume that two points $x \in \mathcal{P}$ and $y \in \mathcal{P}$ are connected if $d(x, y) \leq \sigma$ holds. This is the *communication constraint*. With this threshold, we can address the fading effects with our model in which σ is a communication distance threshold. Moreover, this model implicitly addresses the occlusion (shadowing) effects. If there exists a line-of-sight between x and y the geodesic distance is same as the Euclidean distance. However, when the polygon or an obstacle occludes between x and y , the geodesic distance increases.

Any subset of points A in $\mathcal{P}$ is said to be geodesically convex if for every x and y in A , the geodesic segment $[x, y]$ is contained in A . We define the geodesic convex hull of a set of points X in $\mathcal{P}$ as the minimal geodesically convex set that contains all of the points in X .

To simplify the notation, we scale all distances by the communication distance threshold σ . Throughout the paper, without loss of generality, we assume that the *communication distance* is the unit distance. Let D be the longest geodesic shortest path from b to any point $\in \mathcal{P}$; $m^* = \lceil D - 1 \rceil + 1 = \lceil D \rceil$ is a lower bound on the minimum number of robotic routers necessary to connect any point in $\mathcal{P}$ to b , including the base station as a robotic router.

We define the number of robots used as follows. For a given user trajectory $\mu = u(t)$, let $n(\mu)$ be the number of robots required to connect the user to the base station. For a given environment, the number of robots required is the maximum number over all possible user trajectories, i.e. $n = \max_{\mu} n(\mu)$. When computing n , we do not require that the routers know the user's trajectory in advance. However we assume that the robotic routers in the network are all continuously made aware of the current position of the user¹, and they can instantaneously choose their movements based on this information.

Problem Statement: Given an environment $\mathcal{P}$ (possibly with obstacles) and a base station $b \in \mathcal{P}$, find the minimum number of robotic routers and their motion strategies such that the user u is connected to the base station at all times, and the motion and communication constraints are satisfied.

III. ENVIRONMENTS WITH NO OBSTACLES

In this section, we present a strategy to maintain connectivity using an optimal number of $m^* = \lceil D \rceil$ routers. The strategy, which we call *EQ-DIST*, involves maintaining an equidistant separation along $SP(t)$. We show that this can be achieved without violating communication and motion constraints.

We say that the *Evenly Spaced Property (ESP)* holds at time t if all the routers are positioned uniformly along $SP(t)$. We will refer to this chain of routers as an *arm*. We assume that ESP holds at time 0 (i.e. the user is willing to wait until the initial connection is established).

When the environment $\mathcal{P}$ is a polygon, it is well known that $SP(t)$ is a polygonal path $\{p_0 = b, p_1, p_2, \dots, p_j, u\}$ from b to u , where any p_i for $i > 0$ is a vertex of $\mathcal{P}$. In this path, the *parent* of any point z on SP is defined as the closest vertex to z in the shortest path between b and z (Figure 3). If no such vertex exists, then the parent of z is b .

First, we prove the following lemma.

Lemma 1. *For any t there exists a sufficiently small $dt > 0$ and a shared point $s \in SP(t) \cap SP(t+dt)$ such that $SP(t)$ and $SP(t+dt)$ only differ along a single line segment, from their respective endpoints to s .*

¹For example, this information can be provided by the user.

Fig. 2. For a small enough time step, the shortest path $SP(t)$ changes only along the line segment farthest from b .

Fig. 3. **Left:** Robots r_i only translate along SP while robots r_j rotate about s and translate. **Right:** Similar triangles used to find the relation between $|\dot{u}_\perp(t)|$ and $|\dot{r}_{i\perp}(t)|$.

Proof: First, observe that for all t and for all dt , $SP(t)$ and $SP(t+dt)$ share their start point b , by definition. Therefore s always exists. We now show that we can find a dt and a shared point s such that the two shortest paths differ only along their last line segment.

Take any time interval $dt > 0$. The user moves from $u(t)$ to $u(t+dt)$. Let $p_j(t) \in SP(t)$ be the parent of $u(t)$, and $p_j(t+dt) \in SP(t+dt)$ be the parent of $u(t+dt)$. There are two possibilities, as shown in Figure 2.

Case 1. (Figure 2(a)) $p_j(t) = p_j(t+dt)$, i.e. the parent of the user remained the same over the interval dt . Then $s = p_j(t) = p_j(t+dt)$ is the shared point.

Case 2. (Figure 2(b)) $p_j(t) \neq p_j(t+dt)$, i.e. the parent of the user changed over the interval dt . The parent for any point on a shortest path to b changes only when the new shortest path wraps (or unwraps) around a vertex of $\mathcal{P}$. Here, we discuss the case when it unwraps, i.e. $p_j(t+dt)$ is an ancestor of $p_j(t)$. The case when it wraps is similar.

Let $p_{j-1}(t)$ be the parent of $p_j(t)$. Shoot ray L along line segment $[p_{j-1}(t) p_j(t)]$. There are two possibilities. If L does not intersect $[u(t) u(t+dt)]$, then the parent of the user did not change, which contradicts our assumption in Case 2. Let v be the point of intersection of L and $[u(t) u(t+dt)]$. Since the user follows a continuous path from $u(t)$ through v to $u(t+dt)$, there exists $0 < dt' < dt$ for which $u(t+dt') = v$. Now $s = p_j(t)$ is the desired shared point common to

$SP(t)$ and $SP(t+dt')$. Hence, $SP(t)$ and $SP(t+dt)$ only differ in their last line segment when $dt = dt'$. ■

We now show that the robots can maintain connectivity using *EQ-DIST*:

For any point $z(t) \in SP(t)$, let $z^\dagger(t)$ denote its parent. Parameterize the velocity $\dot{z}(t)$ into a radial component $\dot{z}_\parallel(t)$, along $[z^\dagger(t) z(t)]$, and a tangential component $\dot{z}_\perp$ orthogonal to $\dot{z}_\parallel(t)$. We have $|\dot{z}(t)| = \sqrt{|\dot{z}_\perp(t)|^2 + |\dot{z}_\parallel(t)|^2}$. For any robot $r_i(t)$, we denote its velocity components as $\dot{r}_{i\parallel}(t)$ and $\dot{r}_{i\perp}(t)$ (see Figure 3).

Only the radial component of the user's velocity affects the length of SP . Let λ be a differential change in the length of SP . We have $\lambda = \dot{u}_\parallel(t)$. To satisfy ESP, the robotic routers should move proportional to λ along the radial component.

$$\dot{r}_{i\parallel}(t) = \frac{i}{n+1} \dot{u}_\parallel(t) \quad (1)$$

When the user moves, some line segments along SP rotate, while others remain the same. The tangential velocity of any robot is thus a function of which side of the shared point s the robot lies on.

If r_i lies between s and u , we can show using similar triangles (see Figure 3) that

$$\frac{|\dot{r}_{i\perp}(t)|}{|\dot{u}_\perp(t)|} = \frac{||s r_i(t)||}{||s u(t)||}$$

Where $||s r_i(t)||$ is the length of line segment $[s r_i(t)]$ and $||s u(t)||$ is the length of line segment $[s u(t)]$. Since $r_i(t)$ is closer to b than $u(t)$, we have $||s r_i(t)|| \leq ||s u(t)||$, i.e.

$$|\dot{r}_{i\perp}(t)| \leq |\dot{u}_\perp(t)| \quad (2)$$

The robots between s and b have a tangential component of zero.

Therefore, for any robot, (1) and (2) show that the robot only needs to move at most as fast as the user to stay on the geodesic from b to u while maintaining ESP.

Theorem 1. *In a simply-connected polygon, the number of mobile robots that the EQ-DIST strategy requires is optimal.*

Proof: Recall that the cost of the optimal solution is the required number of robots to connect any user trajectory. When the user goes to a location where SP is maximized, the optimum solution has to use at most $m = \lceil D \rceil$ robots. We have seen that *EQ-DIST* can maintain connectivity using the same number of robots. ■

IV. ENVIRONMENTS CONTAINING A SINGLE OBSTACLE

In this section, we present robotic router strategies where a user is connected to b in a polygonal environment with a single obstacle. First we present a solution for circular obstacles, then we show how to extend this solution to convex and non-convex obstacles.

Fig. 4. Illustration of the first case of our strategy.

A. Circular obstacle

Let $\mathcal{O}$ be a circular obstacle and let c and r be the center and radius of $\mathcal{O}$. Our strategy is as follows. First, we connect every point on $\mathcal{O}$ to b by extending an arm starting from b and wrapping it around $\mathcal{O}$. We call this our *wrapping arm*, and the robotic routers in it are stationary (see Figure 4). After connecting $\mathcal{O}$ to b , we use a *connecting arm* which rotates around $\mathcal{O}$ and connects the user to $\mathcal{O}$ which is then connected to b through the wrapping arm.

We place robots so that ESP is satisfied; these locations can be easily found by using geometric properties of lines and circles. The bounds on the length of the arm and the number of robots used will be obtained in Theorem 2.

The connecting arm's responsibility is to connect user to $\mathcal{O}$ and consequently to b . We achieve this by moving robotic routers on the SP between u and $\mathcal{O}$. To guarantee that the connecting arm is always connected to $\mathcal{O}$, we use an additional robotic router q which moves along the boundary of $\mathcal{O}$. Robot q acts as a base station for the connecting arm. Let $SP_{\mathcal{O}}(t)$ be the shortest geodesic path between $\mathcal{O}$ and $u(t)$ (this path is the subset of SP between c and $u(t)$). Robot q always remains at the beginning of this path on $\mathcal{O}$. Recall that the parent of $u(t)$ is the closest vertex to $u(t)$ in $SP_{\mathcal{O}}(t)$.

We analyze the connecting arm strategy in two cases: (i) $u(t)$ has a parent different than $q(t)$ (ii) $u(t)$ has $q(t)$ as the parent.

Case (i): If there exists a parent s of $u(t)$ such that $s \neq q$, then we can find a dt and a shared point s such that the shortest paths $SP_{\mathcal{O}}(t)$ and $SP_{\mathcal{O}}(t+dt)$ differ only along their last line segment (Lemma 1). Since both shortest paths pass through s and the shortest path from $\mathcal{O}$ to s is same, q does not move, i.e. $\dot{q}(t) = 0$ (See Figure 4). In this case, the connecting arm can execute EQ-DIST and maintain connectivity.

Case (ii): If the parent of $u(t)$ is q , we can move q and the robots on the connecting arm in such a way that they maintain ESP without violating motion and communication constraints. As we did in Section III, we divide the velocity $\dot{u}(t)$ into two components: radial velocity $\dot{u}_{\parallel}$ and tangential velocity $\dot{u}_{\perp}$. Since q is moving on the boundary of $\mathcal{O}$ its radial velocity is 0. If q is the common parent for $u(t)$ and

Fig. 5. Illustration of the second case of our strategy.

$u(t+dt)$, these shortest paths rotate around c and rotation is due to the tangential component of u (see Figure 5). As angular velocity is the same for the user and each robot on $SP_{\mathcal{O}}$, we can conclude that $|\dot{q}_{\perp}(t)| \leq |\dot{r}_{i\perp}(t)| \leq |\dot{u}_{\perp}(t)|$. If $u(t)$ and $u(t+dt)$ do not have q as their common parent, we can show that a time interval $dt' < dt$ can be found such that the above condition holds. The proof is the same as in Lemma 1.

Suppose $\dot{u}_{\parallel}(t)$ is positive; in this case $SP_{\mathcal{O}}$ increases in length. To satisfy ESP, the robotic routers have to move towards u . The distance from q to r_i must increase by $\frac{i}{n_c} |\dot{u}_{\parallel}(t)|$ where n_c is the number of robots in the connecting arm, including q , and i is the robot index in the connecting arm (the 0th robot refers to q). Hence, $0 = |\dot{q}_{\parallel}(t)| \leq |\dot{r}_{i\parallel}(t)| \leq |\dot{u}_{\parallel}(t)|$ holds. Together with the constant angular velocity observation, we conclude that $|\dot{q}(t)| \leq |\dot{r}_i(t)| \leq |\dot{u}(t)|$ and the motion constraint is satisfied.

Theorem 2. *In a polygon with single circular obstacle, let m^* be the minimum number of robots required to maintain connectivity. The strategy presented in this section uses at most $5m^*$ robots.*

Proof: Let p_c and p_f be the closest and furthest points on $\mathcal{O}$ from b , respectively. By definition, we know that $d(b, p_c) \leq d(b, p_f) \leq D$ where D is the maximum SP from b .

We find an upper bound on the length of the connecting arm using triangle inequality. For any point x in the polygon, due to triangle inequality, we have $d(c, x) \leq d(c, b) + d(b, x)$. We subtract r from both sides: $(d(c, x) - r) \leq d(b, x) + (d(b, c) - r)$. The connecting arm has length $d(q, x)$ where q is the closest point on $\mathcal{O}$ from x . This distance is equal to $d(c, x) - r$ and it is upper bounded by $d(b, p_c) + D \leq 2D$ (Figure 6).

Next, we find a bound on the length of the wrapping arm (Figure 7). The length of the wrapping arm is equal to the sum of the SP distance from b to $\mathcal{O}$ and the circumference of

Fig. 6. Bounding the length of the connecting arm using triangle inequality.

Fig. 7. The shortest geodesic distance from b to p_f and the length of wrapping arm in a convex polygon.

$\mathcal{O}$, i.e. $2\pi r + d(b, p_c)$. Since $d(b, p_c)$, the length of wrapping arm is upper bounded by $2\pi r + D$. Now we will find a bound on $2\pi r$ by finding a bound on the ratio: $\frac{\pi r}{D}$. Although we do not know D , we use the lower bound on $d(b, p_f)$ to find an upper bound on this ratio. Hence we will show the following inequality holds: $\frac{\pi r}{D} \leq \frac{\pi r}{d(b, p_f)} \leq 1$. First, we calculate the maximum value of this ratio for a special case where $\mathcal{P}$ does not intersect with the tangents $[b p_t]$ and $[b p'_t]$. In this case, the ratio is $\frac{\pi r}{r(\theta + \cot(\theta) + \pi/2)}$. Using basic calculus, we can show that the maximum value of the ratio is 1.

We now show that if $\mathcal{P}$ intersects with one or both of the tangents, the ratio is reduced. Hence, the upper bound found in the special case is valid for any case.

Two types of vertices of $\mathcal{P}$ exist which can intersect with one or both of the tangents. Figure 8 shows these cases. The furthest point p_f on obstacle has the property that it has two shortest paths from the opposite sides of the obstacle. Any other point on the obstacle has a unique shortest path. For example, in Figure 7, these shortest paths are $SP_1 = \{b, p_t, p_f\}$ and $SP_2 = \{b, p'_t, p_f\}$. Since both shortest paths are equal, we can find $d(b, p_f) = SP_1 = SP_2$. Moreover, since both shortest paths start from b and end at p_f , their union yields a convex hull around $\mathcal{O}$ and b . Hence, we can say that $d(b, p_f)$ is half of $\text{perim}(\mathcal{H})$ where $\text{perim}(\mathcal{H})$ is the perimeter of the convex hull $\mathcal{H} = \{b, p_t, p_f, p'_t\}$.

First, we consider the first type of vertex (top Figure 8). Let s be a vertex which interferes with $[b p_t]$. The distance of the furthest point on $\mathcal{O}$ from b is half of $\text{perim}(\mathcal{H})$ where $\mathcal{H} = \{b, p_t, p_f, p'_t\}$. Now assume that we remove s ; the distance of furthest point on $\mathcal{O}$ from b becomes the half of $\text{perim}(\mathcal{H}^{new})$ where $\mathcal{H}^{new} = \{b, p_t^{new}, p_f^{new}, p'_t, b\}$. By triangle inequality ($d(b, p_t^{new}) \leq d(b, s) + d(s, p_t) + d(p_t, p_t^{new})$), we can show that $\text{perim}(\mathcal{H})$ is longer than

Fig. 8. Two types of vertices of $\mathcal{P}$ which can obscure with $[b p_t]$ and $[b p'_t]$

$\text{perim}(\mathcal{H}^{new})$. Hence, by introducing s , we reduce the ratio: $\frac{\pi r}{d(b, p_f)}$.

Let s be a vertex of the second type (bottom Figure 8) which interferes with both $[b p_t]$ and $[b p'_t]$. Observe that $\frac{\pi r}{d(s, p_f)} \leq 1$ holds due to the special condition that we discussed before. Because $d(b, p_f) = d(b, s) + d(s, p_f)$, the following inequalities hold: $\frac{\pi r}{d(b, p_f)} \leq \frac{\pi r}{d(s, p_f)} \leq 1$.

If there are multiple vertices of $\mathcal{P}$ interfering with $\mathcal{H}$, this ratio still holds. Because introducing additional vertices between s and b only increases $d(b, p_f)$.

Finally, the ratio between the number of robots used by our strategy and the optimal solution including the base station is:

$$\frac{\lceil d(b, p_c) + 2\pi r \rceil + \lceil d(b, p_c) + D \rceil}{\lceil D \rceil} \leq \frac{\lceil 3D \rceil + \lceil 2D \rceil}{\lceil D \rceil} \leq 5$$

■

B. Convex polygonal obstacle

In this section, we extend our strategy for single circular obstacles to single convex polygonal obstacles. The main idea of the strategy remains the same. The wrapping arm stretches from b to convex obstacle $\mathcal{O}$ and wraps around $\mathcal{O}$. The connecting arm stays on the SP from b to $\mathcal{O}$ and an extra robot q moves on the boundary of $\mathcal{O}$, bridging the connecting and wrapping arms.

The strategy of the robots on the wrapping arm is straightforward: We place robots so that ESP holds and robots do not move.

Let k be the number of vertices of $\mathcal{O}$. Let $V = \{v_0, v_1, \dots, v_{k-1}\}$ and $E = \{e_0, e_1, \dots, e_{k-1}\}$ be the vertex and edge sets of $\mathcal{O}$, respectively. We split $\mathcal{P}$ into $2k$ regions by shooting two rays from each vertex v_i : one perpendicular to e_i and one perpendicular to $e_{mod(i+1, k)}$ (See Figure 9).

There are two types of regions in this partitioning. The regions labeled as $\mathcal{V}_i$ has the property that the shortest path

Fig. 9. The partitioning for single convex obstacle case.

Fig. 10. The strategies for convex polygon case.

from any point $x \in \mathcal{V}_i$ to $\mathcal{O}$ starts at v_i . Similarly, $\mathcal{E}_i$ has the property that the shortest path from any point $x \in \mathcal{V}_i$ to $\mathcal{O}$ starts at some interior point of e_i (See Figure 9).

Let $SP_{\mathcal{O}}(t)$ be the shortest geodesic shortest path between $u(t)$ and $\mathcal{O}$ and parent of $u(t)$ be the closest vertex to $u(t)$ in $SP_{\mathcal{O}}(t)$. We investigate the connecting arm's strategy in three cases: (i) $u(t)$ has a parent which is different than $q(t)$ (ii) $u(t)$ has $q(t)$ as its parent and $q(t)$ is on one of the vertices of $\mathcal{O}$ (iii) $u(t)$ has $q(t)$ as its parent and $q(t)$ is on one of the edges of $\mathcal{O}$.

Case (i) If $u(t)$ has a parent other than $q(t)$ then as shown in Section IV-A, we can show that $\dot{q}(t) = 0$ (see bottom strategy in Figure 10). The proof is identical to the first case in Section IV-A. Finally, we use EQ-DIST strategy to keep the user connected when this condition holds.

Case (ii) If $q(t)$ is the parent for SP from $u(t)$ to $\mathcal{O}$ and $q(t) \in \mathcal{V}$, we can think of $q(t)$ as a base station (i.e. $\dot{q}(t) = 0$) and execute EQ-DIST strategy as long as this condition holds (see right strategy in Figure 10).

Case (iii) When $q(t)$ is the parent of $u(t)$ and $q(t)$ is on an edge e_j , then SP becomes a perpendicular line segment to e_j . Let $SP_{\mathcal{O}}(t)$ and $SP_{\mathcal{O}}(t+dt)$ be the shortest paths at time t and $t+dt$. We divide the velocity of u into two components: radial velocity $\dot{u}_{\parallel}(t)$ and tangential velocity $\dot{u}_{\perp}(t)$. We make the same division for any robot r_i on the connecting arm including q . Since both shortest paths are perpendicular to

Fig. 11. We show that in the third case of the strategy ESP and the motion and communication constraints are satisfied.

e_j , we can show that tangential velocities are equal: i.e. $\dot{q}_{\perp}(t) = \dot{r}_{i\perp}(t) = \dot{u}_{\perp}(t)$ (see Figure 11). Since q stays on the boundary, we have $|\dot{q}_{\parallel}(t)| = 0$. Moreover, we can show that $|\dot{r}_{i\parallel}(t)| = \frac{i}{n_c} |\dot{u}_{\parallel}(t)|$ where n_c is the number of robots in the connecting arm and i is the robot index in the connecting arm. Hence, $|\dot{q}_{\parallel}(t)| \leq |\dot{r}_{i\parallel}(t)| \leq |\dot{u}_{\parallel}(t)|$ holds which implies $|\dot{q}(t)| \leq |\dot{r}_i(t)| \leq |\dot{u}(t)|$.

In the above analysis, we assume that in $SP_{\mathcal{O}}(t)$ and $SP_{\mathcal{O}}(t+dt)$ the parent of u is q . If this is not the case, since q is initially the parent, we can find a time interval $dt' < dt$ where q is the parent of u in $SP_{\mathcal{O}}(t+dt')$. After the parent changes, we use EQ-DIST strategy.

Theorem 3. *In an arbitrary polygon with single convex obstacle, let m^* be the number of robots used by optimal solution. Our strategy uses at most $5m^*$ robots.*

Proof: The proof is very similar to the proof of Theorem 2. Recall that by using the triangle inequality, we can show that the length of the connecting arm is less than or equal to $d(b, q) + D$ where D is the length of the maximum SP from b and q be the closest point on $\mathcal{O}$ from u . By definition, we have $d(b, q) \leq D$. The length of the wrapping arm is equal to $d(b, p_c) + \text{perim}(\mathcal{O})$. By definition, we have $d(b, p_c) \leq D$. In the rest of the proof, we show that $\text{perim}(\mathcal{O}) \leq 2D$ holds.

Let $\mathcal{H}_b = \{b, v_i, v_{i+1}, \dots, v_k, v_0\}$ be the convex hull of b and $\mathcal{O}$. The distance between b and its furthest point on $\mathcal{O}$ (i.e. $d(b, p_f)$) is equal to the half of the perimeter of the convex hull (i.e. $\text{perim}(\mathcal{H}_b)$). Hence, if we show that $\text{perim}(\mathcal{O}) \leq \text{perim}(\mathcal{H}_b) = 2d(b, p_f)$ then we have $\text{perim}(\mathcal{O}) \leq 2D$.

We define the above inequality in terms of edge lengths, i.e. $\text{perim}(\mathcal{O}) = e_1 \dots e_i + e_{i+1} \dots e_k + e_0 \leq x + y + e_{i+1} \dots e_k + e_0 = \text{perim}(\mathcal{H}_b)$. As shown in Figure 12, $\mathcal{H}_b$ and $\mathcal{O}$ has common edges and we can subtract these edges from both sides of the inequality, i.e. $e_1 + \dots + e_i \leq x + y$. We show that this inequality holds by induction. The base case where $\mathcal{O}$ differs from $\mathcal{H}_b$ with a single edge is trivial (directly from triangle inequality). Let us assume that this inequality holds for some i , then (i) $e_1 + \dots + e_i \leq x + y$ holds. Using triangle inequality in the triangle $\triangle bb'v_{i+1}$, we have (ii) $e_{i+1} + y \leq (x' - x) + y'$. Combining (i) and (ii), we get $e_1 + \dots + e_i + (e_{i+1} + y) \leq x + y + (x' - x) + y'$ which proves the inductive step.

Fig. 12. We can show that $\text{perim}(\mathcal{O}) \leq \text{perim}(\mathcal{H})$

Hence, we prove that $\text{perim}(\mathcal{O}) \leq \text{perim}(\mathcal{H}) \leq 2D$. Note that $\mathcal{P}$ can occlude with $\mathcal{H}$, however we showed in the proof of Theorem 2 that this only increases the perimeter of $\mathcal{H}$ and the same argument holds for this case too. Hence, we skip the proof.

Finally, the ratio between the number of robots used by our strategy and the optimal solution including the base station is:

$$\frac{\lceil d(b, p_c) + \text{perim}(\mathcal{O}) \rceil + \lceil d(b, q) + D \rceil}{\lceil D \rceil} \leq \frac{\lceil 3D \rceil + \lceil 2D \rceil}{\lceil D \rceil} \leq 5$$

C. Non-convex obstacles

Let $\mathcal{O}$ be a general, possibly non-convex, obstacle. We can extend the strategy for convex obstacles to $\mathcal{O}$ as follows.

Let $\mathcal{H}$ be the geodesic convex hull of $\mathcal{O}$. When u is outside of $\mathcal{H}$, our strategy to maintain connectivity between u and b is similar to what it was when we were dealing with a convex obstacle, but now we treat $\mathcal{H}$ as the obstacle: The wrapping arm stretches from b to $\mathcal{H}$ and wraps around $\mathcal{H}$. The connecting arm stays on the SP from $\mathcal{H}$ to u . We add an extra robot q on the boundary of $\mathcal{H}$ to bridge the connecting and wrapping arms.

When u is inside $\mathcal{H}$, the boundaries of $\mathcal{H}$ and $\mathcal{O}$ create a set of disjoint simply connected regions, and we consider the one that contains u . This region is open only from one side (the boundary of $\mathcal{H}$), and it does not interfere with any edges of $\mathcal{P}$. Hence, when u is inside $\mathcal{H}$, we can use the same strategy as when it is outside, and move our connecting arm so that it is on the SP between u and the boundary of $\mathcal{H}$.

When u is on the boundary of $\mathcal{H}$, q and other robots in the connecting arm are located at the same location. Hence, when the user moves from the inside region to the outside region (or vice versa) the transition point satisfies both the inside and outside strategies.

In both cases we are connecting u to the closest point on the perimeter of a convex region with a robot q as a bridge, and we know from previously that this is possible while satisfying the motion constraint.

Fig. 13. An example of our strategy for dealing with a non-convex obstacle

Theorem 4. In an arbitrary polygon containing a single polygon obstacle, let m^* be the number of robots used by the optimal strategy. Our strategy uses at most $5m^*$ robots.

Proof: The proof is very similar to the proof of the case where the obstacle is convex, except now we use geodesic shortest paths instead of straight lines. Using the triangle inequality as before we can say that the length of the connecting arm is less than or equal to $2D$, where D is the length of the longest $SP \in \mathcal{P}$, and as before it is trivial to see that the part of the wrapping arm from b to the closest point in $\mathcal{H}$ is shorter than D .

We define $\mathcal{H}_b$ as the geodesic convex hull of $\mathcal{H}$ and b . We know that $\text{perim}(\mathcal{H}_b)$ is $2D$, and we want to prove that $\text{perim}(\mathcal{H})$ is not more than that. Similarly to before, $\text{perim}(\mathcal{H})$ and $\text{perim}(\mathcal{H}_b)$ have common edges, and if we subtract these edges from the union of $\text{perim}(\mathcal{H})$ and $\text{perim}(\mathcal{H}_b)$ we end up with a triangle of shortest paths that connect b and the vertices v_0 and v_i where $\mathcal{H}_b$ first touches $\mathcal{H}$. By applying the triangle inequality to this $\triangle bv_0v_i$ we can conclude that $d(v_0, v_i) \leq d(b, v_0) + d(b, v_i)$, and so it follows that $\text{perim}(\mathcal{H}) \leq \text{perim}(\mathcal{H}_b) \leq 2D$.

Therefore the upper bound on the total number of robots required by our strategy is:

$$\lceil D \rceil + \lceil \text{perim}(\mathcal{H}) \rceil + \lceil 2D \rceil \leq \lceil 5D \rceil \leq 5m^*$$

V. POLYGONAL ENVIRONMENTS WITH MULTIPLE OBSTACLES

Let $\mathcal{P}$ be a convex polygonal environment containing two or more non-intersecting obstacles. If the convex hulls of the obstacles are disjoint, we can often extend the strategies for the single obstacle case as follows. First, we partition $\mathcal{P}$ into convex cells such that each cell contains exactly one obstacle.

For each cell we execute a strategy similar to the one we used in domains with a single obstacle: we have a wrapping arm that connects to b and wraps around the obstacle in the cell, and we have a connecting arm that, whenever u is in the cell, connects u to the closest point to u on the obstacle.

Fig. 14. Power diagram edges are shown in blue (light color). Circular obstacles are shaded. The connection from b to cell C_i is shown, along with the construction of the point u_i .

We start by presenting the partitioning strategy for the case when all of the obstacles are circular.

A. Power diagrams for circular obstacles

Our partitioning strategy relies on the concept of power diagrams [3]. The power $\text{pow}(x, s)$ of a point x with respect to a circle (or in our case a circular obstacle) s in the Euclidean space $\mathbb{R}^2$ is given by $d^2(x, z) - r^2$, where d is the Euclidean distance function, and z and r are the center and the radius of s . For a finite set of circles S in $\mathbb{R}^2$, the *power diagram* of S , denoted $PD(S)$, is a cell complex that associates each $s \in S$ with the convex domain $\{x \in \mathbb{R}^2 | \text{pow}(x, s) < \text{pow}(x, t), \forall t \in S - s\}$. An example is shown in Figure 14. When $r = 0$, i.e. the circles degenerate to points, $PD(S)$ becomes the Voronoi diagram. The following properties about $PD(S)$ are relevant to our partitioning strategy (see [3] - §2.2: Observations 1 and 2, and Lemma 1).

- When the circles are non-intersecting, the edges of $PD(S)$ do not intersect any of the circles.
- If the cardinality of S is k , then $PD(S)$ contains at most k cells.

B. Multiple circular obstacles

Let S be the set of finite circular obstacles in our environment. We intersect each cell in $PD(S)$ with $\mathcal{P}$ to get a convex tessellation of $\mathcal{P}$, with each resulting cell C_i containing one obstacle. We include the power diagram edges that bound C_i as part of C_i .

The strategy to maintain connectivity is as follows: At any time, let C_u be the cell that contains the user u . The robots in C_u will move according to the strategy presented in Section IV-A, and maintain the user's connectivity.

The other robots move to “guard” their regions. Let C_i be a region which does not contain u . We project the user onto the boundary of C_i by finding the closest point in C_i to u using the Euclidean distance (i.e. we ignore the obstacles).

Fig. 15. An example valid tessellation for a domain with three obstacles S_1, S_2, S_3 . The obstacles are not circles, but each one is entirely contained within a single circle.

Let u_i be the *closest* point to u in C_i (see Figure 14). Because C_i is convex, u_i cannot move faster than u . Therefore the robots in C_i can maintain u_i 's connectivity to b by executing the strategy presented in Section IV-A. This guarantees that the user's connectivity is maintained by the connecting arm in C_i as soon as the user enters this cell.

We now bound the number of robots.

Lemma 2. *Let m^* be the number of robots used by any optimal solution, including the base station, to guarantee connectivity between u and b in a convex environment with h circular obstacles. Our strategy uses at most $5m^* \cdot h$ robots.*

Proof: The strategy for any C_i is exactly the same as if $\mathcal{P}$ is the geodesic convex hull of C_i and the shortest path from b to S_i and we are guarding u_i . The D_i of that hull cannot be larger than the max distance D of $\mathcal{P}$, so the upper bound on the sum of the lengths of all the arms in C_i is $5D$. Therefore each arm requires at most $5m^*$ robots, for a total upper bound of $5m^* \cdot h$ routers for all of $\mathcal{P}$. ■

C. Extension to non-circular obstacles

When the obstacles are non-circular, the notion of a radius is undefined and power diagrams cannot be applied as such. However, if we can find an enclosing circle for each obstacle such that the circles are disjoint, it is straightforward to extend the previous result. In certain cases, a tessellation exists even if the disks defined by minimum enclosing circles are intersecting. Consider obstacles o_i and o_j with enclosing disks s_i and s_j .

When $s_i \neq s_j$ intersect each other, but do not intersect o_j and o_i respectively, the power diagram edge $p \in PD(S)$ that is formed by s_i and s_j lies in the region $s_i \cap s_j$, which does not contain any obstacle. The resulting partition cells remain convex [3]. Therefore in this situation we can enact our strategy, and because each cell requires up to $\lceil 5D \rceil$ robots, the upper bound on the number of robots we need for the whole domain is $5m^* \cdot h$.

VI. CONCLUSION

In this work, we studied a novel application of robotic sensor networks in which the robots act as mobile routers and maintain the connectivity of a user to a base station. Given a

complex environment where two entities can communicate if the geodesic distance between them is less than a threshold, we presented algorithms to compute the minimum number of necessary robotic routers and the strategies they should use. Specifically, we presented an optimal (in terms of the number of routers) algorithm for simply-connected polygons, a constant factor algorithm for a polygonal environment with a single obstacle, and an $O(h)$ -approximation algorithm for environments with h obstacles.

Our future work includes improving the $O(h)$ approximation ratio. It is easy to see that the lower bound of $\lceil D \rceil$ is loose in some instances. For example, when h obstacles are arranged on a $\sqrt{h} \times \sqrt{h}$ grid, the number of necessary routers is clearly more than $\lceil D \rceil$. Improving the lower bound will yield better a approximation ratio.

ACKNOWLEDGEMENT

The authors would like to thank Professor Ravi Janardan for introducing us power diagrams. This work is supported in part by NSF Grants 0907658, 0917676 and 0936710.

REFERENCES

- [1] N. Atay and B. Bayazit. Mobile wireless sensor network connectivity repair with k-redundancy. In *WAFR*, 2008.
- [2] F. auf der Heide and B. Schneider. Local Strategies for Connecting Stations by Small Robotic Networks. In *Biologically-Inspired Collaborative Computing: IFIP 20th World Computer Congress, Second IFIP TC 10 International Conference on Biologically-Inspired Collaborative Computing, September 8-9, 2008, M*, page 95. Springer Verlag, 2008.
- [3] F. Aurenhammer. Power Diagrams: Properties, Algorithms and Applications. *SIAM Journal on Computing*, 16(1):78–96, 1987.
- [4] J. Butterfield, K. Dantu, B. Gerkey, O. Jenkins, and G. Sukhatme. Autonomous biconnected networks of mobile robots. In *Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks and Workshops, 2008. WiOPT 2008. 6th International Symposium on*, pages 640–646, April 2008.
- [5] P. I. Corke, S. Hrabar, R. A. Peterson, D. Rus, S. Saripalli, and G. S. Sukhatme. Autonomous deployment and repair of a sensor network using an unmanned aerial vehicle. In *ICRA*, pages 3602–3608, 2004.
- [6] J. Cortes, S. Martinez, T. Karatas, F. Bullo, et al. Coverage control for mobile sensing networks. *IEEE Transactions on Robotics and Automation*, 20(2):243–255, 2004.
- [7] K. Dantu, M. Rahimi, H. Shah, S. Babel, A. Dhariwal, and G. Sukhatme. Robomote: enabling mobility in sensor networks. In *Proceedings of the 4th international symposium on Information processing in sensor networks*, page 55. IEEE Press, 2005.
- [8] E. D. Demaine, M. Hajiaghayi, H. Mahini, A. S. Sayedi-Roshkhar, S. Oveisgharan, and M. Zadimoghaddam. Minimizing movement. In *SODA '07: Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 258–267, Philadelphia, PA, USA, 2007. Society for Industrial and Applied Mathematics.
- [9] C. Dixon and E. W. Frew. Maintaining optimal communication chains in robotic sensor networks using mobility control. In *First International Conference on Robot Communication and Coordination (ROBOCOMM)*, Athens, Greece, 15-17 Oct. 2007.
- [10] D. Goldenberg, J. Lin, A. Morse, B. Rosen, and Y. Yang. Towards mobility as a network control primitive. In *Proceedings of the 5th ACM international symposium on Mobile ad hoc networking and computing*, page 174. ACM, 2004.
- [11] A. Kansal, M. Rahimi, D. Estrin, W. Kaiser, G. Pottie, and M. Srivastava. Controlled mobility for sustainable wireless sensor networks. In *Proc. IEEE SECON*, volume 4, 2004.
- [12] G. Kantor, S. Singh, R. Peterson, D. Rus, A. Das, V. Kumar, G. Pereira, and J. Spletzer. Distributed search and rescue with robot and sensor teams. In *Proc. of the 4th Intl. Conf. on Field and Service Robotics, Japan*. Springer, 2003.
- [13] V. Kumar, D. Rus, and S. Singh. Robot and sensor networks for first responders. *IEEE Pervasive Computing*, pages 24–33, 2004.
- [14] J. Kutylowski and F. Meyer auf der Heide. Optimal strategies for maintaining a chain of relays between an explorer and a base camp. *Theoretical Computer Science*, 410(36):3391–3405, 2009.
- [15] E. Meisner, W. Yang, and V. Isler. Network formation through coverage and freeze-tag. In *Eighth International Workshop on the Algorithmic Foundations of Robotics*, 2008.
- [16] P. Ogren, E. Fiorelli, and N. Leonard. Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment. *IEEE Transactions on Automatic Control*, 49(8):1292–1302, 2004.
- [17] R. Shah, S. Roy, S. Jain, and W. Brunette. Data mules: Modeling and analysis of a three-tier architecture for sparse sensor networks. *Ad Hoc Networks*, 1(2-3):215–233, 2003.
- [18] E. Stump, A. Jadbabaie, and V. Kumar. Connectivity management in mobile robot teams. In *Proceedings IEEE International Conference on Robotics & Automation*, pages 1525–1530, 2008.
- [19] O. Tekdas and V. Isler. Robotic routers. In *Robotics and Automation, 2008. ICRA 2008. IEEE International Conference on*, pages 1513–1518, May 2008.
- [20] O. Tekdas, Y. Kumar, V. Isler, and R. Janardan. Building a communication bridge with mobile hubs. In *5th International Workshop on Algorithmic Aspects of Wireless Sensor Networks (ALGOSENSORS)*, 2009.
- [21] O. Tekdas, J. Lim, A. Terzis, and V. Isler. Using mobile robots to harvest data from sensor fields. *IEEE Wireless Communications*, page 3, 2009.