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On the Rearrangeability of Shuffle-Exchange Networks

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Abstract

Let $m(n)$ be the minimum positive integer k so that the Shuffle-Exchange network with k stages, $N = 2^n$ inputs and N outputs is rearrangeable. Beneš conjectured that $m(n) = 2n - 1$. The best bounds known so far are $2n - 1 \leq m(n) \leq 3n - 4$. In this paper, we verify Beneš conjecture for $n = 4$, and use this result to show that $m(n) \leq 3n - 5$. The $n = 4$ case is considerably more complex than the $n = 3$ case, which have been done in the literature. We believe that hidden in our proof there is some general technique that would help improve the bound further.

1 Introduction

Shuffle-Exchange networks (SE networks for short) were initially proposed by Stone (1971, [1]) to be an efficient interconnecting architecture for parallel processors. Various applications benefit from this interconnecting pattern such as FFT, matrix transposition, polynomial evaluation, ... A k -stage SE network with parameter n , denoted by $(SE_n)^k$, is a network with $N = 2^n$ inputs and N outputs, consisting of k SE stages, where each SE stage includes a *perfect shuffle* pattern (see [1]) followed by an array of $\frac{N}{2} \times 2$ crossbars. A typical drawing of a 7-stage SE network with $n = 4$ (i.e. $(SE_4)^7$) is shown in Figure 1.

A standard question to be addressed on any multistage interconnection network (MIN) is that if the network is *rearrangeable*. An N -input N -output MIN is rearrangeable if and only if any one to one mapping from the inputs to the outputs is routable by the network. *Universality* is another term that is often used synonymously with *rearrangeability*. In the context of SE networks, a long standing question was that how many SE stages are necessary and sufficient for a SE network to be rearrangeable. In fact, it is not entirely clear that increasing the number of stages would increase the rearrangeability of a SE network. There has been a very slow progress toward answering this question. For convenience, let us use $m(n)$ to denote the minimum positive integer so that $(SE_n)^{m(n)}$ is rearrangeable. The algorithm discussed by Stone (1971, [1]) showed that $m(n) \leq n^2$, thus $m(n)$ is well defined. Beneš conjectured in 1975 [2] that $2n - 1$ SE stages is necessary and sufficient to route all $N!$ perfect matchings from the inputs to the outputs, i.e. $m(n) = 2n - 1$. Parker (1980,[3]) showed that $n + 1 \leq m(n) \leq 3n$, where the lower bound was obtained by a counting argument, and the upper bound by group calculations plus the rearrangeability of the Beneš network [4]. Wu and Feng (1981, [5]) gave an explicit algorithm to route all matchings, proving $m(n) \leq 3n - 1$. Huang and Tripathi (1986, [6]) improved the bound to $m(n) \leq 3n - 3$. Raghavendra and Varma (1987, [7]) verified the conjecture for $N = 8$. They used that result to show $m(n) \leq 3n - 4$ [8]. They also specified a permutation

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Figure 1: The 7-stage SE network for $N = 16$, i.e. $(SE_4)^7$

which $(SE_n)^k$ can not route if $k \leq 2n - 2$, in effect showing $2n - 1 \leq m(n)$. With a different formulation, Linial and Tarsi (1989, [9]) also verified the conjecture for $N = 8$ and showed $m(n) \leq 3n - 4$. From their formulation it is easy to see that at least $2n - 1$ stages are needed to route all permutations. Feng and Seo (1994, [10]) gave a proof of the conjecture, which was incomplete as pointed out by Kim, Yoon, and Maeng (1997, [11]).

In this paper, we give a proof that $m(4) = 7$ using a new method, and then adapting Linial and Tarsi's results to show that $m(n) \leq 3n - 5$. As we shall see, the $n = 4$ case is considerably more difficult than the $n = 3$ case. We believe that hidden in our proof there is some general technique(s) that would help improve the bound further.

2 Preliminaries

This section presents related concepts and previous results on the problem. Throughout the paper, we shall assume that $n \in \mathbb{N}$ and $N = 2^n$. The following definitions and lemmas are from Linial and Tarsi [9].

Definition 2.1. For $k \in \mathbb{N}$, a $N \times k$ 01-matrix A , denoted by $A_{N \times k}$ is said to be *balanced* if

- (i) Either $k \leq n$ and every row vector $v \in \mathbb{F}_2^k$ occurs exactly 2^{n-k} times as rows of A .
- (ii) or $k > n$ and every n consecutive column vectors of A form a balanced matrix.

Definition 2.2. Given a balanced matrix $A_{N \times (n-1)}$, a column vector $x \in \mathbb{F}_2^N$ is said to *agree* with A if appending x into A yields an $N \times n$ balanced matrix (the matrix $[A, x]$).

Lemma 2.3. If A and B are two $N \times (n - 1)$ balanced matrices, then there exists a vector $x \in \mathbb{F}_2^N$ that agrees with both A and B .

Lemma 2.4. Let $A_{N \times n}$ be a 01-matrix such that deleting any column of A yields a balanced $N \times (n - 1)$ matrix. Then, either (i) A is balanced, or (ii) each row of A has an even number of 1's, or (iii) each row of A has an odd number of 1's.

Lemma 2.5. Let $A_{N \times k}$ be a balanced matrix with $k \leq n$, and let T be a non-singular $k \times k$ 01-matrix, then AT is also balanced, where all the arithmetic is done modulo 2.

Notice that when x agrees with A , we can insert x into any position between the columns of A to get a balanced matrix. It is also easy to see that $(SE_n)^m$ ($m > n$) is rearrangeable if and only if for every two given balanced matrices $A_{N \times n}$ and $B_{N \times n}$ there exists an $N \times (m - n)$ balanced matrix M such that the matrix $[A, M, B]$ is balanced. Here the rows of A are binary representations of the inputs and the corresponding rows of B are binary representations of the matched outputs.

3 Main Results

To illustrate the idea and introduce notations needed for the main theorem ($m(4) = 7$), we first reproduce a known result (see [9, 7]) using the new method.

Lemma 3.1. $m(3) = 5$, namely the network $(SE_3)^5$ is rearrangeable.

Figure 2: The 5-stage SE network for $N = 8$, i.e. $(SE_3)^5$

Proof. We use the same approach as that of Raghavendra and Varma [7], namely from first principles. However, the method is different and more intuitive. Figure 2 shows a typical drawing of a 5-stage SE network for $N = 8$. For convenience, the network can be redrawn and the switches can be labeled as shown in Figure 3. In the figure, the inputs and outputs have been

Figure 3: A redrawing of the $(SE_3)^5$ network

numbered in decimals for convenience. We write $x \in I_i$ ($y \in O_j$) if input x (output y) is connected to input switch I_i (output switch O_j). Given a perfect matching π (or permutation) from the inputs $\{0, 4, 2, 6, 1, 5, 3, 7\}$ to the outputs $\{0, 1, 2, 3, 4, 5, 6, 7\}$, we first construct a 4×4 , 2-regular multi-bipartite graph $G(\pi) = (I, O, E)$ whose bipartitions are $I = O = \{0, 1, 2, 3\}$. I and O correspond to the set of input and output switches respectively. We shall refer to $G(\pi)$ as G if π is clear from the context. $(i, j) \in E$ iff $\pi(x) = y$ for some $x \in I_i$ and $y \in O_j$, introducing multiple edges if necessary. We now need some notations. Suppose we have colored the edges of G with colors in $C = \{0, 1, 2, 3\}$. For each $c \in C$, let $L(c)$ ($R(c)$) be the multi-set of the vertices in I (O) which are incident to an edge colored c . For each subset $S \subseteq C$, let

$L(S) = \bigcup_{c \in S} L(c)$ and $R(S) = \bigcup_{c \in S} R(c)$, where the union is multiset union. For each $e \in E$, let $l(e)$ ($r(e)$) denote the vertex in I (O) incident to e . Similarly, for any subset $A \subseteq E$, let $L(A) = \{l(e) \mid e \in A\}$ and $R(A) = \{r(e) \mid e \in A\}$.

To this end, we observe from Figure 3 that the realizability of the matching is equivalent to the existence of a coloring of G with colors in C such that

- (P_1) For each $c \in C$, $L(c)$ has a representative from each of $\{0, 1\}$ and $\{2, 3\}$.
- (P_2) $L(\{0, 1\}) = L(\{2, 3\}) = \{0, 1, 2, 3\}$. In other words, $L(\{0, 1\})$ and $L(\{2, 3\})$ have distinct elements.
- (P'_1) For each $c \in C$, $R(c)$ has a representative from each of $\{0, 1\}$ and $\{2, 3\}$.
- (P'_2) $R(\{0, 2\}) = R(\{1, 3\}) = \{0, 1, 2, 3\}$. In other words, $R(\{0, 2\})$ and $R(\{1, 3\})$ have distinct elements.

Note that these conditions imply that each color appears exactly twice. The conditions are chosen so that the two edges colored $c \in \{0, 1, 2, 3\}$ can be routed through middle switch M_c . We will not state and prove the correctness of any routing algorithm based on the coloring here, as it is straightforward.

We now describe a procedure to properly color all 4×4 2-regular multi-bipartite graphs G as follows. Along the way, we shall also prove that our procedure works.

Phase 1. As G is 2 regular and multi-bipartite, it is the union of even cycles. G thus can be decomposed into two 4×4 perfect matchings by taking alternate edges on each cycle. Let the matchings be M_1 and M_2 (whose vertex sets are the same as G .)

Phase 2. From each M_i ($i = 1, 2$), construct a 2×2 2-regular bipartite graph G_i by combining within each bipartition of M_i the pairs of vertices $\{0, 1\}$ and $\{2, 3\}$. Figure 4 illustrates the results of our first two phases. Obviously, $L(E(G_i)) = R(E(G_i)) = \{0, 1, 2, 3\}$, for $i = 1, 2$. Here and henceforth the L and R functions are applied in the context of the original graph G .

Figure 4: An illustration of the first two phases

We call the graphs G_1 and G_2 the *basic components* of G . Since the basic components are 2×2 2-regular bipartite graphs, they can only be either a 4-cycle or a union of two 2-cycles. A basic component is said to be of *type 1* if it is a 4-cycle and of *type 2* otherwise. In Figure 4, G_1 is of type 1 and G_2 is of type 2.

Phase 3. As each coloring of G_1 and G_2 induces uniquely a coloring of G , we are to color G_1 and G_2 so that the coloring satisfy conditions P_i and P'_i , $1 \leq i \leq 2$. We call an edge whose color is $c \in C$ a c -edge. Consider two cases:

Case 1 Both G_1 and G_2 are of type 1. In this case we color the graphs as shown in Figure 5a.

It is easy to see that the coloring satisfies all prescribed conditions. The basic idea is that as we have used each color exactly twice, to enforce P_1 and P'_1 we need to make sure that if there is a c -edge going from $\{0, 1\}$ to $\{0, 1\}$, then the other c -edge must go from $\{2, 3\}$ to $\{2, 3\}$ in either basic components, and similarly if a c -edge going from $\{0, 1\}$ to $\{2, 3\}$ then the other c -edge must go from $\{2, 3\}$ to $\{0, 1\}$. To enforce P_2 and P'_2 , on the left side (the I side) we *separate* each color pair $\{0, 1\}$ (i.e. $L(0) \cap L(1) = \emptyset$) and $\{2, 3\}$ (i.e. $L(2) \cap L(3) = \emptyset$), while on the right (the O side) we separate the pairs $\{0, 2\}$ ($R(0) \cap R(2) = \emptyset$) and $\{1, 3\}$ ($R(1) \cap R(3) = \emptyset$).

Figure 5: Illustration of the colorings when $n = 3$

Case 2 There is one graph of type 2. Without loss of generality, assume G_2 is of type 2 as illustrated in Figures 5b and 5c. In this case we color G_1 with $\{0, 2\}$ and G_2 with $\{1, 3\}$. Notice that P_1 , P'_1 , and P'_2 are satisfied even if we switch colors in one (or both) 2-cycles of G_2 . To ensure P_2 , we do this switching if necessary at each 2 cycle of G_2 to separate each pair $\{0, 1\}$ and $\{2, 3\}$ on the left.

□

Secondly, we use the idea to derive a more elaborate proof for the case where $N = 16$. Firstly, we redraw the network as shown in Figure 6, so that it is easier to derive the conditions similar to the P_i and P'_i . Given any perfect matching π from the inputs to the outputs, we first construct the 8×8 2-regular multi-bipartite graph G in a similar way as the G in Lemma 3.1. The bipartitions of G are $I = O = \{0, \dots, 7\}$, and $(i, j) \in E(G)$ if for some $x \in \{0, \dots, 15\}$ we have $x \in I_i$ and $\pi(x) \in O_j$. From Figure 6, the following proposition is easy to see. We reuse all notations introduced in the proof of Lemma 3.1. Again, as a valid coloring induces a routing algorithm in a straightforward way, we shall not describe the algorithm here.

Proposition 3.2. *The fact that $(SE_4)^7$ is rearrangeable is equivalent to the fact that for any 8×8 2-regular multi-bipartite graph $G = (I, O)$ with bipartitions $I = O = \{0, \dots, 7\}$, there exists an edge coloring of G using colors in $C = \{0, \dots, 7\}$ satisfying the following conditions:*

Figure 6: A redrawing of the $(SE_4)^7$ network

- (P_1) For each $c \in C$, $L(c)$ has a representative from each of $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$.
- (P_2) For each pair $\{c_1, c_2\} \in \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}$, $L(\{c_1, c_2\})$ has a representative from each of $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$, and $\{6, 7\}$.
- (P_3) $L(\{0, 1, 2, 3\}) = L(\{4, 5, 6, 7\}) = \{0, 1, \dots, 7\}$. In other words, the elements of $L(\{0, 1, 2, 3\})$ and $L(\{4, 5, 6, 7\})$ are all distinct.
- (P'_1) For each $c \in C$, $R(c)$ has a representative from each of $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$.
- (P'_2) For each pair $\{c_1, c_2\} \in \{\{0, 4\}, \{2, 6\}, \{1, 5\}, \{3, 7\}\}$, $R(\{c_1, c_2\})$ has a representative from each of $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$, and $\{6, 7\}$.
- (P'_3) $R(\{0, 4, 2, 6\}) = R(\{1, 5, 3, 7\}) = \{0, 1, \dots, 7\}$. In other words, the elements of $R(\{0, 4, 2, 6\})$ and $R(\{1, 5, 3, 7\})$ are all distinct.

Note that these conditions imply that each color appears exactly twice. Again, the conditions were specifically chosen so that each pair of edges with the same color $c \in C$ can be routed through middle switch M_c without causing any conflict. From now on, we shall refer to a *valid* coloring of G as the coloring satisfying the prescribed conditions in Proposition 3.2. The following proposition further explores properties of a graph G which can be validly colored.

Proposition 3.3. G can be validly colored if and only if the graph G' obtained from G by applying one of the following operations can also be validly colored. Let X be either I or O , the operations are:

1. Switch labels of the vertices $\{v_1, v_2\}$ in X , where $\{v_1, v_2\} \in \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}$.
2. Switch labels of each pair $\{0, 2\}$ and $\{1, 3\}$ in X .
3. Switch labels of each pair $\{4, 6\}$ and $\{5, 7\}$ in X .
4. Switch labels of each pair $\{0, 4\}$, $\{1, 5\}$, $\{2, 6\}$, and $\{3, 7\}$ in X .

5. Flip G horizontally, i.e. switch labels of each pair $\{0, 7\}$, $\{1, 6\}$, $\{2, 5\}$, and $\{3, 4\}$ in both I and O .
6. Flip G vertically, i.e. G' is the mirror image of G .

Proof. It's not difficult to see that the valid coloring of G induces a valid coloring of G' under all cases except the operation of flipping G vertically. In this case, from the coloring of G we can obtain a coloring of G' by the following mapping of colors: if in G an edge is colored c , whose binary representation is $\overline{c_1 c_2 c_3}$, then we use $\overline{c_3 c_2 c_1}$ to color the edge in G' . The verification that this is indeed a valid coloring of G' is mechanical and we shall not attempt to do so here. $\square$

Theorem 3.4. $m(4) = 7$, namely the network $(SE_4)^7$ is rearrangeable.

Proof. Let G be an 8×8 2-regular multi-bipartite graph. To color G properly, i.e. the coloring satisfies the conditions of Proposition 3.2, we decompose G into 4 basic components. The decomposition is formally described below. Figure 7 illustrates the decomposition procedure.

Phase 1 Decompose G into two edge disjoint 8×8 perfect matchings M_1 and M_2 .

Phase 2 For each $i = 1, 2$, construct the graph G_i by collapsing the pairs of vertices $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$, and $\{6, 7\}$ on each bipartition of M_i . It is clear that the graphs G_i are 4×4 2-regular bipartite graphs.

Phase 3 For each $i = 1, 2$, decompose G_i into two edge disjoint 4×4 perfect matchings M_{i1} and M_{i2} .

Phase 4 For each $i = 1, 2$ and $j = 1, 2$, construct the graph G_{ij} by collapsing the pairs of vertices $\{01, 23\}$ and $\{45, 67\}$ on each bipartition of M_{ij} . As before, the G_{ij} are called *basic components* of G , and can only be one of two types: (a) type 1 corresponds to a 4-cycle and (b) type 2 corresponds to two 2 cycles. We are now ready to color the basic components so that the (uniquely) induced coloring on G is valid.

As we have seen in the proof of Lemma 3.1, the number of type-2 basic components can roughly be thought of as the degree of flexibility in finding a valid coloring for G . When there is no type-2 basic component, we color the edges of G_{ij} as shown in Figure 8. The coloring clearly satisfies conditions P_i and P'_j ($0 \leq i \leq 3, 1 \leq j \leq 3$).

The rest of the cases are considered in Lemmas 3.6, 3.7, and 3.8 with the help of Lemma 3.5 and Proposition 3.3. The basic idea is that as these cases involve at least one type-2 component, Lemma 3.5 allows us to color the other three basic components using certain set of 6 colors, without worrying about coloring the last basic component. Proposition 3.3 helps simplify case analysis. Roughly, the graphs G s could be partitioned into “equivalent” classes, where graphs from each of the class can be obtained from one another by applying a sequence of operations described in Proposition 3.3. $\square$

Lemma 3.5. Assume G has at least one basic component of type 2, say G_{22} . Let

$$\begin{aligned} C_L &= \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}, \\ C_R &= \{\{0, 4\}, \{2, 6\}, \{1, 5\}, \{3, 7\}\} \end{aligned}$$

and $\{c_1, c_2\} \in C_L \cup C_R$ be an arbitrary pair of colors in the set. If the other three basic components can be colored using colors in $C - \{c_1, c_2\}$ so that none of the properties P_i ($0 \leq i \leq 3$) and P'_j ($1 \leq j \leq 3$) are violated, then G_{22} can be colored with c_1 and c_2 to form a completely valid coloring of G .

Figure 7: An illustration of the basic component decomposition

Figure 8: The coloring when there is no type-2 basic component

Proof. We only show the lemma for the case where $\{c_1, c_2\} = \{0, 1\}$. Other cases are easily seen to be similar. We shall try to color each 2-cycle of G_{22} with 0 and 1, switching the colors if necessary.

Firstly, we claim that by coloring each 2-cycle of G_{22} with 0 and 1, properties P_i ($0 \leq i \leq 3$), and P'_1 hold, no matter which edge in each 2 cycle gets which color. Indeed, P_1 , and P'_1 hold trivially. Let e_1 and e_2 be two edges in any 2-cycle of G_{22} , then P_2 holds because $L(\{e_1, e_2\})$ has **either** a representative from each of $\{0, 1\}$ and $\{2, 3\}$ **or** $\{4, 5\}$ and $\{6, 7\}$. Since we have assumed that P_3 was not violated, before the new colors 0 and 1 arrives, $L(\{4, 5, 6, 7\}) = \{0, 1, \dots, 7\}$ and $L(\{2, 3\}) \subset \{0, 1, \dots, 7\}$. In fact, $L(\{2, 3\})$ has 4 distinct members since P_2 was not violated. Thus, as G is 2-regular $L(\{0, 1\}) = \{0, 1, \dots, 7\} - L(\{2, 3\})$, no matter how we assign 0 and 1 to the edges of G_{22} . Hence, P_3 holds.

Notice that $R(4)$ and $R(5)$ each has a representative from each of $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$. We claim that $R(\{4, 5\})$ has a representative from each of $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$, and $\{6, 7\}$. Assume for contradiction, without loss of generality, that $R(\{4, 5\}) \cap \{0, 1\} = \emptyset$. Let's look at the 4 edges whose right end points are 2 or 3. Two of them are colored 4 and 5 as $R(\{4, 5\}) \cap \{0, 1\} = \emptyset$. The third one is one of the 2 edges in a 2-cycle of G_{22} . The fourth edge must have had a color from $\{2, 6, 3, 7\}$, say 2 or 6. Moreover, of the four edges whose right end points are 0 or 1, one of them is the other edge in the 2-cycle of G_{22} , one of them must have been colored 6 or 2 because P'_2 was not violated, the last two have to get colors 3 and 7 as P'_1 holds for 3 and 7. However, this makes P'_2 invalid for the color pair $\{3, 7\}$. Contradiction!

Now, we try to switch colors in each 2-cycle of G_{22} if necessary to achieve P'_2 . Let e_1 , and e_2 be the two edges at the 2-cycle whose right end points are in $\{0, 1, 2, 3\}$. By construction, $R(\{e_1, e_2\})$ has a representative from each of $\{0, 1\}$ and $\{2, 3\}$. Assign colors 0 and 1 to e_1 and e_2 so that $R(\{0, 4\})$ has a representative from each of $\{0, 1\}$ and $\{2, 3\}$. Notice that this implies $R(\{1, 5\})$ has a representative from each of $\{0, 1\}$ and $\{2, 3\}$, too. The same procedure is done with the other 2-cycle of G_{22} .

Lastly, we show that P'_3 holds automatically. Notice that P'_3 is equivalent to the condition that each vertex i of the bipartition O of G is incident to two edges whose colors have different parities. Let $\{v_1, v_2\}$ be any pair of right side vertices in $\{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}$. It is easy to see that after we have colored the edges of G_{22} as above, of the four edges whose right end points are in $\{v_0, v_1\}$, two have even colors and the other two have odd colors. So if after the coloring either v_0 or v_1 is incident to colors of the same parity, then so is the other. However, this means that there must have been a violation of P'_3 even before the coloring of G_{22} , because there is only one edge of G_{22} whose right end point is in $\{v_0, v_1\}$. $\square$

Lemma 3.6. *G can be validly colored if there is at most 1 basic component of type 1.*

Proof. G could either have one or zero type-1 basic component. Without loss of generality, we assume the three type-2 components are G_{12} , G_{21} , and G_{22} . The coloring is roughly shown in Figure 9. The idea is to fix the coloring of G_{11} by two colors $\{0, 4\}$ as shown in the figure, then switch the assigned pair of colors at each 2-cycle if necessary to form a valid coloring of G .

Obviously, properties P_1, P'_i ($1 \leq i \leq 3$) hold. We now do the switching on each 2-cycle of G_{12} so that $|\{0, 1\} \cap L(\{0, 2\})|$ is either 0 or 2, and that $|\{4, 5\} \cap L(\{0, 2\})|$ is either 0 or 2. This is certainly possible. Notice that this implies $|\{0, 1\} \cap L(\{4, 6\})|$ and $|\{4, 5\} \cap L(\{4, 6\})|$ are also either 0 or 2. Secondly, we do the switching on each 2-cycle of G_{21} and G_{22} so that $|\{0, 1\} \cap L(\{0, 2\})| + |\{0, 1\} \cap L(\{1, 3\})| = 2$ and $|\{4, 5\} \cap L(\{0, 2\})| + |\{4, 5\} \cap L(\{1, 3\})| = 2$. Intuitively, we want to “separate” the left end points of the edges having colors in $\{0, 2\}$ from the left end points of edges having colors in $\{1, 3\}$, in order to maintain properties P_2 . The same assertion holds for the pairs $\{4, 6\}$ and $\{3, 5\}$. Lastly, P_3 is assured by the fact that G_{i1} and G_{i2} have edges which form an 8×8 matching, for each $i = 1, 2$. $\square$

Figure 9: The cases when there is at most one type-1 component

The proofs of the following lemmas are put in separate sections, as they are long and thus would distract the reader from the main line of reasoning.

Lemma 3.7. *G can be validly colored if there are exactly 2 basic components of type 1.*

Lemma 3.8. *G can be validly colored if there are exactly 3 basic components of type 1.*

Now, we use the formulation of Linial and Tarsi to first show an auxiliary lemma and then combine the lemma with Theorem 3.4 to improve the upper bound of $m(n)$. The following lemma has been shown by Varma and Raghavendra in [8], however the proof was rather long. We straightforwardly extend Theorem 3.1 in [9] to obtain a much shorter proof.

Lemma 3.9. *If $m(k) = 2k - 1$ for a fixed $k \in \mathbb{N}$, then $(SE_n)^{3n-k-1}$ is rearrangeable whenever $n \geq k$.*

Proof. The assertion in the lemma is equivalent to the fact that if we know $m(k) = 2k - 1$ for some fixed $k \in \mathbb{N}$, then for every two $N \times n$ balanced matrices $A = [a_1, \dots, a_n]$ and $B = [b_1, \dots, b_n]$, there exists an $N \times (2n - k - 1)$ balanced matrix M such that the matrix $[A, M, B]$ is balanced. Here a_i and b_i are the i^{th} columns of A and B respectively. We shall construct the $(2n - k - 1)$ column vectors of M . The construction takes several steps as follows.

Step 1 Repeatedly apply Lemma 2.3 to construct vectors $\{u_1, \dots, u_{n-k}\}$ such that for $i = 1, \dots, n - k$, u_i agrees with $[a_{i+1}, \dots, a_n, u_1, \dots, u_{i-1}]$ and $[u_{i-1}, \dots, u_1, b_n, \dots, b_{i+1}]$. Let $U = [u_1, \dots, u_{n-k}]$ and $U^R = [u_{n-k}, \dots, u_1]$, then after this step both $[A, U]$ and $[U^R, B]$ are balanced.

Step 2 We want to construct vectors $x_1, \dots, x_{k-1}$ such that if we let $X = [x_1, \dots, x_{k-1}]$, then $[A, U, X]$ and $[X, U^R, B]$ are both balanced. Notice that as U is an $N \times (n - k)$ balanced matrix, each row of U occurs exactly 2^k times, and so do the rows of U^R in the same positions. Hence, the rows of U and U^R can be partitioned into 2^{n-k} classes of 2^k identical row vectors in each partition. For v be any column of U or U^R , let $v^{(i)}$ be the subvector of v with entries in the i^{th} partition, where $0 \leq i \leq 2^{n-k} - 1$. Notice that $v^{(i)} \in \mathbb{F}_2^k$ for each i . Also, for each $i = 0, \dots, 2^{n-k} - 1$, let

$$A^{(i)} = [a_{n-k+1}^{(i)}, \dots, a_n^{(i)}]$$

and

$$B^{(i)} = [b_n^{(i)}, \dots, b_{n-k+1}^{(i)}]$$

Then, since Beneš conjecture is true for k (i.e. $m(k) = 2k - 1$), there exist vectors $x_1^{(i)}, \dots, x_{k-1}^{(i)}$ such that $[A^{(i)}, X^{(i)}, B^{(i)}]$ is balanced. The vectors $x_1, \dots, x_{k-1}$ are obtained by pasting together the $x_j^{(i)}$ preserving the positions of the partitions.

After this step, $[A, U, X]$ is balanced because at the positions where the rows of U are identical we have $[A^{(i)}, X^{(i)}]$ being a $2^k \times k$ balanced matrix. The fact that $[X, U^R, B]$ is balanced can be shown similarly.

Step 3 Now we define an $N \times (n - k)$ matrix W from U such that $[A, W, X, U^R, B]$ is balanced. Define W as follows (all arithmetics are done over $\mathbb{F}_2$).

$$w_i = \begin{cases} u_i & 1 \leq i \leq \frac{n-k}{2} \\ u_i + u_{n-k-i} & \frac{n-k}{2} + 1 \leq i \leq n - k - 1 \\ u_{n-k} + a_n & i = n - k \end{cases}$$

We are left to show that $[A, W, X, U^R, B]$ is balanced. The balancedness of $[X, U^R, B]$ has already been established, so we only need to show that $[A, W, X, U^R]$ is balanced. We do this by considering the following types of submatrices:

- (a) Submatrices of the form $[a_i, \dots, a_n, w_1, \dots, w_{i-1}]$ where $2 \leq i \leq n - k + 1$. We apply Lemma 2.5 and use the fact that $[a_i, \dots, a_n, u_1, \dots, u_{i-1}]$ is balanced. $[a_i, \dots, a_n, w_1, \dots, w_{i-1}]$ can be obtained from $[a_i, \dots, a_n, u_1, \dots, u_{i-1}]$ by an invertible linear transformation with the invert map preserves the a_j ($i \leq j \leq n$) and

$$u_j = \begin{cases} w_j & 1 \leq j \leq \frac{n-k}{2} \\ w_j + w_{n-i-k} & \frac{n-k}{2} + 1 \leq j \leq n - k - 1 \\ w_{n-k} + a_n & j = n - k \end{cases}$$

- (b) Submatrices of the form $[a_i, \dots, a_n, w_1, \dots, w_{n-k}, x_1, \dots, x_{k+i-n-1}]$ where $n - k + 2 \leq i \leq n$. Similarly, in this case we use Lemma 2.5 and the balancedness of the matrix $[a_i, \dots, a_n, u_1, \dots, u_{n-k}, x_1, \dots, x_{k+i-n-1}]$
- (c) Submatrices of the form $[w_i, \dots, w_{n-k}, x_1, \dots, x_{k-1}, u_{n-k}, \dots, u_{n-k-i+1}]$ where $1 \leq i \leq n - k$. Here we use the fact that $[a_n, u_1, \dots, u_{n-k}, x_1, \dots, x_{k-1}]$ is balanced.

□

Theorem 3.10. For $n \in \mathbb{N}$ and $n \geq 4$, a SE network with $3n - 5$ stages is rearrangeable.

Proof. This is immediate from Theorem 3.4 and Lemma 3.9. □

4 Proof of Lemma 3.7

In this section we shall present a proof of Lemma 3.7 and also set up most of the basic techniques and notations needed for the proof of Lemma 3.8, which is more complicated. In the proof of this Lemma and the next, we assume that G_{22} is of type-2. Without loss of generality, we also assume that the 2-cycles of G_{12} go horizontally, i.e. one goes from 0123 to 0123, and the other from 4567 to 4567, applying operation 4 of Proposition 3.3 if necessary.

The basic idea behind the proofs of these Lemmas is to start from a coloring of G_{11} , G_{12} , and G_{21} using 6 colors $\{0, 4, 2, 6, 1, 5\}$ which does not violate P_1 and P'_i ($i = 1, 2, 3$), and then

modify this coloring so that none of the properties in Proposition 3.2 is violated. The Lemmas then hold as a consequence of Lemma 3.5.

In the course of modifying the original coloring, we shall need 5 basic color transformations: $UU(c_1, c_2)$, $UL(c_1, c_2)$, $LU(c_1, c_2)$, $LL(c_1, c_2)$, and $A(c_1, c_2)$, where c_1 and c_2 are two distinct colors in C . Transformation $UU(c_1, c_2)$ (for Upper-Upper) switches colors of a c_1 -edge e_1 and a c_2 -edge e_2 where $l(e_i) \in \{0, 1, 2, 3\}$, and $r(e_i) \in \{0, 1, 2, 3\}$, $i = 1, 2$. $UL(c_1, c_2)$ (for Upper-Lower) switches colors of a c_1 -edge e_1 and a c_2 -edge e_2 where $l(e_i) \in \{0, 1, 2, 3\}$, and $r(e_i) \in \{4, 5, 6, 7\}$, $i = 1, 2$. LU and LL are defined similarly in the obvious way. $A(c_1, c_2)$ (for All) changes color of all c_1 -edges to c_2 and vice versa. We will see that the transformations UU, UL, LU, LL are well defined from the associated context where they are applied.

We now need to introduce a concept called the *color incidence vectors* (or CIV for short) associated with a coloring of G . To each subset of vertices $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$ of either I or O , we associate a CIV of 4 components, where the component corresponding to vertex i consists of 2 colors of the edges incident to i . For example, Figure 10 shows a coloring of a graph G and the four associated CIVs. Note that the coloring shown is a **valid** one. It is immaterial if

Figure 10: An example of color incidence vectors

the CIVs are row vectors or column vectors, so we will adopt the convention that in the figures we use column vectors, and in the texts we use row vectors. Moreover, the order of the color pair in each component of a CIV is not important. In the figure, we have specifically chosen the order so that it is easier to see the validity of the coloring.

Corresponding to a coloring of G , there are 4 CIVs. We shall use v_1 and v_2 to denote the CIVs corresponding to the subsets $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$ of I , respectively. Similarly, u_1 and u_2 are the CIVs corresponding to the subsets $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$ of O , respectively. The following definitions relate the CIVs to the conditions P_i and P'_i .

Definition 4.1. Let v be a CIV corresponding to a coloring of G , then

- v is said to *respect* P_1 or P'_1 if the components of v contain all 8 colors $\{0, \dots, 7\}$.
- Let A be the set of 4 colors in the first two components of v , and B be the set of 4 colors in the last two components of v . Then, v is said to *respect* P_2 if each pair $\{c_1, c_2\} \in \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}$ has a representative from each of A and B . Similarly, v is said to *respect* P'_2 if each pair $\{c_1, c_2\} \in \{\{0, 4\}, \{2, 6\}, \{1, 5\}, \{3, 7\}\}$ has a representative from each of A and B .
- v is said to *respect* P_3 if each component of v has a representative from each of $\{0, 1, 2, 3\}$

and $\{4, 5, 6, 7\}$. Similarly, v respects P'_3 if each component of v has a representative from each of $\{0, 4, 2, 6\}$ and $\{1, 5, 3, 7\}$.

The following Proposition is fairly straightforward, thus we omit the proof.

Proposition 4.2. *Let v_1, v_2, u_1, u_2 be the CIVs corresponding to some coloring of G as defined above. Then, the coloring is valid if and only if*

- (i) v_1 and v_2 respect P_i ($i = 1, 2, 3$).
- (ii) u_1 and u_2 respect P'_i ($i = 1, 2, 3$).

To show Lemma 3.7, we consider two cases as follows.

Case 1. *The two type-1 components are G_{i1} and G_{i2} for some $i \in \{1, 2\}$. Without loss of generality, we assume G_{11} and G_{12} are of type-1. We start from the coloring shown in Figure 11, which clearly does not violate P_1 and P'_i ($i = 1, 2, 3$). Moreover, applying both $A(0, 6)$ and $A(2, 4)$ would yield another coloring where P_1 and P'_i are not violated.*

Figure 11: Case 1 of Lemma 3.7

After the tentative coloring introduced in Figure 11, and after applying a sequence of the first 3 operations in Proposition 3.3, the possible CIVs are the union of two sets:

$$F_1 = \left\{ \begin{bmatrix} \bullet 0 \\ \circ 2 \\ \bullet 4 \\ \circ 6 \end{bmatrix}, \begin{bmatrix} \bullet 0 \\ \circ 2 \\ \circ 4 \\ \bullet 6 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ \bullet 2 \\ \bullet 4 \\ \circ 6 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ \bullet 2 \\ \circ 4 \\ \bullet 6 \end{bmatrix} \right\} \quad F_2 = \left\{ \begin{bmatrix} \bullet 0 \\ \circ 6 \\ \bullet 2 \\ \circ 4 \end{bmatrix}, \begin{bmatrix} \bullet 0 \\ \circ 6 \\ \circ 2 \\ \bullet 4 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ \bullet 6 \\ \bullet 2 \\ \circ 4 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ \bullet 6 \\ \circ 2 \\ \bullet 4 \end{bmatrix} \right\}$$

Here, the $\bullet$'s stand for the colors of the edges of a 2-cycle of G_{21} , which are to be determined. In each CIV, one $\bullet$ gets colored 1 and the other gets 5. Obviously, the choice of 5 and 1 on one CIV will affect another. Specifically, the choice of 5 and 1 on one of the CIVs v_i will affect one of the u_j , but not the other v_i . Similarly, the $\circ$'s correspond to colors of G_{22} , which we are not concerned about, as we shall apply Lemma 3.5. Strictly speaking, each F_i consists of four classes of CIVs. However, we shall not differentiate between individual CIV and its class as this is immaterial.

We shall also extend the notion of a condition P_i or P'_i being respected by a CIV to a class of CIV. A class of CIV *respects* P_i or P'_i if the corresponding condition in Definition 4.1 is not violated yet. In this sense, each class of CIVs in $F_1 \cup F_2$ respects all P_i and P'_i . Notice that no matter how we assign 1 and 5 in each 2-cycle of G_{21} , the resulting CIVs in $F_1 \cup F_2$ (u_1 and u_2 in

particular) still respect P'_i . Moreover, u_1 and u_2 keep respecting P'_i even after we apply $A(2, 6)$ or $A(0, 4)$.

If both v_1 and v_2 belong to F_1 , then in both v_i we choose the $\bullet$ that goes with 0 or 2 to be 5 and the other $\bullet$ to be 1. Clearly, the P_i are respected by v_1 and v_2 .

If both v_1 and v_2 belong to F_2 , then we apply $A(2, 6)$ which move v_1 and v_2 to F_1 again.

Hence, we could now assume without loss of generality that $v_1 \in F_1$ and $v_2 \in F_2$, flipping G horizontally if necessary (operation 5 of Proposition 3.3). Furthermore, due to operation 6 of flipping G vertically, we can also assume that u_1 and u_2 don't belong to the same F_i .

As the $\bullet$'s colors in v_1 can be chosen easily so that v_1 respects P_i ($i = 1, 2, 3$), we try to modify the coloring so that v_2 does, too. If $v_2 = [\bullet 0 \circ 6 \circ 2 \bullet 4]$, we can pick colors so that $v_2 = [5 0 \circ 6 \circ 2 \bullet 4]$. When $v_2 = [\circ 0 \bullet 6 \bullet 2 \circ 4]$, we apply $A(0, 6)$ and $A(2, 4)$ to make $v_2 = [\bullet 0 \circ 6 \circ 2 \bullet 4]$ and choose $\bullet$'s colors similarly. Note that the availability of the $\bullet$'s colors in v_1 are not affected.

If $v_2 = [\bullet 0 \circ 6 \bullet 2 \circ 4]$ and $u_1 \in F_1$, we apply $LU(2, 4)$, making $v_2 = [\bullet 0 \circ 6 \circ 2 \bullet 4]$, but keeping P_i respected by the v_j and P'_i respected by the u_j ($i = 1, 2, 3, j = 1, 2$). The $\bullet$'s colors in v_2 could now be picked as before. When $v_2 = [\bullet 0 \circ 6 \bullet 2 \circ 4]$ and $u_2 \in F_1$, we apply $LU(2, 4)$, $UU(0, 6)$, $UL(2, 4)$, which turns v_2 into $[\bullet 0 \circ 6 \circ 2 \bullet 4]$ but keeps v_1 in F_1 , still.

Lastly, when $v_2 = [\circ 0 \bullet 6 \circ 2 \bullet 4]$, we apply $A(0, 6)$ and $A(2, 4)$ and return to the previous case.

Case 2. It is not the case that the two type-1 components are G_{i1} and G_{i2} for any $i \in \{1, 2\}$. Without loss of generality, we assume G_{11} and G_{21} are of type-1. We start from the coloring shown in Figure 12, which clearly does not violate P_1 and P'_i ($i = 1, 2, 3$).

Figure 12: Case 2 of Lemma 3.7

Again, there are 8 possible CIVs divided into two classes:

$$F_1 = \left\{ \begin{bmatrix} 05 \\ \bullet \circ \\ 41 \\ \bullet \circ \end{bmatrix}, \begin{bmatrix} 05 \\ \bullet \circ \\ 4 \circ \\ \bullet 1 \end{bmatrix}, \begin{bmatrix} 0 \circ \\ \bullet 5 \\ 41 \\ \bullet \circ \end{bmatrix}, \begin{bmatrix} 0 \circ \\ \bullet 5 \\ 4 \circ \\ \bullet 1 \end{bmatrix} \right\} \quad F_2 = \left\{ \begin{bmatrix} 01 \\ \bullet \circ \\ 45 \\ \bullet \circ \end{bmatrix}, \begin{bmatrix} 01 \\ \bullet \circ \\ 4 \circ \\ \bullet 5 \end{bmatrix}, \begin{bmatrix} 0 \circ \\ \bullet 1 \\ 45 \\ \bullet \circ \end{bmatrix}, \begin{bmatrix} 0 \circ \\ \bullet 1 \\ 4 \circ \\ \bullet 5 \end{bmatrix} \right\}$$

Remark 4.3. A special property of F_1 is that for each vector $v \in F_1$, there is a *proper assignment* of colors 6 and 2 to the $\bullet$, i.e. one $\bullet$ gets 6, the other gets 2 such that v respect all P_i ($i = 1, 2, 3$). For each $v \in F_2$, a proper assignment exists if 5 and 1 get exchanged in v . Moreover, for any vector $v \in F_1 \cup F_2$, if we exchange colors of one of the pairs $\{4, 1\}$ or $\{0, 5\}$, we could properly assign colors 6 and 2 to the $\bullet$'s of v to maintain P_i .

If both v_1 and v_2 are in F_1 , then we properly assign 2 and 6 to the $\bullet$'s. While, if v_1 and v_2 are in F_2 we apply $A(5, 1)$ and pick $\bullet$'s colors in the same way.

Now, assume $v_1 \in F_1$ and $v_2 \in F_2$. Also, by flipping G vertically if necessary, we can assume that u_1 and u_2 are representatives from F_1 and F_2 .

If $v_1 = [0\circ \bullet 5 \ 4\circ \bullet 1]$, then apply $A(5, 3)$ and $A(1, 7)$ so that $v_1 = [0\circ \bullet 3 \ 4\circ \bullet 7]$. In this case, F_2 becomes

$$F_2 = \left\{ \begin{bmatrix} 07 \\ \bullet\circ \\ 43 \\ \bullet\circ \end{bmatrix}, \begin{bmatrix} 07 \\ \bullet\circ \\ 4\circ \\ \bullet 3 \end{bmatrix}, \begin{bmatrix} 0\circ \\ \bullet 7 \\ 43 \\ \bullet\circ \end{bmatrix}, \begin{bmatrix} 0\circ \\ \bullet 7 \\ 4\circ \\ \bullet 3 \end{bmatrix} \right\}$$

Next, choose the $\bullet$'s colors in v_1 to turn it into $v_1 = [0\circ \bullet 3 \ 4\circ \bullet 27]$. While, in v_2 the $\bullet$ in the first or second component gets 2 and the other gets 6. We have used up 6 colors $\{0, 4, 2, 6, 3, 7\}$. Only 1 and 5 are left for the type-2 component G_{22} , hence Lemma 3.5 applies.

If $v_2 = [0\circ \bullet 1 \ 4\circ \bullet 5]$, then apply $A(5, 7)$ and $A(1, 3)$. The rest is similar.

Consequently, we only need to consider $v_1 \in A_1 := F_1 - \{[0\circ \bullet 5 \ 4\circ \bullet 1]\}$, and $v_2 \in A_2 := F_2 - \{[0\circ \bullet 1 \ 4\circ \bullet 5]\}$. u_1 and u_2 can be assumed to be representatives of A_1 and A_2 for the same reason as before.

If $v_1 = [05 \ \bullet\circ \ 41 \ \bullet\circ]$, then we flip G vertically so that $u_1 = [05 \ \bullet\circ \ 41 \ \bullet\circ]$, and that v_1 and v_2 are now representatives of A_1 and A_2 . Apply $UU(0, 5)$ and $LU(1, 4)$, then P'_i are still respected by v_1 and v_2 . Moreover, 0 and 5 get exchanged in v_1 , while 1 and 4 get switched in v_2 . Proper assignments of 2 and 6 now exist for v_1 and v_2 by Remark 4.3.

Similarly, when $v_2 = [01 \ \bullet\circ \ 45 \ \bullet\circ]$ we apply $LL(0, 5)$ and $UL(1, 4)$ and proceed in the same manner.

Hence, we are left to consider the case where

$$v_1 \in B_1 := \left\{ \begin{bmatrix} 05 \\ \bullet\circ \\ 4\circ \\ \bullet 1 \end{bmatrix}, \begin{bmatrix} 0\circ \\ \bullet 5 \\ 41 \\ \bullet\circ \end{bmatrix} \right\} \quad v_2 \in B_2 := \left\{ \begin{bmatrix} 01 \\ \bullet\circ \\ 4\circ \\ \bullet 5 \end{bmatrix}, \begin{bmatrix} 0\circ \\ \bullet 1 \\ 45 \\ \bullet\circ \end{bmatrix} \right\}$$

and, u_1 and u_2 are representatives of B_1 and B_2 .

If $v_1 = [05 \ \bullet\circ \ 4\circ \bullet 1]$ and $u_1 \in B_2$, $u_2 \in B_1$, then we flip G vertically, apply $UU(0, 5)$ and Remark 4.3. While, if v_1 is the same but $u_1 = [0\circ \bullet 5 \ 41 \ \bullet\circ] \in B_1$ and $u_2 \in B_2$, then we apply $LU(1, 4)$ and Remark 4.3.

When $v_1 = [0\circ \bullet 5 \ 41 \ \bullet\circ]$ and $u_1 \in B_1$, $u_2 \in B_2$, then we flip G vertically, apply $LU(1, 4)$ and Remark 4.3. While, when v_1 is the same but $u_2 = [05 \ \bullet\circ \ 4\circ \bullet 1] \in B_1$ and $u_1 \in B_2$, then we apply $LL(0, 5)$ and Remark 4.3.

Therefore, there are 8 cases left, 4 of which are when $v_1 = u_1 = [05 \ \bullet\circ \ 4\circ \bullet 1]$, and $v_2, u_2 \in B_2$. The other 4 cases are when $v_1 = u_2 = [0\circ \bullet 5 \ 41 \ \bullet\circ]$, and $v_2, u_1 \in B_2$. We shall consider these 8 cases in turn as follows.

(2a) $v_1 = u_1 = [05 \ \bullet\circ \ 4\circ \bullet 1]$, and $v_2 = u_2 = [01 \ \bullet\circ \ 4\circ \bullet 5]$.

This case introduces a new technique which will be used in later cases and the proof of Lemma 3.8. Let us first take a look at Figure 13. The graph shown represents all graphs considered in this case. The two dashed edges at the upper half is a 2-cycle of G_{12} , and the other two dashed edges are from the other 2-cycle. The end points of the dashed edges go to the $\bullet$'s, but we don't know which edge goes to which $\bullet$.

Figure 14 presents our solution to this case. The figure shows a "proof without words".

Figure 13: Representative figure for case 2a of Lemma 3.7

Figure 14: Case 2a of Lemma 3.7

Basically, we consider 3 sub-cases represented sequentially by 3 drawings from left to right.

In the first sub-case, we fix the end points (in the square boxes) of a dashed edge in the upper half, letting all other dashed edges go freely. The second dashed edge in the upper half has only one choice to go, of course. The doubly headed arrow specifies that we could exchange colors 0 and 4 of those two dashed edges so that v_2 looks as shown. It is straightforward to check that in this sub-case v_1 and v_2 respect the P_i and u_1 and u_2 respect the P'_i .

In the second sub-case, we fix the end points of a dashed edge in the lower half. Assign colors to all edges as shown. The doubly headed arrow on the side means that no matter which dashed edge goes to which $\bullet$, we still have v_1 respecting the P_i .

The last drawing considers the only sub-case left. The figure is self-explaining.

$$(2b) \quad v_1 = u_1 = [05 \quad \bullet\circ \quad 4\circ \quad \bullet 1], v_2 = [01 \quad \bullet\circ \quad 4\circ \quad \bullet 5], \text{ and } u_2 = [0\circ \quad \bullet 1 \quad 45 \quad \bullet\circ].$$

Figure 15 presents our solution to this case.

Figure 15: Case 2b of Lemma 3.7

$$(2c) \quad v_1 = u_1 = [05 \quad \bullet\circ \quad 4\circ \quad \bullet 1], v_2 = [0\circ \quad \bullet 1 \quad 45 \quad \bullet\circ], \text{ and } u_2 = [01 \quad \bullet\circ \quad 4\circ \quad \bullet 5].$$

Figure 16 presents our solution to this case.

Figure 16: Case 2c of Lemma 3.7

$$(2d) \quad v_1 = u_1 = [05 \quad \bullet\circ \quad 4\circ \quad \bullet 1], v_2 = u_2 = [0\circ \quad \bullet 1 \quad 45 \quad \bullet\circ].$$

Figure 17 presents our solution to this case.

Figure 17: Case 2d of Lemma 3.7

Before proceeding to the proof of case (2e), we need another proposition, which is very useful later on in the proof of Lemma 3.8.

Proposition 4.4. *Let G be our graph to be colored as usual. For any $j \in \{1, 2\}$, let $\bar{j}$ be the element in $\{1, 2\} - \{j\}$. Assume the following hold:*

- (i) G_{22} is a type-2 component of G with the two 2-cycles going horizontally, i.e. one 2-cycle of G_{22} goes from v_1 to u_1 and the other goes from v_2 to u_2 .
- (ii) There is a partial coloring of G , in which all edges of G_{11} , G_{12} , and G_{21} are colored using twice each color in $\{0, 4, 2, 6, 1, 5\}$.
- (iii) (The partially colored) u_1 and u_2 respect the P'_i ($i = 1, 2, 3$.)
- (iv) (The partially colored) $v_{\bar{j}}$ respects the P_i ($i = 1, 2, 3$.)
- (v) v_j respects P_1 and P_2 but does not respect P_3 because one component of v_j contains some $c \in \{2, 6\}$ and another color c' , but c and c' are not representatives of $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$.
- (vi) The edges get color c go horizontally from v_j to u_j and $v_{\bar{j}}$ to $u_{\bar{j}}$. Moreover, c goes with $\circ$ in a component of u_j , i.e. the edge e that gets colored c has the same right end point as another edge e' in the 2-cycle that goes from v_j to u_j .

Then, G can be properly colored.

The last drawing in Figure 18 is an example of such a situation. In this drawing, $j = 2$, $c = 2$ and $c' = 1$.

Proof. Firstly, assume $c = 2$. Clearly, just as in the proof of Lemma 3.5, we can assign 3 and 7 to edges of the 2-cycle going from $v_{\bar{j}}$ to $u_{\bar{j}}$ so that $v_{\bar{j}}$ respects the P_i and $u_{\bar{j}}$ respects the P'_i . Suppose

$$v_j = [cc' \quad c_1 \circ \quad c_2 c_3 \quad c_4 \circ]$$

where $\{c, c', c_1, c_2, c_3, c_4\} = \{0, 4, 2, 6, 1, 5\} = \{0, 1, 2, 4, 5, 6\}$ because v_j respects P_1 . As cc' is the only component that does not respect P_3 (i.e. $c = 2$ and c' are in $\{0, 1, 2, 3\}$), $c_2 c_3$ are representatives of $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$. Thus, $c_1, c_4 \in \{4, 5, 6\}$. In fact, as v_j respects P_2 , $c_1 \in \{4, 5\}$, $c' \in \{0, 1\}$ and thus $6 \in \{c_2, c_3, c_4\}$.

Now, assign 7 to e' and 3 to the other edge of the 2-cycle that goes from v_j to u_j . We claim that after exchanging colors (7 and 2) of e and e' , we have a valid coloring of G . Indeed, the CIVs v_j and u_j are not affected. u_j respects P'_i after the first assignment of 7 and 3, and keeps respecting P'_i as we have just exchanged colors of edges having the same right end point. We only need to be concerned about v_j after this exchange. There are two cases for v_j before the exchange:

$$v_j = [2c' \quad c_1 \mathbf{7} \quad c_2 c_3 \quad c_4 \mathbf{3}]$$

or

$$v_j = [2c' \quad c_1 \mathbf{3} \quad c_2 c_3 \quad c_4 \mathbf{7}]$$

The reader can easily check that v_j does respect P_i after the exchange of 2 and 7.

The case where $c = 6$ is done similarly. The only difference is that e gets 3 this time. $\square$

Remark 4.5. This proposition will prove to be very useful in the proof of Lemma 3.8. It could be stated in a much more general fashion, however we did not do so because we will need only this instance of the proposition, and because the general statement would be too notationally heavy, thus hard to grasp.

$$(2e) \ v_1 = u_2 = [0\circ \quad \bullet 5 \quad 41 \quad \bullet \circ], \text{ and } v_2 = u_1 = [01 \quad \bullet \circ \quad 4\circ \quad \bullet 5].$$

Figure 18 presents the *partial* solution to this case. The reason this case was only partially solved is due to the last drawing, i.e. the last sub-case, in which there is a violation of P_3 in v_2 . To resolve this violation, we reason as follows. If the 2-cycles of G_{22} go horizontally,

Figure 18: Case 2e of Lemma 3.7

we apply Proposition 4.4. Otherwise, the 2-cycle going from v_1 to u_2 could be colored with 3 and 7 properly, as in Lemma 3.5. The other 2-cycle has edges going from the blacken vertices at u_1 to the $\circ$ of v_2 . We assign 7 to the edge whose right end point is the second component of u_1 and 3 to the other edge.

There are two cases after this assignment, depending on what v_2 ends up being. If v_2 ends up to be $v_2 = [12 \quad 74 \quad 36 \quad 05]$, then we apply $LU(1, 7)$. Obviously, u_1 still respects the P'_i , and v_2 respects all the P_i now. If v_2 becomes $v_2 = [12 \quad 34 \quad 76 \quad 05]$, then we apply $LU(1, 7)$ and $LL(0, 2)$ to get the same result. In these two situations, the relative positions of 4 and 5 are not important.

Lastly, note that we have used 6 colors $\{0, 1, 2, 3, 4, 5\}$ in the first sub-case, leaving $\{6, 7\}$. Lemma 3.5 still applies.

(2f,2g,2h) When $v_1 = u_2 = [0\circ \bullet 5 \ 41 \ \bullet\circ]$, we have three more cases, which are simple and close enough to be considered at once: (2f) $v_2 = [01 \ \bullet\circ \ 4\circ \ \bullet 5]$, and $u_1 = [0\circ \ \bullet 1 \ 45 \ \bullet\circ]$; (2g) $v_2 = [0\circ \ \bullet 1 \ 45 \ \bullet\circ]$, and $u_1 = [01 \ \bullet\circ \ 4\circ \ \bullet 5]$; and (2h) $v_1 = u_2 = [0\circ \ \bullet 5 \ 41 \ \bullet\circ]$, $v_2 = u_1 = [0\circ \ \bullet 1 \ 45 \ \bullet\circ]$.

Figure 19 presents the solutions all three cases.

Figure 19: Case 2f, 2g and 2h of Lemma 3.7

5 Proof of Lemma 3.8

We assume that the 2-cycles of G_{22} go horizontally, i.e. one goes from 0123 to 0123, and the other from 4567 to 4567, applying operation 4 of Proposition 3.3 if necessary. We start from the coloring shown in Figure 20, whose CIV v_j respect P_1 and u_j respect P'_i ($i = 1, 2, 3, j = 1, 2$). Moreover, applying both $A(0, 2)$ and $A(4, 6)$ would yield another coloring where P_1 , and P'_i are not violated. So does applying any of $A(0, 4)$, $A(1, 5)$, or $A(2, 6)$.

Figure 20: The initial coloring for Lemma 3.8

The possible CIVs are the union of four sets:

$$F_1 = \left\{ \begin{bmatrix} 50 \\ \circ 2 \\ 14 \\ \circ 6 \end{bmatrix}, \begin{bmatrix} 50 \\ \circ 2 \\ \circ 4 \\ 16 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ 52 \\ 14 \\ \circ 6 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ 52 \\ \circ 4 \\ 16 \end{bmatrix} \right\} \quad F_2 = \left\{ \begin{bmatrix} 10 \\ \circ 2 \\ 54 \\ \circ 6 \end{bmatrix}, \begin{bmatrix} 10 \\ \circ 2 \\ \circ 4 \\ 56 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ 12 \\ 54 \\ \circ 6 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ 12 \\ \circ 4 \\ 56 \end{bmatrix} \right\}$$

$$F_3 = \left\{ \begin{bmatrix} 50 \\ \circ 6 \\ 14 \\ \circ 2 \end{bmatrix}, \begin{bmatrix} 50 \\ \circ 6 \\ \circ 4 \\ 12 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ 56 \\ 14 \\ \circ 2 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ 56 \\ \circ 4 \\ 12 \end{bmatrix} \right\} \quad F_4 = \left\{ \begin{bmatrix} 10 \\ \circ 6 \\ 54 \\ \circ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ \circ 6 \\ \circ 4 \\ 52 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ 16 \\ 54 \\ \circ 2 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ 16 \\ \circ 4 \\ 52 \end{bmatrix} \right\}$$

Notice that for all $i = 1, 2, 3$, the vectors in F_1 respect P_i , F_2 respect P_i after applying $A(1, 5)$, F_3 respect P_i after applying $A(2, 6)$, and F_4 respect P_i after applying $A(0, 4)$. Hence, if v_1 and v_2 belong to the same F_i then we are done. We could thus also assume u_1 and u_2 are from different F_i 's. Consider 4 cases.

Case 1. $v_1 \in F_1$ and $v_2 \in F_3 \cup F_4$. If $v_2 \in F_4$, applying $A(0, 2)$ and $A(4, 6)$ would keep v_1 in F_1 , while move v_2 to F_3 . Hence, without loss of generality we can assume $v_1 \in F_1$ and $v_2 \in F_3$. Consider 4 sub-cases as follows, which are ordered in increasing in level of complexity.

(1a) $v_2 = [50 \ 06 \ 14 \ 02]$. In this case, v_2 and v_1 already respect the P_i .

(1b) $v_2 = [00 \ 56 \ 14 \ 02]$. In this case, we can not apply Lemma 3.5 directly, but have to go further. There is no problem with picking colors 3 and 7 for the 2 cycle of G_{22} connecting v_1 to u_1 . For the other 2-cycle, we pick colors and do certain transformation as follows. Notice that no matter what u_2 is, the 5 in u_2 is not in the same component as a 0. There is exactly an edge e of this 2-cycle where $r(e)$ and 5 are both in the first two components or the last two components of u_2 . Assign 3 to e and 7 to the other edge. There are two cases depending on how v_2 ends up to be after this assignment.

If v_2 becomes $v_2 = [30 \ 56 \ 14 \ 72]$, then we apply $LL(5, 3)$, which keeps u_2 respecting the P'_i and makes v_2 respect the P_i .

If v_2 becomes $v_2 = [70 \ 56 \ 14 \ 32]$, then we apply $LL(5, 3)$, $LU(4, 6)$ and $UU(0, 2)$. Effectively, this transformation makes $v_2 = [70 \ 34 \ 16 \ 52]$, respecting the P_i , exchanges each pair $\{0, 2\}$ and $\{4, 6\}$ in u_1 , and exchanges $\{0, 2\}$ in v_1 . Hence, after the transformation u_1 still respects the P'_i , and v_1 is still in F_1 , respecting the P_i .

(1c) $v_2 = [50 \ 06 \ 04 \ 12]$.

If u_2 has a 02 as a component (we will write $02 \in u_2$ for short), then apply Proposition 4.4. If $52 \in u_2$ and $00 \in u_2$, then we apply $LL(5, 2)$, then $A(0, 2)$ and $A(4, 6)$, so that $v_2 = [20 \ 04 \ 06 \ 51]$. Proposition 4.4 could be applied now with $c = 2$.

If $14 \in u_1$, then apply $LU(1, 4)$. If $16 \in u_1$, then apply $LU(1, 6)$, $LL(0, 2)$, and $UL(4, 6)$, turning v_2 into $[52 \ 01 \ 04 \ 06]$, while keeping all other CIVs respecting their corresponding conditions.

Hence, we are left to consider the case where $02 \notin u_2$, not both 52 and 00 are in u_2 , and 14 and 16 are not in u_1 .

As we have seen, the CIVs are useful in classifying how each subset of vertices $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$ of I and O are connected to the others. There is another way to classify the “shape” of these connecting patterns. Ignoring the edges of G_{22} , each subset $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$ of I and O are connected to three horizontal edges (i.e. edges connecting v_j and u_j) and three diagonal edges (i.e. edges connecting v_j and $u_{\bar{j}}$). The connecting patterns to each of these vertex subsets can be classified based on the relative end points of the horizontal and diagonal edges, up to the application of the first 3 operations of Proposition 3.3. Figure 21 shows all possible *shapes* of u_1 . In the figure, the thickened edges represent edges that go from u_1 to v_1 (horizontally). The other three edges go from u_1 to v_2 (diagonally). It is straightforward to check that there are only 5 possible shapes as shown, up to applying the first three operations of Proposition 3.3. For example, if $u_1 = [50 \ 02 \ 04 \ 16]$, then it is of shape 1. While, $[00 \ 12 \ 54 \ 06]$ is of shape 3, and $[00 \ 16 \ 54 \ 02]$ is of shape 2. Similarly, Figure 22 shows all possible shape of u_2 . These are just mirror images of the shapes of u_1 . We have tentatively label the edges in the shapes shown. Clearly, some ambiguity exists with the labels of two horizontal or diagonal

Figure 21: All possible shapes of u_1

Figure 22: All possible shapes of u_2

edges which share the same end point. However, as we shall see later, these ambiguities are not important when we refer to these labels.

Because 14 and 16 are not in u_1 , u_1 can only be of shape 3 or 5, because given the way we construct and color G_{11} and G_{12} , the even-numbered edges can not share an end point. Similarly, as not both 52 and $\circ 0$ are in u_2 and $\circ 2 \notin u_2$, u_2 can only be of shape 3 or 5. For, if u_2 is of shape 1 or 4, then either h_1 was colored 2 which makes $\circ 2 \in u_2$, or h_1 was colored 0 and thus both 52 and $\circ 0$ are in u_2 . If u_2 is of shape 2, then h_2 must have been colored 5, which makes $\circ 2 \in u_2$ (2 is the color of h_1 or h_3 .)

Now, we describe a way to recolor the 6 edges of G in G_{11} , G_{12} , and G_{21} . The idea is to use color 1, 2 and 5 to color the horizontal edges, and 0,4,6 to color the diagonal edges. The reader might find it useful sketching several figures while we are discussing the coloring.

Let's start by coloring the edges connected to u_1 . When u_1 is of shape 3, we assign 2 to h_1 , and $\{5, 1\}$ to $\{h_2, h_3\}$. What we mean by this is that one of $\{h_2, h_3\}$ will get 1 and the other will get 5, but which edge actually gets which color is to be decided later. h_1, h_2 and h_3 are connected to the first two components of v_1 (vertices 0 and 1), because $v_1 \in F_1$. If some vertex $j \in \{0, 1\}$ is incident to two of these h_i ($i = 1, 2, 3$), j must be incident to some h_i where $i \in \{2, 3\}$. Assign 5 to h_i , 1 to the edge in $\{h_2, h_3\} - \{h_i\}$. Clearly, this way we get a (partially colored) v_1 respecting the P_i . Now we have to assign 0, 4, and 6 to the diagonal edges connected to u_1 . Assign 0 to the edge d_i that was originally colored 6. If $i = 1$, then assign 4 and 6 to d_2 and d_3 arbitrarily. If $i \neq 1$, the assign 6 to the edge in $\{d_2, d_3\} - \{d_i\}$, and 4 to d_1 . Either way, u_1 respects the P'_j ($j = 1, 2, 3$), and v_2 looks

like

$$v_2 = \begin{bmatrix} c_1 c_2 \\ \circ 0 \\ \circ 4 \\ \mathbf{6} c_3 \end{bmatrix} \quad \text{or} \quad v_2 = \begin{bmatrix} c_1 c_2 \\ \circ 0 \\ \circ 6 \\ \mathbf{4} c_3 \end{bmatrix}$$

where $\{c_1, c_2, c_3\} = \{1, 2, 5\}$ are to be precisely determined when we color the horizontal edges connected to u_2 . The case where u_1 is of shape 5 is done in exactly the same way.

Secondly, we need to color the edges connected to u_2 . When u_2 is of shape 3, we assign 1 to the edge h_i that was originally colored 2. Clearly, $i \neq 3$ because $\circ 2 \notin u_2$. Assign 2 to h_3 and 5 to the last edge. When u_2 is of shape 5, we also assign 1 to the edge h_i that was originally colored 2. If $i \in \{2, 3\}$, then assign 2 to the edge in $\{h_2, h_3\} - \{h_i\}$ and 5 to the last edge. If $i = 1$, then assign 2 and 5 to h_2 and h_3 arbitrarily. Either way, u_2 still respects the P'_j and v_2 must either be

$$v_2 = \begin{bmatrix} \mathbf{52} \\ \circ 0 \\ \circ 4 \\ \mathbf{61} \end{bmatrix} \quad \text{or} \quad v_2 = \begin{bmatrix} \mathbf{52} \\ \circ 0 \\ \circ 6 \\ \mathbf{41} \end{bmatrix}$$

which clearly respects the P_j .

The diagonal edges of u_2 can be assigned 0,4 and 6 in much the same way. d_1 or d_2 will get 6, the other two get 0 or 4 which could be exchanged to ensure v_1 respects the P_j .

(1d) $v_2 = [\circ 0 \quad 56 \quad \circ 4 \quad 12]$.

If $14 \in u_1$, then apply $LU(1, 4)$ to make $v_2 = [\circ 0 \quad 56 \quad \circ 1 \quad 42]$ and then proceed similar to case (1b).

If $16 \in u_1$, then apply $LU(1, 6)$, then $LL(0, 2)$ and $UL(4, 6)$. Hence, we could now assume that u_1 is in shape 3 or 5.

If $52 \in u_2$, then applying $LL(2, 5)$, $UU(0, 2)$ and $LU(4, 6)$ would do.

Now, suppose $\circ 2 \notin u_2$. As $52 \notin u_2$, u_2 can only be of shape 3 or 5. Proceeding in the same manner as case (1c) completes the case. Consequently, we could now assume that $52 \notin u_2$ and $\circ 2 \in u_2$. If $50 \in u_2$, then apply $LL(5, 0)$, making $v_2 = [\circ 5 \quad 06 \quad \circ 4 \quad 12]$. Proposition 4.4 proves to be useful here again. When $52 \notin u_2$ and $\circ 2 \in u_2$ and $50 \notin u_2$, it is not hard to show that u_2 is of shape 2 or 3.

To summarize, we are left with the cases where u_1 is of shape 3 or 5, and u_2 is of shape 2 or 3. We will consider these 4 possible sub-cases in turn.

Firstly, suppose u_1 is of shape 3 and u_2 is of shape 3. We will color the horizontal edges with 0,2 and 4, the diagonal edges with 1, 5 and 6. For u_1 's edges, assign 2 to h_1 and $\{0, 4\}$ to $\{h_2, h_3\}$. Which edge gets 0 or 4 can be decided in the same manner as in case (1c) with 1 and 5. Assign 6 to d_3 and $\{1, 5\}$ to $\{d_1, d_2\}$. Note that one of d_1 or d_2 must have been colored 1 originally. Assign 1 to that originally-1 edge and 5 to the other. After this assignment, v_2 looks like:

$$v_2 = \begin{bmatrix} \circ c_1 \\ c_2 \mathbf{6} \\ \circ \mathbf{5} \\ \mathbf{1} c_3 \end{bmatrix} \quad \text{or} \quad v_2 = \begin{bmatrix} \circ c_1 \\ c_2 \mathbf{5} \\ \circ \mathbf{6} \\ \mathbf{1} c_3 \end{bmatrix}$$

For u_2 's diagonal edges, assign 6 to d_1 , and $\{1, 5\}$ to $\{d_2, d_3\}$. For u_2 's horizontal edges, assign 2 to h_3 (which must have been originally colored 2 because $\circ 2 \in u_2$), and 4 to h_2 and 0 to h_1 . After this assignment, v_2 can only be one of four forms:

$$v_2 = \begin{bmatrix} \circ 4 \\ \mathbf{06} \\ \circ 5 \\ 12 \end{bmatrix} \text{ or } v_2 = \begin{bmatrix} \circ 0 \\ \mathbf{46} \\ \circ 5 \\ 12 \end{bmatrix} \text{ or } v_2 = \begin{bmatrix} \circ 4 \\ \mathbf{05} \\ \circ 6 \\ 12 \end{bmatrix} \text{ or } v_2 = \begin{bmatrix} \circ 0 \\ \mathbf{45} \\ \circ 6 \\ 12 \end{bmatrix}$$

The first form is good due to Proposition 4.4. The second form is good after applying $LL(0, 4)$ and Proposition 4.4. The third and fourth form is good after applying $LL(2, 4)$.

Secondly, suppose u_1 is of shape 3 and u_2 is of shape 2. We can assume that the edge h_1 of u_2 was originally colored 2, otherwise h_3 was and hence the previous trick still applies. Moreover, h_2 must have been 5 originally as an originally odd-number edge must have shared an end point with another edge. Let us first try to assign colors to u_1 's edges. This time we use 0, 2, and 4 for horizontal and 1, 5, and 6 for diagonal edges. As "usual", assign 2 to h_1 of u_1 , $\{0, 4\}$ to $\{h_2, h_3\}$. Starting from here, there are two sub-cases: (i) when d_3 was originally 4; and (ii) when d_3 was originally 6.

If d_3 was originally 4, then assign 6 to d_3 and $\{1, 5\}$ to $\{d_1, d_2\}$ appropriately so that

$$v_2 = \begin{bmatrix} \circ c_1 \\ c_2 \mathbf{5} \\ \circ 6 \\ 1c_3 \end{bmatrix}$$

Then, for the diagonal edges of u_2 , assign 6 to d_1 and $\{1, 5\}$ to $\{d_2, d_3\}$ with 1 and 5 exchangeable to make v_1 valid. For the horizontal edges of u_2 , assign 4 to h_1 (which was originally 2) and $\{0, 2\}$ to $\{h_2, h_3\}$ arbitrarily. After this assignment, v_2 becomes either

$$v_2 = \begin{bmatrix} \circ 0 \\ \mathbf{25} \\ \circ 6 \\ 14 \end{bmatrix} \text{ or } v_2 = \begin{bmatrix} \circ 2 \\ \mathbf{05} \\ \circ 6 \\ 14 \end{bmatrix}$$

which obviously respect the P_i .

When d_3 was originally 6, the situation is more complicated. We will use 0, 6 and 5 for horizontal and 1, 2 and 4 for diagonal edges. For u_1 's edges, assign 5 to h_3 and $\{0, 6\}$ to $\{h_1, h_2\}$. 0 and 6 could be exchanged to make v_1 respect the P_i . Next, assign 2 to d_3 , 1 to d_1 and 4 to d_2 . After this assignment, v_2 can be one of two forms:

$$v_2 = \begin{bmatrix} \circ c_1 \\ c_2 \mathbf{2} \\ \circ 1 \\ 4c_3 \end{bmatrix} \text{ or } v_2 = \begin{bmatrix} \circ c_1 \\ c_2 \mathbf{2} \\ \circ 4 \\ 1c_3 \end{bmatrix}$$

Now it is u_2 's edges' turn. Assign 2 to d_3 , $\{1, 4\}$ exchangeable to $\{d_1, d_2\}$. Next, assign 6 to h_1 , 5 to h_2 (which was 5 originally), and 0 to h_3 . v_2 could now only be:

$$v_2 = \begin{bmatrix} \circ 0 \\ \mathbf{52} \\ \circ 1 \\ 46 \end{bmatrix} \text{ or } v_2 = \begin{bmatrix} \circ 0 \\ \mathbf{52} \\ \circ 4 \\ 16 \end{bmatrix}$$

The second form already respects the P_i . The second form is good too, due to Proposition 4.4.

Thirdly, suppose u_1 is of shape 5 and u_2 is of shape 2. We handle this case differently: we will color the edges so that v_1 and v_2 respect the P'_i , and u_1 and u_2 respect the P_i . A vertical flip of G would complete the case. Notice that at u_1 the original color of d_2 must have been 1, and at u_2 the original color of h_2 must have been 5. We will use 0, 2 and 5 for horizontal edges and 1, 4 and 6 for diagonal edges.

For u_1 's diagonal edges, assign 4 to d_2 , 1 to the edge that was 6 and 6 to the edge that was 4 originally. After this assignment, u_1 is of one of two forms:

$$u_1 = \begin{bmatrix} c_1 c_2 \\ \mathbf{1} \circ \\ c_3 \mathbf{4} \\ \circ \mathbf{6} \end{bmatrix} \quad \text{or} \quad u_1 = \begin{bmatrix} c_1 c_2 \\ \mathbf{6} \circ \\ c_3 \mathbf{4} \\ \circ \mathbf{1} \end{bmatrix}$$

and v_2 must be $[\circ 2 \quad 01 \quad \circ 6 \quad 45]$, which respects the P'_i . Where $\{c_1, c_2, c_3\} = \{1, 4, 6\}$ whose exact assignment will be decided later. If u_2 is of the first form, assign 0 to h_3 and $\{2, 5\}$ to $\{h_1, h_2\}$ exchangeable for v_1 to respect the P'_i . If u_2 is of the second form, assign 2 to h_3 and $\{0, 5\}$ to $\{h_1, h_2\}$ exchangeable for v_1 to respect the P'_i .

For u_2 's horizontal edges, assign 0 to h_2 , 5 to the edge that was 2, and 2 to the edge that was 0 originally. After this assignment, u_2 is one of two forms:

$$u_2 = \begin{bmatrix} c_1 c_2 \\ \mathbf{2} \circ \\ c_3 \mathbf{0} \\ \circ \mathbf{5} \end{bmatrix} \quad \text{or} \quad u_2 = \begin{bmatrix} c_1 c_2 \\ \mathbf{5} \circ \\ c_3 \mathbf{0} \\ \circ \mathbf{2} \end{bmatrix}$$

Here $\{c_1, c_2, c_3\} = \{0, 2, 5\}$ whose exact assignment will be decided later. If u_1 is of the first form, assign 6 to d_3 and $\{1, 4\}$ to $\{d_1, d_2\}$ exchangeable for v_1 to respect the P'_i . If u_1 is of the second form, assign 4 to d_3 and $\{1, 6\}$ to $\{d_1, d_2\}$ exchangeable for v_1 to respect the P'_i . It is also easy to check that v_2 respects the P'_i .

Lastly, suppose u_1 is of shape 5 and u_2 is of shape 3. In this case, if d_2 and d_3 of u_2 don't share the left end point, then we can reuse the previous trick. If d_2 and d_3 of u_2 have the same left end point, we start coloring in the usual way, namely for the v_j to respect the P_i and u_j to respect the P'_i . Note that at u_1 the color of d_2 originally was 1, and at u_2 the original color of h_3 was 2 (because $\circ 2 \in u_2$). We have to consider two cases, depending on the original color of d_1 at u_1 : (i) d_1 was 6; and (ii) d_1 was 4.

If at u_1 , d_1 was originally 6 then we use 0, 5, and 6 for the horizontal edges and 1, 2, and 4 for the diagonal edges. We start with u_1 's horizontal edges, assigning 6 to h_3 and $\{5, 0\}$ to h_1 and h_2 . Then, 2 to d_1 , 1 to d_2 and 4 to d_3 . After this assignment, we must have

$$v_2 = \begin{bmatrix} \circ c_1 \\ c_2 \mathbf{2} \\ \circ \mathbf{4} \\ \mathbf{1} c_3 \end{bmatrix}$$

Now, let's assign colors to the horizontal edges of u_2 . We assign 6 to h_3 , whose original color was 2, and $\{5, 0\}$ to $\{h_1, h_2\}$, whose exact assignment is such that v_2 respects P_i .

This is certainly possible. After the assignment, we either have

$$u_2 = \begin{bmatrix} \circ c_1 \\ c_2 \mathbf{0} \\ c_3 \mathbf{5} \\ \circ \mathbf{6} \end{bmatrix} \quad \text{or} \quad u_2 = \begin{bmatrix} \circ c_1 \\ c_2 \mathbf{5} \\ c_3 \mathbf{0} \\ \circ \mathbf{6} \end{bmatrix}$$

where, c_i is the color of d_i to be chosen from $\{1, 2, 4\}$. We assign 2 to d_1 , 4 to d_2 and 1 to d_3 if u_2 is in the first form. Assign 2 to d_1 , 1 to d_2 and 4 to d_3 if u_2 is in the second form. Recall that d_2 and d_3 shares the left end point, hence v_1 keeps respecting the P_i .

If at u_1 , d_1 was originally 4 then we use 1,5, and 6 for horizontal and 0,2, and 4 for vertical edges. Starting first with u_1 's edges, we assign 6 to h_1 , $\{1, 5\}$ to $\{h_2, h_3\}$, which are exchangeable to make v_1 respect the P_i . Then, assign 2 to d_3 , 4 to d_1 and 0 to d_2 , making

$$v_2 = [\circ c_1 \quad c_2 \mathbf{2} \quad \circ \mathbf{4} \quad \mathbf{0} c_3]$$

With u_2 's edges, assign 6 to h_3 and $\{5, 1\}$ to $\{h_1, h_2\}$ such that

$$v_2 = [\circ \mathbf{1} \quad \mathbf{5} \mathbf{2} \quad \circ \mathbf{4} \quad \mathbf{0} \mathbf{6}]$$

which respects the P_i . After this assignment, u_2 is either

$$u_2 = \begin{bmatrix} \circ c_1 \\ c_2 \mathbf{1} \\ c_3 \mathbf{5} \\ \circ \mathbf{6} \end{bmatrix} \quad \text{or} \quad u_2 = \begin{bmatrix} \circ c_1 \\ c_2 \mathbf{5} \\ c_3 \mathbf{1} \\ \circ \mathbf{6} \end{bmatrix}$$

If u_2 is in the first form, assign 0 to d_3 and $\{2, 4\}$ to $\{d_1, d_2\}$, which are exchangeable to make v_1 valid. If u_2 is in the second form, assign 4 to d_3 and $\{0, 2\}$ to $\{d_1, d_2\}$ arbitrarily.

Case 2. $v_1 \in F_1$ and $v_2 \in F_2$. In this case, we flip G vertically so that $u_1 \in F_1$ and $u_2 \in F_2$. Clearly u_1 and u_2 respect P'_i . We need to modify the coloring so that v_1 and v_2 respect P_i . (P_1 is already respected.) Since $u_1 \in F_1$ and $u_2 \in F_2$, we can apply any sequence of $UU(0, 2)$, $LU(4, 6)$, $UL(4, 6)$ and $LL(0, 2)$ while keeping P'_i respected by the u_j . As case 1 has been done, we can assume that v_1 and v_2 are not representatives of F_1 and $F_3 \cup F_4$. Consider 3 sub-cases left:

- (2a) $v_1 \in F_1$ and $v_2 \in F_2$. In this case, apply $UU(0, 2)$ and/or $UL(4, 6)$ if necessary to make $v_1 = [50 \quad \circ 2 \quad 14 \quad \circ 6]$. Apply $LU(4, 6)$ as needed so that 14 is one component of u_1 . Apply $LL(0, 2)$ if necessary so that v_2 is either $[10 \quad \circ 2 \quad 54 \quad \circ 6]$ or $[10 \quad \circ 2 \quad \circ 4 \quad 56]$. Next, as 14 is a component of u_1 , we can apply $LU(1, 4)$, so that v_2 is either $[40 \quad \circ 2 \quad 51 \quad \circ 6]$ or $[40 \quad \circ 2 \quad \circ 1 \quad 56]$. The first vector already respects P_i . The second vector respects P_i after we switch 2 and 6, so does v_1 . Hence, we apply $A(2, 6)$ if necessary to make both v_1 and v_2 respect P_i .
- (2b) $v_1 \in F_2$ and $v_2 \in F_1$. This is quite similar to the previous case. Apply $LL(0, 2)$ and/or $LU(4, 6)$ if necessary to make $v_2 = [50 \quad \circ 2 \quad 14 \quad \circ 6]$. Apply $UU(0, 2)$ as needed so that 50 is one component of u_1 . Apply $UL(4, 6)$ if necessary so that v_1 is either $[10 \quad \circ 2 \quad 54 \quad \circ 6]$ or $[\circ 0 \quad 12 \quad 54 \quad \circ 6]$. Next, as 50 is a component of u_1 , we can apply $UU(0, 5)$, so that v_1 is either $[15 \quad \circ 2 \quad 04 \quad \circ 6]$ or $[\circ 5 \quad 12 \quad 04 \quad \circ 6]$. The first vector already respects P_i . The second vector respects P_i after we switch 2 and 6, so does v_2 . Hence, we apply $A(2, 6)$ if necessary to make both v_1 and v_2 respect P_i .

(2c) v_1 and v_2 are representatives of F_2 and $F_3 \cup F_4$.

Suppose $v_1 \in F_3 \cup F_4$, so that $v_2 \in F_2$. Apply $LU(4, 6)$ and/or $LL(0, 2)$ as needed so that $v_2 = [10 \ 02 \ 54 \ 06]$. If $v_1 \in F_3$, then apply $UU(0, 2)$ and $UL(4, 6)$ to move v_1 to F_4 . Finally, we apply $A(0, 4)$, making v_1 and v_2 both respect P_i .

The situation when $v_1 \in F_2$ and $v_2 \in F_3 \cup F_4$ is done similarly.

Case 3. $v_1 \in F_2$ and $v_2 \in F_3 \cup F_4$. As $A(0, 2)$ and $A(4, 6)$ transform a vector from F_4 to F_3 , while keeping a vector from F_2 in F_2 , we only need to consider $v_1 \in F_2$ and $v_2 \in F_3$.

If $v_1 = [00 \ 12 \ 04 \ 56]$, then we first apply $A(0, 2)$ and $A(4, 6)$. This would make $v_1 = [10 \ 02 \ 54 \ 06]$, and move v_2 into F_4 . Secondly, $A(0, 4)$ makes both v_1 and v_2 respect the P_i .

If $v_1 = [10 \ 02 \ 04 \ 56]$ and $u_1 \in F_2$, then apply these in order: $LU(4, 6)$, $LL(0, 2)$, $UL(4, 6)$, and $A(0, 4)$. While if v_1 is the same but $u_2 \in F_2$, we do $UL(4, 6)$, and $A(0, 4)$.

If $v_1 = [00 \ 12 \ 54 \ 06]$ and $u_1 \in F_2$, we apply a sequence of transformations: $UU(0, 2)$, then $A(0, 2)$ and $A(4, 6)$, then $A(0, 4)$. While if v_1 is the same but $u_2 \in F_2$, we do $UU(0, 2)$, $LU(4, 6)$, $LL(0, 2)$ and lastly $A(0, 4)$.

Consequently, the cases left to be considered are:

- $v_1 = [10 \ 02 \ 54 \ 06]$, $v_2 \in F_3$, u_1 and u_2 are representatives of some F_i and F_j with $i \neq j$, $i \neq 1$ and $j \neq 1$.
- $v_1 = [10 \ 02 \ 04 \ 56]$, u_1 and u_2 are representatives of F_3 and F_4 .
- $v_1 = [00 \ 12 \ 54 \ 06]$, u_1 and u_2 are representatives of F_3 and F_4 .

We consider all these sub-cases in turn, using the technique “proof without words” introduced in the previous section.

(3a) $v_1 = [10 \ 02 \ 54 \ 06]$, $v_2 \in F_3$, $u_1 \in F_2$, and $u_2 \in F_3$. Figure 23 shows our solution to this case. The dashed edges are from G_{21} , the thickened edges are from G_{11} , and the

Figure 23: Case 3a of Lemma 3.8

rest of the edges are from G_{12} . As we are considering the case where $v_2 \in F_3$, $u_1 \in F_2$, and $u_2 \in F_3$, there are 6 degrees of flexibility for the 1 and 5 to move within either the first two components or the last two components of the associated CIVs. This corresponds precisely to moving the end points of the dashed edges, within the first two components or the last two components of their CIVs.

In reality, there are totally $2^6 = 64$ cases. In Figure 23, we consider only three cases where the end points of the dashed edges in the boxes are fixed. The doubly headed

arrows indicate that no matter if this end point is at one head or the other, this coloring is still good. Occasionally, we have to explain why when we move the end point of the dashed edge to the other head of the arrow, the coloring is still OK. The explanations are given right on the figure itself, and they are self-explaining.

The CIVs v_1 and v_2 have been put next to the colorings for the ease of referencing which edge gets which color, and checking if the coloring is a valid one. Note that we only color the edges not in G_{22} as usual.

- (3b) $v_1 = [10 \ 02 \ 54 \ 06]$, $v_2 \in F_3$ $u_1 \in F_3$, and $u_2 \in F_2$. Figure 24 shows our solution to this case.

Figure 24: Case 3b of Lemma 3.8

- (3c) $v_1 = [10 \ 02 \ 54 \ 06]$, $v_2 \in F_3$ $u_1 \in F_2$, and $u_2 \in F_4$. Figure 25 shows our solution to this case.

Figure 25: Case 3c of Lemma 3.8

- (3d) $v_1 = [10 \ 02 \ 54 \ 06]$, $v_2 \in F_3$ $u_1 \in F_4$, and $u_2 \in F_2$. This sub-case is a little special, as we have to consider 4 cases shown in Figure 26.

- (3e) $v_1 = [10 \ 02 \ 54 \ 06]$, $v_2 \in F_3$ $u_1 \in F_3$, and $u_2 \in F_4$. This case is even more special. There are 4 sub-cases to be considered as shown in Figure 27. In the last sub-case (see the last drawing of Figure 27), we have provided the coloring so that the v_j respect the P'_i and the u_j respect the P_i . Flipping G vertically would complete the proof. The vectors u_1

Figure 26: Case 3d of Lemma 3.8

and u_2 have been put on the right of the drawing, indicating that we are trying to flip G vertically.

- (3f) $v_1 = [10 \ 02 \ 54 \ 06]$, $v_2 \in F_3$ $u_1 \in F_4$, and $u_2 \in F_3$. Figure 28 shows our solution to this case. In the first and second drawings, we use the vertical flip of G .
- (3g) $v_1 = [10 \ 02 \ 04 \ 56]$, $v_2 \in F_3$ $u_1 \in F_3$, and $u_2 \in F_4$. Figure 29 shows our solution to this case.
- (3h) $v_1 = [10 \ 02 \ 04 \ 56]$, $v_2 \in F_3$ $u_1 \in F_4$, and $u_2 \in F_3$. The coloring shown in case (3f) is also valid here.
- (3i) $v_1 = [00 \ 12 \ 54 \ 06]$, $v_2 \in F_3$ $u_1 \in F_3$, and $u_2 \in F_4$. Figure 30 shows our solution to this case. We use the vertical flip of G in the second, third and fifth drawings.
- (3j) $v_1 = [00 \ 12 \ 54 \ 06]$, $v_2 \in F_3$ $u_1 \in F_3$, and $u_2 \in F_4$. The coloring in case (3e) is also valid here.

Case 4. $v_1 \in F_3$ and $v_2 \in F_4$. Now that all cases 1,2, and 3 have been proven, we can assume that u_1 and u_2 are also representatives of F_3 and F_4 . Otherwise, we flip G vertically and apply one of the previous cases. This case is surprisingly simple. There are 2 sub-cases depending on which of u_1 and u_2 comes from which of F_3 and F_4 . Figure 31 shows the solution to both the sub-cases.

6 Discussions

In this paper, we have verified that the 7-stage SE network for $n = 4$ is rearrangeable. This result and an extension of another formulation were used to show that $3n - 5$ SE stages are sufficient for the rearrangeability of the SE network with 2^n inputs and 2^n outputs.

Figure 27: Case 3e of Lemma 3.8

Figure 28: Case 3f of Lemma 3.8

Figure 29: Case 3g of Lemma 3.8

Figure 30: Case 3i of Lemma 3.8

Figure 31: Case 4 of Lemma 3.8

It was conjectured that $2n - 1$ SE stages are necessary and sufficient for the SE network to be rearrangeable. However, there has been very slow progress on proving the conjecture. Although the proof of the main theorem is tedious and fairly tricky, there are several tricks that are used very often. Some of the them were general enough to be put as lemmas and propositions. We were not able to generalize the others, which are interesting in their own right. The proof, in some sense, also “shows” why this conjecture is so difficult. There is no particular technique that could be used throughout, we need different tricks to solve different subcases. That is not to say there is no nice proof of the main theorem or the conjecture, we just were not able to see the “right” formulation, yet. In particular, any connection of our proof to the formulation of Linial and Tarsi should be helpful, beside the fact that the binary representation of our colors are precisely the row vectors of the matrix M to be constructed in Linial and Tarsi’s formulation.

We hope that our work, besides improving the bound, contributes to the effort of attacking this difficult problem. We believe that an algebraic formulation of the proof would yield better upper bound for $m(n)$.

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