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Abstract. We first show that if A is a self-inverse sign pattern matrix, then every principal submatrix of A that is not combinatorially singular (does not require singularity) is also a self-inverse sign pattern. Next we characterize the class of all n -by- n irreducible self-inverse sign pattern matrices. We then discuss reducible self-inverse patterns, assumed to be in Frobenius normal form, in which each irreducible diagonal block is a self-inverse sign pattern matrix. Finally we present an implicit form for determining the sign patterns of the off-diagonal blocks (unspecified block matrices) so that a reducible matrix is self-inverse.

0. Introduction. A matrix whose entries consist of the symbols $+$, $-$, and 0 is called a *sign pattern matrix*. For a real matrix B , by $\text{sgn } B$ we mean the sign pattern matrix in which each positive (respectively, negative, zero) entry is replaced by $+$ (respectively, $-$, 0). If $A = (a_{ij})$ is an n -by- n sign pattern matrix, then the *sign pattern class* of A is defined by

$$Q(A) = \{B \in M_n(R) \mid \text{sgn } B = A\}.$$

A sign pattern matrix A is said to be combinatorially singular if every $B \in Q(A)$ is singular.

We shall be interested in the cycles in a sign pattern matrix, since every real matrix associated with it has the same qualitative cycle structure. If $A = (a_{ij})$ is an n -by- n sign pattern matrix, then a product of the form $\gamma = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1}$, in which the index set $\{i_1, i_2, \dots, i_k\}$ consists of distinct indices is called a *simple cycle of length k* . A cycle is said to be *negative* (respectively, *positive*) if it contains an *odd* (respectively, *even*) number of negative entries and no entries equal to zero. In the remainder of this paper, when we say simple cycle, we mean a nonzero simple cycle.

A *matching of size k* in an n -by- n matrix $A = (a_{ij})$ corresponds to k entries in the matrix among whose collective initial indices there are no repetitions, and among whose collective terminal indices there are no repetitions. We call a matching *principal* if the set of initial indices is the same as the set of terminal indices. A principal matching in an n -by- n matrix is called a *complete* matching if the cardinality of the set of initial (terminal) indices is n . The product of entries in a principal matching of size k is either a simple k -cycle or a product of simple cycles whose total length is k and whose index sets are mutually disjoint. Henceforth

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when we use the term principal matching, we mean the product of entries in the matching. If γ is a principal matching, $\text{sgn } \gamma$ will refer to the sign that the indices of γ takes as a permutation.

Suppose P is a property a real matrix may or may not have. Then A is said to *require* P if *every* real matrix in $Q(A)$ has property P , or to *allow* P if *some* real matrix in $Q(A)$ has property P . Let A be an n -by- n given sign pattern matrix, and let P be the property “ B in $Q(A)$ implies B^{-1} exists and $\text{sgn } B^{-1} = A$.” We call such a sign pattern *self-inverse*. In particular, B and B^{-1} have the same zero pattern. The question of how to characterize the self-inverse sign patterns was raised in [J]. Our purpose here is to give a characterization.

Let A be an n -by- n sign pattern matrix, and let $A(\alpha, \beta)$ denote the submatrix of A obtained by deleting the rows in the index set α and the columns in the index set β . Let $A[\alpha, \beta]$ be the submatrix that lies in the rows of A indexed by α and the columns indexed by β . If $\alpha = \beta$, then $A[\alpha, \alpha]$, shortened to $A[\alpha]$, and $A(\alpha, \alpha)$, shortened to $A(\alpha)$, are principal submatrices of A . To further simplify notation, we shorten $A(\{i\}, \{j\})$ and $A(\{i\})$ to $A(i, j)$ and $A(i)$, respectively.

The *undirected graph* $G(A)$ of an n -by- n sign pattern matrix A is the undirected graph on n vertices $1, 2, \dots, n$ such that there is an undirected edge in $G(A)$ from i to j , denoted by the unordered pair $\{i, j\}$, if and only if $a_{ij} \neq 0$ or $a_{ji} \neq 0$. The *directed graph* $D(A)$ of A is the directed graph on n vertices $1, 2, \dots, n$, such that there is a directed edge in $D(A)$, denoted by (i, j) , if and only if $a_{ij} \neq 0$. The set of all vertices is called the *vertex set* V , and the set of all edges is called the *edge set* E . An *undirected path* in $G(A)$ (respectively, directed path in $D(A)$) from i to j is a sequence of undirected (respectively, directed) edges $\{i, i_1\}, \{i_1, i_2\}, \dots, \{i_{k-1}, j\}$ ($((i, i_1), (i_1, i_2), \dots, (i_{k-1}, j))$). Here the number k of edges is called the length of the undirected (respectively, directed) path. A directed path $(i, i_1), (i_1, i_2), \dots, (i_{k-1}, j)$ is called a *simple path* if the vertices $i, i_1, \dots, i_{k-1}, j$ are distinct.

We call a graph *bipartite* if it is possible to partition the vertex set V into two subsets V_1 and V_2 such that every element in the edge set E joins a vertex of V_1 to a vertex of V_2 or a vertex of V_2 to a vertex of V_1 . Recall that the lengths of the cycles in a bipartite graph are even.

We say that a sign pattern matrix $A = (a_{ij})$ is a *tree sign pattern*, (*t.s.p.*) matrix (see [JJ]) if A is combinatorially symmetric, that is, $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$; and its undirected graph is connected, but acyclic, that is, the graph is a tree.

If A is a self-inverse sign pattern, then A is sign nonsingular, that is, every $B \in Q(A)$ is nonsingular. Sign nonsingular matrices have been heavily studied, and it is well known that they have unambiguously signed determinants; that is, there is at least one nonzero term in the determinant, and all nonzero terms have the same sign.

1. Results. Let $\mathbf{A}$ be the class of all self-inverse sign pattern matrices. Since the equivalences in our first lemma are clear, it is stated without proof.

1.1 LEMMA. *The class $\mathbf{A}$ is closed under the following operations:*

- (i) *signature similarity (that is, similarity by a diagonal matrix, each of whose diagonal entries is nonzero)*
- (ii) *permutation similarity;*
- (iii) *negation; and*
- (iv) *transposition.*

1.2 LEMMA. *If $A \in \mathbf{A}$ is a matrix of order n , α is a subset of $\{1, 2, \dots, n\}$, and $A[\alpha]$ is not combinatorially singular, then $A[\alpha]$ and $A(\alpha)$ are sign nonsingular.*

Proof. Let $A \in \mathbf{A}$ be of order n and assume $A[\alpha]$ is not combinatorially singular. Then there is some $B \in Q(A)$ such that $B[\alpha]$ is nonsingular. Consequently $\det B[\alpha] \neq 0$, and since $A \in \mathbf{A}$, $\det B \neq 0$. Use of Jacobi's identity gives

$$\det B^{-1}(\alpha) = \frac{\det B[\alpha]}{\det B} \neq 0,$$

that is, $B^{-1}(\alpha)$ is nonsingular. Since the sign pattern A is self-inverse, $\text{sgn}(B^{-1}(\alpha)) = A(\alpha)$ and it follows that $A(\alpha)$ is also not combinatorially singular. The sign nonsingularity of A now implies that both $A[\alpha]$ and $A(\alpha)$ are sign nonsingular. $\square$

1.3 LEMMA. *If $A \in \mathbf{A}$ is a matrix of order n and $\alpha \subseteq \{1, \dots, n\}$ is such that $A[\alpha]$ is not combinatorially singular, then $A[\alpha]$ and $A(\alpha)$ are in class $\mathbf{A}$.*

Proof. Let $A = (a_{ij}) \in \mathbf{A}$, and $\alpha \subseteq \{1, 2, \dots, n\}$. Assume $A[\alpha]$ is not combinatorially singular. From lemma 1.2 we know $A[\alpha]$ and $A(\alpha)$ are sign nonsingular. For contradiction, assume $A[\alpha] \notin \mathbf{A}$, and let $C \in Q(A[\alpha])$ be such that $C^{-1} \notin Q(A[\alpha])$. Let D be a fixed matrix in $Q(A(\alpha))$, and let $B = C \oplus D$. For $\epsilon > 0$, define $B_\epsilon \in Q(A)$ as follows: $B_\epsilon[\alpha] = C$; $B_\epsilon(\alpha) = D$; and all other entries of B_ϵ are chosen to be 0, respectively, $\pm \epsilon$, if a_{ij} is 0, respectively, $\pm$. For sufficiently small ϵ , we regard B_ϵ as a perturbation of B . Since

$$\left(\lim_{\epsilon \rightarrow 0} B_\epsilon \right)^{-1} = (B_\epsilon[\alpha])^{-1} \oplus (B_\epsilon(\alpha))^{-1} = C^{-1} \oplus D^{-1},$$

we have, for sufficiently small ϵ , that $A[\alpha] = \text{sgn}(B_\epsilon^{-1}[\alpha]) = \text{sgn} C^{-1}$, which contradicts the assumption that $C^{-1} \notin Q(A[\alpha])$. $\square$

If a sign pattern matrix $A = (a_{ij})$ contains a simple k -cycle, say, $\gamma = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1}$, then the product $\bar{\gamma} = a_{i_1 i_k} a_{i_k i_{k-1}} \dots a_{i_2 i_1}$ is called the *complementary cycle* of γ . If A contains a simple k -cycle, say, the γ given above, and if $\alpha = \{i_1, i_2, \dots, i_k\}$, then $A[\alpha]$ is not combinatorially singular. From lemma 1.3, we know $A[\alpha]$ is a k -by- k matrix in class $\mathbf{A}$ that also contains a simple k -cycle. The nature of lemmas 1.4, 1.7 and 1.8 allows us to use this fact, and to assume, without loss of generality, that if $A \in \mathbf{A}$ contains a simple k -cycle, then A is a k -by- k matrix.

1.4 LEMMA. *If $A \in \mathbf{A}$ contains a simple k -cycle ($k \geq 2$), then it contains the complementary k -cycle.*

Proof. Without loss of generality, suppose $A = (a_{ij}) \in \mathbf{A}$ is a k -by- k sign pattern matrix that contains a simple k -cycle γ . By the equivalence in lemma 1.1,

we may take $\gamma = a_{12}a_{23} \dots a_{k1}$. For any $0 \neq a_{ij}$ lying on γ , the complementary block $A(i, j)$ contains a complete matching, namely, $a_{12} \dots a_{i-1, j-1} a_{i+1, j+1} \dots a_{k1}$, and thus, is not combinatorially singular. Further, if $A(i, j)$ is not sign nonsingular, then there is some $B_1 \in Q(A)$ and $B_2 \in Q(A)$ such that $\det B_1(i, j) = 0$ and $\det B_2(i, j) \neq 0$. Consequently

$$(B_1)_{ji}^{-1} = \frac{(-1)^{i+j} \det B_1(i, j)}{\det B_1} = 0$$

and

$$(B_2)_{ji}^{-1} = \frac{(-1)^{i+j} \det B_2(i, j)}{\det B_2} \neq 0,$$

which contradicts the assumption that $A \in \mathbf{A}$. Thus $A(i, j)$ is sign nonsingular, and since $A \in \mathbf{A}$, this implies that $a_{ji} \neq 0$ for any a_{ij} lying on the cycle γ . We conclude that the complementary cycle $\bar{\gamma}$ is nonzero. $\square$

1.5 LEMMA. *If $A = (a_{ij}) \in \mathbf{A}$ is a matrix of order n , then $a_{ij}a_{ji} \geq 0$, for all i and j in $\{1, 2, \dots, n\}$.*

Proof. Suppose $A \in \mathbf{A}$, and assume $a_{ij} \neq 0$. Then if we expand about the i^{th} row of A to obtain $\det A$, we know $a_{ij}(-1)^{i+j} \det A(i, j)$ is weakly of the same sign as $\det A$. Consequently

$$(1.5a) \quad a_{ji} = \text{sgn}(A^{-1})_{ji} = \frac{(-1)^{i+j} \det A(i, j)}{\det A}$$

implies that

$$a_{ij}a_{ji} = \frac{a_{ij}(-1)^{i+j} \det A(i, j)}{\det A} \geq 0$$

for all i and j in $\{1, 2, \dots, n\}$. $\square$

1.6 LEMMA. *If $A \in \mathbf{A}$ is a matrix of order n , $\alpha \subseteq \{1, 2, \dots, n\}$, and $A[\alpha]$ is irreducible, then $A[\alpha]$ is sign symmetric.*

Proof. Let $A = (a_{ij}) \in \mathbf{A}$, and $\alpha \subseteq \{1, 2, \dots, n\}$. Assume $A[\alpha]$ is irreducible, and $0 \neq a_{ij}$ is an entry of $A[\alpha]$, ($i \neq j$). Then $A[\alpha]$ irreducible implies there is a simple path in the directed graph of $A[\alpha]$ from j to i , say $j \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \rightarrow i$, whose vertices are in the index set α . Thus $a_{ij}a_{ji_1} \dots a_{i_k i}$ is a nonzero simple cycle in A containing a_{ij} . Now from lemma 1.4, we know $a_{ji} \neq 0$; and from lemma 1.5 we know that $a_{ij}a_{ji} \geq 0$. Thus $a_{ij}a_{ji} > 0$, and it follows that $a_{ij} = a_{ji}$, that is, $A[\alpha]$ is sign symmetric. $\square$

1.7 LEMMA. *If $A \in \mathbf{A}$, then A contains no simple odd cycles of length $\ell \geq 3$.*

Proof. Let $A = (a_{ij}) \in \mathbf{A}$.

Case (i). Let $\ell = 3$. Using equivalence 1.1 (ii), we may assume, without loss of generality, that A contains the simple 3-cycle $\gamma = a_{12}a_{23}a_{31}$. Further we may assume that $\gamma = (+)(+)(*)$, where $*$ is $+$, respectively, $-$, depending upon whether

γ is a positive, respectively, negative 3-cycle. Let $\alpha = \{1, 2, 3\}$. From lemmas 1.4 and 1.6 we know $A[\alpha]$ contains $\bar{\gamma}$, and $\bar{\gamma} = a_{13}a_{32}a_{21} = (*) (+) (+)$. Since $A[\alpha]$ is not combinatorially singular, lemma 1.2 implies that $A[\alpha]$ and $A(\alpha)$ are sign nonsingular. We want to determine if a_{11} is zero or nonzero. To this end, consider the complementary block of the principal submatrix (a_{11}) , namely, $A(1)$. Since $A(\alpha)$ is an $(n-3)$ -by- $(n-3)$ sign nonsingular matrix, it contains a complete matching, say, δ , which is also a principal matching in $A(1)$. Further $a_{23}a_{32} = \beta$ is a simple 2-cycle contained in A as well as the principal submatrix $A(1)$. Consequently $\beta \cdot \delta$ is a complete matching in $A(1)$, and it follows that $A(1)$ is not combinatorially singular. From lemma 1.2, we know the complementary submatrix (a_{11}) is sign nonsingular, that is, $a_{11} \neq 0$. Similarly it can be shown that $a_{22} \neq 0$ and $a_{33} \neq 0$. Thus $A[\alpha]$ is an entrywise nonzero matrix, and it can easily be checked that $A[\alpha]$ cannot be sign nonsingular. However, from lemma 1.2, we know $A[\alpha]$ is sign nonsingular, and we have a contradiction. We conclude that A contains no simple 3-cycles.

Case (ii). Let $A = (a_{ij})$ be a k -by- k (k odd) sign pattern matrix in $\mathbf{A}$, $k \geq 5$. Without loss of generality, assume A contains the simple k -cycle $\gamma = a_{12}a_{23} \dots a_{k1}$. By lemma 1.4, A contains the complementary cycle $\bar{\gamma}$. The submatrix $C = A(2, k)$ is not combinatorially singular since it contains a complete matching, namely,

$$\begin{aligned} & c_{12}c_{24}c_{46} \dots c_{k-3,k-1}c_{k-1,1}c_{33}c_{55} \dots c_{k-2,k-2} = \\ & a_{12}a_{34}a_{56} \dots a_{k-2,k-1}a_{k1}a_{43}a_{65} \dots a_{k-1,k-2}. \end{aligned}$$

Thus there is a matrix B in $Q(A)$ such that $\det B(2, k) \neq 0$, and

$$(B^{-1})_{k2} = \frac{-\det B(2, k)}{\det B}.$$

Since $A \in \mathbf{A}$, we know $\text{sgn}(B^{-1})_{k2} = \text{sgn}(b_{k2}) = a_{k2} \neq 0$. Thus A contains the simple 3 cycle $a_{1k}a_{k2}a_{21}$, which contradicts the assumption that $A \in \mathbf{A}$. $\square$

1.8 LEMMA. *If $A \in \mathbf{A}$, then A contains no simple even cycles of length $\ell \geq 6$.*

Proof.

Case (i). Let $A = (a_{ij})$ be a 6-by-6 matrix in $\mathbf{A}$. Without loss of generality, assume A contains the simple 6-cycle $\gamma = a_{12}a_{23} \dots a_{61} = (+)(+)\dots(+)(*)$. By lemma 1.4, A contains the complementary cycle $\bar{\gamma} = a_{16}a_{65} \dots a_{21} = (*) (+) \dots (+)$. Since $\text{sgn}(a_{12}a_{21})\text{sgn}(a_{34}a_{43})\text{sgn}(a_{56}a_{65}) = -$ is a term in $\det A$, and A is sign nonsingular, it follows that $*$ is $+$. We first show that $a_{25} = -$. To this end, consider the complementary block $C = A(2, 5)$, which contains a complete matching, namely, $a_{16}a_{61}a_{32}a_{43}a_{54}$. Then $a_{16}a_{61}a_{32}a_{43}a_{54} = c_{15}c_{51}c_{22}c_{33}c_{44}$, and $-(c_{15}c_{51})c_{22}c_{33}c_{44} = -$ is a term in $\det A(2, 5)$. Thus $a_{52} = (\mathbf{A}^{-1})_{52} = \frac{(-1)^7 \det \mathbf{A}(2, 5)}{\det A} = -$. Further $a_{25} = a_{52} = -$, since A is sign symmetric. Similar arguments show that $a_{14} = a_{41} = -$ and $a_{36} = a_{63} = -$. Consequently $\text{sgn}(a_{36}a_{63})\text{sgn}(a_{25}a_{54}a_{41}a_{12}) = +$ is a term in $\det A$, which contradicts the sign nonsingularity of A . We conclude that A contains no simple 6-cycles.

Case (ii). Let $A = (a_{ij})$ be a k -by- k (k even) sign pattern matrix in $\mathbf{A}$, and assume $k \geq 8$. Without loss of generality, assume A contains the simple k -cycle

$\gamma = a_{12}a_{23} \dots a_{k1}$. By lemma 1.4, A contains the complementary cycle $\bar{\gamma}$. Further, using induction, assume A has no simple m -cycles for any even integer m such that $6 \leq m \leq k-2$. If $m = k$, then $C = A(2, 5)$ is not combinatorially singular since it contains a complete matching, namely, $c_{12}c_{24}c_{45}c_{56} \dots c_{k-2,k-1}c_{k-1,1}c_{33} = a_{12}a_{34}a_{56}a_{67} \dots a_{k-1,k}a_{k1}a_{43}$. Thus there is a matrix B in $Q(A)$ such that $\det B(2, 5) \neq 0$, and by the same argument used in the proof of lemma 1.7, it follows that $a_{52} \neq 0$. However this implies that A contains the simple $(k-2)$ -cycle $a_{21}a_{1k}a_{k,k-1} \dots a_{65}a_{52}$, which contradicts the induction hypothesis. We conclude that A contains no simple even cycles of length $\ell \geq 6$. $\square$

For any $A \in \mathbf{A}$, using lemmas 1.7 and 1.8, we conclude that the only possible simple cycles in A are 1-cycles, 2-cycles and/or 4-cycles. Consequently the graphs of matrices in class $\mathbf{A}$ are bipartite graphs, in addition to possible loops at the vertices. The bipartite nature of the graph of a matrix in class $\mathbf{A}$ is an important ingredient needed to prove the major result in this section.

1.9 LEMMA. *If A is an irreducible n -by- n sign pattern matrix in $\mathbf{A}$, and A has no simple 4 cycles, then $n = 1$ or $n = 2$.*

Proof. Suppose A is an irreducible sign pattern matrix in $\mathbf{A}$, and has no simple 4 cycles. By lemmas 1.7 and 1.8, it follows that A has no simple cycles of length $\ell \geq 3$. From lemma 1.6, we know that A is sign symmetric, and, therefore is a t.s.p. matrix. For contradiction, assume $n \geq 3$.

Case (i). Suppose the bipartite graph of A (relabelled by permutation similarity, if necessary) contains a vertex that is adjacent to two or more vertices, one of which is adjacent to a second vertex. Without loss of generality, assume the first vertex is adjacent to q ($q \geq 2$) vertices, say, $2, 3, \dots, q$, and the q^{th} vertex is adjacent to the $(q+1)^{\text{st}}$ vertex. Then part of the graph of A is

For $B \in Q(A)$, let $C = B^{-1}$. Then the $(1, q+1)$ off-diagonal entry in the product BC , is given by $(BC)_{1,q+1} = 0 + \dots + 0 + b_{1q}c_{q,q+1} + 0 + \dots + 0 = b_{1q}c_{q,q+1} \neq 0$, which contradicts the assumption that A is in $\mathbf{A}$.

Case (ii). Suppose the bipartite graph of A contains a vertex that is adjacent to two or more vertices, none of which is adjacent to a second vertex. Then part of the graph of A is given by

Consequently for any $B \in Q(A)$, with $C = B^{-1}$, we have $(BC)_{23} = \sum_{k=1}^q b_{2k}c_{k3} = b_{21}c_{13} \neq 0$, which contradicts the assumption that A is in class $\mathbf{A}$. $\square$

1.10 LEMMA. *If A is an irreducible n -by- n sign pattern matrix in $\mathbf{A}$, and A contains a simple 4-cycle, then $n = 4$.*

Proof. Without loss of generality, suppose A is an n -by- n irreducible sign pattern matrix in $\mathbf{A}$ that contains the simple 4-cycle $\gamma = a_{12}a_{23}a_{34}a_{41} = (+)(+)(+)(*_1)$. From lemma 1.4, we know A contains the complementary cycle $\bar{\gamma}$. Let $V = \{1, 2, 3, 4\}$, and assume $n \geq 5$. Then there is some $j \leq n$ such that $j \notin V$. Further A irreducible implies there is a simple path from i to j and from j to i , for each $i \in V$. It suffices to choose any vertex in V , since the arguments for all vertices are the same. Henceforth let $i = 1$.

Case (i). Suppose there is a path of length 1 from vertex 1 to j , say, $a_{1j} = *_2 \neq 0$. Since A is irreducible, we know from lemma 1.6, that $a_{j1} = *_2$. Let $\alpha = \{1, 2, 3, j\}$. Then $A[\alpha]$ is not combinatorially singular, and, thus, is in $\mathbf{A}$. Let $C = A[\alpha - \{3\}, \alpha - \{j\}]$. Then C contains a complete matching, namely, $c_{12}c_{23}c_{31} = a_{12}a_{23}a_{j1}$ which implies that $a_{3j} \neq 0$. Now $a_{3j} \neq 0$ implies that $\gamma_1 = a_{j1}a_{12}a_{23}a_{3j} \neq 0$ in $A[\alpha]$. Since $\gamma_2 = a_{23}a_{32}a_{j1}a_{1j} \neq 0$ in $A[\alpha]$, and $A[\alpha]$ is sign nonsingular, it follows that $(\text{sgn } \gamma_1)\gamma_1 = (\text{sgn } \gamma_2)\gamma_2$. Consequently $a_{3j} = -a_{j1}$. Next let $\alpha' = \{1, 3, 4, j\}$. Then $A[\alpha']$ is not combinatorially singular, and, thus, is in $\mathbf{A}$. Further $\gamma_3 = a_{14}a_{43}a_{3j}a_{j1} = -*_1*_2^2$ and $\gamma_4 = a_{1j}a_{j1}a_{34}a_{43} = *_2^2$ are complete matchings in $A[\alpha']$. By the sign nonsingularity of $A[\alpha']$, we conclude that $(\text{sgn } \gamma_3)\gamma_3 = (\text{sgn } \gamma_4)\gamma_4$, that is, $*_1*_2^2 = *_2^2$, or $*_1 = +$. Now, $A[V]$ is not combinatorially singular, and, hence, is sign nonsingular. However $(\text{sgn } \gamma)\gamma = -$, and if $\gamma' = a_{12}a_{21}a_{34}a_{43}$, then $(\text{sgn } \gamma')\gamma' = +$, which contradicts the sign nonsingularity of $A[V]$.

Case (ii). Assume there is a path of even length $p \geq 2$ from vertex 1, to vertex j , $a_{1k_1}a_{k_1k_2} \dots a_{k_{p-1}j}$. Assume $k_i \notin V$, for all $k_i = k_1, k_2, \dots, k_{p-1}$. Since A is irreducible, we know from lemma 1.6 that $a_{mi} \neq 0$ for each a_{im} in the path from 1 to j . Let $\alpha = \{1, 2, k_1, k_2, \dots, k_{p-1}, j\}$. Then $A[\alpha]$ is not combinatorially singular, and, thus, is in $\mathbf{A}$. Let $C = A[\alpha - \{2\}, \alpha - \{j\}]$. Then C contains a complete matching, namely, $c_{12}c_{21}c_{33} \dots c_{p+1,p+1} = a_{12}a_{k_1,1}a_{k_2k_1} \dots a_{j,k_{p-1}}$, which implies that $a_{j2} \neq 0$. If $p = 2$, then A contains the 6-cycle $a_{1k_1}a_{k_1j}a_{j2}a_{23}a_{34}a_{41}$; if $p > 2$, then A contains

an even cycle of length $(p+2) \geq 6$, namely, $a_{1k_1}a_{k_1k_2}\dots a_{k_{p-1},j}a_{j2}a_{21}$. In both cases, we contradict lemma 1.8.

Case (iii). Suppose there is a path of odd length $p \geq 3$, from vertex 1 to j , say, $a_{1k_1}a_{k_1k_2}\dots a_{k_{p-1},j}$. Assume $k_i \notin V$, for all $k_i = k_1, k_2, \dots, k_{p-1}$. From lemma 1.6, it follows that $a_{mi} \neq 0$ for each a_{im} in the path 1 to j . Let $\alpha = \{1, 2, 3, k_1, \dots, k_{p-1}, j\}$. Then $A[\alpha]$ is not combinatorially singular, and is, therefore, in $\mathbf{A}$. Let $C = A[\alpha - \{3\}, \alpha - \{j\}]$. Then C contains the complete principal matching $c_{12}c_{23}c_{31}c_{44}\dots c_{p+2,p+2} = a_{12}a_{23}a_{k_11}a_{k_2k_1}\dots a_{jk_{p-1}}$, which implies that $a_{j3} \neq 0$. Consequently A contains the even cycle $a_{1k_1}a_{k_1k_2}\dots a_{k_{p-1},j}a_{j3}a_{32}a_{21}$, whose length $\ell \geq 6$, contradicting, lemma 1.8. From cases i – iii, it follows that there is no path from vertex 1 to vertex j , which contradicts the assumption that A is irreducible. We conclude that $n = 4$. $\square$

1.11 THEOREM. *If A is an irreducible n -by- n sign pattern matrix in $\mathbf{A}$, then up to equivalence (as stated in lemma 1.1), A must be one of the following:*

$$(+); \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix}; \begin{pmatrix} + & + \\ + & - \end{pmatrix} \text{ or } \begin{pmatrix} 0 & + & 0 & - \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ - & 0 & + & 0 \end{pmatrix}.$$

Proof. Suppose A is an n -by- n irreducible sign pattern matrix in $\mathbf{A}$. From lemmas 1.9 and 1.10, we know $n = 1$, $n = 2$ or $n = 4$. If $n = 1$, then it is clear that A must be nonzero. If $n = 2$, then up to equivalence, the only sign symmetric sign pattern class (meeting the requirement of lemma 1.2) not listed in the theorem is

$$\begin{pmatrix} + & + \\ + & + \end{pmatrix}.$$

However this matrix is not sign nonsingular, and, consequently, is not in $\mathbf{A}$. Simple algebraic calculations using formula 1.5a show that the two 2-by-2 t.s.p. matrices given in the theorem are in $\mathbf{A}$. The only other 4-by-4 matrices, up to equivalences, that contain no 3-cycles, are irreducible and sign symmetric are represented by the following

$$\hat{A} = \begin{pmatrix} \square_1 & *_1 & 0 & + \\ *_1 & \square_2 & *_2 & 0 \\ 0 & *_2 & \square_3 & *_3 \\ + & 0 & *_3 & \square_4 \end{pmatrix}$$

where the $\square$'s and $*$'s are entries in $\{+, -, 0\}$. If $*_i = +$, for all $i = 1, 2, 3$, then the sign pattern of $\hat{A}$ does not have a signed determinant, therefore, is not in $\mathbf{A}$. If $*_1 = *_2 = -$ and $*_3 = +$, or if $*_1 = +$ and $*_2 = *_3 = -$, then the determinant of $\hat{A}$ is not signed, and, therefore, not in $\mathbf{A}$. If $*_1 = -$, for all $i = 1, 2, 3$, and if $\square_1 \neq 0$, formula 1.5a implies $\square_2 = -\square_1$ or $\square_4 = -\square_1$. In either case, $\det \hat{A}$ is not signed. Similar arguments show that if any diagonal entry is nonzero, then $\det \hat{A}$ is not signed. We conclude that $\hat{A}$ has a zero diagonal and is the negation of the 4-by-4 matrix given in the theorem.

Now suppose the sign pattern of $\hat{A}$ contains exactly one $*_i = -$, for some $i \in \{1, 2, 3\}$. For example, let $*_1 = -$ and let

$$\hat{A} = \begin{pmatrix} \square_1 & - & 0 & + \\ - & \square_2 & + & 0 \\ 0 & + & \square_3 & + \\ + & 0 & + & \square_4 \end{pmatrix}.$$

First we note that if any diagonal entry is nonzero, say, $\square_1 \neq 0$, then formula 1.5a implies that $\square_2 = -\square_1$ or $\square_4 = -\square_1$. In either case, $\det \hat{A}$ is not signed, and we conclude $\hat{A}$ has a zero diagonal. However the resulting $\hat{A}$ matrix is equivalent by signature similarity and permutation similarity to the 4-by-4 matrix given in the theorem. Since similar arguments hold if $*_2 = -$ or $*_3 = -$, we omit these cases. Thus the only sign patterns for $\hat{A}$ that are in $\mathbf{A}$, are equivalent to the matrix A in the theorem. We conclude the matrices listed in the theorem, and their equivalences as stated in lemma 1.1, are the only irreducible matrices in $\mathbf{A}$. $\square$

2. Reducible sign patterns in $\mathbf{A}$. Recall that a reducible matrix is permutation similar to a matrix in Frobenius normal form. If A is in $\mathbf{A}$, then from lemma 1.1, we know every permutation similarity of A is in $\mathbf{A}$. Consequently we may assume, without loss of generality, that a reducible matrix in $\mathbf{A}$ is in Frobenius normal form. Further it is clear that if $A = (A_{ij})$ is in $\mathbf{A}$, then each irreducible block A_{ii} of A is in class $\mathbf{A}$.

For reducible matrices in class $\mathbf{A}$, the key question is: What may the off-diagonal blocks look like given the irreducible diagonal blocks, which must be as in 1.11? To answer this question, is to find all possible solutions to a particular qualitative matrix completion problem. We wish to find the sign patterns of the off-diagonal blocks (unspecified block matrices) so that A is in class $\mathbf{A}$. Clearly, any of the matrices given in theorem 1.11 (up to the equivalences allowed by lemma 1.1) can occur as direct summands. A more interesting question occurs when the off-diagonal blocks are nonzero.

We begin by determining the possible off-diagonal blocks lying on the first superdiagonal, that is, we determine the possible sign patterns of the blocks $A_{i,i+1}$. Because of the way in which block triangular matrices multiply, requirements on these off-diagonal blocks depend only upon A_{ii} and $A_{i+1,i+1}$, and we may assume, then, that A has the form

$$(2.0) \quad \begin{pmatrix} A_{ii} & A_{i,i+1} \\ 0 & A_{i+1,i+1} \end{pmatrix}.$$

2.1 THEOREM. *Let A be a reducible sign pattern matrix in $\mathbf{A}$ of the form (2.0). Then, for each irreducible pair A_{ii} and $A_{i+1,i+1}$ (up to negation) respectively, $A_{i,i+1}$*

must be of the following form:

$$\begin{array}{lll}
& A_{ii} & A_{i+1,i+1} & A_{i,i+1} \\
\text{i)} & (+) & (-) & (*) \\
\text{ii)} & \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} & (-) & \begin{pmatrix} * \\ * \end{pmatrix} \\
\text{iii)} & (-) & \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} & (* *) \\
\text{iv)} & \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} & \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} & \begin{pmatrix} *_1 & *_2 \\ -*_2 & -*_1 \end{pmatrix} \\
\text{v)} & \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} & \begin{pmatrix} 0 & - \\ - & 0 \end{pmatrix} & \begin{pmatrix} *_1 & *_2 \\ *_2 & *_1 \end{pmatrix}
\end{array}$$

where $*$, $*_1$, and $*_2$ are any entries in $\{+, -, 0\}$. All other irreducible A_{ii} and $A_{i+1,i+1}$ block combinations preclude A from being in $\mathbf{A}$, unless $A_{i,i+1} = 0$. In particular,

$$\begin{pmatrix} + & + \\ + & - \end{pmatrix} \text{ and } \begin{pmatrix} 0 & + & 0 & - \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ - & 0 & + & 0 \end{pmatrix}$$

can occur as diagonal blocks only as direct summands.

Proof. Without loss of generality, and, to simplify notation, let $A_{ii} = A_{11}$, $A_{i+1,i+1} = A_{22}$, and $A_{i,i+1} = X$. Take $B \in Q(A)$, and let $C = B^{-1}$. Partition B and C conformably to A as follows:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}.$$

Since $BC = I$, it follows that

$$B_{11}C_{12} + B_{12}C_{22} = 0;$$

or

$$C_{12} = -B_{11}^{-1}B_{12}C_{22};$$

or

$$(2.1a) \quad C_{12} = -C_{11}B_{12}C_{12}.$$

Since (2.1a) holds for all $B \in Q(A)$, we get

$$(2.1b) \quad \text{sgn } X = -\text{sgn } A_{11}XA_{22},$$

which must hold *unambiguously*.

Conversely, suppose X satisfies (2.1b) unambiguously. Choose $C_{11} \in Q(A_{11})$, $C_{22} \in Q(A_{22})$, $B_{12} \in Q(X)$, and let $C_{12} = -C_{11}B_{12}C_{22}$. Then $C_{11}^{-1}C_{12} + B_{12}C_{22} = 0$, so that $X = A_{i,i+1}$.

Consequently, (2.1b) becomes the operative equation for determining $X = A_{i,i+1}$. The proof of the theorem then becomes routine. For example, with $A_{11} = \begin{pmatrix} + & + \\ + & - \end{pmatrix}$, each column of X must have a zero entry since $A_{11}X$ is unambiguous, say, $X = \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}$. Then $A_{11}X = \begin{pmatrix} + & - \\ + & + \end{pmatrix}$. However, with any allowable choice for A_{22} , $\text{sgn } X \neq -\text{sgn } A_{11}X A_{22}$, unambiguously, unless $X = 0$.

With $A_{11} = \begin{pmatrix} 0 & + & 0 & - \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ - & 0 & + & 0 \end{pmatrix}$, and using the second and fourth rows of A_{11} ,

either the second or fourth entry in any column of X must be zero. Similarly, the first or third entry in any column of X must be zero, say, $\begin{pmatrix} *1 \\ *2 \\ 0 \\ 0 \end{pmatrix}$ is a column of X .

Then $\begin{pmatrix} *2 \\ *1 \\ *2 \\ -*1 \end{pmatrix}$ is a column of $A_{11}X$; and, again, (2.1b) will not hold unless $X = 0$.

We omit the details for other allowable choices of an arbitrary column in X since the arguments are similar to the one given above.

Now let $A_{11} = (+)$ and $A_{22} = \begin{pmatrix} 0 & - \\ - & 0 \end{pmatrix}$. Then with $X = (*1 \ *2)$, we have $\text{sgn } A_{11}X A_{22} = (- \ *2 \ -*1)$. In order for $\text{sgn } X = -\text{sgn } A_{11}X A_{22}$ to hold, $*1 = *2$, that is, $X = (* \ *)$. We omit the details of the remaining cases since the arguments are similar to the one above. $\square$

Using (2.1b) and induction, one may prove our main theorem of this section. In Theorem 2.2, Q_{ij} denotes the set of increasing sequences from i to j . If $p = i, k_2, \dots, k_{t-1}, j$ is such a sequence, then $|p| = t$. For simplicity, we present the solution only in an implicit form, since a description as explicit as theorem 2.1 would be prohibitive. Further the implicit form of the solution makes it clear how to check a given example.

2.2 THEOREM. *Let A be a reducible sign pattern matrix in $\mathbf{A}$, and assume A is in Frobenius normal form. Then, up to equivalence, the irreducible diagonal blocks are as given in Theorem 1.11, the first superdiagonal blocks are as indicated in Theorem 2.1; and, in general, for all $i < j$,*

$$A_{ij} = \sum_{p \in Q_{ij}} (-1)^{|p|} A_{ii} A_{ik_2} A_{k_2 k_2} A_{k_2 k_3} \dots A_{k_{t-1}, j} A_{jj}$$

with the sum on the right unambiguous.

Proof. Sketching part of the proof, we take $B = (B_{ij}) \in Q(A)$, and let $C = (C_{ij}) = B^{-1}$, where B and C are partitioned conformably to A . Then for $i < j$, we have

$$(2.1c) \quad C_{ij} = -C_{ii}(B_{i,i+1}C_{i+1,j} + \dots + B_{i,j-1}C_{j-1,j} + B_{ij}C_{jj}).$$

Using induction, we have $C_{i+1,j}, \dots, C_{j-1,j}$, each expressed as an appropriate sum. Substitute these sums into (2.1c) and simplify. Since (2.1c) holds for all $B \in Q(A)$, we get the restrictions on the signs in A_{ij} . $\square$

It should be emphasized that each term in the sum for A_{ij} must be unambiguous. Furthermore all conditions of theorem 2.2 must hold. In particular, for *all* $i < j$, A_{ij} must unambiguously be the above sum. A subtle consequence of this is that there may be completions for certain blocks lying on, say, the k^{th} superdiagonal that lead to ambiguities for blocks lying on a higher order superdiagonal. Clearly such completions preclude the matrix from being in class **A**. We illustrate this phenomenon with the following example.

2.3 Example. Let A be the partial block matrix

$$\left(\begin{array}{cc|cc|cccc|cc} 0 & + & - & & 0 & 0 & 0 & 0 & & A_{14} \\ + & 0 & - & & 0 & 0 & 0 & 0 & & \\ \hline & & - & & 0 & 0 & 0 & 0 & - & - \\ \hline & & & & 0 & + & 0 & - & 0 & 0 \\ & & & & + & 0 & + & 0 & 0 & 0 \\ & & & & 0 & + & 0 & + & 0 & 0 \\ & & & & - & 0 & + & 0 & 0 & 0 \\ \hline & & & & & & & & 0 & + \\ & & & & & & & & + & 0 \end{array} \right)$$

Although the first and second superdiagonals were completed using theorems 2.1 and 2.2, there is no completion for A so that A is in **A**. However if $A_{12} = 0$ or $A_{24} = 0$, then

$$A_{14} = \begin{pmatrix} *_1 & *_2 \\ -*_2 & -*_1 \end{pmatrix},$$

where $*_1, *_2 \in \{0, +, -\}$, completes A so that A is in class **A**.

In [BF] the authors consider the graph whose vertices are sign pattern matrices, and whose edges connect those that are possible sign patterns of a matrix and its inverse. Consequently, self-inverse sign patterns we have characterized are the isolated vertices of the graph considered in [BF].

REFERENCES

- [BMQ] BASSETT, L., JOHN MAYBEE AND JAMES QUIRK, *Qualitative Economics and the Scope of the Correspondence Principle*, *Econometrica* 35 (1967), pp. 544–563.
- [BF] BERGER, M.A. AND A. FELZENBAUM, *Sign Patterns of Matrices and Their Inverses*, *Linear Algebra and Its Applications*, 86 (1987), pp. 161–177.
- [JJ] JEFFRIES, CLARK AND C.R. JOHNSON, *Some Sign Patterns Which Preclude Matrix Stability*, *SIAM J. on Matrix Analysis and Applications*, 9 (1988), pp. 19–25.
- [J] JOHNSON, C.R., *Sign Patterns of Inverse Nonnegative Matrices*, *Linear Algebra and Its Applications*, 24 (1979), pp. 75–83.

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